DISCRETE ANALOGUES IN HARMONIC ANALYSIS

LILLIAN PIERCE AND ELIAS STEIN,
TRANSCRIBED BY IAN TOBASCO

ABSTRACT. These notes were taken in the second week of the 2011 Study Analysis and Geometry summer program at Princeton.

CONTENTS

Date: July 11 to 15, 2011.

1. A Quick Introduction to Harmonic Analysis in $\mathbb{R}^d$

This is meant as an introduction to some material which will be useful in L. Pierce's course. We'll discuss the continuous theory now (instead of the discrete theory). The introduction will include:

(1) Three Continuous Operators
(2) The Role of Curvature.

We assume knowledge of L^p spaces and some elementary knowledge of the Fourier transform, which we'll review now.

Recall the functions f which have finite L^p norm

$$||f||_{L^p} = \left(\int_{\mathbb{R}^d} |f|^p \, dx \right)^{1/p}$$

are said to be in the space of L^p functions. There are two important inequalities at the backbone of these spaces: Minkowski's inequality and Holder's inequality. The latter says that if $1 \leq p \leq \infty$ and if $p^{-1} + q^{-1} = 1$ then

$$\int |fg| \, dx \leq ||f||_{L^p} ||g||_{L^1}.$$

Also recall the Fourier transform

$$\int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx = \hat{f}(\xi)$$

along with the inversion formula

$$\int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi = f(x).$$

Note L^2 is special because $||\hat{f}||_{L^2} = ||f||_{L^2}$. This is Plancherel's theorem. Finally recall the convolution,

$$f \star g(x) = \int f(x-y) g(y) \, dy = \int f(y) g(x-y) \, dy.$$

1.1. Three Continuous Operators.

There are three symmetries of $\mathbb{R}^d$,

(1) translation: $x \mapsto x + h, h \in \mathbb{R}^d$
(2) dilation: $x \mapsto \delta \cdot x, \delta > 0$
(3) rotation: $x \mapsto R(x)$ with R linear and $|R(x)| = |x|$.

We wish to investigate how these symmetries interplay with the Fourier transform. So we'll see three operators now, each of which respect some of these symmetries.

1.1.1. *The Maximal Operator.*

The *maximal operator* $f \mapsto M(f)$ is defined as

$$M(f) = \sup_{r>0} \frac{1}{m(B_x)} \int_{B(x)} |f(y)| \, dy.$$

Note the maximal operator is invariant under all three symmetries. From its definition we see the maximal function Mf is larger a.e. than f, but this next theorem says that it is not that much bigger.

Theorem 1.1. *Let Mf be the maximal function for f. Then,*

(1) $||Mf||_{L^p} \leq A_p ||f||_{L^p}, 1 < p \leq \infty$.
(2) $m\{x : (Mf)(x) > \alpha\} \leq \frac{A}{\alpha} ||f||_{L^1}, \forall \alpha > 0$.

Taught by E. Stein on July 11 & 12, 2011.

This second inequality is called a weak-type inequality, and is the analogue for Chebyshev's inequality to Mf.

Remark. The maximal function is the "mother of all averages." Take Φ to be a radial function, $\Phi(x) = \Phi_0(|x|)$, with Φ_0 positive, decreasing, and $\int_\Phi = 1$. Then,

$$\Big|\int f(x-y)\Phi(y)\,dy\Big| \leq M(f)(x).$$

To see this, approximate Φ by $\sum \frac{c_k}{m(B_k)}\chi_{B_k}$ where B_k are balls cenetered at the origin and $\sum c_k = 1$.

Proof of (2). Let $E_\alpha = x : M(f)(x) > \alpha$ and K compact with $K \subset E_\alpha$. Then $K \subset \cup_{i=1}^N B_i$, $B_i = B_i(x_i)$ with $\frac{1}{m(B_i)}\int_{B_i}|f| > \alpha$. By Vitali's lemma, we can select disjoint $B_{i_1}, \dots, B_{i_k}$ so that $\cup_j B^*_{i_j} \supset K$ with B^* having radius 3 times that of B. Hence,

$$m(k) \leq \sum m(B^*_{i_j}) = 3^d \sum m(B_{i_j}) \leq \frac{3^d}{\alpha}\sum \int_{B_{i_j}} |f| \leq \frac{3^d}{\alpha}|f|$$

and so (2) is proved. □

Recall the *distribution function* of $F \geq 0$,

$$\lambda(\alpha) = m\{x : F(x) > \alpha\}.$$

Fact 1.2. $\int_0^\infty \lambda(\alpha)\,d\alpha = \int_{\mathbb{R}^d} F(x)\,dx$.

This is how (1) is proved.

1.1.2. *The Fractional Integral.* The second operator is the *fractional integral*, or *Riesz potential*:

$$I_\alpha(f)(x) = \frac{1}{\gamma_\alpha}\int_{\mathbb{R}^d} f(x-y)|y|^{-d+\alpha}\,dy$$

for $0 < \alpha < d$ and $\gamma_\alpha = \pi^{d/2}2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)}$. Note that

$$I_\alpha(f) = \int f(x-y)K_\alpha(y)\,dy$$

for locally integrable K_α.

Why is this operator interesting? First, $I_\alpha(f)\hat{(}\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi)$, so $I_\alpha(f) = (-\triangle)^{\alpha/2}f$. We get the identity $\hat{K}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$ in the sense of distributions. Second, when $\alpha = 2$ and $d \geq 3$ then I_2 is the fundamental solution operator for the Laplacian $\triangle$. (When $d = 3$, this is the Newtonian potential.) Third, we can write an expression for f (vanishing at infinity) in terms of $\partial_{x_i} f, i = 1, \dots, d$:

$$f(x) = \frac{1}{\omega}\int \sum_{j=1}^d \partial_{x_j} f(x-y)\frac{y_j}{|y|^d}\,dy$$

where ω is the area of the unit sphere, and hence

$$|f(x)| \leq cI_1\left(|\nabla f|\right)(x).$$

Theorem 1.3. *Suppose* $1 < p < q < \infty$ *and* $q^{-1} = p^{-1} - \alpha/d$. *Then*

$$||I_\alpha(f)||_{L^q} \leq A_{p,q}||f||_{L^p}.$$

Remark. Note that dilation is a "relative"-invariance, in the following sense. If $f_\delta(x) = f(\delta x)$, then
$$I_\alpha(f_\delta) = \delta^{-\alpha}(I_\alpha(f))_\delta.$$
Also $||f_\delta||_{L^p} = \delta^{-d/p}||f||_{L^p}$. These fix the relation $q^{-1} = p^{-1} - \alpha/d$.

There are many proofs of this theorem, each useful in different cases. One in particular follows from

Lemma 1.4. *Set* $0 < \theta < 1$ *and* $\theta = p/q$, *then*
$$I_\alpha(f)(x) \le c(Mf(x))^\theta ||f||_{L^p}^{1-\theta}.$$

To see the lemma, consider the special case $d = 1, \alpha = 1/4, p = 2, q = 4$. Write
$$I_1(f) = c\int_{|y|\le R} |y|^{-3/4} f(x-y)\,dy + c\int_{|y|>R} |y|^{-3/4} f(x-y)\,dy.$$
Estimate the first integral using the fact we chose earlier about the maximal function. Estimate the second via Schwarz's inequality inequality. Then choose R suitably to get the result.

The same examples which show the $p = 1$ case to be bad for Mf show this theorem fails for $p = 1$; a duality argument shows $q = \infty$ will not work either.

1.1.3. *Singular Integral Operators.* Now we come to the third (and trickiest) of the three operators. We come to singular integral operators (Mihlin, Calderón, Zygmund):
$$T(f)(x) = pv\int_{\mathbb{R}^d} f(x-y)K(y)\,dy = \lim_{\epsilon > 0}\int_{|y|>\epsilon} f(x-y)K(y)\,dy.$$
Here

- K is homogeneous of degree $-d$ ($K_\delta = \delta^{-d}K$)
- K is smooth when $x \neq 0$
- $\int_{|x|=1} K(x)\,d\sigma = 0$.

Example 1.5. In $d = 1$ we have the Hilbert transform
$$H(f)(x) = \frac{pv}{\pi}\int_{-\infty}^{\infty} f(x-y)\frac{dy}{y}.$$
Note that H is translation invariant and dilation invariant, but not rotation invariant. In fact, the only bounded operator which commutes with all three is the identity operator (up to constant). The only bounded operator which commutes with the first two is H. Also note that H is unitary on $L^2(\mathbb{R})$, $H^* = -H$, and $H^2 = -I$. And the Hilbert transform provides the link between complex and harmonic analysis.

Example 1.6. $R_{ij}(f) = \partial_{x_i x_j} I_2(f)$

Example 1.7. $R_j(f) = \partial_{x_j} I_1(f)$

Theorem 1.8. *Let T be a singular integral operator. Then,*

(1) $T : L^p \to L^p$, $1 < p < \infty$. *That is, T is a bounded operator on L^p.*
(2) $m\{x : |Tf(x)| > \alpha\} \le \frac{A}{\alpha}||f||_{L^1}$, *for all* $\alpha > 0$.

The usual method of proof is referred to as the "Calderon-Zygmund paradigm." It has two parts: the L^2 theory and the L^1 theory. The L^2 theory is particularly easy (but its generalizations are not). Observe that

$$(Tf)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$$

in the sense of distributions. By Plancherel's theorem it then suffices to prove that $|m(\xi)| \leq c$ for all ξ. For then,

$$||T(f)||_{L^2} = ||(T(f))^\wedge||_{L^2} = ||m(\xi)\hat{f}(\xi)||_{L^2} \leq c||\hat{f}||_{L^2} = c||f||_{L^2},$$

so the L^2 theory follows by application of the Fourier transform.

The L^1 theory does not use the Fourier transform, but introduces two new ideas. First, the "atoms." Suppose $f \in L^1$, supported in a ball B and $\int_B f\,dx = 0$. Let B^* be the "double" of B. Then

$$\int_{(B^*)^c} |Tf|\,dx \leq c\int_B |f|\,dx.$$

Now if B is centered at the origin,

$$\begin{aligned} T\left(f\right)\left(x\right) &= \int K\left(x-y\right) f\left(y\right)\, dy \\ &= \int \left(K\left(x-y\right) - K\left(x\right)\right) f\left(y\right)\, dy, \end{aligned}$$

and hence

$$\left|T\left(f\right)\left(x\right)\right| \leq \int_B \left|K\left(x-y\right) - K\left(x\right)\right| \left|f\left(y\right)\right| dy.$$

If $y \in B$ and $x \in \left(B^*\right)^c$ (B has radius r), then

$$\left|K\left(x-y\right) - K\left(x\right)\right| \leq r \cdot \max_{x\in L} \left|\nabla K\left(x\right)\right|$$

where L is the line segment joinging x to $x-y$. But since $|x| \geq 2r$, $|y| \leq r$, and $|\nabla K| \leq A|x|^{-d-1}$, we have

$$\int_B \left|K\left(x-y\right) - K\left(x\right)\right| dx \leq cr\int_{|x|\geq 2r} |x|^{-d-1}\, dx = c'$$

whenever $|y| \leq r$. So the claim holds on atoms.

Second, the Calderon-Zygmund decomposition. This technique seeks to decompose an arbitrary L^1 function into the sum of a L^2 function and appropriate atoms. Given $f \in L^1$ and $\alpha > 0$, we can write $f = g + b$, $b = \sum_k b_k$ so that $|g(x)| \lesssim \alpha$ and so that $g(x) = f(x)$ except when $x \in E_\alpha$ with $m(E_\alpha) \lesssim \frac{1}{\alpha}\int |f|\,dx$. Also each b_k should be supported in a ball B_k,

$$\int b_k\,dx = 0 \text{ and } \int |b_k|\,dx \lesssim \alpha m(B_k),$$

and

$$\sum m(B_k) \lesssim \frac{1}{\alpha}\int |f|\,dx.$$

Once we have decomposed f the claim follows from our previous work.

How do we know such a decomposition is possible? Here is an example. Place a mesh of equal size cubes Q, and choose their size so that

$$\frac{1}{m(Q)}\int_Q |f|\,dx \leq \alpha.$$

Now take each cube Q and divide it into 2^d equal cubes by bisecting the sides. Now the mean value of f on each subdivided cubes is either $\leq \alpha$, or there is a cube on which the mean value of f is $> \alpha$. Reject the subdivided cubes in the second class, and then subdivide the remaining cubes again. Continue this process indefinitely. Now on the rejected cubes,

$$\alpha < \frac{1}{m(Q)} \int_Q |f| \leq 2^d \alpha.$$

The rejected cubes are disjoint and the sum of their measures satisfies the right type of estimate. On the cubes which were never rejected, there is a sequence of decreasing cubes on which the average is always $\leq \alpha$. So $f \leq \alpha$ a.e. on the non-rejected cubes. So we get the right estimates in both cases.

Some quick remarks on the Calderon-Zygmund decomposition. Here is another way to produce the decomposition. First, define the space into two parts, where $Mf(x) \geq \alpha$ and where $Mf(x) > \alpha$, the latter of which is an open set. Now given any open set, we can decompose it into the union of cubes with disjoint interiors, with side length comparable to the distance from the complement of the open set (Whitney decomposition). On any given cube,

$$c\alpha \leq \frac{1}{m(Q)} \int_Q |f|\, dx \leq c'\alpha.$$

To see this, surrond the given cube with a large ball and compare the radius of this ball to the size of the cube.

Further readings in "Singular Integrals" by Stein. See chapters 1 and 2.

1.2. **The Role of Curvature.** Let's start with the simplest example of curvature: curves. A *curve* γ in $\mathbb{R}^d$ is a smooth map $\gamma : [-1,1] \to \mathbb{R}^d$ with non-vanishing derivative. For simplicity, assume $\gamma(0) = 0$. We can average along the curve with either the Hilbert transform,

$$H_\gamma(f)(x) = p.v. \int_{-1}^{1} f(x - \gamma(t))\, \frac{dt}{t},$$

or the maximal function,

$$M_\gamma\left(f\right)\left(x\right) = \sup_{0<h\leq 1} \frac{1}{2h} \int_{|t|\leq h} \left|f\left(x - \gamma\left(t\right)\right)\right| dt.$$

We ask: is there an $L^p(\mathbb{R}^d)$ theory for H_γ and M_γ? The answer:

(1) Yes if γ is a straight-line segment.
(2) No in general.
(3) Yes, if γ has some "curvature" (near $t = 0$).

Why does the straight-line segment work? After rotating the space, we can assume the straight-line segment is on the x-axis; then for each y the theory in $\mathbb{R}$ applies. Now we'll investigate (3) more.

The precise curvature hypothesis here is that the vectors $\gamma'(0), \ldots, \gamma^{(N)}(0)$ must span $\mathbb{R}^d$. An iconic example is $\gamma(t) = (t, t^2)$ in $\mathbb{R}^2$. Here is the main theorem:

Theorem 1.9. *Under the curvature hypothesis, both H_γ and M_γ are bounded operators on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

Note. The methods for proving the theorems in the previous lecture fail here. In particular, no results (e.g., weak type) are known for $p = 1$ for H_γ and M_γ.

The key tool here is *oscillatory integrals*. Some examples are

$$I(\gamma) = \int_a^b e^{i\lambda\Phi(x)}\,dx$$
$$J(\gamma) = \int_a^b e^{i\lambda\Phi(x)}\psi(x)\,dx.$$

Here Φ is the "phase" and is real-valued (along with λ), and ψ is the "amplitude". The question we'll ask is how $I(\lambda), J(\lambda)$ behave as λ goes to infinity. The question to ask is: What estimates can we make which are independent of the size of the integrals?

Theorem 1.10. *If $\Phi \in C^2$, $|\Phi'(x)| \geq 1$, and Φ' is monotonic, then $|I(\lambda)| \leq c|\lambda|^{-1}$. Alternatively, if $|\Phi''(x)| \geq 1$, then $|I(\lambda)| \leq c|\lambda|^{-1/2}$ for λ real, $|\lambda| \to \infty$.*

Remarks. (1) These results are scale-invariant, i.e., invariant under $x \mapsto \delta x$. So the bound is independent of a, b.
(2) To understand these results consider $\Phi(x) = x$ for the first result and $\Phi(x) = x^2$ for the second.

Proof. For the first, write

$$e^{i\lambda\Phi(x)} = \frac{1}{i\lambda\Phi'(x)}\frac{d}{dx}\left(e^{i\lambda\Phi(x)}\right),$$

so

$$I(\lambda) = \frac{1}{i\lambda} = \int_a^b \frac{1}{\Phi'(x)}\frac{d}{dx}\left(e^{i\lambda\Phi(x)}\right)\,dx,$$

and integrate by parts. For the second, write $I(\lambda) = I + II$ where

$$I = \int_{c-\lambda}^{c+\lambda} e^{i\lambda\Phi}\,dx$$

and II is the complementary part. Here, c is the unique point where $\Phi'(c) = 0$ (w.l.o.g. assume there is one such c). Now $|I| \leq 2\delta$, and by the first result $|II| \leq \frac{2}{|\lambda|\delta}$. Choose $\delta = |\lambda|^{-1/2}$. $\square$

Corollary 1.11. *If $|\Phi'(x)| \geq 1$ and Φ' is monotonic, then*

$$|J(\lambda)| \leq \frac{c}{|\lambda|}\cdot\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

If $|\Phi''(x)| \geq 1$, then

$$|J(\lambda)| \leq \frac{c}{|\lambda|^{1/2}}\cdot\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

Proof. Write

$$I^u(\lambda) = \int_a^u e^{i\lambda\Phi(x)}\,dx$$

and write

$$J(\lambda) = \int_a^b \frac{d}{du}(I^u(\lambda))\psi(u)\,du,$$

then integrate by parts. $\square$

Exercise 1.12. Work out the details of the proof of the corollary.

Now we will prove a theorem about the Hilbert transform on the parabola in $\mathbb{R}^2$. Let $\gamma(t) = (t, t^2)$ and

$$H_\gamma(t) = p.v. \int f(x - t, y - t^2)\, \frac{dt}{t}.$$

Theorem 1.13. *Let*

$$m(\xi, \eta) = p.v. \int_{-\infty}^{\infty} e^{-2\pi i(\xi t + \eta t^2)}\, \frac{dt}{t},$$

then $|m(\xi, \eta)| \le A$ for all (ξ, η).

Corollary 1.14. *The map $f \mapsto H_\gamma(f)$ is bounded on $L^2(\mathbb{R}^2)$.*

Proof of theorem 1.13. The idea is to consider new "dilations", $\delta \circ (\xi, \eta) = (\delta\xi, \delta^2\eta)$, $\delta > 0$, which commute with the Hilbert transform on γ. Also consider a new "norm", $\rho(\xi, \eta) = |\xi| + |\eta|^{1/2}$, which is homogenous with degree one w.r.t. the new dilations. Indeed, $\rho(\delta \circ (\xi, \eta)) = \delta\rho(\xi, \eta)$.

Now write

$$p.v. \int e^{-2\pi i(\xi t + \eta t^2)} \frac{dt}{t} = \sum_{k \in \mathbb{Z}} \int_{2^k \le |t| \le 2^{k+1}} e^{-2\pi i(\xi t + \eta t^2)}\, \frac{dt}{t}.$$

Note that

$$\int_{2^k \le |t| \le 2^{k+1}} e^{-2\pi i(\xi t + \eta t^2)}\, \frac{dt}{t} = \int_{1 \le |t| < 2} e^{-2\pi i(2^k \xi t + 2^{2k} \eta t^2)}\, \frac{dt}{t}.$$

Let

$$m_0(\xi, \eta) = \int_{1 \le |t| < 2} e^{-2\pi i(\xi t + \eta t^2)}\, \frac{dt}{t},$$

then the previous integral is $m_0(2^k\xi, 2^{2k}\eta)$. We want to use the previous theorem, so write

$$m_0(\xi, \eta) = \int_{1 \le |t| < 2} e^{i\lambda\Phi(t)} \psi(t)\, dt.$$

Observe the following facts about $m_0(\xi, \eta)$:

1. $|m_0(\xi, \eta)| \le A$ for all (ξ, η)
2. $m_0(\xi, \eta)$ is a smooth function, with $m_0(0, 0) = 0$, hence $|m_0(\xi, \eta)| \le c\rho(\xi, \eta)$
3. $|m_0(\xi, \eta)| \le c(\rho(\xi, \eta))^{-1/2}$

The first two assertions are easy. Now we prove (3). The first case is $|\xi| \ge 10|\eta|$. Take $\Phi(t) = -2\pi(t + \eta/\xi \cdot t^2), \lambda = \xi, \psi(t) = 1/t$. Then $|\Phi'(t)| \ge 1$, and

$$|m_0(\xi, \eta)| \le \frac{c}{|xi|} \le \frac{c}{\rho(\xi, \eta)^{1/2}}$$

if $|\xi| \ge 10|\eta|$ and $\rho(\xi, \eta) \ge 1$. If $|\xi| \le 10\eta$, write $\Phi(t) = -2\pi(t\eta/\xi + t^2), \lambda = \eta$. Then $|\Phi''(t)| \ge 1$, and

$$|m_0(\xi, \eta)| \le \frac{c}{\eta}^{-1/2} \le c\rho(\xi, \eta)^{-1/2}.$$

Now

$$\begin{aligned} m(\xi, \eta) &= \sum_k m_0(2^k\xi, 2^{2k}\eta) \\ &= \sum_{2^k\rho(\xi,\eta) \le 1} + \sum_{2^k\rho(\xi,\eta) > 1} \end{aligned}$$

The first sum is

$$c\sum \rho(2^k\xi, 2^{2k}\eta) = c\left(\sum_{2^k\rho(\xi,\eta)\le 1} 2^k\right)\rho(\xi,\eta) \le c'.$$

The second sum is bounded by

$$\sum_{2^k\rho(\xi,\eta)>1} \rho(2^k\xi, 2^{2k}\eta)^{-1/2} = \left(\sum_{2^\rho(\xi,\eta)\le 1} 2^{-k/2}\right)\rho(\xi,\eta)^{-1/2} \le c''.$$

□

Here is a more general result involving oscillatory integrals.

Theorem 1.15. *Suppose $P(t) = a_0 + a_1 t + \cdots + a_k t^k$ is a polynomial with real coefficients. Then*

$$\left|p.v. \int_{\mathbb{R}} e^{iP(t)}\,\frac{dt}{t}\right| \le M$$

with M independent of the a_i.

Corollary 1.16. *Suppose $Q(t) = (Q_1(t), \ldots, Q_d(t))$ is a polynomial function from $\mathbb{R}$ to $\mathbb{R}^d$. Then*

$$H_Q(f)(x) = pv \int_{\mathbb{R}} f(x - Q(t))\,\frac{dt}{t}$$

is bounded on $L^2(\mathbb{R}^d)$.

Now we'll prove boundedness of the operator M_γ. To do so we'll consider *non-isotropic dilations* in $\mathbb{R}^d$,

$$x \mapsto \delta \circ x = (\delta^{a_1}x_1, \ldots, \delta^{a_d}x_d)\,,$$

where $x = (x_1, \ldots, x_d)$. Here $a_1, \ldots, a_d$ are fixed strictly positive exponents. The quantity $a_1 + \cdots + a_d = Q$ is the *homogeneous dimension* of $\mathbb{R}^d$ (note that $d(\delta \circ x) = \delta^Q dx$). It's a general fact that many of the basic results (theorems 1.1, 1.3, 1.8) have valid extensions (when properly formulated) where the isotropic dilations $x \mapsto \delta x = (\delta x_1, \ldots, \delta x_d)$ are replaced by non-isotropic dilations.

Example 1.17. Consider the maximal function $\tilde{M}$,

$$\tilde{M}(f)(x) = \sup_{\delta>0} \frac{1}{m(B_\delta)} \int_{B_\delta} |f(x-y)|\,dy,$$

where $B_\delta = \delta \circ B$ with B the unit ball and $m\left(B_\delta\right) = \delta^Q m(B)$.

Proposition 1.18. *Let $\tilde{M}$ be as in the previous example. Then,*

(1) $||\tilde{M}(f)||_{L^p} \le A||f||_{L^p}$ when $1 < p \le \infty$
(2) $m\{x : \tilde{M}(f)(x) > \alpha\} \le \frac{A}{\alpha}||f||_{L^1}$ for all $\alpha > 0$

And there is an analogue to Vitali for the non-isotropic case: There exists $c > 0$ so that if $B_{\delta_1}(x_1) \cap B_{\delta_2}(x_2) \ne \emptyset$ and $\delta_1 \ge \delta_2$, then $B_{c\delta_1}(x_1) \supset B_{\delta_2}(x_2)$.

Theorem 1.19. *Let*

$$M_\gamma(f)(x) = \sup_{0<r} \frac{1}{2r} \int_{|t|\le r} |f(x_1 - t, x_2 - t^2)|\,dt,$$

then

$$||M_\gamma(f)||_{L^2} \le A||f||_{L^2}.$$

Proof. It suffices to assume $f \geq 0$ and prove

$$\big|\big|\sup_{k\in\mathbb{Z}} |A_k(f)|\big|\big|_{L^2} \leq C||f||_{L^2}$$

where

$$A_k f = 2^{-(k+1)} \int_{|t|\leq 2^k} f(x_1 - t, x_2 - t^2)\, dt.$$

(Take $2^{k-1} \leq r \leq 2^k$.) Let ϕ be a smooth positive function on $\mathbb{R}^2$, supported in $|x| \leq 1$ and with $\int \phi(x)\, dx = 1$. Set $\phi_k(x) = 2^{-3k}\phi(2^{-k}x_1, 2^{-2k}x_2)$. Note that $\int \phi_k(x)\, dx = 1$, and ϕ_k is supported in B_{2^k} with $m(B_{2^k}) = 2^{3k}m(B)$.

Now write

$$M_k(f) = f \star \phi_k = \int_{\mathbb{R}^2} f(x-y)\phi_k(y)\, dy,$$

then

$$M_k(f)(x) \leq c\frac{1}{m(B_{2^k})}\int_{B_{2^k}} f(x-y)\, dy \leq c\tilde{M}(f).$$

So

$$\big|\big|\sup_{k} |M_k(f)|\big|\big|_{L^2} \leq c||f||_{L^2}.$$

So we only need to compare M_k with A_k.

Consider the *square function* S,

$$(S(f)(x))^2 = \sum_{k\in\mathbb{Z}}(A_k(f)(x) - M_k(f)(x))^2.$$

Notice

$$|A_k(f)(x) - M_k(f)(x)| \leq S(f)(x),$$

and hence

$$\sup_k\ A_k(f)(x) \leq S(f)(x) + c\tilde{M}(f)(x).$$

So the entire proof (in L^2) is reduced to an estimate on S. Here is the key lemma:

Lemma 1.20. $||S(f)||_{L^2} \leq c||f||_{L^2}$.

Proof. Let $\hat{\phi}(\xi)$ be the Fourier transform of ϕ,

$$\hat{\phi}(\xi) = \int e^{-2\pi i(\xi x_1 + \eta x_2)}\phi(x_1, x_2)\, dx.$$

Observe that

(1) $\hat{\phi}$ is smooth
(2) $\hat{\phi}(0) = 1$
(3) $\hat{\phi}$ is rapidly decreasing at infinity, in that $|\hat{\phi}(\xi,\eta)| \leq c(\rho(\xi,\eta))^{-1/2}$
(4) $\hat{\phi}_k(\xi,\eta) = \hat{\phi}(2^k\xi, 2^{2k}\eta)$.

So then

$$(A_k - M_k)^\wedge(\xi,\eta) = m_k(\xi,\eta) - \hat{\phi}_k(\xi,\eta)$$

with

$$m_k(\xi,\eta) = 2^{-k-1}\int_{|t|\leq 2^k} e^{-2\pi i(\xi t + \eta t^2)}\, dt = m_0(2^k\xi, 2^{2k}\eta).$$

Note that

(1) m_0 is smooth
(2) $m_0(0,0) = 1$
(3) $|m_0(\xi,\eta)| \leq c\rho(\xi,\eta)^{-1/2}$.

In conclusion,

$$((A_k - M_k)f)^\wedge = \triangle_k \cdot \hat{f}$$

where

$$\triangle_k(\xi,\eta) = m_k(\xi,\eta) - \hat{\phi}_k(\xi,\eta) = \triangle_0(2^k\xi, 2^{2k}\eta)$$

and

$$\begin{aligned}|\triangle_0(\xi,\eta)| &\le c(|\xi|+|\eta|) \le c\rho(\xi,\eta) \text{ if } \rho \le 1\\ |\triangle_0(\xi,\eta)| &\le c(\rho(\xi,\eta))^{-1/2} \text{ if } \rho \ge 1.\end{aligned}$$

But by Plancherel's theorem,

$$||(A_k - M_k)f||_{L^2}^2 = \int_{\mathbb{R}^2} |\triangle_k(\xi,\eta)|^2 |\hat{f}(\xi,\eta)|^2 \, d\xi d\eta,$$

so

$$\begin{aligned}||S(f)||_{L^2}^2 &= \sum_k ||(A_k - M_k)f||_{L^2}^2\\ &= \int \left(\sum_k |\triangle_k(\xi,\eta)|^2\right) |\hat{f}|^2 \, d\xi d\eta\\ &\le c^2 \int |\hat{f}(\xi,\eta)|^2 \, d\xi d\eta\\ &= c^2 \int |f(x)|^2 \, dx\end{aligned}$$

since $\sum_k |\triangle_k(\xi)|^2 \le c^2$. □

Applying the lemma finishes the proof of theorem 1.19. And in fact,

$$\begin{aligned}\sum_{2^k\rho\le 1} 2^{2k}\rho(\xi,\eta)^2 &\le c_1\\ \sum_{2^k\rho\ge 1} 2^{-k}\rho(\xi,\eta)^{-1} &\le c_1.\end{aligned}$$

□

There is an L^p theory. Note that the decay estimates used here are better than what was needed, so the L^2 theory has a lot of room. So one could conceive of proving a result in a worse case than L^2 and then interpolating.

Further readings in "Harmonic Analysis" by Stein include chapter 8, sections 1-3 for oscillatory integrals, and chapter 9, sections 1.2 and 2 for the maximal function and singular integrals on curved varieties.

2. Three Classical Discrete Operators

Throughout the course we'll discuss convolution operators, of the type

$$Tf(n) = \sum_{m \in \mathbb{Z}^R} f(m)K(n-m)$$

for f with "compact" (finite) support. The question we'll always be asking is: does T extend to a bounded operator on $\ell^p = L^p(\mathbb{Z}^k)$? For $1 \leq p < \infty$ we have the norm

$$||f||_{\ell^p} = \left(\sum_n \in \mathbb{Z}^k |f(n)|^p \right)^{1/p}$$

and for $p = \infty$,

$$||f||_{l^\infty} = \sup_n |f(n)|.$$

We want

$$||Tf||_{l^q} \leq A_{p,q} ||f||_{\ell^p}.$$

We'll discuss three important examples today:

(1) maximal function
(2) fractional integral operators
(3) Hilbert transform.

Some of the methods we'll use to understand discrete operators include

(1) implication
(2) imitation
(3) circle method
(4) sampling method
(5) method of refinements
(6) not named yet... under development.

2.1. The Maximal Operator.

We'll start our analysis with the *maximal operator*,

$$Mf(x) = \sup_{r \geq 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x-y)|\, dy : L^p \to L^p$$

for $1 < p \leq \infty$. In the discrete case,

$$Mf(n) = \sup_{r>0} \frac{1}{N(r)} \sum_{\substack{|m|<r \\ m \in \mathbb{Z}^k}} |f(n-m)|$$

where $N(r) = \#\mathbb{Z}^k \cap B_r$.

Theorem 2.1. *$M : \ell^p \to \ell^p, 1 < p \leq \infty$, and the weak-type $(1,1)$ estimate holds.*

We'll see two proofs, first by implication (method 1) then by imitation (method 2). Here is the proof via method 1.

Proof. Define the unit cube

$$Q = \{x \in \mathbb{R}^k : -1/2 < x_j \leq 1/2, j = 1, \ldots, k\}.$$

Then cubes tile $\mathbb{R}^k$, in that $\mathbb{R}^k = \cup_{n \in \mathbb{Z}^k} Q + n$. We have a

---

Taught by L. Pierce on July 11, 2011.

Lemma 2.2. *Let $V(r) = Vol(B_r)$ and $N(r)$ as above. Then,*

(1) $V(r - \sqrt{k}/2) \leq N(r) \leq V(r + \sqrt{k}/2)$
(2) $N(r) = V(r) + O(r^{k-1})$

Proof. (1) $\Longrightarrow$ (2): Note that $V(r) = c_k r^k$ and $V(r \pm \sqrt{k}2) = c_k(r + \sqrt{k}2)^k$. □

Now given f we define its *companion function* F as

$$F(x) = \sum_{n \in \mathbb{Z}^k} f(n) \chi_Q(x - n).$$

Note that $||F||_{L^p(\mathbb{R}^k)} = ||f||_{\ell^p(\mathbb{Z}^k)}$. W.l.o.g., we may assume $f \geq 0$. Then for any $x \in Q + n$,

$$\begin{aligned}
\sum_{|m|<r} f(n-m) &= \sum_{|n-m|<r} f(m) \\
&= \sum_{|n-m|<r} \int_{Q+m} F(y)\,dy \\
&\leq \int_{|n-y| \leq r + \sqrt{k}/2} F(y)\,dy \\
&\leq \int_{|n-y| \leq r + \sqrt{k}/2} F(y)\,dy \cdot \frac{V(r+\sqrt{k})}{V(r+\sqrt{k})} \cdot \frac{1}{N(r)}.
\end{aligned}$$

Thus, $Mf(n) \leq c_k MF(x), x \in Q + n$. □

Here is the second proof, by imitiation.

Proof. l^∞ bound is trivial. Now the weak-type $(1,1)$ bound: the goal is to prove

$$|E_\alpha| = \#\{n \in \mathbb{Z}^k : Mf(n) > \alpha\} \leq \frac{A}{\alpha} ||f||_{l^1}.$$

For each $n \in E_\alpha$, $\exists r_n$ so that

$$\sum_{|m|<r} f(n) \geq \alpha N(r_n).$$

Thus $E_\alpha \subset_{n \in E_\alpha} B_{r_n}$. Take any finte set $E \subset E_\alpha$, and apply the Vitali covering lemma. So there exists a disjoint subcollection $\{B_j l\}$ so that

$$\sum_{j=0}^{J} N(r_j(n_j)) \geq 3^- k|E|,$$

and then

$$\sum_m \in \cup_{n_j \in E} B_{r_j(n_j)} |f(m)| \geq \alpha \sum_j N(r_j(n_j)) \geq \alpha \cdot 3^- k|E|.$$

The left is majorized by $||f||_{l^1}$, so we get the $(1,1)$ inequality. Now apply Marcinkiewicz interpolation. □

2.2. **Discrete Fractional Integration.** The next operator we'll study is *fractional integration*,

$$\mathfrak{I}_\lambda f(x) = \int_{\mathbb{R}^k} \frac{f(x-y)}{|y|^{k\lambda}}\, dy$$

for $0 < \lambda < 1$.

Theorem 2.3. *For $1 < p < q < \infty$, $0 < \lambda < 1$,*

$$\mathfrak{I}_\lambda : L^p(\mathbb{R}^k) \to L^q(\mathbb{R}^k)$$

iff $q^{-1} = p^{-1} - (1-\lambda)$ (homogeneity).

The discrete analogue is

$$I_\lambda f(n) = \sum_{\substack{m\in\mathbb{Z}^k \\ m\neq 0}} \frac{f(n-m)}{|m|^{k\lambda}}$$

for $0 < \lambda < 1$.

Theorem 2.4. *For $0 < \lambda < 1$, $I_\lambda : \ell^p(\mathbb{Z}^k) \to l^q(\mathbb{Z}^k)$ iff*

(1) $q^{-1} \le p^{-1} - (1-\lambda)$
(2) $q^{-1} < \lambda$, $p^{-1} > 1-\lambda$.

(1) with equality (??).

Proof. We proceed by method 2, imitation with $f \ge 0$. The goal is a pointwise estimate,

$$I_\lambda f(n) \le c(Mf(n))^{p/q} ||f||_{\ell^p}^{1-p/q}.$$

Split the sum at R:

$$I_\lambda f(n) = \sum_{0<|m|<R} \frac{f(n-m)}{|m|^{k\lambda}} + \sum_{|m|\ge R} \frac{f(n-m)}{|m|^{k\lambda}}.$$

The first term on the right satisfies

$$\begin{aligned}
I &\le c\sum_{j=0}^{\infty} \sum_{\substack{2^{-j-1}R<|m|\le 2^{-j}R \\ |m|\approx 2^-jR}} \frac{f(n-m)}{|m|^{k\lambda}} \\
&\le c\sum_{j=0}^{\infty} (2^{-j}R)^{-k\lambda} \sum_{|m|\approx 2^{-j}R} f(n-m) \\
&\le c\sum_{j=0}^{\infty} (2^{-j}R)^{-k\lambda} \sum_{|m|\approx 2^{-j}R} f(n-m)\cdot \frac{N(2^{-j}R)}{N(2^{-j}R)} \\
&\le c\left(\sum_{j=0}^{\infty} (2^{-j}R)^{-k\lambda+k}\right) Mf(n) \\
&\le cR^{-k\lambda+k} Mf(n).
\end{aligned}$$

The second term satisfies

$$II \le \left(\sum_{|m|\ge R} |f|^p\right)^{1/p} \left(\sum_{|m|\ge R} \frac{1}{|m|^k\lambda p'}\right)^{1/p'}.$$

With some work the right hand parenthesis is approximately $R^{k\left((1/p')-\lambda\right)} = R^{-k/q}$. So we conclude $II \leq R^{-k/q}||f||_{\ell^p}$. Thus

$$I + II \leq c\left(Mf(n)R^{k(1-\lambda)} + R^{-k/q}||f||_{\ell^p}\right)$$

and we can choose R so that $R^{k/q+k(1-\lambda)} = ||f||_{\ell^p}/Mf(n)$. □

Again there is a second proof, by method 1.

Proof. Assume $f \geq 0$, and construct its companion F. We have the following

Fact 2.5. *For $n \neq m \in \mathbb{Z}^k$, for any $u, v \in Q$ there exists a uniform constant C so that*

$$|n-m|^{-k\lambda} \leq C|(n+u)-(m+v)|^{-k\lambda}.$$

Using this fact, for $x \in Q + n$

$$\begin{aligned} I_\lambda f(n) &= \sum \frac{f(n-m)}{|m|^{k\lambda}} \\ &= \sum_{\substack{m\in\mathbb{Z}^k \\ m\neq n}} \frac{f(n-m)}{|m|^{k\lambda}} \\ &\leq \sum_m \int_{Q+m} \frac{C\cdot F(y)}{|x-y|^{k\lambda}}\,dy \\ &\leq C\int_{\mathbb{R}^k} |F(y)||x-y|^{k\lambda}\,dy \\ &= \mathfrak{I}_\lambda F(x). \end{aligned}$$

After applying the continuous result we're done. □

Remarks. (1) There is a nesting property $l^{q_1} \subset l^{q_2}$, for $q_1 < q_2$.
(2) Take $f(0) = 1$ and $f(n) = 0$ if $n \neq 0$, then $f \in \ell^p$ but $I_\lambda f(n) = |n|^{-k\lambda}$ which implies $I_\lambda f \in l^q$ if $\lambda q > 1$, and thus $q^{-1} < \lambda$. Also

$$< I_\lambda f, g >= \sum_{\substack{m,n \\ m\neq n}} \frac{f(n)\bar{g}(m)}{|n-m|^{k\lambda}}$$

and so $I_\lambda : \ell^p \to l^q$, $I_\lambda^k : l^{q'} \to l^{p'}$. So by duality $q^{-1} < \lambda$ then $p^{-1} > 1-\lambda$.

INSERT RIESZ DIAGRAM HERE (D1).

Exercise 2.6. Set $f(n) = n^{-\lambda}$ if $n \geq 1$ and 0 otherwise to prove condition (1) is necessary. Hint: think of $\lambda = k/p + q$.

2.3. **The Hilbert Transform.** The third and last operator we'll discuss today is the *Hilbert transform*. Recall that in the continuous setting this is

$$\mathcal{H}f(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\int_{|x|>\epsilon} \frac{f(x-y)}{y}\,dy : L^p(\mathbb{R}) \to L^p(\mathbb{R}).$$

The discrete version is

$$Hf(n) = \frac{1}{\pi}\sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{f(n-m)}{m}.$$

The goal is to show $H : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ for $1 < p < \infty$. We'll consider the bilinear form

$$B(f,g) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq n}} \frac{f(m)g(m)}{m-n}.$$

Boundedness of H on ℓ^p is equivalent to showing

$$|B(f,g)| \leq A_p ||f||_{\ell^p} ||g||_{l^{p'}}$$

for $p^{-1} + p'^{-1} = 1$. The continuous version of this is

$$\mathcal{B}(F,G) = \int_{|x-y|\geq 1} \frac{F(x)G(y)}{x-y}\, dxdy.$$

The key observation is that since $\mathcal{H}$ is bounded on L^p, then

$$|\mathcal{B}(F,G)| \leq ||F||_{L^p} ||G||_{L^{p'}}.$$

A quick note on truncated operators:

Theorem 2.7. *Let $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, i.e.,*

$$||Tf||_{L^p} \leq A_p ||f||_{L^p},$$

with $T = f \star K$, $|K(x-y)| \leq \frac{A}{|x-y|^n}$. Define

$$K_\epsilon(x-y) = \begin{cases} K(x-y) & |x-y| \geq \epsilon \\ 0 & \textit{else} \end{cases}$$

and set $T_\epsilon f = f \star K_\epsilon$. Then,

$$||T_\epsilon f||_{L^p} \leq A'_p ||f||_{L^p}$$

where A'_p is indepedent of ϵ.

Now we'll try to acheive the goal. Here is a lemma:

Lemma 2.8. $|\mathcal{B}(F,G) - B(f,g)| = O(||f||_{\ell^p} ||g||_{l^{p'}})$.

And now we'll show the result.

Proof. We claim

$$\int_{\substack{|x-y|\geq 1 \\ x\in Q+n \\ y\in Q+m}} \frac{dxdy}{x-y} = \frac{1}{n-m} + O\left(\frac{1}{(n-m)^2}\right)$$

if $n \neq m$ and 0 if $n = m$. Suppose this were true, then

$$\begin{aligned}
|\mathcal{B}(F,G) - B(f,g)| &\leq c\Big|\sum_{n,m} \int_{\substack{|x-y|\geq 1 \\ x\in Q+n \\ y\in Q+m}} \frac{F(y)G(x)}{x-y}\, dxdy - \sum_{\substack{n,m \\ n\neq m}} \frac{f(m)g(n)}{n-m}\Big| \\
&\leq c\sum_{n,m} |f(m)g(n)| \Big| \int_{\substack{|x-y|\geq 1 \\ x\in Q+n \\ y\in Q+m}} \frac{dxdy}{x-y} - \frac{1}{n-m}\Big| \\
&\leq c\sum_{n-m} \frac{f(m)g(n)}{1+|n-m|^2} \\
&\leq c\Big|\Big|\frac{1}{1+x^2}\Big|\Big|_{L^1} ||fg||_{l^1} \\
&\leq c||f||_{\ell^p} ||g||_{l^{p'}}
\end{aligned}$$

by Young's and Holder's inequality.

So we prove the claim. If $n = m$, the integral is zero so we're done. Say $n \neq m$ ($|n-m| \geq 2$), then

$$\begin{aligned}
\left|\int_{m-1/2}^{n+1/2}\int_{n-1/2}^{n+1/2}\frac{dxdy}{x-y}-\frac{1}{n-m}\right| &\leq \int_{m-1/2}^{n+1/2}\int_{n-1/2}^{n+1/2}\left|\frac{1}{x-y}-\frac{1}{n-m}\right| \\
&\leq \int_{m-1/2}^{n+1/2}\int_{n-1/2}^{n+1/2}\frac{|x-n|+|y-m|}{|x-y||n-m|} \\
&= O\left(\frac{1}{|n-m|^2}\right).
\end{aligned}$$

$\square$

3. The Discrete Hilbert Transform on a Parabola

We aim to prove the discrete version of L^2 boundedness of Hilbert transform on the parabola. Ours is a number theoretic approach; first we'll discuss some properties of translation invariant discrete operators.

3.1. **The Fourier Multiplier.**

An operator T is said to be *translation invariant* if it commutes with $\tau_h : f(x) \mapsto f(x-h)$, i.e., if $T \circ \tau_h = \tau_h \circ T$.

Example 3.1. The operator

$$Tf(n) = \sum_{m \in \mathbb{Z}^k} f(n - P(m))K(m)$$

with $P : \mathbb{Z}^k \to \mathbb{Z}^l$ a polynomial is translation invariant. The operator

$$Tf(n,t) = \sum_{m \in \mathbb{Z}^k} f(n-m, t - n \cdot m)K(m)$$

with $(n,t) \in \mathbb{Z}^k \times \mathbb{Z}$ is not translation invariant. But it is "quasi-translation invariant," as it is still translation invariant in one coordinate.

We'll define a Fourier transform $\mathcal{F} : f \to \hat{f}$, where f is defined on $\mathbb{Z}^k$ and $\hat{f}$ is defined on $\mathbb{T}^k$, by

$$\hat{f}(\theta) = \sum_{n \in \mathbb{Z}^k} f(n) e^{-2\pi i n \theta}.$$

The corresponding inversion formula is

$$h^\vee(n) = \int_0^1 h(\theta) e^{2\pi i n \cdot \theta}\, d\theta.$$

Proposition 3.2. *Let $Tf(n) = \sum_m f(m)K(n,m)$ and suppose T is translation invariant, then*

1. *There exists a function K_0 so that $K(n,m) = K_0(n-m)$.*
2. *T is bounded on ℓ^2 iff there exists $m \in L^\infty(\mathbb{T}^k)$ so that $\tilde{T} = \mathcal{F}T\mathcal{F}^{-1}$ is of the form $\tilde{T}f(x) = m(x)f(x)$, and moreover, then $||m(x)||_{L^\infty} = ||T||_{\ell^2 \to \ell^2}$.*
3. *If T is bounded on ℓ^p for some $1 < p < \infty$ then T is bounded on ℓ^2 and $||T||_{\ell^p \to \ell^p} = ||T||_{\ell^2 \to \ell^2}$.*

Remark. Property (2) can be written as $(Tf)\hat{}(\theta) = m(\theta)\hat{f}\theta$. Here, m is called the *Fourier multiplier.*

Example 3.3. Again let P be a polynomial. Then

$$(Tf)\hat{}(\theta) = \sum_n \sum_m f(n - P(m))K(m)e^{-2\pi i n \cdot \theta} = m(\theta)\hat{f}(\theta)$$

where

$$m(\theta) = \sum_m e^{-2\pi i P(m)\theta} K(m).$$

The *parabolic Hilbert transform* is given by

$$H_{\text{par}} f(n) = \sum_{m \in \mathbb{Z}, m \neq 0} \frac{f(n_1 - m, n_2 - m^2)}{m}.$$

We want to prove

Taught by L. Pierce on July 12 & 13, 2011.

Theorem 3.4. $H_{par} : \ell^2 \to \ell^2$.

From the above it's enough to show $||m||_L^\infty[0,1] \leq c$. Explicitly, we have

$$m(\theta, \phi) = \sum_{m \neq 0} \frac{e^- 2\pi i(m^2\theta + m\phi)}{m},$$

and we need to show

$$|\sum_{m \neq 0} \frac{e^- 2\pi i(m^2\theta + m\phi)}{m}| \leq A$$

independent of θ, ϕ. Observe the following:

- We can't pass the absolute value inside
- It's okay if $(\theta, \phi) = (0, 0)$ by cancellation, $(\theta, \phi) \approx (0, 0)X$
- We have to worry about $(\theta, \phi) = (1/2, 1/2)$
- And we worry about $(\theta, \phi) \approx (a/q, b/q)$, the disk of $(m^2, m) \mod q$.
- But, as q gets large we worry less.

With these in mind, we'll say (θ, ϕ) lie in a *major arc* whenever $(\theta, \phi) \approx (a/q, b/q)$ with q small; if $(\theta, \phi) \approx (a/q, b/q)$ with q large, we'll say (θ, ϕ) lie in a *minor arc.* But of course, the notions of $\approx$, small, and large are vague at this point. Now we'll see how to clarify these.

3.2. **Tools from Number Theory.** We'll discuss

(1) Diophantine (rational) approximation
(2) Exponential sums and bounds
(3) Theta functions.

We'll discuss the first two today.

3.2.1. *Diophantine Approximation.* Item (1) is taken care of via

Proposition 3.5 (Dirichlet's Approximation Principle)**.** *Let* $\theta \in [0, 1]$*,* $N \geq 1$*. Then there exists integers* $1 \leq a \leq q \leq N, (a, q) = 1$ *so that*

$$\left|\theta - \frac{a}{q}\right| \leq \frac{1}{qN}.$$

Proof. Look at the $N + 1$ numbers $0 \cdot \theta, \ldots, N\theta \mod 1$. There exist a, b so that $|a\theta - b\theta \mod 1| \leq 1/N$. So there exists $r \in \mathbb{Z}$ so that $|a\theta - b\theta - r| \leq 1/N$. □

Example 3.6. Complete the proof above.

3.2.2. *Exponential Sums.* The first type we'll consider are *linear sums*,

$$S = \sum_{1 \leq n \leq N} e^{2\pi i n\theta} = e^{\pi i(N+2)\theta} \frac{\sin(\pi N\theta)}{\sin(\pi\theta)}.$$

So we know

$$|S| \leq \min\left(N, \frac{1}{|\sin \pi\theta|}\right).$$

We can make this more clear via the

Fact 3.7. $|\sin \pi\alpha| \geq \frac{1}{2}||\alpha||$ *where* $||\alpha||$ *is the distance from* α *to the nearest integer.*

Thus,

$$|S| \leq c \min\left(N, \frac{1}{||\theta||}\right).$$

Note that if $\theta = a/q$ then

$$\sum_{1 \leq n \leq q} e^{2\pi i n a/q} = \begin{cases} q & a \equiv 0\,(q) \\ 0 & a \not\equiv 0\,(q) \end{cases}.$$

The next type are *quadratic sums*, such as the Gaussian sum

$$S(a,q) = S(a/q) = \sum_{1 \leq n \leq q} e^{2\pi i a n^2/q}$$

for $(a,q) = 1$, or the Weyl sum

$$S = \sum_{1 \leq n \leq N} e^{2\pi i P(n)\theta}$$

for $P(n) : \mathbb{Z} \to \mathbb{Z}$ and $\theta \in [0,1]$. First we discuss the Gauss sums. We'll produce the bound

$$|S(a/q)| \leq c q^{1/2}$$

also called "square-root cancellation." To show this we perform the squaring trick:

$$|S(a/q)|^2 = S(a/q)\overline{S(a/q)} = \sum_n \sum_m e^{2\pi i (m^2 - n^2) a/q}.$$

Setting $m \mapsto n + l$ we get

$$|S(a/q)|^2 = \sum_l \left(\sum_{1 \leq n \leq q} e^{2\pi i (2nla/q)} \right) e^{2\pi i l^2 a/q}.$$

Consider the inner sum. If q is odd, then $|S|^2 = q$. If q is even, there are two cases. If also $q \equiv 2 \mod 4$ then

$$|S|^2 = q(e^{2\pi i q a} + e^{\pi i q a/2}).$$

Since $q/2$ is odd and so is a, the terms here cancel, so q even and $q \equiv 2 \mod 4$ yield $S = 0$. Finally, suppose $q \equiv 0(4)$. Then $2l \equiv 0(q)$ and thus

$$|S|^2 = q(e^(s\pi i q a) + e^{\pi i q a/2}) = 2q.$$

This proves the bound.

Now we discuss the *Weyl sum*. Weyl was interested in the equidistribution of $\{n\theta\}_{n=1}^{\infty} \mod 1$ for $\theta \in \mathbb{R}/\mathbb{Q}$. He asked: does

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N, n\theta \mod 1 \in [a,b]\} \to b - a$$

as $N \to \infty$? (It turns out the answer is yes.) Then Weyl asked about the equidistribution of a_n. So arose Weyl's criterion:

$$\sum_{n=1}^{N} e^{2\pi i a_n l} = o(N)$$

as $N \to \infty$ for all $l \neq 0$. We'll use the latter.

Proposition 3.8 (Weyl Bound (deg 2))**.** *Let $\theta \in \mathbb{R}$ and suppose there exists $(a, q) = 1$ so that $|\theta - a/q| \leq 1/q^2$. Then for*

$$S(\theta, q) = \sum_{1 \leq n \leq N} e^{2\pi i(n^2\theta + n\phi)}$$

the bound holds

$$|S(\theta, \phi)| \leq c(\frac{N}{q^{1/2}} + q^{1/2})(\log q)^{1/2}.$$

Proof. Apply the squaring trick again:

$$\begin{aligned} |S(\theta, \phi)|^2 &= \sum_{1 \leq m,n \leq N} e^{2\pi i((m^2-n^2)\theta + (m-n)\phi)} \\ &= \sum_{\substack{1 \leq n \leq N \\ 1-n \leq l \leq N-n}} e^{2\pi i((l^2+2nl)\theta + l\phi)} \\ &= \sum_{|l|<N} e^{2\pi i(l^2\theta + l\phi)} \sum_{1 \leq n, n+l \leq N} e^{2\pi i(2nl\theta)} \\ &\leq \sum_{|l|<2N} \min\left(N, \frac{1}{||l\theta||}\right). \end{aligned}$$

This is hard to understand if θ is not rational. We want to say $\theta \approx a/q$ and pull the approximation through. Write

$$|S|^2 \leq N + 2 \sum_{1 \leq l \leq 2N} \min\left(N, \frac{1}{||l\theta||}\right).$$

Now we need a

Lemma 3.9.

$$\sum_{M \leq n \leq M+q} \min\left(N, \frac{1}{||l\theta||}\right) \leq N + q \log q,$$

with $\theta = a/q + \gamma, |\gamma| \leq 1/q^2$.

First we make a

Claim. For any real number u, there exists at most three choices of n $M \leq n \leq M + q$ so that $||n\theta - u|| \leq \frac{1}{2q}$.

Proof. Let $n = M + m$, and write

$$\frac{1}{2q} \geq ||n\theta - u|| = ||(M = m)\theta - u|| = ||m\theta - (M\theta - u)|| = ||m\theta - v||.$$

Then conclude if $||n\theta - u|| \leq \frac{1}{2q}$ then $||m\theta - v|| \leq \frac{1}{2q}$. As a result,

$$||m\frac{a}{q} - v|| = ||m\theta - m\gamma - v|| \leq ||m\theta - v|| + ||m\gamma||,$$

and hence $||ma/q - v|| \leq 3/2$. Thus there are only three choices of m and hence of n. □

Proof of lemma. Let $\mathcal{S} = 1/q, 2/q, \dots, (q/2)/q$, then $|S| = q/2$. Write

$$\sum_{M\le n\le M+q} \min\left(N, \frac{1}{||n\theta||}\right) = \frac{1}{|S|}\sum_{u\in S}\sum_{M\le n\le M+q} \min\left(N, \frac{1}{||n\theta||}\right)$$
$$\le \frac{1}{|S|}\sum_{u\in\mathcal{S}}\left(3N + \sum_{\substack{M\le n\le M+q \\ ||n\theta-u||\le 1/(2q)}} \frac{1}{n\theta}\right).$$

Now $u = b/q$ for $1 \le b \le q/2$ so with some more work one can show

$$\sum_{M\le n\le M+q} \min\left(N, \frac{1}{||n\theta||}\right) \le 3N + \frac{1}{|S|}\sum_{u\in\mathcal{S}}\sum_{n,||n\theta-u||\le 1/(2q)} \frac{1}{||u|| - 1/(2q)}$$

and using the fact we get

$$\sum_{M\le n\le M+q} \min\left(N, \frac{1}{||n\theta||}\right) \le 3N + \frac{q}{q/2}\sum_{m=1}^{q/2} \frac{1}{m/q - 1/(2q)}$$
$$\le 3N + q\sum_{m=1}^{q/2}\frac{1}{m}$$
$$\approx N + q\log q.$$

□

Using the lemma, we can write

$$|S(\theta,\phi)|^2 \le N + (\frac{N}{q} + 1)(N + q\log q)$$

and after square-rooting, we get the proposition. □

This was only the degree two case. Here is the real Weyl bound:

Theorem 3.10 (Weyl Bound)**.** *Under the same conditions as in the degree two case but now with a polyonmial P of degree k, the following bound holds:*

$$|S| \le c_{\epsilon,k} N^{1+\epsilon}\left(\frac{1}{N} + \frac{1}{q} + \frac{q}{N^k}\right)^{1/(2^{k-1})}.$$

3.3. **Bounding the Fourier Multiplier.** Now we'll apply the results from the previous section to the parabolic Hilbert transform. Recall

$$m(\theta,\phi) = \sum_{n\neq 0} \frac{e^{2\pi i(n^2\theta + n\phi)}}{n}.$$

We need a discrete bump function. Define

$$K = \sum_{j=0}^{\infty} K_j(x)$$

with

$$K_j(x) = \chi_{2^j\le|x|<2^{j+1}}(x)\cdot\frac{1}{x}.$$

This is an analog of $1/x$ in the discrete case. Now write

$$m(\theta,\phi)=\sum_{j=0}^{\infty}\left[\sum_{2^j\le|n<2^{j+1}|}\frac{e^{2\pi i(n^2\theta+n\phi)}}{n}\right].$$

Fix j and $\epsilon\le 1/4$. Given $\theta, 10\cdot 2^j=N\ge 1$ there exists $1\le a\le q\le N$ so that $|\theta-a/q|\le\frac{1}{qN}$. If $q\le N^{1-\epsilon}$ then we'll say θ is in a *major arc*. So the major arcs are

$$\mathcal{M}_j\left(\frac{a}{q},\frac{b}{q}\right)=\left\{(\theta,q)\in[0,1]^2:\left|\theta-\frac{a}{q}\right|\le\frac{1}{qN},\left|\phi-\frac{b}{q}\right|\le\frac{1}{2q}\right\}.$$

Then we define the *minor arcs* (for fixed j) as the complement of the union of the major arcs. The key point here is the following

Proposition 3.11. *The major arcs for a fixed j are disjoint.*

Proof. Suppose $\mathcal{M}_j(a/q,b/q)\cap\mathcal{M}_j(a'/q',b'/q')$, then there exists at least one θ so that

$$|\frac{q}{q}-\frac{q'}{q'}|\le|a/q-\theta|+|a'/q'-\theta|\le 1/(qN)+(1/q'N).$$

On the other hand

$$\frac{1}{qq'}\le|\frac{aq'-a'q}{qq'}|=|\frac{q}{q}-\frac{q'}{q'}|.$$

Suppose $q\le q'$. Then we get $1/(qq')\le 2/(qN)$ and thus $q'\ge N/2$. This is a contradiction, however, as by assumption $q\le N^{1-\epsilon}$. $\square$

Note, however, that the minor arcs may not be disjoint, but we'll treat them in a way for which this will not matter. As it turns out, it will be convenient to consider a new version of the major arcs, which are independent of j:

$$I\left(\frac{a}{q},\frac{b}{q}\right)=\left\{(\theta,\phi):|\theta-\frac{a}{q}|\le\frac{1}{10q^2},|\phi-\frac{b}{q}|\le\frac{1}{2q}\right\}.$$

These are disjoint in a weaker sense: if $I\left(\frac{a'}{q'},\frac{b'}{q'}\right)\cap I\left(\frac{a}{q},\frac{b}{q}\right)$ then q and q' must be "widely separated." We can write

$$\begin{aligned}\frac{1}{qq'}&\le|\frac{aq'-a'q}{qq'}|\\&=|\frac{a}{q}-\frac{a'}{q'}|\\&\le|\theta-\frac{a}{q}|+|\theta-\frac{a'}{q'}|\\&\le\frac{1}{10q^2}+\frac{1}{10\,(q')^2}\end{aligned}$$

and so

$$10\le\frac{q'}{q}+\frac{q}{q'}$$

which tells us either $q\ge 5q'$ or $q'\ge 5q$, $q_1,5q_1,5^2q_1,5^3q_1,\ldots$. Now we have a new key fact: $\mathcal{M}_j\left(\frac{a}{q},\frac{b}{q}\right)\subset I\left(\frac{a}{q},\frac{b}{q}\right)$.

Let's discuss the minor arcs. Assume θ, ϕ lie in a minor arc, then

$$|\sum_{|n|\approx 2^j} \frac{e^{2\pi i\left(n^2\theta+n\phi\right)}}{n}| = |\sum_{|n|\approx 2^j} \frac{S_n - S_{n-1}}{n}|$$

where

$$|S_n| = \Big|\sum_{1\leq k\leq n} e^{2\pi i\left(k^2\theta+k\phi\right)}\Big|.$$

Partial summation gives

$$|\sum_{|n|\approx 2^j} \frac{e^{2\pi i\left(n^2\theta+n\phi\right)}}{n}| \approx \sum_{|n|\approx 2^j} |\frac{1}{n} - \frac{1}{n+1}||S_n|.$$

The first term in the summand is $O\left(n^{-2}\right)$, so

$$|\sum_{|n|\approx 2^j} \frac{e^{2\pi i\left(n^2\theta+n\phi\right)}}{n}| \approx 2^j \cdot 2^{-2j} \sup_{|n|\approx 2^j} |S_n|.$$

Recall the Weyl bound: If $|\theta - \frac{a}{q}| \leq \frac{1}{q^2}, = (a,q) = 1$ then

$$|S_n| \leq \left(nq^{-1/2} + q^{1/2}\right)(\log q)^{1/2}.$$

Since θ is in a minor arc, any such a/q must have q large, i.e., $N^{1-\epsilon} \leq q \leq N$. So now

$$|\sum_{|n|\approx 2^j} \frac{e^{2\pi i\left(n^2\theta+n\phi\right)}}{n}| \approx 2^j \cdot 2^{-2j}\left(Nq^{-1/2} + q^{1/2}\right)(\log q)^{1/2}.$$

It's a fact that $N = 2^j \cdot 10$, $N^{1-\epsilon} \leq q \leq N$, so

$$|\sum_{|n|\approx 2^j} \frac{e^{2\pi i\left(n^2\theta+n\phi\right)}}{n}| \leq 2^{-j/2} + \epsilon' j.$$

So the total contribution from $\theta \in$minor arc is

$$\sum_{j=0}^{\infty} 2^{-j\left(\frac{1}{2}-\epsilon'\right)} < C$$

independent of θ (as long as it lies in a minor arc).

Let's go back to the major arcs. Let $\theta \in \mathcal{M}_j\left(\frac{a}{q}, \frac{b}{q}\right)$, and write $\theta = \frac{a}{q} + \alpha, \phi = \frac{b}{q} + \beta$. We want to split the sum

$$\sum_{\substack{|n|\approx 2^{-j} \\ 2^j \leq |n| < 2^{j+1}}} \frac{e^{-2\pi i\left(n^2\left(\frac{a}{q}+\alpha\right)+n\left(\frac{b}{q}+\beta\right)\right)}}{n}$$

into an arithmetic part times an integral. Write $n = mq + l$, then write

$$\sum_{\substack{|n|\approx 2^{-j} \\ 2^j \le |n| < 2^{j+1}}} \frac{e^{-2\pi i\left(n^2\left(\frac{a}{q}+\alpha\right)+n\left(\frac{b}{q}+\beta\right)\right)}}{n}$$
$$= \sum_{1\le l\le q} \sum_{\frac{2^j}{q}\le m<\frac{2^{j+1}}{q}} \frac{e^{-2\pi i\left((mq+l)^2\left(\frac{a}{q}+\alpha\right)+(mq+l)\left(\frac{b}{q}+\theta\right)\right)}}{mq+l} + O\left(q2^{-j}\right).$$

Let's show the error is acceptable:

$$\sum_{j=0}^{\infty} q2^{-j} \le \sum_{j=0}^{\infty} \left(10\cdot 2^j\right)^{1-\epsilon} 2^{-j} \approx \sum_{j=0}^{\infty} 2^{-j\epsilon} < C < \infty,$$

so we'll stop writing the error. Now

$$\left(\frac{a}{q}+\alpha\right)(mq+l)^2 = \left(m^2q^2+2mql+l^2\right)\left(\frac{a}{q}+\alpha\right)$$
$$= e^{2\pi i l^2 a/q} e^{2\pi i (mq+l)^2\alpha},$$

and a similar thing happens with the linear term. So the original dyadic sum satisfies

$$\text{dyadic sum} = \sum_{l=1}^{q} e^{2\pi i\left(l^2 a + lq\right)/q} \sum_{|m|\approx\frac{2^j}{q}} \frac{e^{2\pi i\left((mq+l)^2\alpha+(mq+l)\beta\right)}}{mq+l}.$$

The first sum looks like a Gauss sum; if we call it $S\left(\frac{a}{q},\frac{b}{q}\right)$ then it's a fact that

$$|S\left(\frac{a}{q},\frac{b}{q}\right)| \le cq^{1/2}.$$

To handle the second sum we'll approximate it by an integral.

Theorem 3.12 (Van der Corput)**.** *Suppose $f \in C^2$, real-valued, and*

(1) f' is monotonic
(2) $|f'(x)| \le \gamma < 1$.

Also suppose ϕ is differentiable, and

(1) $|\phi(x)| \le 1$ for all x
(2) $\int_{\mathbb{R}} |\phi'(x)|\, dx \le 1$.

Then

$$\sum_{n=a}^{b} e^{2\pi i f(n)}\phi(n) = \int_a^b e^{2\pi i f(x)}\phi(x)\ dx + O\left(\int_a^b |\phi'(x)|\, dx\right)$$
$$= \int_a^b e^{2\pi i f(x)}\phi(x)\ dx + O(1).$$

Let's apply this here. Our $f(x) = (xq+l)^2\alpha + (xq+l)/3$, and we have

$$\begin{aligned} |f'(x)| &= |2\alpha q(xq+l)| + |q\beta| \\ &\le 2q\left(\frac{1}{qN}\right)\left(\frac{2^{j+1}}{q}\cdot q + q\right) + q\cdot\frac{1}{2q} \\ &\le 2\left(\frac{1}{10\cdot 2^j}\right)\left(2^{j+1} + \left(1-\cdot 2^j\right)^{1-\epsilon}\right) + \frac{1}{2} \\ &< 1. \end{aligned}$$

So our f is legitimate for the theorem. Now $\phi(x) = (xq+l)^{-1}$. One can check this satisfies the conditions in the theorem as well. So we get

$$\text{dyadic sum} = \sum_{l=1}^{a} e^{2\pi i\left(l^2 a + lb\right)/q}\int_{|y|\approx\frac{2^j}{q}} e^{2\pi i\left((yq+l)^2\alpha + (yq+l)\beta\right)}\frac{dy}{yq+l} + O\left(q^{-2j}\right).$$

Set $x = yq+l$, then

$$\begin{aligned} \text{integral} &= \frac{1}{q}\int_{|x-l|\approx 2^j} e^{2\pi i\left(x^2\alpha + x\beta\right)}\frac{dx}{x} \\ &= \frac{1}{q}\int_{|x|\approx 2^j} e^{2\pi i\left(x^2\alpha + x\beta\right)}\frac{dx}{x} + \left(\frac{1}{q}\cdot q\cdot 2^{-j}\right). \end{aligned}$$

Finally we've shown

$$\sum_{|n|\approx 2^j}\frac{e^{2\pi i\left(n^2\theta + n\phi\right)}}{n} = \frac{1}{q}S\left(\frac{a}{q},\frac{b}{q}\right)\Phi\left(2^{2j}\alpha, 2^j\beta\right) + \text{acceptable error.}$$

with S the Gauss sum and

$$\Phi(u,v) = \int_{1\le|x|\le 2} e^{2\pi i\left(ux^2+vx\right)}\frac{dx}{x}.$$

Notice Φ looks familiar: it is exactly the dyadic part of the multiplier of the continuous Hilbert transform on the parabola.

We already know $|S\left(\frac{a}{q},\frac{b}{q}\right)| \le cq^{1/2}$. Now recall the

Lemma 3.13. *Let Φ be as above. Then,*

(1) $|\Phi(u,v)| \le A\left(|u|^{1/2} + |v|\right)$ *if* $|u|^{1/2} + |v| \le 1$

(2) $|\Phi(u,v)| \le A\left(|u|^{1/2} + |v|\right)^{-1/2}$ *if* $|u|^{1/2} + |v| \ge 1$.

Recall to prove (1) we used the fact that $\Phi(0,0) = 0$; to prove (2) we used stationary phase.

Now we'll fix θ,ϕ, and try to sum over all j. Let I_{a_j/q_j} denote the major arc θ lies in for j (if any). We'll sum over all j so that a_j/q_j is the same fraction a/q:

$$\sum_{\substack{j=0 \\ \frac{a_j}{q_j} = \text{ fixed } \frac{a}{q}}}^{\infty}\Phi\left(2^{2j}\alpha, 2^{2j}\beta\right) \le \sum_{j=0}^{\infty}|\Phi\left(2^{2j}\alpha, 2^{2j}\beta\right)| \le c$$

as a consequence of (i) and (ii) from the lemma. The point here is that in the sum

$$\sum_{\frac{a}{q},\frac{b}{q}}\frac{1}{q}S\left(\frac{a}{q},\frac{b}{q}\right)\chi_{I\left(\frac{a}{q},\frac{b}{q}\right)}(\theta)\sum_{\substack{j \\ \frac{a_j}{q_j}=q}}|\Phi|$$

we in fact only need to sum over one a and one b, since θ, ϕ are fixed. This is because the major arcs are disjoint. Thus

$$\begin{aligned}\sum_{\frac{a}{q},\frac{b}{q}} \frac{1}{q} S\left(\frac{a}{q},\frac{b}{q}\right) \chi_{I\left(\frac{a}{q},\frac{b}{q}\right)}(\theta) \sum_{\substack{j \\ \frac{a_j}{q_j}=q}} |\Phi| &\leq c \sum_q \frac{1}{q^{1/2}} \chi_{\left(\frac{a}{q},\frac{b}{q}\right)}(\theta) \\ &\leq c \sum_{j=0}^{\infty} \left(5^j\right)^{-1/2} \\ &\leq A,\end{aligned}$$

for by disjointness again, $q \geq 5q'$ (the q's are geometrically spaced). This completes the bound on the multiplier of the discrete parabolic Hilbert transform, and hence the boundedness of that transform.

3.4. **The Circle Method.** The proof we just gave is an example of the circle method. We had an operator

$$Tf(n) = \sum_{m \in \mathbb{Z}^k} f(n - P(m)) K(m)$$

with $P : \mathbb{Z}^k \to \mathbb{Z}^l$ a polynomial. Since this was a translation invariant convolution, it had a Fourier multiplier

$$m(\theta) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} e^{-2\pi i P(m) \cdot \theta} K(m)$$

where $\theta \in \mathbb{T}^l \simeq [0,1]^l$. To prove boundedness of T, we bounded the multiplier; for this we needed to understand the distribution of $P(m) \mod q$ for "lots of" q. So we constructed major/minor arcs, hence the name of the method.

Number theorists worked on problems like this historically. They would ask questions such as: For how many $x \in \mathbb{Z}^s$ is $F(x) = N$? A specific example is Waring's problem, where

$$F(x) = x_1^k + \cdots + x_s^k.$$

It's easier just to solve this for $|x_i| \leq N^{1/k} =: B$, so for $x \in \mathcal{B} = [-B, B]^s$. Then for $F(x) = N$, we would try to understand

$$\mathcal{N} = \sum_{x \in \mathcal{B}} 1 = \int_0^1 \sum_{x \in \mathcal{B}} e^{2\pi i (F(x) - N)\alpha} \, d\alpha.$$

Our work above is an example of this problem. So we've seen how number theorey contributes to analysis – perhaps we can go the other direction!

4. Discrete Fractional Integral Operators

In this section we'll discuss the (unfinished) theory of discrete fractional integral operators.

4.1. **The Main Conjecture.**

Consider the operator

$$I_{k,\lambda}f(n)=\sum_{m=1}^{\infty}\frac{f\left(n-m^k\right)}{m^\lambda}$$

for $k\geq 1$ and integer and $0<\lambda<1$. This is called a *fractional integral operator*. We've already seen that $I_{1,\lambda}:\ell^p\to\ell^q$ iff $q^{-1}\leq p^{-1}-(1-\lambda)$. For general k we can consider the continuous operator

$$\mathfrak{I}_{k,\lambda}f(x)=\int_{|y|\geq 1}\frac{f\left(x-y^k\right)}{y^\lambda}\,dy.$$

Under the change of variables $u=y^k$ we get

$$\mathfrak{I}_{k,\lambda}f(x)=\frac{1}{k}\int_{|u|\geq 1}f(x-u)\,\frac{du}{u^{1-(1-\lambda)/k}}.$$

Recall the

Theorem 4.1. *$\mathfrak{I}_{k,\lambda}:L^p\to L^q$ iff $q^{-1}=p^{-1}-(1-\lambda)/k$ with $1<p<q<\infty$.*

So we have the following conjecture.

Conjecture 4.2. *For $0<\lambda<1$, $I_{k,\lambda}:\ell^p\to\ell^q$ iff*

(1) $q^{-1}\leq p^{-1}-(1-\lambda)/k$
(2) $q^{-1}<\lambda, p^{-1}>1-\lambda$.

Example 4.3. Show the neccesary condition with the function

$$f(n)=\begin{cases}n^{-\gamma} & n\geq 1\\ 0 & n\leq 0\end{cases}.$$

Think of $\gamma=p^{-1}+\epsilon$. Also consider

$$\begin{cases}f(0)=1 & \\ f(n)=0 & n\neq 0\end{cases}.$$

In the conjecture, the $k=1$ case is easy (we've done it already). The case $k=2$ is hard, and was solved by Stein, Wainger, Oberlin, and Ionescu. The case $k\geq 3$ is unkown, and is equivalent to a 100 year old unsolved number theory conjecture (Pierce). We'll explore these results throughout the rest of this course.

The operatorour assumption implies $I_{k,\lambda}$ is translation invariant, so it has a Fourier multiplier:

$$\left(I_{k,\lambda}f\right)^{\wedge}(\theta)=m_{k,\lambda}(\theta)\,\hat{f}(\theta)$$

with

$$m_{k,\lambda}(\theta)=\sum_{m=1}^{\infty}\frac{e^{-2\pi i m^k\theta}}{m^\lambda}.$$

Note, however, that it's not enough for our purposes to produce a uniform bound on $m_{k,\lambda}$ (in contrast to what we've done thus far). Instead we must show a stronger property:

Taught by L. Pierce on July 13-15, 2011.

Proposition 4.4. *For all* $0 < \lambda < 1$,

$$|\{\theta : m_{k,\lambda}(\theta) > \alpha\}| \leq \alpha^{-r}$$

for all $\alpha > 0$ *and* $r = k/(1-\lambda)$.

Intuitively, this says that $m_{k,\lambda}$ cannot be very large very often. Technically, this condition means $m_{k,\lambda} \in L^{r,\infty}[0,1]$, which is a *Lorentz space* or *weak* L^r *space.*

Lemma 4.5 (Stein & Wainger)**.** *Let* T *be a convolution operator acting on functions* $f : \mathbb{Z} \to \mathbb{C}$ *with* F *and multiplier* m,

$$(Tf)^{\wedge}(\theta) = m(\theta)\hat{f}(\theta).$$

Then if $m \in L^{r,\infty}[0,1]$, *then* $T : \ell^p \to \ell^q$ *for all* $q^{-1} = p^{-1} - r^{-1}$ *and* $1 < p \leq 2 \leq q < \infty$.

Remark. The proof utilizes certain properties of Lorentz spaces and interpolation arguments. We won't do it here, but see L. Pierce's thesis for an exposition.

Let's make one more reduction. We claim it's sufficient to prove $m_{k,\lambda} \in L^{r,\infty}[0,1]$ for $\lambda_k^* < \lambda < 1$ where $\lambda_k^* = 1 - k/(2k-1)$. The proof of this is by standard interpolation arguments. With these observations, we now discuss the current approaches to the conjecture.

There are two methods to show $m_{k,\lambda} \in L^{r,\infty}[0,1]$ for $r = k/(1-\lambda)$ and $\lambda_k^* < \lambda < 1$. The first method is a circle method argument (as in the previous section). Write

$$m_{k,\lambda}(\theta) = \sum_{j=0}^{\infty} \sum_{|m| \approx 2^j} \frac{e^{-2\pi i m^k \theta}}{m^\lambda}$$

then consider θ in minor/major arcs w.r.t. j. Use a Weyl bound on the minor arcs. For the major arcs, decompose into an arithmetic component and an integral component. The arithmetic component will be

$$|S_k\left(\frac{a}{q}\right)| = |\sum_{1 \leq n \leq q} e^{-2\pi i a n^k/q}| \leq q^{1-1/k},$$

but herein lies the problem – the bound is exponential in k. With some work we can get the result in a restricted range, $1 - 1/2^k < \lambda < 1$. And with some harder work, we can get the result for small k in $1 - 1/\left(2^{k-1}+1\right) < \lambda < 1$ and for large k in $1 - 1/\left(\frac{3}{2}k^2 \log k\right) < \lambda < 1$. This approach is outlined in the supplied exercises.

Now we'll explore the second approach in detail. And we'll see how it leads to a historic problem from number theory.

4.2. **Bounding the Multiplier, A Second Approach.** For now we'll lower our expectations, and only try to understand the conjecture for $1/2 < \lambda < 1$. The following claim shows why we might chose such a range for λ.

Claim. $L^\infty[0,1] \subset \cdots \subset L^r[0,1] \subset L^{r,\infty}[0,1] \subset L^p[0,1]$ as long as $r > p$.

Proof. Suppose $m \in L^{r,\infty}[0,1]$. Set

$$\Lambda(\alpha) = |\{x \in [0,1] : |m(x)| > \alpha\}|,$$

then $|\Lambda(\alpha)| \leq 1$ and by assumption $|\Lambda(\alpha)| \leq \alpha^{-r}$. So

$$||m||_{L^p[0,1]} = \frac{1}{p}\int_0^\infty \alpha^{p-1}\Lambda(\alpha)\, d\alpha = \int_0^1 + \int_1^\infty .$$

We have

$$\int_0^1 \leq \int_0^1 \alpha^{p-1}\, d\alpha = O(1)$$

because $\Lambda(\alpha)$ is bounded and also

$$\int_1^\infty \leq \int_1^\infty \alpha^{p-1-r}\, d\alpha = O(1)$$

as $r > p$. Hence the result. $\square$

How does this claim help us? Consider $r = k/(1-\lambda)$ for $1/2 < \lambda < 1$. For such λ, $2k < r < \infty$ always, so we get the nesting property ensured in the lemma automatically. Hence we'll only need to show $m_{k,\lambda} \in L^{2k}[0,1]$ for $1/2 < \lambda < 1$. We have significantly lowered our expectations, but we'll get a weak version of the conjecture.

Let's see what it means to show $m_{k,\lambda} \in L^{2k}[0,1]$ for $1/2 < \lambda < 1$. So assume this to be true: this is equivalent to having $(m_{k,\lambda})^k \in L^2$ for $1/2 < \lambda < 1$, i.e.,

$$\left(\sum_{n=1}^\infty \frac{e^{-2\pi i n^k \theta}}{n^\lambda}\right)^k = \sum a_l e^{-2\pi i l\theta} \in L^2[0,1]$$

with

$$a_l = \sum_{\substack{n_1,\dots,n_k \\ l = n_1 k + \dots + n_k^k}} \frac{1}{n_1^\lambda \cdots n_k^\lambda}.$$

Note that $n_j^k \leq l \implies n_1^\lambda \cdots n_k^\lambda \leq l^\lambda \implies a_l \geq l^{-\lambda} r_{k,k}(l)$ where

$$r_{k,k}(l) = \#\text{ representations of } l = n_1^k + \dots + n_k^k.$$

So by Parseval's theorem $\sum |a_l|^2 < \infty$ and hence

$$\sum_{l=1}^\infty (r_{k,k}(l))^2 l^{-2\lambda} < \infty.$$

Now via partial summation,

$$\sum_{l=1}^N (r_{k,k}(l))^2 l^{-2\lambda} = \sum_{l=1}^N (r_{k,k}(l))^2 N^{-2\lambda} - \int_1^N \sum_{l=1}^u (r_{k,k}(l))^2 (-2\lambda) u^{-2\lambda-1}\, du,$$

and by assumption the first term on the right is bounded for all $1/2 < \lambda < 1$. So our assumption on the Fourier multiplier implies

$$\sum_{l=1}^N (r_{k,k}(l))^2 = O\left(N^{1+\epsilon}\right)$$

as $N \to \infty$.

What does this mean? This is saying that "on average" $r_{k,k}(l) \leq l^\epsilon$. Let's generalize a bit. Let

$$r_{s,k}(l) = \#\left\{x_1, \dots, x_s \in \mathbb{N} : l = x_1^k + \dots + x_s^k\right\}.$$

We know from Waring's problem that

$$r_{s,k}(l) \sim c_{s,k}(l)\, l^{s/k-1} + o\left(l^{s/k-1}\right)$$

for s sufficiently large w.r.t. k. Consider the case $k=2$. Number theory says

$$l = x^2 - y^2 = Q(x,y),$$
$$l = (x+iy)(x-iy) = d_1 d_2.$$

So for $k=2$ the "on average" statement is absolutely true (already known). For $k=3$, we consider

$$l = x_1^3 + x_2^3 + x_3^3.$$

Is it true that $r_{3,3}(l) \ll l^{\epsilon}$? This is not true, via Mahler in the 1930's. No one knows if the result is true for $k \geq 4$.

But really we're discussing averages. The first type of average is a sum

$$\sum_{l=1}^{N} r_{k,k}(l) = \sum_{l=1}^{N} \#\left\{l = x_1^k + \cdots + x_k^k\right\}.$$

This restricts $x_j \leq l^{1/k} \leq N^{1/k}$. So

$$\sum_{l=1}^{N} r_{k,k}(l) \leq \left(N^{1/k}\right)^k = N$$

by a simple combinatorics argument. This is a trivial estimate. Instead we should look at the second moment,

$$\sum_{l=1}^{N} \left(r_{k,k}(l)\right)^2.$$

Conjecture 4.6 (Hypothesis K^*)**.**

$$\sum_{l=1}^{N} \left(r_{k,k}(l)\right)^2 = O\left(N^{1+\epsilon}\right)$$

as $N \to \infty$ for all $k \geq 2$.

Is such a result known? For $k=2$,

$$\sum_{l=1}^{N} \left(l^{\epsilon}\right)^2 = O\left(N^{1+2\epsilon}\right)$$

and so the result. For $k=3$, Hooley and Heat-Brown assumed the generalized Reimann hypothesis and HW_6 and showed the $k=3$ result. But $k \geq 4$ is again unknown.

What about the other direction? It's true that Hypothesis K^* is equivalent to knowing $m_{k,\lambda}(\theta) \in L^{2k}[0,1]$ for all $1/2 < k < 1$. But now let's assume something slightly weaker than Hypothesis K^*; call

$$\sum_{l=1}^{N} \left(r_{k,k}(l)\right)^2 = O\left(N^{\beta+\epsilon}\right)$$

as $N \to \infty$ for $\beta \geq 1$ "Property $K^*(\beta)$." One can show $\beta \leq 2$. A more general property is

$$\sum_{l=1}^{N} \left(r_{s,k}(l)\right)^2 = O\left(N^{\beta+\epsilon}\right),$$

called "Property $K^*_{s,k}(\beta)$." Given $\beta > 0$, $m_{k,\lambda} \in L^{2s}[0,1]$ for all $\beta k/(2s) < \lambda < 1$ is equivalent to Property $K^*_{s,k}(\beta)$ being true. We know for large s w.r.t. k that

$$r_{s,k}(l) \sim c_{s,k}(l)\, l^{s/k-1} + \text{error}$$

for $s \ggg k$. This was improved to $s \gg 2^k$ and recently for $s \geq 2k^2 - s\left[\frac{\log k}{\log 2}\right]$ (Wooley, 2011). For $s = k$ we can apply Hua's inequality, etc. For small s w.r.t. k we get

$$\sum_{l=1}^{N} \left(r_{s,k}(l)\right)^2 = O\left(N^{s/k}\right)$$

if $s < k$, and in fact if $s \leq \frac{1}{4}\frac{\log k}{\log\log k}$ (Salsberger and Wooley, 2010).

The moral of the story: given an operator

$$\sum f(n - P(m))\, K(m)$$

with $\deg(P) > 2$, the problem is much, much harder. But we can discuss the quadratic case.

4.3. **The Quadratic Case and Theta Functions.** To keep things notationally simple, we'll write the $k = 2$ operator simply as

$$I_\lambda f(n) = \sum_{\substack{m \neq 0 \\ m \in \mathbb{Z}}} \frac{f\left(n - m^2\right)}{m^\lambda}.$$

Theorem 4.7 (Stein, Wainger, Oberlin, Ionescu)**.** *For $0 < \lambda < 1$, $I_\lambda : \ell^p \to \ell^q$ iff*

(1) $q^{-1} \leq p^{-1} - (1-\lambda)/k$
(2) $q^{-1} < \lambda_1, p^{-1} > 1 - \lambda$.

In fact we have shown this for $2/3 < \lambda < 1$; it turns out $1/2 < \lambda < 1$ is much more tractable than $0 < \lambda < 1$, and we'll discuss it now. First, recall it's sufficient to prove

$$m_\lambda(\theta) = \sum_{m \neq 0} \frac{e^{-2\pi i m^2 \theta}}{m^\lambda}$$

lies in $L^{r,\infty}[0,1]$ with $r = 2/(1-\lambda)$, for

$$\left|\{\theta : |m_\lambda(\theta)| > \alpha\}\right| \leq A\alpha^{-r}$$

for all $\alpha > 0$. Next, consider the following fact:

$$m^{-\lambda} = (2\pi)^{\lambda/2}\, \Gamma\left(\frac{\lambda}{2}\right) \int_0^\infty e^{-2\pi m^2 y} y^{\lambda/2-1}\, dy,$$

so we can write

$$m_\lambda(\theta) = c_{k,\lambda} \int_0^\infty \sum_{m \neq 0} e^{-2\pi m^2 (y + i\theta)} y^{\lambda/2-1}\, dy.$$

The sum

$$\sum_{m \neq 0} e^{-2\pi m^2 (y + i\theta)}$$

is our first example of a *Theta function.*

Historically, Jacobi considered the Theta function

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau}$$

with $\tau = x + iy$, $y > 0$. Note that by restricting the domain to $\mathbb{H}$ we ensure

$$\Theta(\tau) \leq \sum_n |e^{-\pi n^2 y}|$$

converges absolutely. The first thing to notice is periodicity: $\Theta(\tau + 2) = \Theta(\tau)$. Then the next thing to think of is $\Theta\left(-\frac{1}{\tau}\right)$. Historically, the aim was to show Θ is a "modular form."

To understand $\Theta(-1/\tau)$, we apply Poisson summation. Assume $\tau = iy$, for once we show a property on the imaginary axis, we can analytically continue to the rest of $\mathbb{H}$ and retain the property. Write

$$f(u) = e^{i\pi u^2 (iy)} = e^{-\pi u^2 y} = e^{-\pi\left(u\sqrt{y}\right)^2}$$

and so

$$\begin{aligned}
\hat{f}(v) &= \int_{-\infty}^{\infty} e^{-\pi\left(u\sqrt{y}\right)^2} e^{-2\pi i u v}\, du \\
&= \frac{1}{\sqrt{y}} \int_{-\infty}^{\infty} e^{-\pi w^2} e^{-2\pi i w\left(\frac{v}{\sqrt{y}}\right)}\, dw \\
&= \frac{1}{\sqrt{y}} e^{-\pi\left(\frac{v}{\sqrt{y}}\right)^2} \\
&= \frac{1}{\sqrt{y}} e^{-\pi v^2 / y}.
\end{aligned}$$

Then $\sum f(n) = \sum \hat{f}(n)$ says

$$\sum e^{i\pi n^2 (iy)} = \frac{1}{\sqrt{y}} \sum_n e^{i\pi n^2 (i/y)} = \left(\frac{i}{iy}\right)^{1/2} \sum_n e^{-i\pi n^2/(iy)}.$$

By analytic continuation to $\mathbb{H}$ (domain of analyticity) we get

$$\sum e^{i\pi n^2 \tau} = \left(\frac{i}{\tau}\right)^{1/2} \sum_n e^{i\pi n^2 (-1/\tau)}$$

or equivalently

$$\Theta(\tau) = \left(\frac{i}{\tau}\right)^{1/2} \Theta\left(-\frac{1}{\tau}\right).$$

This is called the "Jacobi inversion formula." We can rewrite as

$$\Theta\left(-\frac{1}{\tau}\right) = \left(\frac{i}{\tau}\right)^{-1/2} \Theta(\tau).$$

Note this result hinged on the fact that the Fourier transform of a Gaussian is again a Gaussian (i.e., we needed $k = 2$).

Going back to the proof, recall we have

$$m_\lambda(\theta) = c_{k,\lambda} \int_0^{\infty} \sum_{m \neq 0} e^{-2\pi m^2 (y + i\theta)} y^{\lambda/2 - 1}\, dy.$$

We'll make a reduction: we'll prove (1) $\int_1^\infty$ has absolutely convergent Fourier series, and hence (2) the corresponding operator maps $\ell^p \to \ell^q$ for all $q \geq p$. To compute the Fourier coefficients of $\int_1^\infty$ write

$$\sum_m e^{-2\pi i m^2 \theta} \int_1^\infty e^{-2\pi m^2 y} y^{\lambda/2-1}\, dy = \sum c_m e^{-2\pi i m^2 \theta}.$$

For $|m| = 1$,

$$c_1 = \int_1^\infty e^{-2\pi y} y^{\lambda/2-1}\, dy = O(1),$$

but this will not affect the convergence of the whole series. For $|m| \geq 2$,

$$\begin{aligned} |c_m| &\leq \int_1^\infty e^{-2\pi m^2 y} e^{2\pi y}\, dy \\ &\leq c \int_1^\infty e^{-2\pi\left(|m|^2-1\right)y}\, dy \\ &= O\left(\frac{e^{-|m|^2-1}}{|m|^2-1}\right) \end{aligned}$$

which has rapid decay. So (1) is proved. Now suppose T, a translation invariant operator, has multiplier m with absolutely convergent Fourier series. Then $Tf = f \star K$ where $K = m^\vee$. Herem

$$K(n) = \int_0^1 m(\theta)\, e^{2\pi i n\theta}\, d\theta.$$

By Young's inequality,

$$||Tf||_{\ell^p} \leq ||K||_{\ell^1} ||f||_{\ell^p},$$

so consider

$$\begin{aligned} ||K||_{\ell^1} &= \sum |K(n)| \\ &= \sum_n \left| \int_0^1 m(\theta)\, e^{2\pi i n\theta}\, d\theta \right| \\ &= \sum_n |c_{-n}(m)| \\ &\leq O(1). \end{aligned}$$

Thus $T : \ell^p \to \ell^q$ for all p, and by the nesting property we get this for all $q \geq p$.

We'll make one more reduction. By the nesting property

$$L^\infty[0,1] \subset \cdots \subset L^r[0,1] \subset L^{r,\infty}[0,1] \subset L^p[0,1]$$

we can consider the full sum $\sum_{m\in\mathbb{Z}} e^{-2\pi m^2 \tau}$, for

$$m_\lambda(\theta) + \int_0^1 y^{\lambda/2-1}\, dy = O(1)$$

when $0 < \lambda < 1$. Hence it's sufficient to study the Fourier multiplier

$$\nu_\lambda(\theta) = \int_0^1 \Theta(y + i\theta)\, y^{\lambda/2-1}\, dy$$

with

$$\Theta(z) = \sum_{m \in \mathbb{Z}} e^{-2\pi m^2 z}.$$

Note this is absolutely convergent for $\Re(z) > 0$ and uniformly convergent on $\Re(z) \geq \delta > 0$. Before we continue the proof, we'll consider a general Theta function to understand the transformation law.

4.4. **A General Theta Function.** Consider the Theta function associated to a posivite definite quadratic form Q,

$$\Theta_Q(z) = \sum_{m \in \mathbb{Z}^k} e^{-2\pi Q(m) z},$$

for $\Re(z) > 0$. One simple quadratic form is $Q(x_1, \ldots, x_k) = x_1^2 + \cdots + x_k^2 = |x|^2$, and has associated matrix

$$A = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}.$$

A more general quadratic form is $Q(x) = \frac{1}{2} x^t A x$ for a symmetric, positive definite matrix A with even diagonal entries. Given this general quadratic form Q, we'll define its adjoint quadratic form to be $Q^*(x) = \frac{1}{2} x^t A^{-1} x$. As it turns out,

$$\Theta_Q(z) \longleftrightarrow \Theta_{Q^*}\left(-\frac{1}{z}\right).$$

Let's try to understand this transformation law. Make the rational approximation $\theta = a/q + \alpha$. Then

$$\Theta_Q(y + i\theta) = \frac{1}{q^k |A| (y + i\alpha)^{k/2}} \sum_{m \in \mathbb{Z}^k} S_Q(a, -m; q)\, e^{\frac{-2\pi Q^*(m)}{q^2 (y + i\alpha)}}$$

where we have introduced the Gauss sum

$$S_Q(a, b; q) = \sum_{r \in (\mathbb{Z}/q\mathbb{Z})^k} e^{-2\pi i [a Q(r) + b \cdot r]/q}.$$

We'll set $S_Q(a; q) = S_Q(a, b; q)$ for convenience. Here is a key fact:

Fact 4.8. *If $(a, q) = 1$, then $|S_Q(a, b; q)| \leq c q^{k/2}$.*

The key step is to write the Jacobi inversion formula for Θ_Q in the following way:

$$\sum_{l \in \mathbb{Z}^k} e^{-2\pi Q(lq + r) z} = \frac{1}{q^k |A|^{1/2} z^{k/2}} \sum_{l \in \mathbb{Z}^k} e^{2\pi i r \cdot l/q} e^{-2\pi \frac{Q^*(l)}{q^2 z}}.$$

To show this identity holds on $\Re(z) > 0$:

(1) Assume $z = y$, $\Re(y) > 0$, for it's sufficient by analytic continuation.
(2) Use the Poisson summation formula with

$$f(x) = e^{-2\pi Q(xq + r) y}.$$

Then

$$\hat{f}(\xi) = \frac{e^{2\pi i r \cdot \xi/q}}{q^k |A|^{1/2} y^{k/2}} \cdot e^{-2\pi Q^*\left(\frac{\xi}{q\sqrt{y}}\right)}.$$

To compute this, write out the definition of the Fourier transform, diagonalize Q as $Q(x) = \sum_{i=1}^{k} \gamma_i x_i^2$, split up the integral into k one-dimensional integrals, and identify each as a normalized Gaussian.

Now to use this identity, we write

$$\begin{aligned}\Theta_Q\left(y + i\left(\frac{a}{q} + \alpha\right)\right) &= \sum_m e^{-2\pi Q(m)\left(y+i\left(\frac{a}{q}+\alpha\right)\right)} \\ &= \sum_{r\in(\mathbb{Z}/q\mathbb{Z})^k} \sum_{l\in\mathbb{Z}^k} e^{-2\pi Q(lq+r)\left(y+i\left(\frac{a}{q}+\alpha\right)\right)} \\ &\sum_{r\in(\mathbb{Z}/q\mathbb{Z})^k} e^{-2\pi i Q(r)a/q} \sum_{l\in\mathbb{Z}^k} e^{-2\pi Q(lq+r)(y+i\alpha)}.\end{aligned}$$

Calling $y + i\alpha = z$ and applying the identity, we get the aforementioned transformation law. And then we have isolated the Guass sum, and so we can bound it.

4.5. **Application to the Theorem.** In our simple case, $A = [2]$, $A^{-1}[1/2]$, $Q(m) = m^2$, and $Q^*(m) = m^2/4$. For $\theta = a/q + \alpha$,

$$\Theta(y + i\theta) = \frac{1}{q2^{1/2}(y+i\alpha)^{1/2}} \cdot \sum_{m\in\mathbb{Z}} S(a, -m; q)\, e^{-\frac{\pi m^2}{4q^2(y+i\alpha)}}$$

with

$$S(a, b; q) = \sum_{r \mod q} e^{-2\pi i r^2 a/q} e^{-2\pi i b r/q}.$$

And we have written $S(a/q) = S(a; q) = S(a, 0; q)$. We want write the sum as

$$\sum f(n) = \sum \hat{f}(n) = \hat{f}(0) + \sum_{n\neq 0} \hat{f}(n)$$

and identify the remaining sum as a small error. Let's assume the following the conditions:

(1) $(a, q) = 1$, $1 \leq a \leq q$
(2) $q \leq y^{-1/2}$
(3) $q|\alpha| = y^{1/2}$, $\theta = a/q + \alpha$.

Given these, we'll now prove the "approximate identity"

$$\Theta(y + i\theta) = \frac{S(a;q)}{q\sqrt{2}}(y + i\alpha)^{-1/2} + O\left(y^{-1/4}\right).$$

The first term is from $m = 0$, so we only need to consider the sum on $m \neq 0$:

$$\Big|\sum_{m\neq 0} \frac{|S|}{q\sqrt{2}(y+i\alpha)^{1/2}} e^{-\frac{\pi m^2}{4q^2(y+i\alpha)}}\Big| \leq \frac{C}{q^{1/2}(y^2+\alpha^2)^{1/4}} \sum_{m\neq 0} e^{-\frac{\pi m^2 y}{4q^2(y^2+\alpha^2)}}.$$

The problem here is that y could be very close to zero, but we want to show exponential decay. However, we know

$$\frac{y}{q^2(y^2+\alpha^2)} \gtrsim 1$$

because (2) gives $y \gtrsim q^2 y^2$ and (3) gives $\alpha^2 q^2 \lesssim y$. So

$$\sum_{m\neq 0} \leq c \sum_{m\neq 0} e^{-Cm^2 u} \leq ce^{-Cu} \leq u^{-1/4}$$

since $u \gtrsim 1$. In total, the error term satisfies

$$\text{error} \leq \frac{C}{q^{1/2}\left(y^2+\alpha^2\right)^{1/4}}\left(\frac{y}{q^2\left(y^2+\alpha^2\right)}\right)^{-1/4} = O\left(y^{-1/4}\right),$$

and hence the approximate identity.

What do the assumptions mean? The third means that as y gets closer to zero, the error α must get really small. But the second means the denominator cannot get very large. To keep track of the interplay here we'll define major/minor arcs. So write

$$\nu_\lambda(\theta) = \sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{-j+1}} \Theta(y+i\theta)\, y^{\lambda/2-1}\, dy.$$

Now the major arcs. Fix j and consider $2^{-j} \leq y < 2^{-j+1}$. For each $1 \leq q \leq \frac{1}{10}2^{j/2}$, consider $1 \leq a \leq q$ with $(a,q)=1$, and define the major arcs to be

$$\mathcal{M}_j\left(\frac{a}{q}\right) = \left\{\theta \in [0,1] : |\theta - \frac{a}{q}| \leq \frac{1}{q2^{j/2}}\right\}.$$

The minor arcs (for this j) are $[0,1]\setminus \cup_{a,q}\mathcal{M}_j(a/q)$. Again, the major arcs are disjoint, for if not and $(a,q)\neq(a',q')$ we get

$$\frac{1}{qq'} \leq |\frac{a}{q}-\frac{a'}{q'}| \leq |\theta-\frac{a}{q}| + |\theta - \frac{a'}{q'}| \leq \frac{1}{q2^{j/2}} + \frac{1}{q'2^{j/2}}$$

which leads to contradiction via our choice of upper bound on q. And again, it's convenient to have a version that is independent of j. Define for $(a,q)=1$,

$$\mathcal{M}^*\left(\frac{a}{q}\right) = \left\{\theta : |\theta - \frac{a}{q}| \leq \frac{1}{10q^2}\right\}.$$

The key properties:

(1) $\mathcal{M}_j\left(\frac{a}{q}\right) \subset \mathcal{M}^*\left(\frac{a}{q}\right)$
(2) $\mathcal{M}^*\left(\frac{a}{q}\right) \cap \mathcal{M}^*\left(\frac{a'}{q'}\right) = \emptyset$ if $q' \leq a \leq 2q'$.

If the q's are widely spaced, we might get intersections, but we'll try to avoid this in our final argument. And one can check that the three assumptions are satisfied if $2^{-j} \leq y < 2^{-j+1}$ and $\theta = a/q+\alpha$ with $|\alpha| \leq 1/\left(q2^{j/2}\right)$ and $1 \leq q \leq 2^{j/2}$.

There are three types of terms in $\nu_\lambda(\theta)$ we need to consider:

(1) main term of Θ on major arcs
(2) remainder term of Θ on major arcs
(3) all of Θ on the minor arcs.

We'll start with (3). The minor arcs are

$$\left\{\theta : |\theta - \frac{a}{q}| \leq \frac{1}{q2^{j/2}} \implies q \geq \frac{1}{10}2^{j/2}\right\}.$$

For fixed j,

$$
\begin{aligned}
|\Theta(y+i\theta)| &\le c\frac{|S(a;q)|}{q}|(y+i\alpha)|^{-1/2} + O\left(y^{-1/4}\right) \\
&\le q^{-1/2}y^{-1/2} + O\left(y^{-1/4}\right) \\
&\le c2^{-j/4}2^{j/2} + O\left(2^{j/4}\right) \\
&= O\left(2^{j/4}\right).
\end{aligned}
$$

Plugging this into $\nu_\lambda(\theta)$ yields

$$
\begin{aligned}
\sum_{j=1}^{\infty}\chi_{\min_j}(\theta)\int_{2^{-j}}^{2^{-j+1}}\Theta(y+i\theta)\,y^{\lambda/2-1}\,dy &= \sum_{j=1}^{\infty}\chi_{\min_j}(\theta)\int_{2^{-j}}^{2^{-j+1}}2^{j/4}y^{\lambda/2-1}\,dy \\
&\to \sum_{j=1}^{\infty}2^{j/4}2^{-j(\lambda/2)} < \infty
\end{aligned}
$$

if $\lambda > 1/2$. So we've proved $\nu_{\lambda,\min}(\theta) \in L^\infty[0,1]$. And (2) is even easier: write

$$
\sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{-j+1}} O\left(y^{-1/4}\right)y^{\lambda/2-1}\,dy = O(1)
$$

if $\lambda > 1/2$ and hence $\nu_{\lambda,\text{remainder}}(\theta) \in L^\infty[0,1]$.

Now we get to (1), the main term of the theta function on the major arcs. Let $\chi^j_{a/q}$ be the characteristic function of $\mathcal{M}_j(a/q)$, and $\chi_{a/q}$ the characteristic function of $\mathcal{M}^*(a/q)$. As usual, we'll fix a a/q pair and sum over all j (with fixed θ). The contribution of the main term of Θ is

$$
\frac{1}{q\sqrt{2}}S(a;q)\sum_{j=1}^{\infty}\chi^j_{a/q}(\theta)\int_{2^{-j}}^{2^{-j+1}}|y+i\left(\theta-\frac{a}{q}\right)|^{-1/2}y^{\lambda/2-1}\,dy.
$$

Now

$$
|\text{main contrib.}| \le cq^{-1/2}\chi_{a/q}(\theta)\int_0^1 y^{\lambda/2-1}|y+i\left(\theta-\frac{a}{q}\right)|^{-1/2}\,dy.
$$

The steps are (1) to show $\int_0^1 \le \int_0^\infty$and (2) to change variables $y \mapsto \alpha y$. The integral becomes

$$
|\alpha|^{\lambda/2-1/2}\int_0^\infty y^{\lambda/2-1}|y+i|^{-1/2}\,dy.
$$

Splitting it up:

$$
\int_0^1 \le \int_0^1 y^{\lambda/2-1}\,dy = O(1)
$$

since $\lambda > 0$ and $|y+i| \ge 1$;

$$
\int_1^\infty \le \int_1^\infty y^{\lambda/2-1}y^{-1/2}\,dy = O(1)
$$

since $\lambda < 1$, $|y+i| \ge y$. So the contribution to the Fourier multiplier from a fixed a/q pair is

$$
q^{-1/2}\chi_{a/q}(\theta)\,|\theta-\frac{a}{q}|^{-\frac{1}{2}(1-\lambda)}.
$$

Note. There could be a problem with the change of variables when $\alpha = 0$. But we could have just defined our major arcs to leave out an ϵ-neighborhood of $\alpha = 0$. After redoing the whole argument we get the result independent of ϵ, then take $\epsilon \to 0$.

Finally, we sum

$$\sum_{s=0}^{\infty} \sum_{2^s \le q < 2^{s+1}} \sum_{\substack{(a,q)=1 \\ 1 \le a \le q}} q^{-1/2} \chi_{a/q}(\theta) |\theta - \frac{a}{q}|^{-\frac{1}{2}(1-\lambda)}.$$

We wanted to check $m \in L^{r,\infty}[0,1]$, i.e., $|\{\theta : |m(\theta)| > \beta\}| < \beta^{-r}$ for $r = 2/(1-\lambda)$. As a first check, let's see if $g(u) = |u|^{-\frac{1}{2}(1-\lambda)} \in L^{r,\infty}[0,1]$. Since

$$|\left\{u : |u| < \beta^{-\left(\frac{2}{1-\lambda}\right)}\right\}| \le \beta^{-r},$$

the check passes. Now we need to sum this up.

Lemma 4.9. *Given N functions $f_1, \ldots, f_N$ with disjoint support and f_j uniformly in $L^{r,\infty}[0,1]$, then $F_N = N^{-1/r} \sum_{j=1}^{N} f_j$ belons to $L^{r,\infty}[0,1]$ uniformly in N.*

Proof. Since

$$\{|F_N| > \alpha\} = \cup_{j=1}^{N} \left\{N^{-1/r}|f_j| > \alpha\right\},$$

we get

$$|\{|F_N| > \alpha\}| = \sum_{j=1}^{N} |\left\{N^{-1/r}|f_j| > \alpha\right\}| \le \sum_{j=1}^{N} \left(N^{-1}\alpha^{-r}\right) \le \alpha^{-r}$$

uniformly in N. $\square$

Applying the lemma to the sum shows the $L^{r,\infty}$ norm of

$$\sum_{2^s \le q < 2^{s+1}} \sum_{\substack{(a,q)=1 \\ 1 \le a \le q}} q^{-1/2} \chi_{a/q}(\theta) |\theta - \frac{a}{q}|^{-\frac{1}{2}(1-\lambda)}$$

is

$$O\left(N^{1/r} 2^{-s/2}\right) = O\left(\left(2^{2s}\right)^{\frac{1-\lambda}{2}} 2^{-s/2}\right) = O\left(2^{s\left(\frac{1}{2}-\lambda\right)}\right).$$

Finally, our multiplier has $L^{r,\infty}[0,1]$ norm

$$\sum_{s=0}^{\infty} O\left(2^{s\left(\frac{1}{2}-\lambda\right)}\right) = O(1)$$

if $\lambda > 1/2$. So we have proved $I_\lambda : \ell^p \to \ell^q$ iff (i) $q^{-1} \le p^{-1} - (1-\lambda)/k$ and (ii) $q^{-1} < \lambda_1, p^{-1} > 1-\lambda$, under the condition $1/2 < \lambda < 1$. Note that the result holds for $0 < \lambda < 1$, but the proof for $0 < \lambda \le 1/2$ is much more difficult.

Appendix A. Discrete Operators, The Big Picture

Why should one care about discrete operators? In this addendum we will not be concerned with details; instead, we will focus on the "big picture." We'll discuss the relation of discrete operators to

(1) Ergodic Theory,
(2) PDE,
(3) Fourier Series,
(4) Number Theory.

A.1. **Ergodic Theory.** Consider a measure space (X, μ) with $\mu(X) = 1$. Let $T : X \to X$ be an invertible, measure-preserving map, i.e., $\mu\left(T^{-1}(E)\right) = \mu(E)$ for all sets $E \subset X$. We'll say T is *ergodic* if for every set E with $T^{-1}(E) = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. Intuitively, ergodic transformations "mush everything around."

Example A.1. Take $X = \text{circle} = \mathbb{R}/\mathbb{Z}$ and $\mu = ds$ ($\frac{1}{2\pi} ds$ on the circle). Then $T : x \mapsto x + \theta$ is ergodic iff θ is irrational.

There are the so-called "ergodic theorems." These consider different averages: the *time average*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right)$$

and the *space average*

$$\frac{1}{\mu(X)} \int f \, d\mu.$$

Theorem A.2 (Pointwise Ergodic Theorem). *Let $T : X \to X$ be an invertible, measure-preserving map. Then,*

(1) $\lim \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right)$ exists a.e. and the limit function F is in $L^1(X)$.
  (a) F is T-invariant, i.e., $F \circ T = F$
  (b) if also $0 < \mu(X) < \infty$ then $\int F \, d\mu = \int f \, d\mu$.

(2) If T is also ergodic, then

$$\lim_{n\to\infty} \textit{time average} = F = \frac{1}{\mu(X)} \int f \, d\mu.$$

Conclusion (2) follows from (1) via

Lemma A.3. *T is ergodic iff the T-invariant measurable functions f are constant a.e.*

Remark. There are also limiting theorems in L^p.

Now define

$$A_r f(x) = \frac{1}{2r+1} \sum_{|m| \le r} f\left(T^{P(m)} x\right)$$

with $P : \mathbb{Z} \to \mathbb{Z}$ a polynomial. This is a different type of average. Is there a similar pointwise ergodic theorem? Does there exists a limit F as $r \to \infty$? If $f \in L^p(X, \mu)$ is $L \in L^p(X, \mu)$?

Taught by L. Pierce on July 13, 2011.

Theorem A.4 (Bourgain, 1980's)**.** *If $f \in L^p(X,\mu)$ then there exists a function $F \in L^p(X,\mu)$ so that*

$$\lim_{r\to\infty} A_r f(x) = F$$

a.e. and in L^p. Moreover, if T is ergodic, then

$$F = \frac{1}{\mu(X)} \int f \, d\mu.$$

This is a corollary of the following key theorem:

Theorem A.5. *Define a discrete maximal function*

$$M_p f(n) = \sup_{r>0} \frac{1}{2r+1} \sum_{|m|\le r} |f(n-m)|.$$

Then $M_p : \ell^p \to \ell^p$ for $1 < p \le \infty$.

The proof of this theorem is uses similar methods as the proofs we've been seeing so far (e.g., the circle method). Indeed, each of the operators (for fixed r) have Fourier multipliers

$$\frac{1}{2r+1} \sum_{|m|\le r} e^{-2\pi i P(m)\theta}.$$

Remark. Similar results exist for operators

$$A_r f(x) = \frac{1}{2r+1} \sum_{|m|\le r} f\left(T^{P_m(m)} x\right)$$

where $P_m = T^m P$. Also, one can start to ask questions about

$$A_r f(x) = \frac{1}{2r+1} \sum_{|m|\le r} f\left(T^{P(m,n)} x\right),$$

but now there is a loss of translation-invariance.

A.2. **PDE's for Periodic Functions.** We'll consider the non-linear Schrodinger operator in a periodic setting:

$$\triangle_x u + i\partial_t u + u|u|^{p-2} = 0$$

where $u(x,t), x \in \mathbb{R}^d, t \in R$ is periodic of period one in x and with initial data $u(x,0) = \phi(x)$ for some function at $t=0$. In the non-periodic case, one first needs the "Strichartz inequality," which says that (for $p = 2(d+2)/d$)

$$\begin{aligned} ||e^{it\triangle}\phi||_{L^p(\mathbb{R}^{d+1})} &:= \left|\left| \int \hat{\phi}(\xi)\, e^{2\pi i\left(x\cdot\xi + t|\xi|^2\right)} d\xi \right|\right|_{L^p(\mathbb{R}^{d+1})} \\ &\le c||\phi||_{L^2(\mathbb{R}^d)}. \end{aligned}$$

In the periodic setting, instead one wants to show

$$\Big|\Big| \sum_{\substack{|n|\le N \\ n\in\mathbb{Z}^d}} a_n e^{2\pi i\left(n\cdot x + t\cdot|n|^2\right)} \Big|\Big|_{L^p(\mathbb{T}^{d+1})} \le C||\{a_n\}||_{\ell^2} := C\left(\sum_{|n|\le N} |a_n|^2\right)^{1/2}.$$

where the a_n is the nth Fourier coefficient of ϕ.

A.3. **Convergence of Fourier Series.** This is the oldest application of discrete analogues. Consider the partial sum

$$S_N f(\theta) = \sum_{n=0}^{N} a_f(n)\, e^{2\pi i n\theta}$$

where

$$a_f(n) = \int_0^1 e^{-2\pi i n\theta} f(\theta)\, d\theta.$$

The usual question to ask is: when does $S_N \to f$? Let's make this harder. Instead, we'll study periodic functions so that $a_f(n) = 0$ unless $n \in \{n_k\}$ for some sequence $\{n_k\}$. Now the Fourier series is

$$f(\theta) \sim \sum_{k=0}^{\infty} a_f(n_k)\, e^{2\pi i n_k\theta}.$$

Question: For what types of sequences $S = \{n_k\}$ must the partial sums of the Fourier series converge uniformly for all such f? Arkhipov and Oskolkov studied this problem, and saw that the key bound was

$$\Big| \sum_{1\le k\le M} \frac{e^{2\pi i n_{k+N}\theta}}{n} \Big| \le A$$

uniformly in $M, N, \theta \in [0,1]$. This looks like the Fourier multiplier of the Hilbert transform

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f(n - S(k))}{k}.$$

A.4. **Number Theory.** Let's step back to the continuous world. Define

$$\mathcal{A}_* f(x) = \sup_{0<\lambda<\infty} \frac{1}{S(\lambda)} \int_{|y|=\lambda} |f(x-y)|\, dy.$$

Here is a deep theorem.

Theorem A.6 (Stein, Stein & Wainger, Bourgain)**.** *$\mathcal{A}_*$ is bounded on $L^p\left(\mathbb{R}^k\right)$ iff $k \ge 2$, $p > \frac{k}{k-1}$.*

Remark. This is an amazing theorem, and its proof took three papers. Consider that $f \in L^p$ will remain in L^p even if its values are changed on a set of measure zero. In particular, $|y| = \lambda$ has measure zero! The proof heavily utilizes the idea of Fourier transform on surfaces with curvature.

In the discrete setting, we have

$$A_* f(x) = \sup_{r>0} \frac{1}{N(r)} \sum_{\substack{|m|=r \\ m\in\mathbb{Z}^k}} |f(n-m)|$$

where $N(r) = \#\left\{m \in \mathbb{Z}^k : |m|^2 = r^2\right\}$. And the sup is taken over only r so that $N(r) \neq 0$.

Theorem A.7 (Magyar-Stein-Waingar, 2002)**.** *A_* is bounded on $\ell^p\left(\mathbb{Z}^k\right)$ iff*

(1) $k \ge 5$ and $p > \frac{k}{k-2}$
(2) $k \le 4$ and $p = \infty$.

Let's try to understand condition (2). Make the notation $r_{k,2}\left(r^2\right) = N(r)$. For $k \geq 5$,

$$r_{k,2}(n) \sim c_k(n)\, n^{k/2-1} + o\left(n^{k/2-1}\right)$$

where $c_k(n) \geq \delta > 0$. What about $k = 4$ (and lower)? We know that every positive integer can be written as the sum of four squares. But this is not unique! So c_k in the above can be zero sometimes. But consider

$$r_{4,2}(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d$$

and consider that $r_{4,2}\left(2^{2t}\right) = 8(1+2) = 24$ for all t. So set $f(0) = 1$ and $f(n) = 0$ for all $n \neq 0$. Then for $|n| = 2^t$,

$$A_* f(n) \geq \sup_t \frac{1}{N(2^t)} \sum_{|m|=2^t} |f(n-m)| \geq \sup_t \frac{1}{N(2^t)} \geq \frac{1}{24},$$

where $N(2^t) = r_{4,2}\left((2^t)^2\right)$. This completes the counter-example.

What is really going on here? We have

$$\mathrm{Vol}(B_R) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n,$$

$$\mathrm{SA}(B_R) = \frac{d\mathrm{Vol}}{dR} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} R^{n-1}.$$

Renormalize with $R = N^{1/2}$ and write

$$\mathrm{Vol}(B_R) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} N^{n/2},$$

$$\mathrm{SA}(B_R) = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} N^{\frac{n}{2}-\frac{1}{2}}.$$

The discrete analog is

$$\#\{m \in \mathbb{Z}^n : |m| \leq N\} = \mathrm{Vol}(B_{N^{1/2}}) + \text{error},$$

$$\#\{m \in \mathbb{Z}^n : |m| = N\} \approx c_n(N)\, N^{\frac{n}{2}-1} + \text{error}.$$

The power here is different than in the continuous case! Gauss looked at this (Gauss' circle problem). Gauss was able to show the number of lattice points inside the circle goes like $\pi r^2 + O(r)$, but Hardy and Landau showed the error $\geq r^{1/2+\epsilon}$. Recently we've gotten to $O\left(r^{0.37}\right)$.