HARMONIC ANALYSIS AND NONLINEAR DISPERSIVE EQUATIONS

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Abstract. These notes were taken in the third week of the 2011 Study Analysis and Geometry summer program at Princeton.

Contents

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1. The Linear Schrodinger Equation

We begin with the linear Schrodinger equation,

$$(i\partial_t + \triangle_x)\, u = 0$$

for $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$, and with initial data $u(x,0) = \phi(x)$. Taking the Fourier transform on $\mathbb{R}^d$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi}\, dx,$$

the equation becomes

$$i\partial_t \hat{u}(\xi,t) - |\xi|^2 \hat{u}(\xi,t) = 0.$$

This has the solution

$$\hat{u}(\xi,t) = \hat{\phi}(\xi)\, e^{-it|\xi|^2}.$$

So for $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ we may write

$$u(x,t) = \left(e^{it\Delta}\phi\right)(x) = \int_{\mathbb{R}^d} \phi(x-y)\, K(y,t)\; dy$$

with kernel

$$K(y,t) = Ct^{-d/2} e^{i|x|^2/(4t)}.$$

Here are some properties of the linear Schrodinger flow:

Proposition 1.1. *For initial data ϕ:*

1. $||e^{it\Delta}\phi||_{L^2(\mathbb{R}^d)} = ||\phi||_{L^2(\mathbb{R}^d)}$ *for all* $t \in \mathbb{R}$.
2. $e^{it\Delta} \circ e^{is\Delta} = e^{i(t+s)\Delta}$ *for all* $t, s \in \mathbb{R}$.
3. $e^{it\Delta} \circ \tau_{x_0} = \tau_{x_0} \circ e^{it\Delta}$ *for all* $t \in \mathbb{R}, x_0 \in \mathbb{R}^d$, *where* $(\tau_{x_0} f)(x) = f(x - x_0)$.
4. $e^{it\Delta} \circ \delta_r = \delta_r \circ e^{i\left(t/r^2\right)\Delta}$ *for all* $t \in \mathbb{R}, r \in \mathbb{R}_+$, *where* $(\delta_r f)(x) = f(x/r)$.

Exercise 1.2. (a) Prove these identities.
(b) Is it true that $\lim_{t\to 0} e^{it\Delta}\phi = \phi$? In what sense and for what kind of functions ϕ?

2. The Nonlinear Schrodinger Equation (NLS)

The simplest nonlinearity to introduce is the so-called *power nonlinearity*. The equation is now

$$(i\partial_t + \triangle_x)\, u = \mu u |u|^{p-1}$$

with initial data $u(x,0) = \phi(x)$. Here, $\mu = \pm 1$ and $p > 1$. Taking the Fourier transform and solving the ode results in the Duhamel formula:

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta} N(s)\; ds$$

where

$$N(s) = \mu u(t)\,|u(t)|^{p-1}.$$

This is not an explicit formula for u. But we can approximate u via a perturbative iteration scheme,

$$u_1(t) = e^{it\Delta}\phi,$$

$$u_{n+1}(t) = e^{it\Delta\phi} - i\int_0^t e^{i(t-s)\Delta}\left[\mu u_n(s)\,|u_n(s)|^{p-1}\right]\, ds.$$

We can only hope for this scheme to work for short enough time and for small enough functions. This is to ensure the second term does nothing more than correct small errors in the initial guess. But sometimes a different approximation scheme is needed. Then we'd write

$$i\partial_t u - u|u|^{p-1} = -\triangle_x u,$$

and with the anzatz $|u| = ct$ (??) we'd get

$$u(x,t) = u(x,0)\, e^{-it|u(x)|^{p-1}}.$$

This would produce an iterative scheme to deal with functions which are large in short time. But in this lecture we'll assume we're working where the first scheme is convergent.

It will be useful to understand which quantities are conserved under the time evolution of NLS.

Exercise 2.1. Prove (formally) the conservation of mass and energy for solutions of the NLS: if u is a "nice" solution on an interval I, then the quantities

$$\begin{aligned} M(t) &= \int_{\mathbb{R}^d} |u(x,t)|^2\, dx, \\ E(t) &= \frac{1}{2}\int_{\mathbb{R}^d} |\triangle_x u(x,t)|^2\, dx + \frac{\mu}{p+1}\int_{\mathbb{R}^d} |u(x,t)|^{p+1}\, dx \end{aligned}$$

are conserved.

Now assume $\phi \in H^\sigma(\mathbb{R}^d)$. For the iteration scheme to work, we need a space $X^\sigma = X^\sigma(I)$ so that

$$e^{it\Delta}\phi \in X^\sigma(I) \text{ if } \phi \in H^\sigma$$

and so that

$$\int_0^t e^{i(t-s)\Delta}\left(f(s)\,|f(s)|^2\right) ds \in X^\sigma(I) \text{ if } f \in X^\sigma(I).$$

Exercise 2.2. Assume $d = 3, p = 3$. Show the perturbative scheme produces a solution in $X^\sigma(I) = C(I : H^\sigma)$, provided that σ is sufficiently large and $|I|$ is sufficiently small. Hint: Sobolev embedding.

The perturbative scheme only produces local solutions. To go from local to global we have to use the conserved quantities. Another useful tool is the Strichartz estimates:

(1) If (q,r) is an admissible pair, i.e., $(q,r) \in (2,\infty] \times [2,\infty]$, $2/q + d/r = d/2$, then

$$||e^{it\Delta}\phi||_{L_t^q L_x^r} \lesssim_q ||\phi||_{L^2}.$$

(2) If $(q,r), (\overline{q},\overline{r})$ are admissible pairs, then

$$\Big|\Big|\int_{-\infty}^t e^{i(t-s)\Delta} N(s)\Big|\Big|_{L_t^q L_x^r} \lesssim_{q,\overline{q}} ||N||_{L_t^{\overline{q}'} L_x^{\overline{r}'}}.$$

Let's try to understand what (1) means. Let $d = 4$, and $q = r$, then we get

$$||e^{it\Delta}\phi||_{L^3_{x,t}} \lesssim ||\phi||_{L^2}.$$

This says the L^3-norm of $e^{it\Delta}\phi$ is finite for a.e. t, so long as we have $\phi \in L^2$. So the forward-time evolution has a smoothing effect. And this also says that the L^3-norm of $e^{it\Delta}\phi$ has to decay forward in time. In the next section we'll prove the Strichartz estimates.

3. Proof of the Strichartz Estimates

First we write the Strichartz estimates in a more conceptual way:

(1) If (q,r) is an admissible pair, i.e., $(q,r) \in (2,\infty] \times [2,\infty]$, $2/q + d/r = d/2$, then

$$||e^{it\Delta}\phi||_{L_t^q L_x^r} \lesssim_q ||\phi||_{L^2}.$$

(2) If (q,r) is an admissible pair, then

$$||\int_{-\infty}^{t} e^{i(t-s)\Delta} N(s)||_{L_t^\infty L_x^2 \cap L_t^q L_x^r} \lesssim_q ||N||_{L_t^1 L_x^2 + L_t^{q'} L_x^{r'}}.$$

Exercise 3.1. (a) Show that if $f : \mathbb{R}^d \times I \to \mathbb{C}$ is a measurable function and $p,q \in [1,\infty]$ then

$$||f||_{L_t^p L_x^q} = \sup_{||g||_{L_t^{p'} L_x^{q'}}=1} \Big|\int_{\mathbb{R}^d \times I} fg\, dxdt\Big|.$$

(b) Show that if $(q,r), (\overline{q},\overline{r})$ are admissible pairs, $q \le \overline{q}$, then

$$||f||_{L_t^{\overline{q}} L_x^{\overline{r}}} \le ||f||_{L_t^q L_x^r \cap L_t^\infty L_x^2},$$
$$||f||_{L_t^{\overline{q}'} L_x^{\overline{r}'}} \ge ||f||_{L_t^{q'} L_x^{r'} + L_t^1 L_x^2}.$$

Our goal is to prove the Strichartz estimates. First we recall the concept of fractional integration.

Exercise 3.2 (Fractional Integration)**.** Prove that

$$||f \star |y|^{-\gamma}||_{L^q(\mathbb{R}^d)} \le C_{p,q}||f||_{L^p(\mathbb{R}^d)}$$

if $0 < \gamma < d$, $1 < p < q < \infty$, and $q^{-1} = p^{-1} - (d-\gamma)/d$.

And next, complex interpolation.

Exercise 3.3 (Complex Interpolation)**.** Let $S = \{z \in \mathbb{C} : \Re(z) \in [0,1]\}$ and assume that $T_z : S(\mathbb{R}^d) \to L^1_{\text{loc}}(\mathbb{R}^d)$, $z \in S$, is an analytic family of operators, i.e., the map

$$z \mapsto \int_{\mathbb{R}^d} T_z(f) \cdot g\, dx, \quad f,g \in S(\mathbb{R}^d)$$

is analytic (and bounded) in $\text{int}(S)$ and continuous in S. Assume that $p_0, q_0, p_1, q_1 \in [1,\infty]$ and

$$||T_{iy}||_{L^{p_0} \to L^{q_0}} \le M_0,$$
$$||T_{1+iy}||_{L^{p_1} \to L^{q_1}} \le M_1.$$

Then, for all $\theta \in [0,1]$

$$||T_\theta||_{L^{p_\theta} \to L^{q_\theta}} \le M_0^{1-\theta} M_1^\theta$$

where $p_\theta^{-1} = (1-\theta)/p_0 + \theta/p_1$, $q_\theta^{-1} = (1-\theta)/q_0 + \theta/q_1$.

Now we prove the first Strichartz estimate. We have the dispersive estimates

$$||e^{it\Delta}||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim 1,$$
$$||e^{it\Delta}||_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}.$$

By interpolation,

$$||e^{it\Delta}||_{L^{r'}(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} \lesssim |t|^{-d(1/2-1/r)}.$$

Now we have the TT^* argument. We want

$$||e^{it\Delta}\phi||_{L_t^q L_x^r} \lesssim_q ||\phi||_{L^2}.$$

This is equivalent to

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}} \left(e^{it\Delta}\phi\right)(x)\, g\,(x,t)\ dxdt\Big| \lesssim_q ||\phi||_{L^2}||g||_{L_t^{q'}L_x^{r'}},$$

which is equivalent to

$$\Big|\Big|\int_{\mathbb{R}^d\times\mathbb{R}} e^{-it|\xi|^2}e^{ix\cdot\xi}g\,(x,t)\ dxdt\Big|\Big|_{L^2_\xi} \lesssim_q ||g||_{L_t^{q'}L_x^{r'}},$$

which is equivalent to

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}}\int_{\mathbb{R}^d\times\mathbb{R}} g\,(x,t)\,\overline{g\,(y,s)}K\,(x-y,t-s)\ dxdtdyds\Big| \lesssim_q ||g||^2_{L_t^{q'}L_x^{r'}}$$

where K is the kernel of the free Schrodinger flow. Note that to use Fubini's theorem here we needed absolute integrability. This does not hold in general, but we can instead consider g to be a simple function, or perhaps in the Schwartz class. Then we can use a density argument to extend to more general g.

But by the dispersive estimate,

$$\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\,(x,t)\,h\,(y,x)\,K\,(x-y,t-s)\ dxdy\Big| \lesssim G\,(t)\,G\,(s)\,|t-s|^{-d(1/2-1/r)}$$

where

$$G\,(t) = \left[\int_{\mathbb{R}^d} |g\,(x,t)|^{r'}\ dx\right]^{1/r'}.$$

The desired bound now follows from Holder's inequality and fractional integration. The admissibility condition will drop out of this as an algebraic condition. And note that to apply the fractional integration result we need $q>2$. The theorem turns out to be true for $q=2, d\geq 3$, but this is a more difficult (and recent) result due to Tao.

Exercise 3.4. Prove the remaining Strichartz estimates.

4. Global Well-Posedness of Cubic NLS

In this section we'll discuss global well-posedness of the cubic nonlinear Schrodinger equation,

$$(i\partial_t + \triangle_x)\,u = u|u|^2$$

with initial data $u\,(x,0) = \phi\,(x)$ on $\mathbb{R}^3$. Recall the Duhamel formula,

$$u\,(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}N\,(s)\ ds$$

where

$$N\,(t) = u\,(t)\,|u\,(t)|^2.$$

We're looking to find a solution via the perturbative iteration scheme,

$$u_1(t) = e^{it\Delta}\phi,$$
$$u_{n+1}(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(u_n(s)\,|u_n(s)|^2\right)\,ds.$$

We need to construct a space in which the u_n form a Cauchy sequence. Throughout we'll use the Strichartz estimates which were proved above. And we also have the Sobolev imbedding: if $3/s = 3/r - 1$, $r, s \in (1, \infty)$, then

$$||f||_{L^s(\mathbb{R}^3)} \lesssim ||f||_{L^r(\mathbb{R}^3)} + ||\nabla f||_{L^r(\mathbb{R}^3)}.$$

Now assume $\phi \in H^1(\mathbb{R}^3)$ and fix an admissible pair (q, r). Define

$$X^1(I) = \left\{ f \in C\left(I : H^1\left(\mathbb{R}^3\right)\right) : ||f||_{X^1(I)} := ||f||_{L_t^\infty L_x^2 \cap L_t^q L_x^r} + ||\nabla f||_{L_t^\infty L_x^2 \cap L_t^q L_x^r} < \infty \right\}.$$

Exercise 4.1. Prove that $X^1(I)$ is a Banach space.

Lemma 4.2. *If $\phi \in H^1(\mathbb{R}^3)$ then $e^{it\Delta}\phi \in X^1(I)$ and $||e^{it\Delta}\phi||_{X^1(I)} \le C_0||\phi||_{H^1}$.*

Lemma 4.3. *If $f, g, h \in X^1(I)$ then*

$$F := \int_0^t e^{i(t-s)\Delta}\left(f(s)\,g(s)\,h(s)\right)\,ds \in X^1(I)$$

and

$$||F||_{X^1(I)} \le C|I|^\delta ||f||_{X^1(I)}||g||_{X^1(I)}||h||_{X^1(I)}.$$

Now we construct the nonlinear solution. Let u_n be as given above. Assume $||\phi||_{H^1} \le A$. The two lemmas show that $\{u_n\}_{n\ge1}$ is a Cauchy sequence in the complete metric space

$$Y(I) := \left\{ f \in X^1(I) : ||f||_{X^1(I)} \le 2C_0 A \right\}$$

if $|I|$ is sufficiently small (depending on A). The first conclusion to draw is that, assuming $\phi \in H^1(\mathbb{R}^3)$, there is $\epsilon = \epsilon(||\phi||_{H^1})$ small and a unique solution $u \in X^1(-\epsilon, \epsilon)$ of the equation

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(u(s)\,|u(s)|^2\right)\,ds.$$

Moreover, the mapping $\phi \mapsto u$ is continuous from $H^1 \to X^1(-\epsilon, \epsilon)$.

Exercise 4.4. Prove conservation of mass and energy for the solution u constructed above.

Theorem 4.5 (Global Well-Posedness of Cubic NLS)**.** *Assuming that $\phi \in H^1(\mathbb{R}^3)$ and $I \subset \mathbb{R}$ is a bounded interval, there is a unique solution $u \in X^1(I)$ of the equation $u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(u(s)\,|u(s)|^2\right)ds$. Moreover, the mapping $\phi \mapsto u$ is continuous from $H^1 \to X^1(-\epsilon, \epsilon)$.*

5. Behavior of Solutions at Infinity

Our main result thus far was global well-posedness for the cubic nonlinear Schrodinger equation. Motivated by the Duhamel formula, we proposed a perturbative iterative scheme. This led to local existence and uniqueness, and then global existence and uniqueness.

In this section we'll try to understand how the solution behaves at $t = \infty$. Recall that

$$X^1(I) = \left\{ f \in C\left(I; H^1\left(\mathbb{R}^3\right)\right) : ||f||_{X^1(I)} := ||f||_{L_t^\infty L_x^2 \cap L_t^q L_x^r} + ||\nabla f||_{L_t^\infty L_x^2 \cap L_t^q L_x^r} < \infty \right\}.$$

Definition 5.1. We say that the solution u *scatters at* $+\infty$ if there is $u_+ \in H^1\left(\mathbb{R}^3\right)$ such that

$$\lim_{t \to +\infty} e^{-it\Delta} u(t) = u_+$$

in H^1. Similarly, we say that the solution u *scatters at* $-\infty$ if there is $u_- \in H^1\left(\mathbb{R}^3\right)$ such that

$$\lim_{t \to -\infty} e^{-it\Delta} u(t) = u_-$$

in H^1.

Since $e^{it\Delta}$ is an isometry, scattering at $+\infty$ means that u behaves like a solution of the linear equation cubic

$$(i\partial_t + \triangle) v = 0$$

with initial data $v(0) = u_+$, and similarly for scattering at $-\infty$. We'd like to demonstrate scattering for solutions of the cubic NLS, i.e., that

$$u_+ = \phi - i \lim_{t \to +\infty} \int_0^t e^{-is\Delta} \left(u(s) |u(s)|^2\right) ds.$$

Theorem 5.2. *For every asymptotic state $u_+ \in H^1$ there is a unique H^1 solution u that scatters to u_+. Moreover, the* wave operater $\Omega_+ : H^1 \to H^1$, $\Omega_+(u_+) = u(0)$ *is well-defined and continuous.*

Proof. Step 1. Evolution from $t = +\infty$ to $t = t_0$: solve

$$u(t) = e^{it\Delta} u_+ + i \int_t^\infty e^{i(t-s)\Delta} \left(u(s) |u(s)|^2\right) ds$$

using a fixed point argument in the metric spcae $X^1 \cap B_\delta\left(Y^1\right)$ with norm

$$||f||_{Y^1} := ||f||_{L^5_{x,t}} + ||f||_{L_t^q L_x^r} + ||\nabla f||_{L_t^q L_x^r}.$$

Step 2. Evolve from $t = t_0$ to $t = 0$: solve

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} \left(u(s) |u(s)|^2\right) ds$$

as in the global well-posedness theory. □

Is the wave operator surjective? This is called *asymptotic completeness*. The answer is "yes" in the defocusing case ($\mu = 1$) and "no" in the focusing case ($\mu = -1$).

Exercise 5.3. Let

$$M_a(t) = 2\Im \int_{\mathbb{R}^3} \partial_j a(x) \cdot \overline{u(x,t)} \partial_j u(x,t)\, dx.$$

Show that

$$\partial_t M_a(t) = 4\Re \int_{\mathbb{R}^3} \partial_j \partial_k a \cdot \partial_j \overline{u} \partial_k u \, dx - \int_{\mathbb{R}^3} \triangle^2 a \cdot |u|^2 \, dx + \int_{\mathbb{R}^3} \triangle a \cdot |u|^4 \, dx$$

in the defocusing case.

Exercise 5.4 (Spacetime bound implies asymptotic completeness)**.** Prove scattering assuming that $||u||_{L^4_{x,t}} \lesssim 1$.

Exercise 5.5. In the defocusing case, prove the spacetime bound $||u||_{L^4_{x,t}} \lesssim 1$.

6. Analysis of the Periodic Case

In this section we'll discuss some results for periodic NLS. We begin with a review of Fourier analysis on the torus.

6.1. **Fourier Analysis on $\mathbb{T}^d$.** First we recall Fourier analysis on $\mathbb{T}^d = [\mathbb{R}/(2\pi\mathbb{Z})]^d$. For $f \in L^1(\mathbb{T}^d)$ we set

$$\mathcal{F}f(n) = \hat{f}(n) = \int_{\mathbb{T}^d} f(x)\, e^{-in\cdot x}\, dx$$

with $n \in \mathbb{Z}^d$.

Theorem 6.1 (Fourier Inversion Formula)**.** *If $f \in L^1(\mathbb{T}^d)$ and $\hat{f} \in L^1(\mathbb{Z}^d)$, then*

$$f(x) = (2\pi)^{-d} \sum_{n \in \mathbb{Z}^d} \hat{f}(n)\, e^{in\cdot x}$$

for $x \in \mathbb{Z}^d$.

Theorem 6.2 (Plancherel's Theorem)**.** *The operator $(2\pi)^{-d/2}\mathcal{F}$ defines an isometry $L^2(\mathbb{T}^d) \to L^2(\mathbb{Z}^d)$.*

Exercise 6.3. Prove the Fourier inversion formula, the Plancerel theorem, and three more interesting identities related to the Fourier transform on $\mathbb{T}^d$. (Think of products, convolutions, derivatives, action of isometries, etc.)

Exercise 6.4. (a) Assume $1 \leq p \leq q \leq \infty$. Prove that

$$L^p(\mathbb{Z}^d) \hookrightarrow L^q(\mathbb{Z}^d),$$
$$L^q(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d).$$

(b) Prove the Sobolev imbedding inequality: if $d/q = d/p - 1$ with $p, q \in (1, \infty)$ then

$$||f||_{L^q(\mathbb{T}^d)} \lesssim ||f||_{L^p(\mathbb{T}^d)} + ||\nabla f||_{L^p(\mathbb{T}^d)}.$$

Now we consider the Littlewood-Paley projections. Assume that $\eta : \mathbb{R}^d \to [0,1]$ is a smooth function with $\eta(\xi) = 1$ if $|\xi| \leq 1$ and $\eta(\xi) = 0$ if $|\xi| \geq 2$. For $k = 0, 1, \ldots,$ define

$$\eta_{\leq k}(\xi) = \eta\left(\frac{\xi}{2^k}\right),$$
$$\eta_k(\xi) = \eta_{\leq k}(\xi) - \eta_{\leq k-1}(\xi),$$

and

$$P_{\leq k}f = \mathcal{F}^{-1}\left[\eta_{\leq k}(\xi)\cdot\hat{f}(\xi)\right],$$
$$P_k = P_{\leq k} - P_{\leq k-1},$$

where, by definition, $\eta_{\leq -1} = 0$ and $P_{\leq -1} = 0$. So

$$P_0 + P_1 + \cdots + P_k = P_{\leq k}$$
$$P_0 + P_1 + \ldots = \mathrm{id}\,.$$

Exercise 6.5. (a) Prove the Poisson summation formula: if $f, \hat{f} \in L^1\left(\mathbb{R}^d\right)$ then

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\,.$$

(b) Prove that for all $k \geq 0$ and $p \in [1,\infty]$,

$$||P_{\leq k}f||_{L^p(\mathbb{T}^d)} \lesssim ||f||_{L^p(\mathbb{T}^d)}.$$

6.2. **The Defocusing Periodic NLS.** We'll use these tools to understand the defocusing periodic NLS:

$$(i\partial_t + \triangle_x)\,u = u|u|^{p-1}$$

with initial data $u(x,0) = \phi(x)$ and where $\mu = \pm 1$, $p > 1$. The quantities

$$M(t) = \int_{\mathbb{T}^d} |u(x,t)|^2\,dx,$$
$$E(t) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla_x u(x,t)|^2\,dx + \frac{1}{p+1}\int_{\mathbb{T}^d} |u(x,t)|^{p+1}\,dx$$

are conserved. Again we set up a perturbative iteration scheme:

$$u_1(t) = e^{it\Delta}\phi,$$
$$u_{n+1}(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(\mu u_n(s)\,|u_n(s)|^{p-1}\right)\,ds.$$

As in the Euclidean case, we have local well-posedness in H^σ for large enough σ (via Sobolev embedding). Aiming towards global well-posedness, we'd like to prove local well-posedness in $H^1\left(\mathbb{T}^d\right)$. But this is not so easy – it depends on the Strichartz estimates.

Recall the Strichartz estimates on Euclidean space:

$$||P_{\leq k}e^{it\Delta}f||_{L^{(2d+4)/d}_{x,t}} \lesssim ||f||_{L^2(\mathbb{R}^d)},$$
$$||P_{\leq k}e^{it\Delta}f||_{L^\infty_{x,t}} \lesssim 2^{kd/2}||f||_{L^2(\mathbb{R}^d)}.$$

And by interpolation,

$$||P_{\leq k}e^{it\Delta}f||_{L^p_{x,t}} \lesssim 2^{k[d/2-(d+2)/p]}||f||_{L^2(\mathbb{R}^d)}$$

for $p \in [(2d+4)/d, \infty]$.

Theorem 6.6 (Periodic Strichartz Estimates)**.** *On $\mathbb{T}^d$,*

$$||P_{\leq k}e^{it\Delta}f||L^4_{x,t}\left(\mathbb{T}^d\times\mathbb{T}\right) \lesssim A_d(k)\,||f||_{L^2(\mathbb{R}^d)},$$

where

$$\begin{aligned} A_2(k) &= C_\epsilon 2^{k\epsilon}, \\ A_3(k) &= C_\epsilon 2^{k(1/4+\epsilon)}, \\ A_4(k) &= C_\epsilon 2^{k(1/2+\epsilon)}, \\ A_d(k) &= C 2^{k(d-2)/4}, \quad d \geq 5. \end{aligned}$$

The L^∞ estimate still holds,

$$||P_{\leq k} e^{it\Delta} f||_{L^\infty_{x,t}(\mathbb{T}^d \times \mathbb{T})} \lesssim 2^{kd/2} ||f||_{L^2(\mathbb{T}^d)}.$$

The key obstruction to global well-posedness is that the Euclidean dispersive bound

$$||P_{\leq k} e^{it\Delta}||_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \lesssim \left[\frac{2^k}{1 + 2^k |t|^{1/2}} \right]^d$$

fails in the periodic case.

Exercise 6.7. Prove the $\Lambda(4)$ bound for the squares

$$|| \sum_{|n| \leq N} a_n e^{in^2 x} ||_{L^4(\mathbb{T}^1)} \lesssim_\epsilon N^\epsilon || \sum_{|n| \leq N} a_n e^{in^2 x} ||_{L^2(\mathbb{T}^1)}$$

for any $\epsilon > 0$ and $N \geq 1$.

Exercise 6.8. Let

$$K_k(x,t) = \sum_{n \in \mathbb{Z}^d} e^{-i|n|^2 t} e^{ix \cdot n} \eta_{\leq k}(n)$$

denote the kernel of $P_{\leq k} e^{it\Delta}$, $x \in \mathbb{T}^d$, $t \in \mathbb{R}$. Prove that

$$|K_k(x,t)| \lesssim \left[\frac{2^k}{\sqrt{q} \left(1 + 2^k |t/(2\pi) - a/q|^{1/2}\right)} \right]^d$$

if $t/(2\pi) = a/q + \beta$, $(a,q) = 1$, $q \in \{1, \ldots, 2^k\}$, $|\beta| \leq (2^k q)^{-1}$.

6.3. **Global Well-Posedness on $\mathbb{T}$.** For the remainder of the course we'll restrict our attention to the defocusing periodic (cubic) NLS on $\mathbb{T} = \mathbb{T}^1$:

$$(i\partial_t + \triangle_x) u = u|u|^2$$

with initial data $u(x,0) = \phi(x)$. Again, the quantities

$$\begin{aligned} M(t) &= \int_{\mathbb{T}} |u(x,t)|^2 \, dx \\ E(t) &= \frac{1}{2} \int_{\mathbb{T}} |\nabla_x u(x,t)|^2 \, dx + \frac{1}{4} \int_{\mathbb{T}} |u(x,t)|^4 \, dx \end{aligned}$$

are conserved.

Theorem 6.9. *The initial value problem above is globally well-posed for small data in $L^2(\mathbb{T})$.*

Exercise 6.10. Use Sobolev imbedding to prove that the initial-value problem is well-posed in $H^1(\mathbb{T})$

Again we have a perturbative iteration scheme:

$$u_1(t) = e^{it\Delta}\phi,$$

$$u_{n+1}(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left[u_n(s)\,|u_n(s)|^2\right]\,ds.$$

Step 1: The homogenous Strichartz estimate

$$||e^{it\Delta}\phi||_{L^4(\mathbb{T}\times\mathbb{T})} \lesssim ||\phi||_{L^2(\mathbb{T})}.$$

Step 2: The inhomogeneous Strichartz estimate

$$\Big|\Big|\int_0^t e^{i(t-s)\Delta}\left(f(s)\right)\,ds\Big|\Big|_{L_t^\infty L_x^2(\mathbb{T}\times I)} \lesssim ||f||_{L^{4/3}(\mathbb{T}\times I)}.$$

Exercise 6.11. Prove the Strichartz estimate with loss

$$\big|\big|P_{\leq k}e^{it\Delta}\phi\big|\big|_{L^6(\mathbb{T}\times\mathbb{T})} \lesssim_\epsilon 2^{\epsilon k}||\phi||_{L^2(\mathbb{T})}.$$

State and prove the analogue of the inhomogeneous Strichartz estimate above.

Step 3: The inhomogeneous Strichartz estimate

$$\Big|\Big|\int_0^t e^{i(t-s)\Delta}\left(f(s)\right)\,ds\Big|\Big|_{L^4(\mathbb{T}\times I)} \lesssim ||f||_{L^{4/3}(\mathbb{T}\times I)}$$

for any $f \in L^{4/3}(\mathbb{T}\times I)$, $0 \in I$.

Step 4: Prove that $\{u_n\}_{n\geq 1}$ is a Cauchy sequence in the space

$$X(I) = C\left(I : L^2(\mathbb{T})\right) \cap L^4(\mathbb{T}\times I).$$

Then use the L^2 conservation law to extend the solution to $\mathbb{T}\times\mathbb{R}$. Conclude that for any $\phi \in L^2$ with $||\phi||_{L^2} \leq \epsilon_0$ there is a unique solution

$$u \in C\left(\mathbb{R} : L^2(\mathbb{T})\right) \cap L^4_{\text{loc}}(\mathbb{T}\times\mathbb{R})$$

of the equation

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(u(s)\,|u(s)|^2\right)\,ds.$$

The mapping $\phi \mapsto u$ is continuous from $L^2(\mathbb{T})$ to $X(I)$ for any bounded interval I.