Topology I

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Abstract. This is part one of a two semester course on algebraic topology. The course was offered in Fall 2011 at the Courant Institute for Mathematical Sciences, a division of New York University. Though the course is not taken from any particular text, some references are *Introduction to Algebraic Topology* by Munkres, Hatcher's series of texts on algebraic topology, Greenberg & Hauper, Spanier, etc. Milnor's *Topology from the Differentiable Viewpoint* may be useful in the manifolds portion of the course; Kelly's text may be useful for the review of point-set. There will be a final and suggested homework problems.

Contents

CHAPTER 1

Introduction

Lecture 1, 9/12/11

We begin with a discussion of some famous topological questions. Historically, topologists studied the classification of spaces. Soon enough we'll make the notion of "space" rigorous; a first example is any $X \subset \mathbb{R}^n$ with the Euclidean distance

$$d_{\mathbb{R}^n}(u,v) = ||u-v|| = \left(\sum_{i=1}^{n}(u_i - v_i)^2\right)^{1/2}.$$

Some "nice" spaces are manifolds, e.g. S^2. We call this a manifold as it is locally Eucliean. Another example is the surface of a donut, or T^2. Notice $T^2 \subset \mathbb{R}^3$ has one hole – we say it has genus $g = 1$. A surface with two holes is genus $g = 2$, and so on. There is an infinite family of spaces in this manner, each with different genus.

One of the basic classification results in topology is that of one-manifolds. Obviously $\mathbb{R}^1$ is a one-manifold. Another is $S^1 \subset \mathbb{R}^2$, for it is locally homeomorphic to $\mathbb{R}^1$. It is a classic result that S^1 is the only compact connected one-manifold.

A second problem in topology concerns embeddings of manifolds into Euclidean space. Given a manifold M^n, can one construct a copy of M^n in $\mathbb{R}^N$ for some N? This naturally gets easier as N increases. The interesting question is: What is the lowest N for which this is possible for all n-dimensional manifolds? Whitney answered this question with the astounding result that $M^n \hookrightarrow \mathbb{R}^{2n}$ for any n. Actually, Whitney first proved $M^n \hookrightarrow \mathbb{R}^{2n+1}$, then worked very hard to reduce the dimension of the ambient space by one.

There are more two-manifolds than one-manifolds. One interesting example is the real projective space $\mathbb{R}P^2$. Here is a first description. We can view the two-sphere as the result of gluing two hemispheres together along their boundary (so along S^1). We write $S^2 = D^2_+ \cup_{S^1} D^2_-$ to mean this. Now consider the Mobius band. It is a two-dimensional manifold with boundary, but it has only one boundary component, and in fact $\partial\,(\text{Mobius}) = S^1$. To construct $\mathbb{R}P^2$, we'll glue $\mathbb{D}^2$ to the Mobius band along S^1. So $\mathbb{R}P^2 = \mathbb{D}^2 \cup_{S^1} \text{Mobius}$. We can't carry this out physically in $\mathbb{R}^3$; $\mathbb{R}P^2$ is a closed, compact two-dimensional manifold which does not embed in $\mathbb{R}^3$. But of course, it will embed in $\mathbb{R}^4$.

Here is a second description of $\mathbb{R}P^2$. We have $\mathbb{R}P^1 = S^1$, so perhaps we can think along the same lines for $\mathbb{R}P^2$. We proceed by identifying each $v \in S^2$ with $-v$, its antipode. This yields $\mathbb{R}P^2$. An informal proof is as follows: consider S^2 as consisting of north and south polar caps along with an equatorial band. After identifying the two caps, there is only one cap left. And the space we're left with is (cap) $\cup$ (Mobius). Somehow, we've arrived at our previous construction.

A third question from topology stems from the observation that $S^n = \partial\mathbb{D}^{n+1}$ for each n. The question is: Given a manifold without boundary, M^n, does there exists W^{n+1} such that $\partial W = M$? The answer in general is no. For example, $\mathbb{R}P^2$ is not a boundary. Thom answered this question by determining a set of numerical invariants that determine completely wheter a manifold is a boundary.

A large and powerful part of topology consists of the so-called "fixed point theorems". We ask: Given continuous $f : X \to X$, does there exist $x \in X$ with $f(x) = x$? It is a standard trick to turn an equation of interest into a question about fixed points. One famous result is the Brouwer fixed point theorem, which says that every continuous $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point. We can prove this with calculus in the case $n = 1$. If $f(x) \neq x$ for all x, define

$$g(x) = \frac{f(x) - x}{|f(x) - x|}$$

which is (pointwise) either $+1$ or -1. Since g is continuous, we have the two cases $g \equiv 1$ and $g \equiv -1$. In the first case we get $f(1) > 1$ and in the second $f(-1) < 1$, both of which are contradictions.

Here is the idea for arbitrary n. Assuming $f(x) \neq x$ for all $x \in \mathbb{D}^n$, construct g as in the following picture: (D1) Clearly g is continuous, so we've exhibited a continuous map $g : \mathbb{D}^n \to \partial\mathbb{D}^n = S^{n-1}$ which restricts to the map $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$.

Lecture 2, 9/19/11

As it turns out, this is impossible, a fact which we'll prove later in the course.

Analysis and topology are deeply connected. There are many results in which analytical quantities are topological quantities as well. A 19th century example is the Riemann-Rach Theorem, which gives a topological formula for the dimension of the space of meromorphic functions with prescribed zeros/poles on a Riemann surface. In the 20th century, the theorem was extended to all dimensions, to singular varieties, to differential operators (Atiyah-Singer Index Theorem for elliptic operators). But perhaps this is too grand for now.

Here is a more naive story – Morse Theory. This subject consists of the study of topological consequences of critical points of functions on a given manifold. A critical point of a function f is where f looks constant to first order. Consider the height function $h : T^2 \subset \mathbb{R}^3 \to \mathbb{R}$ given by $h(x) = x_3$. h then has four crtical points. Now observe that we can always increase the number of critical points of h by deforming the manifold T^2. A central result of Morse Theory is that no matter how we deform T^2, h will always have at least four critical points. The justification of this fact is in the "Morse inequalities," which are formulae for the minimum number of critical points. Another example is S^2: the height function always will have two critical points, no matter how S^2 is deformed. Somehow, the topology of a space (a global result) is determined by the critical points of functions (a local, analytic result).

In this course we'll discuss continuous and differentiable functions. In topology, there is a vast interest in generalizing these notions to manifolds. So one could ask: how unique is the idea of differentiability on a manifold? More explicitly, suppose we have a function f defined on a manifold. Can we give that manifold two different differential structures, one on which f is considered differentiable and the other on which f is not? Remarkably, the answer is yes! It is a theorem of Milnor that S^7 has 28 differential structures. And $\mathbb{R}^4$ has uncountably many differential structures (though $\mathbb{R}^n$ for $n \neq 4$ only has one). $\mathbb{R}P^4$ also has more than one differential structure (due to Cappell and Sharean). It is unknown whether S^4 has a unique differential structure.

CHAPTER 2

Point-Set Topology

1. Metric Space

The first thing to do is make the notion of "space" rigorous. Recall the notion of a *metric space*, (X, d), a set X and a distance $d : X \times X \to \mathbb{R}^{\geq 0}$ which satisfies

(1) $d(u, v) = 0 \iff u = v$ in X
(2) $d(u, v) = d(v, u)$ for all $u, v \in X$
(3) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

The third requirement is called the "triangle inequality".

Examples 1.1.

(1) $\mathbb{R}^n$ with the Euclidean metric $d(u, v) = \left(\sum_i (u_i - v_i)^2\right)^{1/2}$.
(2) If (X, d_X) is a metric space, then any subset $A \subset X$ can be made into a metric space by setting $d_A = d_X|_A$.
(3) Function spaces, e.g. $C[0, 1]$ with $d(f, g) = \max_{[0,1]}|f(x) - g(x)|$.
(4) We can give $\mathbb{R}^n$ a different metric, $d_{\max}(u, v) = \max_{i=1,\ldots,n}|u_i - v_i|$.

    Exercise 1.2. Check $d_{\max}$ is a metric.

(5) Any set X can be made into a metric space with $d(u, v) = \delta_{u=v}$.

    Exercise 1.3. Check d is a metric.

(6) The p-adic metric on $\mathbb{Z}$. The idea is to compare $\mathbb{Z}$ with $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$. Given a prime number p, we'll define a metric d_p on $\mathbb{Z}$ with small $||p||$. Here $||v|| = d(v, 0)$. Given $n \in \mathbb{Z}$, write $n = p^a b$ with b prime to n, and set $||n||_p = p^{-a}$. (E.g. $||12||_2 = 1/4$.) Then define $d_p(u, v) = ||u - v||_p$ for $u, v \in \mathbb{Z}$. (E.g. $d_2(4, 12) = 1/8$, $d_2(11, 12) = 1$.)

    Exercise 1.4. Prove d_p is a metric. In fact, prove the "ultra-metric inequality" which says

$$d_p(u, w) \leq \max\{d_p(u, v), d_p(v, w)\}.$$

(Hint: This follows from $||a+b||_p \leq \max\{||a||_p, ||b||_p\}$. Let $a = u - v$ and $b = u - w$, then $a + b = u - w$. The point here is that the power of p which divides $a + b$ is at least as big as the minimum of the powers of p dividing a and b.)

We can extend the definition of p-adics to $\mathbb{Q}$. We have $\mathbb{Q} = \left\{u/v \,\middle|\, v \neq 0\right\}$. So define $||u/v||_p = ||u||_p/||v||_p$. In other words, given $u/v = p^\alpha m$ where m has no powers of p (α can be negative), set $||u/v||_p = p^{-\alpha}$. (E.g. $||1/12||_2 = 4$.)

Exercise 1.5. Check this satisfies the metric axioms and the ultra-metric condition.

We can extend the p-adics even further by completing $\mathbb{Q}$, via limits in $||\cdot||_p$. This produces the *p-adic numbers* $\mathbb{Q}_p$. These look like power series in p:

$$\mathbb{Q}_p = \left\{ \frac{a_{-i}}{p^i} + \frac{a_{-(i-1)}}{p^{i-1}} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots \right\}.$$

Of course, we can also consider objects such as $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$, and so on.

Now let $f : (X, d_X) \to (Y, d_Y)$ be a function.

DEFINITION 1.6. We say f is *continuous at a point* u if for all $\epsilon > 0$ there exists $\delta > 0$ so that if $d_X(u, v) < \delta$ then $d_Y(f(u), f(v)) < \epsilon$. We say f is *continuous* if it is continuous at all points $x \in X$.

EXERCISE 1.7. Consider $f : (\mathbb{R}^n, d_{\text{Euclidean}}) \to (\mathbb{R}^n, d_{\max})$ given by $f(x) = x$. Is f continuous? Is f^{-1} continuous?

DEFINITION 1.8. We call f an *isometry* if it is a 1-1 correspondance which has $d_X(u, v) = d_Y(f(u), f(v))$ for all $u, v \in X$.

Isometry is too strict a condition for our purposes. For example, all circles in the plane should be topologically equivalent. But only those circles with the same radii will be isometric. Eventually we'll see a weaker notion of equivalence.

DEFINITION 1.9. $B_r(u) = \left\{ v \in X \,\middle|\, d(u, v) < r \right\}$ is the *open ball* of radius $r > 0$ around u.

PROPOSITION 1.10. *f is continuous at $u \in X$ iff for all $\epsilon > 0$ there is $\delta > 0$ so that $f(B_\delta(u)) \subset B_\epsilon(f(u))$.*

DEFINITION 1.11. A subset $U \subset X$ is *open* if it contains an open ball around each of its points.

PROPOSITION 1.12. *The collection of open sets satisfy:*

(1) X is open, $\emptyset$ is open.
(2) The union of open sets is open.
(3) If $A, B \subset X$ are open then $A \cap B$ is open.

REMARK. Usually people write (3) in its alternate form, which asserts the finite intersection of open sets is open. Naturally, they are equivalent. But observe that the infinite intersection of open sets need not be open, e.g. $\cap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$.

PROPOSITION 1.13. *$f : (X, d_X) \to (Y, d_Y)$ is continuous iff $f^{-1}(U)$ is open in X whenever $U \subset Y$ is open in Y.*

An immediate consequence is

PROPOSITION 1.14. *If $f : X \to Y$, $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

The ease with which this follows from proposition 1.13 suggests reformulating the subject in terms of open sets, and not a metric. This brings us to the idea of an abstract topological space.

2. Topological Space

DEFINITION 2.1. Let X be a set. $\mathcal{T}$ is called a *topology* on X if it is a collection of subsets of X, called the *open sets*, so that

(1) $\emptyset, X \in \mathcal{T}$
(2) If $A_\alpha \in \mathcal{T}$ for all α then $\cup_\alpha A_\alpha \in \mathcal{T}$.
(3) If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.

EXAMPLES 2.2.

(1) Any metric space with $\mathcal{T}$ consisting of the open sets as determined by the metric.
(2) Given a topological space $(X, \mathcal{T}_X)$ and any $A \subset X$ we can define the *relative topology* on A by taking $\mathcal{T}_A = \left\{B \cap A \,\middle|\, B \in \mathcal{T}_X\right\}$.

   EXERCISE 2.3. $(A, \mathcal{T}_A)$ is a topological space.

   REMARK. The open sets in $\mathcal{T}_A$ are not necessarily the open sets in $\mathcal{T}_X$. Consider the open sets on $\mathbb{R}^1$ and the open sets in $A = [0, 1]$ in the relative topology.

   EXERCISE 2.4. $\mathcal{T}_A \subset \mathcal{T}_X \iff A \in \mathcal{T}_X$

   EXERCISE 2.5. Let $B \subset A \subset X$ and let $\mathcal{T}_X$ be a topology on X. We can give A the relative topology from X, then B the relative topology from A. Show this is the same as going directly from X to B.

(3) Given any set X, we gave it the discrete metric $d(u, v) = \delta_{u=v}$. Thus any set X gets a topology from d, and its not hard to show $\mathcal{T}_X = \mathcal{P}(X)$. We'll call this the *discrete topology.* It is the largest topology. The smallest topology is $\mathcal{T}_X = \{\emptyset, X\}$, which we'll call the *indiscrete topology.* Note that with any space X and any topology $\mathcal{T}_X$ we have $\mathcal{T}_{\text{indiscrete}} \subset \mathcal{T}_X \subset \mathcal{T}_{\text{discrete}}$, so the terms "smallest" and "largest" are meaningful.

   EXERCISE 2.6. Consider $\text{id} : X \to X_{\text{discrete}}$, $\text{id} : X_{\text{discrete}} \to X$, $\text{id} : X \to X_{\text{indiscrete}}$, and $\text{id} : X_{\text{indiscrete}} \to X$. Which of these is continuous, no matter what topology X has?

(4) The Zariski topology. The idea (from algebraic geometry) is to only study open sets defined by algebraic equations. For example, we could take the open sets to be $\left\{\mathbb{C}^N \setminus \{\text{solutions of systems of polynomial equations}\}\right\}$ along with the empty set. These are "fat" open sets, in that they are very big. To be precise, if A, B are non-trivial open sets, then $A \cap B$ will also be a non-trivial open set.

   THEOREM 2.7. *The Zariski topology is a topology on* $\mathbb{C}^N$.

   This is a highly non-trivial theorem. But we can prove it in $\mathbb{C}^1$. By definition, the open sets are of the form $A = \mathbb{C} \setminus \{\text{roots of some poly}\}$, but as we can write $p(z) = \Pi_i (z - a_i)$ for any polynomial p the result follows.

Lecture 3, 9/26/11

(5) On any set X we can produce the finite-complement topology by setting $\mathcal{T} = \left\{A \subset X \,\middle|\, \#A^c < \infty\right\}$ and adding the emptyset. Note that this topology does not derive from a metric if X is infinite, because any two non-trivial sets have non-trivial intersection (one cannot separate open

sets). The Zariski topology with $X = \mathbb{C}$ is a specific example of this topology; in fact, the finite-complement topology is sometimes called the "Zariski topology" on X. But note that the two Zariski topologies only coincide on $\mathbb{C}$.

One can also put topologies on finite sets. Recall on any set X there are two "extreme" topologies: the discrete topology $\mathcal{T} = \mathcal{P}(X)$ (which is a metric topology) and the indiscrete topology $\mathcal{T} = \{\emptyset, X\}$ (which is not). On a finite set, there are a number of topologies; the number of possible topologies grows rapidly with the number of points. Suppose we have two points a, b. The possible topologies are

$$\begin{aligned} \mathcal{T}_1 &= \text{discrete} \\ \mathcal{T}_2 &= \text{indiscrete} \\ \mathcal{T}_3 &= \{\emptyset, \{a\}, X\} \\ \mathcal{T}_4 &= \{\emptyset, \{b\}, X\}. \end{aligned}$$

In some sense, $\mathcal{T}_3$ and $\mathcal{T}_4$ should be the same. We'll discuss a way to realize this now.

Definition 2.8. A function $f : X \to Y$ is said to be *continuous* if $f^{-1}(V) \in \mathcal{T}_X$ whenever $V \in \mathcal{T}_Y$.

As we saw in the previous section, this is equivalent to the usual $\epsilon - \delta$ definition of continuity if X is a metric space.

Definition 2.9. A map $f : X \to Y$ is called a *homeomorphism* if it is a 1-1 correspondance and if both it and its inverse are continuous. If there exists such a map then we say X and Y are *homeomorphic*.

Example 2.10. A small circle and a large circle are homeomorphic. So are a small circle and a large square.

Proposition 2.11. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps then $g \circ f : X \to Z$ is continuous.*

Proposition 2.12. *Homeomorphism on spaces is an equivalence relation.*

Let's go back to the finite set with two points. The map $f : (\{a, b\}, \mathcal{T}_3) \to (\{a, b\}, \mathcal{T}_4)$ with $f(a) = b$ and $f(b) = a$ is a homeomorphism. So although there are four topologies on two points, there are only three up to homeomorphism. Of course if we go to three points, there are many more topologies.

Exercise 2.13. How many topologies does a three point set have? How many up to homeomorphism? (There is a trick.)

Note that the only topology on the two point set which is a metric topology is the discrete topology. In general, a metric topology on a finite set is the discrete topology. (Is it the only metric topology?) There is no formula in general for the number of topologies on a finite set with n elements.

3. The Separation Axioms

One fundamental problem in topology is to classify spaces (e.g. manifolds) up to homeomorphism. Depending on what classification means, this problem may be algorithmically unsolvable. We'll clarify what we mean later. One way to classify spaces is with the so-called "separation axioms". Here are two of the possibilites:

- For each pair of distinct points $p, q \in X$, there is an open set with one and not the other.
- For each pair of distinct points $p, q \in X$, there is an open set containing p but not q (and one containing q but not p).

The first condition is very weak. The second is still weak – it is even satisfied by the Zariski topology on a set X (take $U = X \setminus \{p\}$ and $V = X \setminus \{q\}$). A stronger condition is the so-called *Hausdorff condition (T_2 condition)*:

- Given $p, q \in X$ there exist open sets U, V containing p, q with $U \cap V = \emptyset$.

This is the condition one would use to do geometry.

Proposition 3.1. *Metric spaces are Hausdorff.*

But the Zariski topology is not Hausdorff, which really shows we have found a stronger condition. Throughout we'll assume our spaces are T_2.

Exercise 3.2. If a finite topological space is Hausdorff, then it is the discrete topology.

Consequently, the only metric topology on a finite set is the discrete topology.

4. Compactness

Let's go back to metric spaces to build some intuition. Let (X, d) be a metric space. As usual, a sequence of points $\{x_j\}$ in X is said to be *convergent* if there exists a point $x_\infty \in X$ so that $d(x_j, x_\infty) \to 0$ as $j \to \infty$. In this case, we write $x_\infty = \lim_{j\to\infty} x_j$ or $x_j \to x_\infty$.

Exercise 4.1. x_∞ is unique.

Also recall

Definition 4.2. A sequence of points $\{x_j\}$ in X is said to be *Cauchy* if for every $\epsilon > 0$ there exists N so that if $k, l > N$ then $d(x_k, x_l) < \epsilon$.

Exercise 4.3. If $x_j \to x_\infty$ then $\{x_j\}$ is Cauchy.

Remark. The converse does not hold in general. Consider $X = (0, 1)$ and the sequence $x_j = j^{-1}$.

Definition 4.4. A space is called *complete* if a sequence is Cauchy iff it is convergent.

Example 4.5. $\mathbb{R}$ is complete, but $(0, 1)$ is not complete.

Observe, however, that $(0, 1)$ is homeomorphic to $\mathbb{R}$. (Send $1 \mapsto \infty$ and $0 \mapsto -\infty$.)

Definition 4.6. A property of a space is called *topological* if it is equivalent for homeomorphic spaces.

Thus completeness is not topological.

Definition 4.7. A metric space (X, d) is said to be *pre-compact* if for every $\epsilon > 0$ there are finitely many points $x_1, \ldots, x_N$ such that each $u \in X$ satisfies $d(u, x_j) < \epsilon$ for some $j \in \{1, \ldots, N\}$.

In the previous definition, we have $X = \cup_{j=1}^N B_\epsilon(x_j)$. So a pre-compact X is covered by finitely many balls of radius ϵ.

EXAMPLE 4.8. $\mathbb{R}$ is not pre-compact, but $(0,1)$ is pre-compact.

So precompactness is not topological. But consider the following proposition.

PROPOSITION 4.9. *The following are equivalent for a metric space X:*

1. *X is pre-compact and complete.*
2. *Every sequence has a convergent subsequence.*
3. *Every cover of X by open sets has a finite subcover.*

What does three mean? Three says that if $X = \cup_\alpha A_\alpha$ for open A_α, then actually $X = \cup_{j=1}^N A_j$. In some sense, this says that the space does not go on to infinity. A proof of this proposition can be found in Dieudonne's *Foundation of Modern Analysis*, or many other standard texts.

DEFINITION 4.10. A space satisfying any of the properties in the previous proposition is said to be *compact*.

EXAMPLE 4.11. $\mathbb{R}$ is not compact (it is not pre-compact). $(0,1)$ is not compact (it is not complete). $[0,1]$ is compact.

Note that condition three is the only topological condition, as it only refers to the open sets. So for an abstract topological space we'll use three to characterize the compact sets.

DEFINITION 4.12. A subset $A \subset X$ of a topological space is called *compact* if whenever $A \subset \cup_\alpha U_\alpha$ with U_α open in X then $A \subset U_{j=1}^N U_j$.

EXERCISE 4.13. Let X be a topological space. $A \subset X$ is compact as a subset of X iff A is compact in the relative topology.

PROPOSITION 4.14. *If X is Hausdorff, then compact sets are closed.*

REMARK. The converse is false in general. Just take $X = A = \mathbb{R}$.

PROOF. Fix $y \in X \backslash A$. For all $x \in A$ we can find open and disjoint $U_x, V_x \subset X$ with $x \in U_x$ and $y \in V_x$. Now $A \subset \cup_x U_x$ so by assumption $A \subset \cup_{j=1}^N U_j$. But then $\cap_{j=1}^N V_j$ is an open set which contains y, and is disjoint from A by construction, hence $X \backslash A$ is open. $\square$

PROPOSITION 4.15. *If X is compact, then $A \subset X$ is closed iff it is compact.*

PROOF. We have compact implies closed. If A is closed, let $\{U_\alpha\}$ be an open cover of A. Then $\{U_\alpha, X \backslash A\}$ is a cover of X, so $\{U_1, \ldots, U_N, X \backslash A\}$ covers X. Hence $\{U_1, \ldots, U_N\}$ covers A, so we are done. $\square$

This has some useful implications for topology.

PROPOSITION 4.16. *On $\mathbb{R}^n$, compact means bounded and closed.*

PROOF. It's easy to see compact implies bounded and closed. Conversely, if A is bounded, then it sits inside a closed ball of some radius. This ball is compact, so by the previous proposition, A is compact as well. $\square$

PROPOSITION 4.17. *Suppose $f : X \to Y$ is continuous. If X is compact, then Y is compact as well.*

PROPOSITION 4.18. *Suppose X, Y are compact and Hausdorff. If $f : X \to Y$ is a continuous 1-1 correspondence, then f is a homeomorphism.*

EXERCISE 4.19. Prove the previous two propositions.

5. The Product Topology

If $(A, \mathcal{T}_A)$ and $(B, \mathcal{T}_B)$ are (disjoint) topological spaces, we give $A \cup B$ a topology by setting $\mathcal{T} = \{U \cup V \mid U \in \mathcal{T}_A, V \in \mathcal{T}_B\}$.

EXERCISE 5.1. If we have $f : A \cup B \to C$, then f is continuous iff $f|_A$ and $f|_B$ are.

Products arise in the dual situation. Suppose A, B are topological spaces. How do we topologize $A \times B$? Given maps $f : C \to A$ and $g : C \to B$, we want to combine these to form the product map $(f, g) : C \to A \times B$. Then we want to say that (f, g) is continuous iff f, g are continuous. This tells us the topology we want on the product space. Is there such a topology? A first guess might be to take $\mathcal{T} = \{U \times V\}$ with U open in A and V open in B. But this would only give us rectangular open sets. (Consider the picture for $\mathbb{R} \times \mathbb{R}$ and $U, V = (0, 1)$. Should sets like $(0, 1)^2$ be the only open sets?)

DEFINITION 5.2. Given a topological space X with a topology $\mathcal{T}$, a *basis* for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ such that every non-$\emptyset$, non-X set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$.

EXAMPLE 5.3. The most natural example is when $\mathcal{T}$ is a metric topology. Then the open balls form a basis; the open balls of radius $1/n$ for $n = 1, 2, \ldots$ form a basis as well.

PROPOSITION 5.4. *A collection $\mathcal{B}$ is a basis for a topology on X iff given $U, V \in \mathcal{B}$, $U \cap V$ is a union of sets in $\mathcal{B}$. Equivalently, $\mathcal{B}$ is a basis if for any point $p \in U \cap V$, there exists $W \in \mathcal{B}$ with $p \in W \subset U \cap V$.*

EXAMPLE 5.5. If X is the discrete topology, we could take $\mathcal{B} = \{\{u\} \mid u \in X\}$.

DEFINITION 5.6. The *product topology* on $A \times B$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_A, V \in \mathcal{T}_B\}$.

PROPOSITION 5.7. *If $f : C \to A$ and $g : C \to B$, then f, g are continuous iff $(f, g) : C \to A \times B$ is continuous.*

PROOF. Notice that $A \times B$ comes with the coorindate projections p_1, p_2 onto A, B. First we claim p_1, p_2 are continuous. To see this, let U be open in A, then $p_1^{-1}(U) = U \times B$ which is open. It's the same for p_2. Now if (f, g) are continuous, then $f = p_1 \circ (f, g)$ and $g = p_2 \circ (f, g)$ are the composition of continuous functions and hence continuous.

Conversely, suppose f, g are continuous. We only need check $(f, g)^{-1}(U \times V)$ is open, but $(f, g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ and so we're done. □

NOTE. We have developed a satisfactory theory of continuity for maps into product spaces. There is no characterization in general for maps from product spaces. This extends beyond topology: one can take a whole course on multivariable calculus, where the results do not follow simply from single-variable calculus!

For the infinite product $\Pi_{\alpha \in I} X_\alpha$, the first guess would be to say that the open sets are of the form $\Pi_{\alpha \in I} U_\alpha$ with U_α open in X. This fails for the same reason as in the finite product.

DEFINITION 5.8. The *product topology* on $\Pi_{\alpha \in I} X_\alpha$ has as basis sets of the form $U_1 \times U_2 \times \cdots \times U_k \times \Pi_{\alpha \neq 1, \ldots, k} X_\alpha$.

THEOREM 5.9. *The product of compact Hausdorff spaces is also compact Hausdorff.*

The proof of this in the case of two sets is not hard, and neither is the proof in the finite case. But the infinite product requires some non-trivial logic.

EXAMPLE 5.10. The solid cubes are compact. Since solid cubes and solid spheres are homeomorphic, solid spheres are compact as well.

Many spaces (not most) are product spaces.

EXAMPLES 5.11.
(1) $I \times I \simeq \mathbb{D}^2$
(2) $S^1 \times I$, the cylinder.
(3) $S^1 \times S^1 \simeq T^2 \subset \mathbb{R}^3$. This is the only closed two-manifold which has a product decomposition.
(4) $X \times I \simeq X \times S^1$ by a gluing construction. For $S^2 \times I$ we would have to glue the inner surface to the outer surface. This is possible in $\mathbb{R}^4$.

Lecture 4, 10/3/11

6. Connectedness

What do we mean when we say a space is disconnected? Think of two disjoint blobs A, B which together make up a space X. This is the intuitive idea. How do we formalize this? Note that both A and B are open in the topology of X (each point in A has a ball around it, and the same with B), but so they are both closed.

DEFINITION 6.1. A space X is said to be *connected* if we cannot decompose X as a union of disjoint, non-trivial open subsets $A, B \subset X$. Equivalently, there does not exist $A \subset X$ with $A \neq \phi, X$ and A is both open and closed.

There is a stronger notion of connectedness. Intuitatively, a space should be path-connected if any two points have a path between them. What is a path?

DEFINITION 6.2. A *(parameterized) path* in a space X is a continuous map $\omega : I = [0, 1] \to X$. If $u = \omega(0)$,$v = \omega(1)$ then ω is said to be the *path from u to v*, which themselves are the *initial* and *terminal* points of ω (the *endpoints* of ω).

Given a path ω from u to v in X and a path γ from v to w in X, we can splice them together to form a new path $\omega{\cdot}\gamma : I \to X$. But note we need to reparametrize:

$$\omega\gamma(t) = \begin{cases} \omega(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

We have what begins to look like an algebra among paths – we can combine two paths to form a new path. But there is a difficulty, as the product is only defined if $\omega(1) = \gamma(0)$.

DEFINITION 6.3. A space X is said to be *path-connected* if for any points $u, v \in X$, there exists a path from u to v.

Given a space X, define a relation $\sim$ among points of X by setting $u \sim v$ if there exists a path from u to v in X. Observe that $\sim$ is an equivalence relation among points. We have $u \sim u$ by the trivial path $e_u : I \to \{u\}$ with $e_u(t) = u$ for all $0 \leq t \leq 1$, and $u \sim v \implies v \sim u$ for if we have ω from u to v then we have $\omega^{-1}(t) = \omega(1-t)$ from v to u. Finally, we have $u \sim v + v \sim w \implies u \sim w$ by the splicing technique above. So X decomposes into equivalence classes, which we call the *path-components* of X.

PROPOSITION 6.4. *Path-connected implies connected.*

REMARK. The converse is false. Consider the space consisting of the points $\left\{x, \sin\frac{1}{x}\right\} \cap \{y\text{-axis}\} \subset \mathbb{R}^2$.

PROOF. If not, then we can write $X = A \cup B$ for $A, B \neq \emptyset$, $A \cap B = \emptyset$, and A, B open. Let $u \in A$ and $v \in B$ and let ω be a path in X from u to v. Consider $\omega^{-1}(A)$ and $\omega^{-1}(B)$. These are open disjoint subsets which seperate $I = [0, 1]$, which is not possible. $\square$

The situation can be remedied as follows. Recall a neighborhood about a point is an open set containing that point.

DEFINITION 6.5. Say X is *locally path-connected* if each point $u \in X$ has a path-connected neighborhood.

PROPOSITION 6.6. *If X is connected and locally path-connected, then it is path-connected.*

REMARK. This result applies to manifolds, for instance.

PROOF. Decompose X into path-components. The components are open subsets of X, by local path-connectedness. So X decomposes into disjoint open subsets. As X is connected, all but one component is null, and the remaining one is all of X. So X is path-connected. $\square$

7. Quotient Spaces and Gluing

We want to be able to take an interval from 0 to 1 and glue the points $0, 1$ together with an equivalence relationship, so that we get a circle. That is, we want to say $S^1 = I/0 \sim 1$. We'll formalize this notion now.

Given a space X and an equivalence relation $\sim$ among points of X, we can form a new space, called the *quotient space* $X/\sim$, whose points are the equivalence classes of $\sim$. Given $a \in X$, we'll write $[a]$ to mean its equivalence class.

NOTE. When writing an equivalence relationship $\sim$, we often omit that $u \sim u$.

Given an equivalence relation $\sim$, we always have a quotient map $q : X \to X/\sim$ which maps $m \mapsto q(u) = [u]$. So we give $X/\sim$ a the *quotient topology* by setting $U \subset X/\sim$ to be open if $q^{-1}(U)$ is open in X. This makes q is continuous.

EXAMPLE 7.1. $S^1 = I/0 \sim 1$. This is easy to see for the quotient map is a 1-1 correspondence between compact, Hausdorff spaces and is clearly continuous in one direction.

Now we can describe a nice way to construct new spaces from old, by "gluing". Let A, B be disjoint spaces and let $f : C \to A$ and $g : C \to B$ be maps from a third space C into A and B. We might insist that there is a copy of C in A, B, i.e. that f and g are injective. But this is too restrictive. Similarly, it would be too much to allow both to be not injective. So let's only insist, say, that f is injective. Given all this, we can form a new space X by gluing:

$$X = A \cup B/f(u) \sim g(u) \text{ for } u \in C.$$

If X is a space formed in such a way, we write $X = A \cup_C B$ to mean A glued to B along C.

EXAMPLE 7.2.

(1) $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$.
(2) $\mathbb{R}P^2 = S^2/u \sim (-u)$. The quotient map $q : S^2 \to \mathbb{R}P^2$ is two-to-one.
(3) $\mathbb{R}P^2 =$ Mobius $\cup_{S^1}$ disc.
(4) $\mathbb{R}P^2 = D^2/u \sim (-u)$ for $u \in S^1$.
(5) $\mathbb{R}P^2 = \left\{\text{lines through 0in } \mathbb{R}^3\right\} \longleftrightarrow \left\{\text{pairs of antipodal points on } S^2\right\}$.
(6) $\mathbb{R}P^n = \left\{\text{lines through 0in } \mathbb{R}^{n+1}\right\} \longleftrightarrow S^n/u \sim (-u)$. Then, $\mathbb{R}P^0 = \{1 \text{ point}\}$ and $\mathbb{R}P^1 = S^1/u \sim (-u) = D^1_+/\text{endpoints identified}$. So $\mathbb{R}P^1 \backsimeq S^1$. This looks familiar! Consider the map $f(z) = z^2$ in the complex plane, which maps S^1 2-to-1 to S^1.
(7) $\mathbb{C}P^n = \left\{\mathbb{C}\text{-lines through 0in } \mathbb{C}^{n+1}\right\}$. It turns out that $\mathbb{C}P^1 \backsimeq S^2$.

CHAPTER 3

The Fundamental Group

1. Homotopy of Paths

Consider a two dimensional space X with a hole cut out. Given points $u, v \in X$ on opposite sides of the hole, there is a family of paths which can be deformed into each other. But there is a second such family, and any path in that second family cannot be deformed into a path in the first family. Let's formalize this.

Definition 1.1. Given two paths α, β from u to v in X, a *homotopy* of α to β is a continuous map $H : I \times I \to X$ with

$$\begin{aligned} H(0,s) &= u,\ 0 \le s \le 1 \\ H(1,s) &= v,\ 0 \le s \le 1 \\ H(t,0) &= \beta(t),\ 0 \le t \le 1 \\ H(t,1) &= \alpha(t),\ 0 \le t \le 1. \end{aligned}$$

There is another way to think of what homotopy means. Think of H_s as the the path from u to v given by $H_s(t) = H(t,s)$. So H_s is a parameterized family of paths (from u to v) from $H_0 = \beta$ to $H_1 = \alpha$.

Exercise 1.2. The relation $\alpha \underset{h}{\sim} \beta$ ($\alpha \sim \beta$ if α is homotopic to β) is an equivalence relation among paths from u to v. (Hint: The tricky one is transitivity.)

One of our goals in this course is to decide when paths are homotopic to each other. We're not ready for that now, though.

Exercise 1.3. Recall a subset $A \subset \mathbb{R}^n$ is said to be *convex* if for all $p, q \in A$ we have $\left\{ tp + (1-t)q \,\middle|\, 0 \le t \le 1 \right\} \subset A$. Show that if A is convex, then any two paths from u to v in A are homotopic.

Recall the product operation between paths. A nice fact is that the product operation is compactible with homotopy.

Proposition 1.4. *Suppose $\alpha \underset{h}{\sim} \beta$ with*

$$\begin{aligned} \alpha(0) &= \beta(0) = u \\ \alpha(1) &= \beta(1) = v \end{aligned}$$

and $\gamma \underset{h}{\sim} \delta$ with

$$\begin{aligned} \gamma(0) &= \delta(0) = v \\ \gamma(1) &= \delta(1) = w, \end{aligned}$$

Then $\alpha \cdot \gamma \underset{h}{\sim} \beta \cdot \delta$.

Exercise 1.5. Prove the proposition above.

Remark. But associativity fails. Even if $(\alpha \cdot \beta) \cdot \gamma$ makes sense (the endpoints have to match), it is usually the case that $(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$. The images will be the same, but the parametrizations are different.

Proposition 1.6. *Under the endpoint matching conditions above,* $(\alpha \cdot \beta) \cdot \gamma \underset{h}{\sim} \alpha \cdot (\beta \cdot \gamma)$.

Exercise 1.7. Prove the proposition above.

So we have associativity up to homotopy of paths. Now if we are given a path α from u to v in X, then its inverse α^{-1} will be given by $\alpha^{-1}(t) = \alpha(1-t)$, the reverse path from v to u. Note that $\alpha \cdot \alpha^{-1}$ is a path from u to itself.

Definition 1.8. A *loop* is a path from a point u to itself. It is said to be *fixed* at the point u.

Proposition 1.9. *Let α be a path from u to v in X. Then* $\alpha \cdot e_v \underset{h}{\sim} \alpha \underset{h}{\sim} e_u \cdot \alpha$.

Exercise 1.10. Prove the proposition.

Here is a slightly less obvious proposition, where homotopy becomes interesting.

Proposition 1.11. $\alpha \cdot \alpha^{-1} \underset{h}{\sim} e_u$.

This result really cannot be acheived by reparameterization.

Proposition 1.12. *Let* $\pi_1(X, u) = \{\text{loops in } X \text{ based at } u\} / \underset{h}{\sim}$. *This is a group.*

Definition 1.13. $\pi_1(X, u)$ is called the *(first) fundamental group* for X at u.

As the notation suggests, there are $\pi_2, \pi_3, \ldots$ – we'll get to these later. We'll see that π_1 is a measure of the one-dimensional holes in a space.

Remark. We proved homotopy is compatible with the product. So $[\alpha\beta] = [\alpha] \cdot [\beta]$. But $[\alpha] \cdot [\beta] \neq [\beta] \cdot [\alpha]$ in general. We'll see why later on by considering a figure eight.

Here is the easiest example of a fundamental group:

Example 1.14. If X is convex, then $\pi_1(X, u) = \{[e_u]\}$ for all $u \in X$.

Definition 1.15. A space is called *simply connected* if it is connected and if $\pi_1 = \{0\}$.

Example 1.16. $\pi_1(S^1) = \mathbb{Z}$. Proof is by winding number.

Lecture 5, 10/17/11

2. Independence of Basepoint

Having defined the (first) fundamental group $\pi_1(X, x)$ of a space X with basepoint x, the first natural question to ask is: how important is the choice of basepoint? The answer is not at all, so long as we stay within a single path-connected component of X.

Theorem 2.1. *If x, x' are points in the same path-component of X, then* $\pi_1(X, x) \simeq \pi_1(X, x')$.

As a result,

COROLLARY 2.2. *If X is path-connected, $\pi_1(X, x)$ is independent of the choice of $x \in X$.*

REMARK. As we'll see, there is not a natural isomorphism. We'll have to make a choice. But there is not always a natural isomorphism: $\mathbb{Z} \simeq \mathbb{Z}$ in (at least) two ways (via $x \mapsto x+1$ or $x \mapsto x-1$).

To prove the theorm, we'll first need some ideas from algebra.

DEFINITION 2.3. If G is a group, then ϕ is an *automorphism* of G if $\phi : G \to G$ is an isomorphism. We'll write $\mathrm{Auto}(G)$ to mean the set of automorphisms of G to itself.

EXAMPLE 2.4. $x \mapsto x+1$ and $x \mapsto x-1$ on $\mathbb{Z}$ are automorphisms.

PROPOSITION 2.5. *$\mathrm{Auto}(G)$ forms a group under composition.*

EXAMPLE 2.6. $\mathrm{Auto}(\mathbb{Z}) \simeq \{\pm 1\} \simeq \mathbb{Z}_2$. Also, $\mathrm{Auto}(\mathbb{Z}_p) \simeq \mathbb{Z}_p^* (\simeq \mathbb{Z}_{p-1})$. The last result is a theorem from number theory.

As in the previous example, given a ring R we write R^* to mean the multiplicative invertible elements of R. So R^* forms a group under multiplication.

DEFINITION 2.7. G is *cyclic* if it is generated by one element, $G = \left\{ g^k \,\middle|\, k \in \mathbb{Z} \right\}$.

PROPOSITION 2.8. *If G is infinite cyclic, then $G \simeq \mathbb{Z}_{\pm 1}$. If G is finite cyclic, then $G \simeq \mathbb{Z}_{m^+}$ (generated by 1).*

DEFINITION 2.9. Given a group G and an element $g \in G$, define *conjugation* of u by g to be the map $\phi_g : G \to G$, $u \mapsto gug^{-1}$. We call $\phi_g(u)$ the *conjugate* of u by g.

PROPOSITION 2.10. *$\phi_g = \mathrm{id}_G$ if G is abelian.*

We call conjugation by an element an *inner automorphism* of G. The set of all inner automorphisms of G is denoted $\mathrm{Inner}(G)$.

EXERCISE 2.11. $\phi_g(uv) = \phi_g(u)\,\phi_g(v)$. Also ϕ_g is 1-1, as $\phi_{g^{-1}} = (\phi_g)^{-1}$.

EXERCISE 2.12. $\phi_{hg} = \phi_h \circ \phi_g$.

As a consequence, we get

PROPOSITION 2.13. *$\mathrm{Inner}(G) \subset \mathrm{Auto}(G)$.*

EXERCISE 2.14. G is abelian iff $\mathrm{Inner}(G) = \{e\}$.

We can relate the group of inner automorphisms to G very easily, by the map $\Phi : G \to \mathrm{Inner}(G)$, $g \mapsto \phi_g$.

EXERCISE 2.15. Show Φ is a homomorphism. Also show $\mathrm{Ker}(\Phi) = Z(G) = \left\{ g \in G \,\middle|\, gh = hg \ \forall h \in G \right\}$.

DEFINITION 2.16. $Z(G) = \left\{ g \in G \,\middle|\, gh = hg \ \forall h \in G \right\}$ is called the *center* of G.

G is abelian iff $Z(G) = G$. And if A is abelian then $Z(G \times A) \supset A$. So we can easily produce centers out of any group in this way.

Now let's go back to topology. Our goal is to see why the group of homotopic loops is independent of the basepoint. The proof is almost geometric.

PROOF OF THEOREM. Draw a path γ from x' to x. Define a map $\chi_\gamma : \pi_1(X,x) \to \pi_1(X,x')$, $[\alpha] \mapsto [\gamma\alpha\gamma^{-1}]$. (This is not a conjugation, since γ is not in the group. But it is close.) Now it's easy to check

(1) This depends only, up to homotopy of loops, on the homotopy class of α. So it is well-defined.
(2) It it a homomorphism. Check $\chi_\gamma(\alpha\beta) = \chi_\gamma(\alpha)\chi_\gamma(\beta)$. The LHS is $[\gamma\alpha\beta\gamma^{-1}]$ and the RHS is $[\gamma\alpha\gamma^{-1}][\gamma\beta\gamma^{-1}]$.
(3) It has an inverse, hence χ_γ is an isomorphism. This proves the proposition.

□

EXERCISE 2.17. Check step three. Specifically, show that $(\chi_\gamma)^{-1} = \chi_{\gamma^{-1}}$.

How canonical was the choice of isomorphism in the proof? Notice χ_γ depends only on the choice of the homotopy class of paths from x' to x. One can check that this isomorphism of $\pi_1(X,x)$ and $\pi_1(X,x')$ is "almost" canonical, in that any two isomorphisms differ by an inner automorphism. If $\chi_{\gamma_1}, \chi_{\gamma_2} : \pi_1(X,x) \to \pi_1(X,x')$ are two choices of isomorphism via paths γ_1, γ_2 from x' to x, then the composite $\chi_{\gamma_2}^{-1}\chi_{\gamma_1}$ is an automorphism of $\pi_1(X,x)$.

CLAIM. This is an inner automorphism, in fact given by $\chi_{\gamma_2\gamma_1^{-1}}$. So $\chi_{\gamma_1} = (\text{inner})\,\chi_{\gamma_2}$.

EXERCISE 2.18. Prove the claim. This is a harder exercise than the others.

We'll never use this fact in this course. But it does come up sometimes.

Recall we called X *simply connected* if X is path-connected and if $\pi_1(X,x) = \{e\} = \{0\}$. (The first notation indicates multiplication, and the second addition.) And we saw that if X is convex, then $\pi_1(X,x) = \{0\}$. More generally, we have

DEFINITION 2.19. $X \subset \mathbb{R}^n$ is called *star-like* if it has a *star-point* $x \in X$ such that for any $u \in X$, the ray $tu + (1-t)x,\ 0 \le t \le 1$ lies in X.

EXAMPLE 2.20. If X is star-like, then $\pi_1(X,x) = \{0\}$ for all $x \in X$. First start at the starpoint then use the fact that there is one path-component.

3. Comparing Spaces Algebraically

Now given spaces and basepoints we can produce groups. The next question is: given a function between such spaces, can we produce a homomorphism? We'll write $X_{,x}$ to show the basepoint x explicitly.

DEFINITION 3.1. A continuous map $f : X_{,x} \to Y_{,y}$ is called *basepoint-preserving* if $f(x) = y$.

We'll use the notation space$_+$ to mean a space with a basepoint. We'll write map$_+$ to mean a basepoint preserving map.

Let f be a map$_+$ from $X_{,x}$ to $Y_{,y}$. Then f induces a homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ via the following

DEFINITION 3.2. Given a loop α in X based at x, define the *induced map* $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ by $f_*[\alpha] = [f \circ \alpha]$.

This is well-defined, for $\alpha \underset{h}{\sim} \beta$ in X implies $f \circ \alpha \underset{h}{\sim} f \circ \beta$ in Y. To see this, take the homotopy and compose it with f.

CLAIM. f_* is a homomorphism, i.e. $f_*([\alpha]\cdot[\beta]) = f_*([\alpha])\cdot f_*([\beta])$.

PROOF. The claim holds on the level of loops, so it holds on the homotopy classes. □

Now we have a way to linearize the problems of topology. We have a way to turn the problems of topology into problems of groups, and the problems of maps between spaces to homomorphisms between groups. This subject can get quite close to linear algebra.

EXAMPLE 3.3. Consider $\mathrm{id}_X : X \to X$. Clearly $(\mathrm{id}_X)_*[\alpha] = [\alpha]$, as it holds on the level of paths.

As we saw in the previous example, our construction has the fundamental property that

$$\mathrm{id}_{\pi_1(X,x)} = (\mathrm{id}_X)_* : \pi_1(X,x) \to \pi_1(X,x). \tag{3.1}$$

A second fundamental property is compatibility with composition: If $f : X_{,x} \to Y_{,y}$ and $g : Y_{,y} \to Z_{,z}$ are map_+s, then

$$(g\circ f)_* = g_*\circ f_*. \tag{3.2}$$

This is a consequence of associativity of composition. Relations (3.1) and (3.2) are essential to our algebraic view of topology. Whenever we look at topology in an algebraic way, we'll look for these properties. Some people call this *naturality*; others call it *functoriality*.

EXAMPLE 3.4. Consider the map $\mathrm{id}_{S^1} : S^1 \to S^1$. This is a loop in the circle, so $[\mathrm{id}_{S^1}] \in \pi_1(S^1_{,1})$. (We're using complex numbers in the plane here to denote the basepoint.) There is an isomorphism by winding number between $\pi_1(S^1,1)$ and $\mathbb{Z}$. The isomorphism sends the generator $[\mathrm{id}_{S^1}]$ to 1.

There is a more geometrical way to see this. Given $z \in S^1$ write $z = e^{2\pi i\theta}$ and take the map $g_k(z) = z^k$ from S^1 to itself. Then $g_k(e^{2\pi i\theta}) = e^{2\pi ik\theta}$ is the function which winds around the circle k times. If we regard $[g_k]$ as an element of $\pi_1(S^1,1)$, then

$$[g_k] = \underbrace{[g_1]\cdot[g_1]\cdots[g_1]}_{k\text{ times}} = [g_1]^k.$$

And here, $g_1 = \mathrm{id}_{S^1}$. If we write it additively, then we have winding $\# : \pi_1(S^1,1) \to \mathbb{Z}$, $[g_k] \mapsto k\cdot 1 = k$.

Now if we regard g_k as a map on S^1, then $(g_k)_* : \pi_1(S^1) \to \pi_1(S^1)$ has

$$(g_k)_*[\alpha(t)] = \left[(\alpha(t))^k\right],\ 0 \le t \le 1.$$

Additively, we have $(g_k) : \mathbb{Z} \to \mathbb{Z}$, $1 \mapsto k$. Note $(g_k)_*$ is only a homomorphism (it is not invertible, think of $k = 3$).

THEOREM 3.5. $\pi_1(X\times Y) = \pi_1(X)\times\pi_1(Y)$ *(i.e. they are isomorphic).*

PROOF. Let p_1, p_2 be the coordinate projections from $X\times Y$ onto X, Y. Then we have induced projections $(p_1)_* : \pi_1(X\times Y) \to \pi_1(X)$ and $(p_2)_* : \pi_1(X\times Y) \to \pi_1(Y)$. Then we have a natural map $(p_1)_*\times(p_2)_* : \pi_1(X\times Y) \to \pi_1(X)\times\pi_1(Y)$. It's automatically a homomorphism, since it is a product of homomorphisms. Now we claim this is an isomorphism. Given paths $\alpha, \beta : I \to X, Y$ we can get $\alpha\times\beta : I \to X\times Y$, $t \mapsto (\alpha(t), \beta(t))$ easily. Just check $p_1\times p_2(\alpha\times\beta) = \alpha\times\beta$ then apply this to homotopy classes to get $(p_1)_* \times (p_2)_* ([\alpha \times \beta]) = ([\alpha], [\beta])$. This is surjectivity.

For injectivity, we apply this principle to homotopy classes (go one dimension higher, it's a general principle in topology when we want to check injectivity). We only need to check that $\ker (p_1)_* \times (p_2)_* = \{0\} = \{e\}$. Suppose for some loop in the product $\alpha \times \beta : I \to X \times Y$ we have $(p_1)_* \times (p_2)_* ([\alpha \times \beta]) = e$ and so

$$(p_1)_* ([\alpha \times \beta]) = e$$
$$(p_2)_* ([\alpha \times \beta]) = e.$$

Hence, $[\alpha] = e$ in $\pi_1(X)$ and $[\beta] = e$ in $\pi_1(Y)$, so we have a homotopy H_x in X of α to the constant loop at x and similarly a homotopy H_y in Y of β to the constant loop at y. Thus $H_x \times H_y$ is a homotopy of $\alpha \times \beta$ in $X \times Y$ to the constant loop at (x, y). □

NOTE. The proofs of surjectivity and injectivity followed the same pattern. Surjectivity was done at the level of paths; injectivity was done at the level of homotopies (one dimension higher).

Now we can compute some fundamental groups.

EXAMPLE 3.6. Consider that $T^2 \simeq S^1 \times S^1$. So $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. This is the group of lattice points in the plane. Similarly, $\pi_1\left(\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}\right) = \mathbb{Z}^n$. In fact, the same proof carries through for infinite products. (Think about how the product topology was defined for infinite products.)

Now suppose we want to describe maps from the n-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ to the m-torus $T^m = \underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}}$. For the circle to the circle, the maps were given by the g_k's. On the torus, we can map each of the circles to each of the circles. Given an $n \times m$ matrix with integer entries A, we can produce a map $g_A : T^n \to T^m$ so that $(g_A)_* : \pi_1(T^n) = \mathbb{Z}^n \to \pi_1(T^m) = \mathbb{Z}^m$ is just matrix multiplication by A. We'll see an explicit version of this later on. Here is an example.

EXAMPLE 3.7. Consider the map $f : S^1 \times S^1 \to S^1 \times S^1$ which has $f(x, y) = (y, x)$. Then $f_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is multiplication by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lecture 6, 10/24/11

Now suppose $X_{,x}$ is a set$_+$ and let $[\alpha] \in \pi_1(X, x)$. What does it mean for $[\alpha]$ to be trivial? Let's give a geometric answer.

DEFINITION 3.8. Let $\alpha : S^1 \to X$ be a (continuous) map and let $i : S^1 \hookrightarrow D^2$ be inclusion. If there exists (continuous) $a : D^2 \to X$ so that $a \circ i = \alpha$, then we say α can be *extended* to a.

PROPOSITION 3.9. $[\alpha] = e$ *iff* $\alpha : S^1 \to X$ *can be extended to a map* $a : D^2 \to X$.

PROOF. There is a homotopy $H : I \times I \to X$ between α and e_x. Think of $I \times I$ as a box with α on the top and x on the left, right, and bottom sides. Identify the left, right, and bottom sides and let $q : I^2 \to I^2/\sim$ be the quotient map. Let $\overline{H}$ be so the diagram commutes.

$$\begin{tikzcd} I^2 \arrow[r, "H"] \arrow[d, "q"'] & X \\ I^2/\sim \arrow[ur, "\overline{H}"'] & \end{tikzcd}$$

Since $I^2/\sim = D^2$ in fact we can think of $\overline{H}$ as a map from D^2 to X. Then $\overline{H}\big|_{S^1} = \alpha$ so this is the extension we want.

The converse is proved similarly. $\square$

EXERCISE 3.10. Work out the details above.

Given an inclusion $i : A_{,x} \hookrightarrow X_{,x}$, note that $i_* : \pi_1(A, x) \to \pi_1(X, x)$ need not be injective, and may even be zero.

EXAMPLE 3.11. If $i : S^1 \hookrightarrow D^2$ is the inclusion, then $i_* : \pi_1(S^1) = \mathbb{Z} \to \pi_1(D^2) = 0$ must be the zero map.

But inclusions can still useful.

DEFINITION 3.12. Given $i : A \hookrightarrow X$, a *retraction* is a map $r : X \to A$ with $r \circ i = \mathrm{id}_A$. If there is a retraction $r : X \to A$, we say A is a *retract* of X.

EXAMPLE 3.13. Let $X_{,x}$ and $Y_{,y}$ be disjoint and define $X_\vee Y = X \cup Y/x \sim y$. Let $i : X \hookrightarrow X_\vee Y$ be inclusion, $q : X_\vee Y \to X_\vee Y/Y$ the quotient map, and $g : X_\vee Y/Y \to X$ the natural identification.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "g"'] & X_\vee Y \arrow[d, "q"] \\ & X_\vee Y/Y \end{tikzcd}$$

Given the above, the map $r = g \circ q^{-1}$ is a retraction. Check that $r \circ i = \mathrm{id}_X$.

As usual, a retract$_+$ map will be a retract that preserves basepoint.

PROPOSITION 3.14. *Suppose A is a retract$_+$ of X and let $i : A \hookrightarrow X$ be the inclusion. Then the induced map $i_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is injective.*

EXAMPLE 3.15. $i_* : \pi_1(X) \hookrightarrow X_\vee Y$ is injective.

PROOF. (D2) $r \circ i : \mathrm{id}_A$ so $(r \circ i)_* = (\mathrm{id}_A)_*$ and hence $r_* \circ i_* = \mathrm{id}_{\pi_1(A)}$ by naturality. So i_* is injective. $\square$

EXAMPLE 3.16. As a result, S^1 is not a retract of D^2. We've already seen the importance of this fact.

4. Homotopy of Maps

(D3) Suppose we have a space $X_{,x}$ and consider $Y = X \times I/x \times I$. There are (at least) two inclusions, $j : X \hookrightarrow Y$ which maps to $X \times \{1\}$ and $i : X \hookrightarrow Y$ which maps to $X \times \{0\}$. The picture suggests that for $[\alpha] \in \pi_1(X, x)$, $i_*([\alpha]) = j_*([\alpha])$ in $\pi_1(Y)$. So we expect $i_* = j_*$. We'll capture this in general with a new notion of deformation for maps.

DEFINITION 4.1. Given two maps $f, g : X \to Y$, a *homotopy* of f to g is a map $H : I \times X \to Y$ which satisfies $H(u,0) = g,(u)$, and $H(u,1) = f(u)$ for all $u \in X$. We write $f \underset{h}{\sim} g$ if there is a homotopy between them.

REMARK. As with paths, another way of thinking about homotopy is to consider the family of maps $H_t(u) = H(u,t)$. Since H is a homotopy, H_t has $H_0 = g$ and $H_1 = f$.

PROPOSITION 4.2. *$\underset{h}{\sim}$ is an equivalence relation on maps from $X \to Y$.*

Topologists use the following notation:

$$[X,Y] = \{\text{continuous maps from } X \text{to } Y\} / \underset{h}{\sim}.$$

As we'll see, $[X,Y]$ will turn out to be countable for reasonable spaces. In many cases of interest, $[X,Y]$ will also turn out to be a group. Let's also introduce the notation $f \underset{h_+}{\sim} g$ to mean $f(x) = y$, $g(x) = y$, and there exists a homotopy$_+$ H with $H(x,t) = y$ for all $0 \leq t \leq 1$. And then

$$[X,Y]_+ = [X,Y] = \{\text{continuous maps}_+ \text{from } X \text{to } Y\} / \underset{h_+}{\sim}.$$

These considerations enormously shrink the subject of maps from X to Y.

EXAMPLE 4.3. If $Y \subset \mathbb{R}^n$ is convex, then $[X,Y]$ (or $[X,Y]_+$) has just one element, represented by the constant map to the basepoint of Y. Proof is by linear interpolation: generic f, g are homotopic by $tf + (1-t)g$.

But we still retain useful information, topologically speaking.

PROPOSITION 4.4. *If $f \underset{h_+}{\sim} g$, then $f_* = g_*$ as homomorphisms from $\pi_1(X,x)$ to $\pi_1(Y,y)$.*

PROOF. The idea is that if we can deform the whole space, then we can deform loops. We want to show that $f_*([\alpha]) = g_*([\alpha])$ for $[\alpha] \in \pi_1(X,x)$. So let H be a homotopy$_+$ from f to g, and let $L : X \times I \to X \times I$ be given by $L(u,t) = (\alpha(u), t)$. Then $H \circ L$ is a homotopy (in the sense of loops) between $f \circ \alpha$ to $g \circ \alpha$. So $[f \circ \alpha] = [g \circ \alpha]$, and so $f_*([\alpha]) = g_*([\alpha])$. □

DEFINITION 4.5. A map$_+$ $f : X \to Y$ is said to be *null-homotopic* if it is homtopic$_+$ to the constant map (to the basepoint of Y).

These maps play the role of zero maps.

PROPOSITION 4.6. *If f is null-homotopic, then f_* is the zero map.*

PROOF. Let c be the constant map. If $f \underset{h_+}{\sim} c$, then $f_* = c_* = 0$. □

EXAMPLE 4.7. If Y (or X) is convex, then all maps of X to Y are null-homotopic.

Notice that in our notation, $\pi_1(X,x) = \left[S^1, X\right]_+$. What makes π_1 so special is that it is a group.

DEFINITION 4.8. The *higher homotopy groups* are the sets $\pi_k(X,x) = \left[S^k, X\right]_+$.

The name here is due to Hurewicz, who discovered them.[1] A priori it's not obvious these are groups, but we'll prove it, and in fact we'll also see they are abelian for $k > 1$. As k increases, π_k provides a way to study higher dimensional (spherical) holes.

DEFINITION 4.9. A is called a *deformation retract*$_{(+)}$ of X if there is an inclusion$_{(+)}$ $i : A \hookrightarrow X$ and a map$_{(+)}$ $r : X \to A$ so that $r \circ i \underset{h\,(+)}{\sim} \mathrm{id}_A$.

PROPOSITION 4.10. *Suppose A is a deformation retract$_+$ of X and let $i : A \hookrightarrow X$ be the inclusion. Then the induced map $i_* : \pi_1(A) \to \pi_1(X)$ is an inclusion.*

EXAMPLE 4.11. Suppose A is convex and $i : A \hookrightarrow X$ is inclusion.

CLAIM. A is a deformation retract of X.

PROOF. Take $r : X \to A$ to be a constant map. As A is convex, any two maps are homotopic. So $r \circ i \underset{h}{\sim} \mathrm{id}_A$. □

5. Homotopy of Spaces

For the purposes of algebraic calculation in topology, we need to loosen the notion of homeomorphism. Recall a continuous map $f : X \to Y$ is a *homeomorphism* if there is a continuous map $g : Y \to X$ so that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

DEFINITION 5.1. A map $f : X \to Y$ is called a *homotopy equivalence*$_{(+)}$ *(h.e.)* if there is a map$_{(+)}$ $g : Y \to X$ so that $g \circ f \underset{h\,(+)}{\sim} \mathrm{id}_X$ and $f \circ g \underset{h\,(+)}{\sim} \mathrm{id}_Y$. If there is such a map, then we say X and Y are *homotopy equivalent*$_{(+)}$ spaces.

PROPOSITION 5.2. *Homotopy equivalence is an equivalence relation among spaces.*

EXERCISE 5.3. Check this in detail. The main point (as usual) is transitivity.

PROPOSITION 5.4. *If $f : X \to Y$ is a homotopy equivalence, then the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

PROOF. Suppose $g \circ f \underset{h\,+}{\sim} \mathrm{id}_X$ and $f \circ g \underset{h\,+}{\sim} \mathrm{id}_Y$ as in the definition. Then $g \circ f \underset{h\,+}{\sim} \mathrm{id}_X$ and $(g \circ f)_* = (\mathrm{id}_X)_*$ so by naturality $g_* \circ f_* = \mathrm{id}_{\pi_1(X)}$. It's the same for $f \circ g$. □

In light of this,

DEFINITION 5.5. Spaces h.e. to X are said to have the *same homotopy type* as X.

EXERCISE 5.6. $[X, Y]$ depends only on the homotopy types (respectively) of X and Y.

REMARK. The converse does not hold. There are counterexamples showing $\pi_k(X)$ and $\pi_k(Y)$ can be the same for all k even though X and Y are not homotopy equivalent.

[1] There is an interesting non-trivial map from S^3 to S^2, discovered by Hurewicz, which shows $\pi_3(S^2) \neq 0$.

THEOREM 5.7 (J.H.C. Whitehead). *Suppose $f : X \to Y$ is a map between "reasonable" path-connected spaces. Then $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism iff f is a homotopy equivalence.*

The hard part is the converse. We won't do this here.

Homotopy equivalence is much weaker (even suprisingly so) than homeomorphism.

EXAMPLES 5.8.

(1) $\mathbb{R}^n$ is homotopy equivalent to a point. In general, suppose X is convex, then X will be homotopy equivalent to a point. Indeed, given $p \in X$, let $f : X \to \{p\}$ be the constant map and let $g : \{p\} \to X$ be a map to any point in X. Then $f \circ g = \mathrm{id}_{\{p\}}$, and $g \circ f$ maps X to (a point in) X so that $g \circ f \underset{h}{\sim} \mathrm{id}_X$.

(2) $\mathbb{R}^2 \setminus \{p\}$ is homotopy equivalent to S^1.

EXERCISE 5.9. Check this. Let $i : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ be inclusion and let $f : \mathbb{R}^2 \setminus \{0\} \to S^1$ be given by $f(v) = v/||v||$.

(3) $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to S^{n-1}.

(4) $\mathbb{R}^n \setminus \{p_1, p_2\}$ with $p_1 \neq p_2$ is homotopy equivalent to a "figure eight".

(5) $\mathbb{R}^n \setminus \{p_1, \ldots, p_k\}$ with $p_i \neq p_j$ for $i \neq j$ is homotopy equivalent to $\vee_k S^{n-1}$.

(6) $\mathbb{R}^2 \setminus \{0\} \simeq S^1 \times \mathbb{R}^+$ which is homotopy equivalent to $S^1 \times \{p\}$ by the next exercise.

EXERCISE 5.10. If X is h.e. X' and Y is h.e. Y', then $X \times Y$ is h.e. $X' \times Y'$.

Similarly, $\mathbb{R}^n \setminus \{p\} \simeq S^{d-1} \times \mathbb{R}^+$ which is homotopy equivalent to $S^{d-1} \times \{p\}$.

Next time, we'll see that $\pi_1(S^n) = 0$ for $n > 1$.

THEOREM 5.11 (Poincare Conjecture). *If M is a closed, simply-connected, three-dimensional manifold, then $M \simeq S^3$.*

It's not that hard (relatively, at least) to prove

PROPOSITION 5.12. *If M is a closed, simply-connected, three-dimensional manifold, then M is homotopy equivalent to S^3.*

Lecture 7, 10/31/11

6. Combinatorial Group Theory and Van Kampen's Theorem

We already know how to compute the fundamental group of a product.

PROPOSITION 6.1. $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

Now let's consider a different type of construction. Suppose we have $X = A \cup B$ with $A \cap B \neq \emptyset$. Can we describe $\pi_1(X)$ in terms of $\pi_1(A)$ and $\pi_1(B)$? First we need some group theory.

DEFINITION 6.2. If G is a group and $S \subset G$ is a set of elements of G, the *subgroup generated by* S (in G) is the smallest subgroup of G containing S.

The definition makes sense by the following easy proposition.

PROPOSITION 6.3. *If $S_1, S_2 \subset G$ are subgroups, then so is $S_1 \cap S_2$.*

So the smallest subgroup of G containing S is the intersection of all such subgroups. And the intersection is non-empty, for at least $S \subset G$.

Can we explicitly describe the subgroup H which is generated by S? Suppose $S = \{s_1, s_2, \ldots, s_k, \ldots\}$, then

$$H = \left\{\text{all finite products } s_{j_1}^{\pm 1} s_{j_2}^{\pm 1} \cdots s_{j_m}^{\pm 1} \,\middle|\, s_{j_i} \in S\right\}.$$

Here we call the product $s_{j_1}^{\pm 1} s_{j_2}^{\pm 1} \cdots s_{j_m}^{\pm 1}$ a *word*. So H is the subgroup which consists of all words built from elements in S. And if $H = G$ then we say S *generates* the group G. S is the group-theory equivalent to the basis from linear algebra.

EXAMPLE 6.4. $\mathbb{Z}$ is generated by $\{+1\}$, $\{-1\}$, $\{2, 7\}$, etc.

Now suppose $A_{,x}, B_{,x}$ are open in $X_{,x}$ and suppose $X = A \cup B$. Then a natural guess is that $\pi_1(X, x)$ is generated by $\pi_1(A, x)$ and $\pi_2(B, x)$. Note this implies that if S is a set generated from $\pi_1(A)$ and T a set generated from $\pi_1(B)$, then $S \cup T$ generates all of $\pi_1(X)$. The next example shows this guess is incorrect.

EXAMPLE 6.5. Let $A = S^1 / \left\{e^{i\theta},\ \theta \in [-\pi/4, \pi/4]\right\}$ and $B = S^1 / \left\{e^{i\theta},\ \theta \in [3\pi/4, 5\pi/4]\right\}$. Then there are loops in $A \cap B$ which are not in A or in B.

The problem here is path-connectedness (In the example above, $A \cap B$ has two path-components.)

THEOREM 6.6 (1/2 of van Kampen's Theorem). *Let $X_{,x} = A_{,x} \cup B_{,x}$ where A, B are open in X, and suppose $A \cap B$ is path-connected. Then $\pi_1(X, x)$ is generated by $\pi_1(A)$ and $\pi_1(B)$.*

COROLLARY 6.7. *Let A, B as in the above theorem. Then if A, B are simply connected, then $\pi_1(A \cup B) = 0$.*

EXAMPLES 6.8.

(1) Thus S^n is simply connected for $n > 1$. Write $S^n = A \cup B$ where A is a bit more than the upper hemisphere and B a bit more than the lower hemisphere. We can choose A, B to be open sets so that $A \cap B$ is the equatorial band. Then $A \cap B \simeq S^{n-1} \times (-1, +1)$ is path-connected, and since $A, B \simeq D^n$, A, B are simply connected. Now apply the corollary.
(2) $\Pi_{i=1}^k S^{n_i}$ for $n_i > 1$ is simply connected. For example, $S^2 \times S^2$ is simply connected.

PROOF OF THEOREM. The proof goes as follows:

(1) Break a loop based at $x \in A \cap B$ into a finite number of pieces which are (alternately) contained in $A \cap B$.
(2) Modify the resulting paths to make them into loops based at x.

The details are not hard. For step one, say $\gamma : I \to X$ is a loop based at x and consider $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$. These are open subsets of I, each of which can be written as a union of open intervals. By compactness, there is a finite subcover. We can combine sequential ones belonging all in $\gamma^{-1}(A)$ (similarly for those in $\gamma^{-1}(B)$). So γ is decomposed into a series of sequentially overlapping open intervals, each contained in $\gamma^{-1}(A)$ or in $\gamma^{-1}(B)$ (and we can assume these alternate). Finally, picking points in the sequential overlaps (which lie in $\gamma^{-1}(A) \cap \gamma^{-1}(B)$), we get $\gamma = \Pi_{i=1}^k \gamma_i$ where γ_i is a path in A or B, alternately.

Now step two. Suppose the points we chose above correspond to times $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < 1 = t_k$. Let γ_i be the path $\gamma|_{[t_{i-1},t_i]}$. Finally,

$$[\gamma] = \left[\left(\gamma_1\delta_1^{-1}\right)\left(\delta_1\gamma_2\delta_2^{-1}\right)\left(\delta_2\gamma_3\delta_3^{-1}\right)\cdots\left(\delta_{k-1}\gamma_k\right)\right]$$

in $\pi_1(X)$ by cancellation up to path homotopy (each parenthesis contains a loop in $A \cap B$). Here δ_i is a path from x to $\gamma_i(t_{i-1})$. (CHECK) $\square$

Before we can compute more complicated fundamental groups, we need to develop some basic combinatorial group theory.

6.1. Group Presentation. First we introduce the notion of a "free group".

Definitions 6.9. Let A be any set, then the set $W = \left\{x_1^{\pm 1}x_2^{\pm 1}\cdots x_n^{\pm 1} \,\middle|\, x_i \in A\right\}$ consists of all the *words* generated from the *alphabet* A. A *reduced word* is a word which does not contain sequentially the letters x, x^{-1} for $x \in A$. The *reduction map* $r = \text{red.} : W = \text{words} \to RW = \text{reduced words}$ sends a word to its reduced form.

Note. For example, $r : xyy^{-1}z^{-1}zzxy^{-1} \mapsto xzxy^{-1}$. One has to prove this is well-defined, i.e. reducing in any order results in the same word. For details, see any book on combinatorial group theory.

We can define a group whose elements are the reduced words, where product is given by concatenation followed by reduction.

Definition 6.10. The *free group* on A, $F(A)$, consists of the reduced words RW from A with the product $w_1 \cdot w_2 = r(w_1 \cdot w_2)$.

Note. $F(A)$ is said to be "free" as it is constructed so that there are no relations between the generating elements.

$F(A)$ depends, up to isomorphism, on $|A|$. If $|A| = k$, we show this explicitly by writing $F(A) = F_k$.

Example 6.11. $F_0 = \{e\}$ and $F_1 = \left\{x^k \,\middle|\, k \in \mathbb{Z}\right\} \simeq \mathbb{Z}$.

$$F_2 = F(a,b) = \left\{\begin{array}{c} e, \\ a, b, a^{-1}, b^{-1}, \\ a^2, a^{-2}, ab, ba, a^{-1}b, a^{-1}b^{-1}, b^{-1}a, ba^{-1}, b^{-1}a^{-1} \\ \vdots \end{array}\right\}.$$

Already F_2 is non-abelian.

By construction, $A \subset F(A)$. A behaves almost like a basis in linear algebra (but now $F(A)$ can be non-abelian).

Proposition 6.12. *Let G be a group and $i : A \hookrightarrow F(A)$ inclusion. Given a map $\phi : A \to G$, there is a unique homomorphism $\Phi : F(A) \to G$ with $\Phi \circ i = \phi$.*

Proof. Set $\Phi\left(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_k^{\epsilon_k}\right) = \phi(x_1)^{\epsilon_1}\phi(x_2)^{\epsilon_2}\cdots\phi(x_k)^{\epsilon_k}$. $\square$

Exercise 6.13. Check the details.

The construction above is very concrete. It would be better, philosophically, to say the free group $F(A)$ on a set A is the unique group with the property that given the inclusion $i : A \hookrightarrow F(A)$ any map $\phi : A \to G$ there exists a unique homomorphism Φ so that $\Phi \circ i = \phi$. (So that the diagram commutes.) (D1) This is the true defining characteristic of the free group. Of course, one can check that everything we've done above holds in the more general view.

PROPOSITION 6.14. *Every group is a quotient of a free group. Every finitely generated group is a quotient of a free group on a finite set.*

PROOF. Say a group G is generated by a set $S \subset G$. Write $S = \{a, b, c, \dots\}$ and let $\overline{S} = \left\{\overline{a}, \overline{b}, \overline{c}, \dots\right\}$. Now define a homomorphism $\Phi : F\left(\overline{S}\right) \to G$ by $\Phi\left(\overline{u}\right) = u$. (By the previous proposition it is enough to specify Φ on the generators.) Since Φ is surjective, $G \simeq F\left(\overline{S}\right) / \ker(\Phi)$. $\square$

EXERCISE 6.15. Let $\Phi : H \to G$ be a surjective homomorphism. Check that $G \simeq H/\ker(\Phi)$.

EXAMPLES 6.16.

(1) $\mathbb{Z}_n = F_1/a^n$ where $F_1 = F(a)$.
(2) Consider the symmetric group (permutation group) on n elements, S_n.

    EXERCISE 6.17. S_n is generated by two elements $a : (1, 2, \dots) \mapsto (2, 1, \dots)$ (permutation of the first two entries) and $b : (1, \dots, n) \mapsto (n, 1, \dots, n-1)$.

    Define $\Phi : F(A, B) \to S_n$ by $\Phi(A) = a$ and $\Phi(B) = b$. Then $S_n = F_2/\ker(\Phi)$. Note $\ker(\Phi)$ contains A^2, B^n, etc.

Now we know we can always write $G = F(A)/\text{something}$. In fact the "something" must be a so-called "normal" subgroup.

DEFINITION 6.18. Let H be a subgroup of G. Define the *right coset* of H in G to be the set $Hg = \left\{hg \,\middle|\, h \in H,\ g \in G\right\}$.

EXERCISE 6.19. Show that if $Hg_1 \cap Hg_2 \neq \emptyset$, then $Hg_1 = Hg_2$.

So we can write write $G = He \cup Hg_1 \cup Hg_2 \cup \cdots$ which is a disjoint union of right cosets. (This is the *right coset decomposition* of G). What about left cosets? Consider that if $Hg = gH$ for all $g \in G$, then $H = gHg^{-1}$ for all g. So we have

DEFINITION 6.20. A subgroup $H \subset G$ is called *normal* if $H = gHg^{-1}$ for all $g \in G$.

PROPOSITION 6.21. *If G is abelian, all subgroups are normal.*

EXAMPLE 6.22. S_3 has $3! = 6$ elements. It has subgroups $\mathbb{Z}_2$ of order two and $\mathbb{Z}_3$ of order three. The first permutes elements (ab) and the second shuffles $(abc) \mapsto (cab)$.

EXERCISE 6.23. Check that $\mathbb{Z}_2$ is not normal and $\mathbb{Z}_3$ is normal.

If $H \triangleright G$ (read: H is normal in G), then the left and right cosets are the same.

DEFINITION 6.24. Supose $H \triangleright G$, then the *quotient group* G/H is the group consisting of the cosets $\left\{Hg \,\middle|\, g \in G\right\}$ with product $(Hg_1) \cdot (Hg_2) = Hg_1g_2$.

PROPOSITION 6.25. $|G/H| = |G|/|H|$.

If we let $[G : H]$ be the number of cosets of H in G, then the proposition reads $[G : H] = |G|/|H|$.

Proposition 6.26. *Given a homomorphism* $\chi : K \to L$, $\ker(\chi) = \{u \in K \mid \chi(u) = e\}$ *is a normal subgroup.*

Proof. Let $H = \ker(\chi)$ so that $\chi(H) = e$. Then $\chi(gHg^{-1}) = \chi(g)\chi(H)\chi(g^{-1}) = \chi(g)e\chi(g)^{-1} = e$ for all $g \in G$. Hence $gHg^{-1} \subset H$ and $H \subset gHg^{-1}$, so we're done. $\square$

Exercise 6.27. Check the proof.

Now suppose we have a surjective homomorphism $\Phi : F(A) \to G$, then as above $G \simeq F(A) / \ker(\Phi)$. Of course, we could describe $\ker(\Phi)$ by listing its generators, but this is in general too difficult and not necessary. Recall $x^g = gxg^{-1}$ is the *conjugate* of x by g. And recall we said a set S generates a subgroup $H \subset G$ if H is the smallest subgroup containing S. Explicitly, H is all the finite products of elements of S or their inverses. Now we have

Definition 6.28. A set S *normally generates* the normal subgroup H of G if H is the smallest normal subgroup of G containing S.

We'll write $\langle S \rangle$ to mean the subgroup H normally generated by S. H now consists of all products of elements of S, their inverses, or conjugates of S.

Example 6.29. Let $F(a,b)$ be the free group on two generators. Then we have $F(a,b) / \langle b \rangle = F(a)$. Lots of elements are killed (not just b)! $\langle b \rangle$ contains b^k, ab^ka^{-1}, $a^jb^ka^{-j}$, etc. The point here is that we did not need to specify that all these elements are killed; we only needed to say b (and the normal subgroup it generates) are killed.

This is a general phenomena: $\ker(\Phi)$ is often easily describable by its normal generators. This gives a more efficient way to describe the quotient $F(A) / \ker(\Phi)$.

Example 6.30. A clock with ten hours (dihedral group of order $2n$). There is an element σ which is rotation by $1/n$. Then $\sigma^n = e$. And there is an element τ which is a flip. Then $\tau^2 = e$ ($\tau = \tau^{-1}$). And one can show $\tau\sigma\tau = \tau^{-1}$ (flip, rotate, then flip). This is equivalent to writing $(\tau\sigma)^2 = e$. These three relations describe the whole multiplication table of the group. There are $2n$ elements in the group; n rotations and n flips. In the notation above, we have $D_{2n} = \text{free}/\langle\text{relations}\rangle = F(s,t) / \langle S^n, T^2, TSTS \rangle$. This is a very convenient way to describe the whole group. We can write this in the notation

$$D_{2n} = \left\{ \underbrace{s,t}_{\text{generators}} \mid \underbrace{S^n, T^2, TSTS}_{\text{relations}} \right\}.$$

This is the "presentation" of the group D_{2n}.

Lecture 8, 11/7/11

We have shown that every group can be described by its *presentation*,

$$G = \left\{ (x_i)_{i \in I} \mid (R_j)_{j \in J} \right\} = F\left\{ (x_i)_{i \in I} \right\} / \left\langle (R_j)_{j \in J} \right\rangle$$

where the x_i are the generators and the R_j = word in x_j's are relations. The angled brackets on the right mean that not only are the R_j set to e, but also all products and conjugates of them and their inverses.

Example 6.31. $\mathbb{Z}_n = \{x \mid x^n\}$.

Note that it is both diffucult in theory and in practice to recognize that a group is trivial from its presentation.

DEFINITION 6.32. A group is *finitely presented* if both $|I|$ and $|J|$ are finite.

EXERCISE 6.33. $\mathbb{Q}$ is not finitely generated. (Hint: Think about denominators.)

EXERCISE 6.34. (Rapaport) Let x, y, z be the generators. Let $yxy^{-1} = x^2$, $zyz^{-1} = y^2$, $xzx^{-1} = z^2$. (When the number of relations equals the number of generators, the group is said to have a *balanced presentation.*) Show this is the trivial group. (Hint: Once we get, say, $x = e$ then by symmetry $y = z = e$.)

EXAMPLE 6.35. There is an analogue due to Serre in four generators x, y, z, w. The relations are are

$$\begin{aligned} yxy^{-1} &= x^2 \\ zyz^{-1} &= y^2 \\ wzw^{-1} &= z^2 \\ xwx^{-1} &= w^2, \end{aligned}$$

but now the conclusion is that the group is infinite.

How could we go about deciding if a group is trivial? Suppose G has the presentation

$$G = \left\{ (x_i)_{i \in I} \;\middle|\; (R_j)_{j \in J} \right\}.$$

The following *Tietze moves* modify the presentation while keeping G the same:

(1) $G = \{(x_i), y \,|\, (R_j), y\}$
(2) $G = \left\{(x_i) \;\middle|\; (R_j), R_\alpha^{\pm 1} R_\beta^{\pm 1}\right\}$
(3) $G = \{(x_i) \,|\, (R_j) \text{ with one replaced by a conjugate}\}$

These moves are the non-commutative analogue to Gaussian elimination from linear algebra. But here, the number of relations can change (in Gaussian elimnation the number of rows/columns stay the same).

THEOREM 6.36. *Given two finite presentations of a group G, they are related by a sequence of such moves.*

PROOF SKETCH. Represent the second set of relations in terms of the first and mimic with a sequence of moves. $\square$

REMARK. There are moves which preseve balance. But the corresponding theorem in this case is unknown (see the Andrews-Curtis conjecture, which people believe is false). This is interesting to topologists since balanced presentations arise naturally for low-dimensional manifolds.

Now suppose G is trivial, but we are given a presentation with n relations. One can show the number of Tietze moves needed to conclude G is trivial grows faster than any computable function. Hence it is not algorithmically decidable if an arbitrary group G is the trivial group!

6.2. Van Kampen's Theorem. Van Kampen's theorem describes the fundamental group of $X = A \cup B$ in terms of the fundamental groups of A and B. So we need a way to combine fundamental groups. We already have the direct product, but this was only useful in describing $\pi_1(X \times Y)$. Here is a second way to combine groups.

DEFINITION 6.37. Suppose $G_1 = \left\{x_i \,\middle|\, R_j\right\}$ and $G_2 = \left\{y_k \,\middle|\, S_l\right\}$ are groups. Their *free product* is the group

$$G_1 \star G_2 = \left\{(x_i)\,,(y_k) \,\middle|\, (R_j)\,,(S_l)\right\}.$$

NOTE. This is independent of the choice of presentations of G_1, G_2.

EXAMPLE 6.38. $F_k \star F_l = F_{k+l}$. So $F_k = \mathbb{Z} \star \mathbb{Z} \star \cdots \star \mathbb{Z}$ where there are k copies of $\mathbb{Z}$.

Here is a more conceptual way to describe the free product. Given groups G_1, G_2, a group K is called a *free product* of G_1 and G_2 if there exists a diagram of group homomorphisms (D1) such that given any homomorphsisms $\phi_1 : G_1 \to L$ and $\phi_2 : G_2 \to L$ into some group L, there exists a unique $\Phi : K \to L$ so that $\Phi \circ i = \phi_1$ and $\Phi \circ j = \phi_2$. (D2) Then one can prove:

1. If K exists it must be unique (use uniqueness of Φ to get, when L, K satisfy the hypotheses, maps both ways, etc.).
2. There exists such a K. (Take $K = G_1 \star G_2$.)

Recall the direct product $G_1 \times G_2$. First, note that $(a, b) \cdot (a', b') = (aa', bb')$ so then (a, e) and (e, b) commute. So $G_1 \star G_2 \neq G_1 \times G_2$, and in fact

$$G_1 \times G_2 = \left\{(x_i)\,,(y_k) \,\middle|\, (R_j)\,,(S_l)\,, x_i y_k x_i^{-1} y_k^{-1}\right\}.$$

But still the two products are related: (D3). Though the existence of Φ is guaranteed by the definition of free product we gave above, we already knew it was possible to find such a homomorphism. Indeed, it is always possible to get a homomorphism just by adding more relations (to get a quotient group).

Why does all of this matter for topology? Here is another weak form of Van Kampen's theorem.

THEOREM 6.39 (Weak van Kampen). *Let $X = A \cup B$ with A, B open and assume $A, B, A \cap B$ are path connected. Choose a common base point $x \in A \cap B$. Assume $A \cap B$ is simply connected, then*

$$\pi_1(X, x) = \pi_1(A, x) \star \pi_1(B, x).$$

REMARK. We usually can apply van Kampen's theorem even when A, B are not open. If A is not open, the trick is to replace A by a slightly larger open set A' such that $i : A \hookrightarrow A'$ is a homotopy equivalence (similarly replace B with B'), then apply van Kampen's theorem to A' and B'.

EXAMPLE 6.40. We know $\pi_1\left(S^1\right) = \mathbb{Z}$. So then $\pi_1\left(\vee_2 S^1\right) = \mathbb{Z} \star \mathbb{Z}$. More generally, $\pi_1\left(\vee_k S^1\right) = F_k$. And $\pi_1\left(\vee_k S^2\right) = \{0\}$ since $S^k = D_+^k \cup_{S^{k-1}} D_-^k$.

Here is the first non-trivial example of a free product.

EXAMPLE 6.41. The infinite dihedral group, $\mathbb{Z}_2 \star \mathbb{Z}_2$. What does this look like? Say the first is generated by a, the second by b, then

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\{a, b \,\middle|\, a^2, b^2\right\}.$$

We can list out the words:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \begin{cases} ababab \cdots a \\ ababab \cdots b \\ bababa \cdots a \\ bababa \cdots b \end{cases}.$$

In fact $\mathbb{Z} \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ as a normal subgroup, look for products of (ab). So then $\mathbb{Z}_2 \times \mathbb{Z}_2 / \mathbb{Z} = \mathbb{Z}_2$.

EXERCISE 6.42. Prove that $D_{2n} = \left\{a, b \,\middle|\, a^2, b^2, (ab)^n\right\}$.

(D4)

EXAMPLE 6.43. $\pi_1 (X_\vee Y) = \pi_1 (X) \star \pi_2 (Y)$.

What if $A \cap B$ is not simply connected? Then the free product is not enough, since $\pi_1 (A) \star \pi_1 (B)$ includes possibly two copies of $\pi_1 (A \cap B)$. We need a way to "count" the overlap once.

DEFINITION 6.44. Let G_1, G_2 be groups and suppose we are given a group $H = \left\{(z_m) \,\middle|\, \cdots\right\}$ with homomorphisms $i : H \to G_1$ and $j : H \to G_2$. Then the *free product of G_1 and G_2 with amalgamation by H* is given by

$$G_1 \star_H G_2 = \left\{(x_i), (y_k) \,\middle|\, (R_j), (S_l), i(z_m) = j(z_m) \ \forall m\right\}.$$

NOTE. At this point, we don't need to assume that i, j are injective.

THEOREM 6.45 (van Kampen). *Suppose $X = A \cup B$ with A, B open and $A, B, A \cap B$ path connected. Choose a common basepoint $x \in A \cap B$. Then*

$$\pi_1 (X) = \pi_1 (A) \star_{\pi_1 (A \cap B)} \pi_1 (B)$$

with the maps $i_* : \pi_1 (A \cap B) \to \pi_1 (A)$ and $j_* : \pi_1 (A \cap B) \to \pi_1 (B)$ induced by the inclusions $i : A \cap B \hookrightarrow A$ and $j : A \cap B \hookrightarrow B$.

EXAMPLES 6.46.

(1) $G \star_G G = G$ with the maps $i, j = \mathrm{id}_G$.
(2) $G_1 \star_{\{e\}} G_2 = G_1 \star G_2$.

As remarked before, i_* and j_* may fail to be injective. Here is a interesting example from geometry.

EXAMPLE 6.47. The solid torus is $D^2 \times S^1$, which has $\pi_1 \left(D^2 \times S^1\right) = \mathbb{Z}$.

CLAIM. $S^3 = D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2$.

Before giving a proof, let's see what this says about π_1. We have $\pi_1 \left(S^3\right) = \{0\}$ and

$$\begin{aligned} \pi_1 \left(D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2\right) &= \pi_1 \left(D^2 \times S^1\right) \star_{\pi_1 (S^1 \times S^1)} \pi_1 \left(S^1 \times D^2\right) \\ &= \mathbb{Z} \times_{\mathbb{Z} \times \mathbb{Z}} \mathbb{Z} \end{aligned}$$

with the respective maps $i_* = (0, 1)$ and $j_* = (1, 0)$ (written as matrices). So

$$\begin{aligned} \pi_1 \left(D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2\right) &= \left\{x, y \,\middle|\, i_* (a) = j_* (a), i_* (b) = j_* (b)\right\} \\ &= \left\{x, y \,\middle|\, e = y, x = e\right\} \\ &= \{e\}. \end{aligned}$$

Now here are two proofs of the claim. The first is symbolic. For manifolds M, N we have

$$\partial(M \times N) = \partial M \times N \cup_{\partial M \times \partial N} M \times \partial N.$$

(Think about $\partial(I \times I)$.) So

$$\partial(D^p \times D^q) = \partial D^p \times D^q \cup_{\partial D^p \times \partial D^q} D^p \times \partial D^q$$

and then

$$S^{p+q-1} = \partial\left(D^{p+q}\right) = S^{p-1} \times D^q \cup_{S^{p-1} \times S^{q-1}} D^p \times S^{q-1}.$$

In particular,

$$S^3 = \partial\left(D^2 \times D^2\right) = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1.$$

And now here is a picture: (D5)

Lecture 9, 11/14/11

7. Application to Knot Theory

This section discusses an interesting geometric application of the fundamental group to knots. A typical knot looks like (D1), known as the *trefoil knot*. (The knot lives in $\mathbb{R}^3$, but we've drawn the "under-over" projection of the knot onto a plane.) The most naive idea of a knot is an injective map $\omega : I \to \mathbb{R}^3$. But we could pull apart this knot completely since the ends are not permanently attached. So instead we have

Definition 7.1. An injective map $\omega : S^1 \to \mathbb{R}^3$ is called a *knot*.

It is a basic question of knot theory to classify the knots. The *trivial knot* (*unknot*) is just the inclusion $S^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$. In this section we hope to prove that the trefoil knot and the unknot are not the same. We'll do so by coming up with invariants and then computing these for the trefoil knot and unknot.

Definition 7.2. The *crossing number* of a knot is the minimum number of crossings in a projection.

The unknot has crossing number zero, and the trefoil knot has crossing number three. Here is another view of the trefoil knot: (D2) The *figure eight knot* has four crossings: (D3) All of these so far are examples of *alternating* knots, or knots where the crossings alternate under then over then under again. Another alternating knot, this time with five crossings: (D4) But have we proven yet that the crossing numbers are what we've asserted above? We only guessed what the numbers are, because the pictures are so simple. In fact, the crossing number is not a useful invariant, as there is no formula in general.

And what exactly do we mean when we say two knots are the same? All of the knots are really just circles, being injective maps defined on S^1. But what makes them different is that they are sitting in $\mathbb{R}^3$ in different ways. Note that $\mathbb{R}^3$ is not essential; a common (classical) way to think of a knot is as a map $S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \infty$. So we have

Definition 7.3. A *knot* is a map $K \hookrightarrow S^3$ where $K \simeq S^1$.

And equivalence is defined via

Definition 7.4. Two knots $K_1 \hookrightarrow S^3_1$, $K_2 \hookrightarrow S^3_2$ are *equivalent* if there is a homeomorphism $\phi : S^3_1 \to S^3_2$ with $\phi(K_1) = K_2$.

NOTE. This particular definition of equivalence does not keep track of direction. For example, it cannot tell a knot from its mirror image.

Usually we restrict further the notion of a "knot." One could have a knot with a infinite amount of "knottedness", (D5) Pictures like this are excluded by making any of several requirements: one could require the knot to be

- piecewise linear (a finite number of line segments)
- a differential submanifold (with tangents non-zero)
- topologically locally flat (near each point it looks standard).

But we won't do any of these, since we only care to make a difference between the trefoil and the unknot.

Step one is to consider everything that's not the knot.

DEFINITION 7.5. The *knot complement* of a knot $K_1 \hookrightarrow S^3$ is the space $S^3 \backslash K_1$.

The knot complement of the unknot is quite simple. But we'll see that the knot complement of the trefoil knot is quite complicated. The idea now is to show that $S^3\backslash$trefoil and $S^3\backslash$unknot are not homeomorphic. Indeed, if there were an equivalence of knots $\phi : S^3 \to S^3$ then restricting it to the knot $\phi| : K_1 \to K_2$ gives a homeomorphism and so does restricting it to the knot complement $\phi| : S^3\backslash K_1 \to S^3\backslash K_2$. This raises a philisophical question: Is a knot uniquely determined by its complement? The answer is yes in dimension three (due to Gordon and Leucke). The answer is no in higher dimensions, e.g. dimension five.

The second step is to realize that it would suffice to show the knot complements are not homotopy equivalent. Are we throwing away more information again? Yes, but we'll skip over this for the discussion here. We have only one technique for showing spaces are not homotopy equivalent. So the third step is to show that the fundamental groups of the complements are not isomorphic.

DEFINITION 7.6. Given a knot $K_1 \hookrightarrow S^3$, the *group of the knot G* is the group $G = \pi_1\left(S^3\backslash K_1\right)$.

We'll show

PROPOSITION 7.7. *The group of the trefoil knot is not abelian, whereas the group of the unknot is $\mathbb{Z}$.*

From this it immediately follows

PROPOSITION 7.8. *The unknot and the trefoil are not equivalent.*

There are generalizations of knot theory to higher dimensions.

THEOREM 7.9 (Zeeman unknotting). *Any knot $S^k \hookrightarrow S^n$, $k \leq n$, is equivalent to the standard knot unless $k = n - 2$.*

The first person to discover knots in higher dimensions was Artin.

It's often annoying to deal with non-compact spaces. So we often work with what's called the *closed complement* of the knot. Suppose we have $S^1 \hookrightarrow S^3$ a knot. Then one can prove there exists a tube $S^1 \times D^2$ in which the knot is included. Then $S^3\backslash\left(S^1\right)^0 \times D^2 \hookrightarrow S^3\backslash S^1 \times \{0\}$. On the left we have the closed complement, on the right the knot complement. These are not very different. The closed complement is a three-dimensional manifold with boundary, but the manifold without its boundary is homotopy equivalent to the knot complement.

Now recall we saw

$$S^3 = D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2.$$

(D6) From the picture it follows that the closed complement of the unknot is

$$S^3 - S^1 \times D^2 = D^2 \times S^1.$$

So

$$\begin{aligned}\pi_1 \text{(closed complement of the unknot)} &= \pi_1\left(D^2 \times S^1\right)\\ &= 0 \times \mathbb{Z}\\ &= \mathbb{Z}.\end{aligned}$$

And the complement of the unknot is

$$S^3 \backslash \text{unknot} \simeq \text{(open disc)} \times S^1 \simeq \mathbb{R}^2 \times S^1$$

so again $\pi_1 = \mathbb{Z}$. (D7)

Now for the trefoil knot. The trefoil can be drawn on the standard torus in $\mathbb{R}^3$. In general, *torus knots* are knots that can be drawn on the torus. (D8) Segments I, II, III, IV altogether form a loop on the torus. We claim $I \cup II \cup III \cup IV =$ trefoil. (D9) So the trefoil is the $(2,3)$-torus knot. In general we have the (p,q)-*torus knots* where p, q are relatively prime.

Now we want to prove $(2,3)$-torus knot is not the unknot. Let's compute $G_{(p,q)} = \pi_1\left(S^3 \backslash (p,q)\text{-torus knot}\right)$. Now

$$S^3 \backslash K_{(p,q)} = \left(D^2 \times S^1 \backslash K_{(p,q)}\right) \cup_{\left(S^1 \times S^1 \backslash K_{(p,q)}\right)} \left(S^1 \times D^2 \backslash K_{(p,q)}\right).$$

By deleting the knot we have only taken away from the boundary of $D^2 \times S^1$ and $S^1 \times D^2$. So $\left(D^2 \times S^1 \backslash K_{(p,q)}\right)$ is still homotopy equivalent to S^1, and the same with $\left(S^1 \times D^2 \backslash K_{(p,q)}\right)$. And $\left(S^1 \times S^1 \backslash K_{(p,q)}\right)$ is homotopy equivalent to $S^1 \times (0,1)$. (D10) So

$$\begin{aligned}\pi_1\left(S^3 \backslash K_{(p,q)}\right) &= \pi_1\left(S^1\right) \star_{\pi_1(S^1)} \pi_1\left(S^1\right)\\ &= \mathbb{Z} \star_{\mathbb{Z}} \mathbb{Z}\end{aligned}$$

where the inclusions are multiplication by p on the first $\mathbb{Z}$ and q on the second $\mathbb{Z}$. So

$$\pi_1\left(S^3 \backslash K_{(p,q)}\right) = \left\{x, y \,\middle|\, x^p = y^q\right\}.$$

To finish the proof we'll check this group is abelian.

Claim. $G_{(2,3)}$ is not abelian. In fact, it has a non-abelian quotient.

To see this, recall that

$$D_6 = \left\{\sigma, \tau \,\middle|\, \sigma^2 = e, \tau^3 = e, (\sigma\tau)^2 = e\right\}$$

is the non-commutative group of order six. These relations are a quotient of the relations on $G_{(2,3)}$. Take the homomorphism $\phi : x \mapsto \sigma, y \mapsto \tau$ since $\sigma^2 = \tau^3$ anyways in D^6. So if we take $G_{(2,3)}$ and add additional relations then we get D_6. So the trefoil knot is not the unknot.

Lecture 10
11/21/11.

8. Application to Topological Groups and H-spaces

This is another application for which the notion of fundamental group is useful. We'll study spaces that are both topological spaces and groups.

EXAMPLES 8.1.

(1) $\mathbb{R}^d$
(2) G, any finite group with the discrete topology.
(3) $S^1 \subset \mathbb{R}^2$ is a topological space because it is sitting in the plane. But also $S^1 = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\}$. So z gets a group structure from complex multiplication, $z_1 \cdot z_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.
(4) The n-torus $S^1 \times S^1 \times \cdots \times S^1$.
(5) Matrix groups. The *general linear group* is the group
$$GL(n, \mathbb{R}) = \{\text{real } n \times n \text{ invertible matrices}\}$$
under matrix multiplication. Since $GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$ this is also a topological space. And as $M \in GL$ iff $\det M \neq 0$, $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$. But $GL(n,\mathbb{R})$ is not compact. There are many familiar subspaces of the general linear group, e.g. the *orthogonal group*
$$O_n = \left\{A \in GL(n,\mathbb{R}) \,\middle|\, A^{-1} = A^t\right\}.$$
One can show $O_n \subset GL(n,\mathbb{R})$ is compact. Note that if $A \in O_n$ then $\det A = \pm 1$. The *special orthogonal group* is
$$SO_n = \left\{A \in O_n \,\middle|\, \det A = +1\right\}.$$
(6) Complex analogues. $\mathbb{C}^n$ is a topological group under addition. $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is a topological group under multiplication. The *complex general linear group* is $GL(n,\mathbb{C})$. The *unitary group*
$$U_n = \left\{A \in GL(n,\mathbb{C}) \,\middle|\, A^{-1} = A^*\right\}$$
is a compact subspace. Note $A \in U_n$ iff $|\det A| = 1$. The *special unitary group* is
$$SU_n = \left\{A \in U_n \,\middle|\, \det A = 1\right\}.$$
This last group is important in physics.

DEFINITION 8.2. A *topological group* is a set G so that

(1) G is a group, i.e. G has multiplication $m : G \times G \to G$ and inversion $i : G \to G$.
(2) G is a topological space.
(3) m and i are continuous.

NOTE. The third condition says that the two structures are compatible.

Topological groups often represent the symmetries of geometrical systems. They can also represent the symmetries of physical systems. Einstein realized that understanding the symmetries of the universe was important to physics.

The previous examples are all topological groups. In fact, they are are also *Lie groups*, in that they are both groups and differentiable manifolds. It is known that all finite dimensional topological groups are Lie groups. This was proved 60-70 years ago.

A natural question is: Which spaces are topological groups? If a space X has a toplogical group structure, then X satisfies a certain "uniformity". Let $e \in X$ be the identity element. Then given any $g \in X$, we can produce a homeomorphism by (left or right) multiplication by g. That is, the map $\times g : X \to X$, $x \mapsto xg$ is a homeomorphism. Notice that $\times g : e \mapsto g$. So given any point $g \in X$, there is a homeomorphism carrying e to g, and therefore there is a homeomorphism between any two points $x, y \in X$. Thus if a space is to be a topological group, then it must look the same at all points. For example, the figure eight, X, and Y are not topological groups.

THEOREM 8.3. *If G is a topological group, then $\pi_1(G, e)$ is abelian.*

This is another way to see that the figure eight is not a topological group, for its fundamental group is the free group on two elements.

EXAMPLE 8.4. $\mathbb{R}^2 \setminus \{p\}$ is a topological group. But $\mathbb{R}^2 \setminus \{p_1, p_2\}$ is not a topological group, for $\mathbb{R}^2 \setminus \{p_1, p_2\} \underset{h.e.}{\sim} S^1 \vee S^1$.

Before we prove the theorem, consider

PROPOSITION 8.5. *Given a group K, the multiplication map $m : K \times K \to K$ is a homomorphism iff K is abelian.*

PROOF. We have $m : (h, g) \mapsto hg$. In the product group we have $(h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, g_1 g_2)$. Apply the homomorphism m to get $(h_1 g_1) \cdot (h_2 g_2) = h_1 h_2 g_1 g_2$ and hence $g_1 h_2 = h_2 g_1$. This happens iff the group is abelian. $\square$

Now we'll see two proofs of the theorem. The first is quick, the second is pictorial.

FIRST PROOF. Consider multiplication $m : G \times G \to G$, which is continuous. It induces a map $m_* : \pi_1(G) \times \pi_1(G) = \pi_1(G \times G) \to \pi_1(G)$ which is given by $m_*([\alpha] \cdot [\beta]) = [\alpha \cdot \beta]$. To see this, write $[\alpha] = [\alpha] \cdot e$ and $[\beta] = e \cdot [\beta]$, so by considering pointwise multiplication we find $m_*(([\alpha], [\beta])) = m_*(([\alpha] \cdot e, e \cdot [\beta])) = [\alpha] \cdot [\beta]$. This says m_* is a homomorphism, and hence $\pi_1(G)$ is abelian. $\square$

SECOND PROOF. In $\pi_1(G)$ there seem to be two products:

(1) The usual loop product, $\alpha \cdot \beta$.
(2) The pointwise product of loops, $\alpha \times \beta(t) = \alpha(t) \cdot \beta(t)$.

We claim $[\alpha \cdot \beta] = [\alpha \times \beta]$ and that both multiplicatoins are abelian. The first claim follows as $\alpha \cdot \beta \underset{h}{\sim} \alpha \times \beta$ in G. The homotopy is depicted below. (D1) To prove the multiplication are abelian, either slide some more, or turn the picture upside down. (D2) $\square$

REMARK. We never used that G has an inverse. What's more, we never used that multiplication is associative. We only used continuous multiplication and the identity element.

This last remark leads to the following generalization of topological groups.

DEFINITION 8.6. An *H-space* is a topological space X equipped with a multiplication $m : X \times X \to X$ satisfying

(1) m is continuous.

(2) There exists an identity element e, with $m(e,u) = u = m(u,e)$ for all $u \in X$.

EXAMPLE 8.7. Any topological group is a H-space. This is the main example.

The proofs above give the next theorem.

THEOREM 8.8. *If X is a H-space, then $\pi_1(X,e)$ is abelian.*

REMARK. Lots of topological spaces have non-abelian fundamental groups. This theorem says that such topological spaces cannot have multiplication.

There are spaces that have more than one H-space structure. There is a large literature on finite dimensional H-spaces. Before we move on, let us see an interesting example of a Lie group.

EXAMPLE 8.9. S^3 is a Lie group. A concrete way to see this is via a system of arithemetic called the *quaternion numbers*, $\mathbb{H} = \left\{a + bi + cj + dk \,\middle|\, a,b,c,d \in \mathbb{R}\right\}$. The quaternions are associative and satisfy $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$. They form a group. One can define a norm via $|z| = \left(a^2 + b^2 + c^2 + d^2\right)^{1/2}$. Then $(|z||\overline{z}|)^{1/2} = |z|$ where $\overline{z} = a - bi - cj - dk$.

CLAIM. $S^3 = \left\{z \in \mathbb{H} \,\middle|\, |z| = 1\right\}$.

So S^3 has a multiplication via the quaternions, so it is a group. The quaternions were discovered by Hamilton.

What spheres are Lie groups? As it turns out, $S^0 = \{\pm 1\}$, S^1, and S^3 are Lie groups. S^7 is an H-space, via the Cayley numbers. But the multiplication is not associative. In fact, S^7 has no *associative H-space* structure, where multiplication is associative up to homotopy. So it cannot be a Lie group. In general, we would hope that "nice" multiplication in a Euclidean space satisfies the property

$$a \neq 0,\ b \neq 0 \implies ab \neq 0. \tag{8.1}$$

As it were, $\mathbb{R}^1$, $\mathbb{R}^2 = \mathbb{C}$, $\mathbb{R}^4 = \mathbb{H}$, and $\mathbb{R}^8 =$ Cayley numbers have such a multiplication. The surprise is that these are the only Euclidean spaces on which such a multiplication can be defined.

THEOREM 8.10 (J. Frank Adams). *S^n is a H-space iff $n = 0, 1, 3, 7$.*

EXERCISE 8.11. Suppose $\mathbb{R}^n$, $n \neq 1, 2, 4, 8$, is made into a H-space via the multiplication $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Show that the above theorem implies that property 8.1 must fail. Therefore show that the only Euclidean spaces $\mathbb{R}^n$ on which a "nice" multiplication is defined are with $n = 1, 2, 4, 8$.

There is strong evidence for the following conjecture.

CONJECTURE 8.12. *Every finite dimensional H-space is homotopy equivalent to a manifold.*

CHAPTER 4

Covering Space Theory

1. Covering Spaces

Intuitively, a covering space gives a way to unwrap a space. As we'll see, there is a way to translate any question about a fundamental group into a question about a covering space, and vice-versa.

Examples 1.1.

(1) Consider the real line $\mathbb{R}$ and take the map $f:\mathbb{R}\to S^1$, $f(x)=e^{2\pi x}$. Then $f^{-1}(1)=\mathbb{Z}\subset\mathbb{R}$. In some sense this map wraps the line around and around the circle. We want to think about the circle being unwrapped to the line.

(2) Wrap a circle k times around a circle. So consider the map $g_k:S^1\to S^1$, $g_k(z)=z^k$. Then $g_k^{-1}(1)=\left\{e^{2\pi il/k}\right\}$, and g_k is a k-to-1 map.

We'll see that these are the only covering spaces of the circle, the first is the infinite cover and the second the finite cover.

Example 1.2. Wrap $\mathbb{C}$ around $\mathbb{C}\setminus\{0\}$. The map is $f:\mathbb{C}\to\mathbb{C}\setminus\{0\}$, $f(z)=e^z$. The inverse map on a peice of this is a log.

Definition 1.3. A continuous map $f:Y\to X$ is said to be a *covering map* if each point $u\in X$ has an open neighborhood U so that $f^{-1}(U)=\cup_{j\in J}V_j$ where the V_j are disjoint open sets so that $f|_{V_j}:V_j\to U$ is a homeomorphism. Such a neighborhood U is said to be *evenly covered* by f.

Note. So a mapping f is a covering map if X is the union of evenly covered open subsets.

Example 1.4. The map $f:(-5,7)\to S^1$ which wraps the open interval around S^1 is a local homeomorphism, for we can find a neighborhood on which f restricts to a homeomorphism. But it does not hit each point the same number of times. So f is not a covering map.

Proposition 1.5. *Let X be a connected space. If $f:Y\to X$ is a covering map, then for all $p,q\in X$ we have $\#f^{-1}(p)=\#f^{-1}(q)$.*

Proof. We claim $\#f^{-1}(p)$ is locally constant, i.e. $\#f^{-1}(p)$ is constant in a neighborhood of every $p\in X$. To see this, pick U an evenly covered neighborhood of p, then $\#f^{-1}(q)=|J|$ for all $q\in U$. Then the set of points q so that $\#f^{-1}(q)=\#f^{-1}(p)$ is an open subset of X. Therefore, X is a disjoint union of open sets (whose non-trivial inverse images have given cardinality). Since X is connected, one of these open subsets is all of X, and the rest are null. $\square$

Definition 1.6. Let X be connected and f a covering map. Its *degree* is the number $\deg(f)=\#f^{-1}(p)$.

NOTE. This is well-defined by the proposition.

EXERCISE 1.7. If $f_1 : Y_1 \to X_1$, $f_2 : Y_2 \to X_2$ are covering maps, then $f_1 \times f_2 : Y_1 \times Y_2 \to X_1 \times X_2$ is a covering map.

EXERCISE 1.8. If $f : Z \to Y$, $g : Y \to X$ are covering maps, then $g \circ f : Z \to X$ is a covering map. And $\deg (g \circ f) = \deg (g) \cdot \deg (f)$.

EXAMPLE 1.9. Let f be the quotient map $f : S^n \to \mathbb{R}P^n$ given by the description $S^n/v \sim (-v) = \mathbb{R}P^n$. Then f is a 2-to-1 covering map. In particular, the quotient map $g : S^1 \to \mathbb{R}P^1 = S^1/v \sim (-v)$ is given by $g(z) = z^2$.

As we'll see, the only covers of $\mathbb{R}P^n$, $n > 1$, are given by the quotient map in the example above, and the identity $\mathrm{id}_{\mathbb{R}P^n}$.

2. Higher Homotopy Groups

Lecture 11
11/28/11

Recall elements of $\pi_1 (X, x)$ can be represented in various ways:

- If $\omega : S^1_{,p} \to X_{,x}$ is a loop with $\omega (p) = x$, then $[\omega] \in \pi_1 (X, x)$. How do we multiply loops? Let $q : S^1 \to S^1/S^0 = S^1 \vee S^1$ be the quotient map, and given two loops α, β, let $\alpha \vee \beta : S^1 \vee S^1 \to X$ be a map defined on the figure eight (α on the first S^1, β on the second). (D1) Then $\alpha \cdot \beta = (\alpha \vee \beta) \circ q$. And we had $[\alpha] \cdot [\beta] = [\alpha\beta]$. This description of loop multiplication will generalize nicely to higher homotopy groups.
- If $\omega : I = [0, 1] \to X_{,x}$ is a path with $\omega (0) = \omega (1)$, then $[\omega] \in \pi_1 (X, x)$. We already saw how to multiply, by splicing.
- If $\omega : \mathbb{R} \to X$ is a continuous map and $\omega (\mathbb{R} \backslash K) = \{x\}$ where $K \subset \mathbb{R}$ is compact, then $[\omega] \in \pi_1 (X, x)$. Once again we splice together to multiply, but now on the whole real line. (D2)
- $\pi_1 (X, x) = \left[S^1, X\right]_+$.

DEFINITION 2.1. The *higher homotopy groups* are $\pi_k (X, x) = \left[S^k, X\right]_+$.

We can describe π_k in ways analogous to the above:

- If $\omega : S^k_{,p} \to X_{,x}$ then $[\omega] \in \pi_k (X, x)$. Let $q : S^k \to S^k/S^{k-1} = S^k \vee S^k$ be the quotient map, and let $\alpha \vee \beta : S^k \vee S^k \to X$ be the wedged map. Then we define $\alpha + \beta = (\alpha \vee \beta) \circ q$, and we define $[\alpha] + [\beta] = [(\alpha \vee \beta) \circ q]$.
- If $\omega : D^k \to X$ with $\omega \left(S^{k-1}\right) = x$ then $\omega \in \pi_k (X, x)$. Multiplication is similar to above.
- If $\omega : \mathbb{R}^k \to X$ with $\omega \left(\mathbb{R}^k \backslash K\right) = \{x\}$ where $K \subset \mathbb{R}^k$ is compact, then $[\omega] \in \pi_k (X, x)$. Multiplication is similar to above.

EXERCISE 2.2. Prove that $\pi_k (X, x)$ is a group under loop multiplication.

So the picture is all the same so far. But in fact, we get more information when $k > 1$.

THEOREM 2.3. *For $k > 1$, $\pi_k (X, x)$ is abelian.*

REMARK. This is why we use additive notation for π_k, $k > 1$.

EXERCISE 2.4. Prove the theorem. (Hint: Use the third description of $\pi_k (X, x)$ above. Rotate the space to get a homotopy between $[\alpha] + [\beta]$ and $[\beta] + [\alpha]$.)

As mentioned in the previous chapter, the homotopy groups do not completely determine the space. But the following facts hold.

PROPOSITION 2.5. *Let X be path-connected. If $\pi_k(X,x) = 0$ for all $k \geq 1$, then X is contractible.*

PROPOSITION 2.6. *If X has only one non-zero homotopy group, say in dimension k, then that group G determines the homotopy type of X.*

REMARK. The *Eilenberg-MacLane spaces*, denoted $K(G,k)$, are those spaces whose kth (and only non-zero) homotopy group is G. By the proposition above, for $k \geq 2$ it is neccesary that G is abelian to ensure the existence of a space.

EXAMPLE 2.7. We'll prove later that

$$\pi_k\left(S^1\right) = \begin{cases} \mathbb{Z} & k=1 \\ 0 & k>1 \end{cases}.$$

So $S^1 = K(\mathbb{Z},1)$.

The following theorems are analogues from before. Their proofs are similar as before.

THEOREM 2.8. *$\pi_k(X \times Y) \simeq \pi_k(X) \times \pi_k(Y)$.*

EXAMPLE 2.9. Combining the previous example with this theorem gives that

$$\pi_k\left(S^1 \times S^1\right) = \begin{cases} \mathbb{Z}\times\mathbb{Z} & k=1 \\ 0 & k>1 \end{cases}.$$

So $S^1 \times S^1 = K(\mathbb{Z}\times\mathbb{Z},1)$.

Note that the torus $T^2 = S^1 \times S^1$ has a two-dimensional hole (in the center). But the example above shows that π_2 fails to capture this hole. This is because the center hole is not spherically shaped (it is shaped like a torus). Since we arrived at the theory of homotopy groups by investigating spherically shaped holes, we should not expect π_2 to see this hole. Next semester, we'll study homology theory, which investigates holes of arbitrary shape. Such a theory would find the center hole in T^2.

THEOREM 2.10. *An element $\omega : S^k \to X$ vanishes in $\pi_k(X,x)$ ($[\omega] = 0$ in $\pi_k(X)$) iff ω extends to a map $\overline{\omega} : D^{k+1} \to X$, i.e. iff there exists a map ϕ so that the following diagram commutes. (D3)*

EXERCISE 2.11. Prove the previous two theorems.

3. The Lifting Problem

Suppose X, Y are path connected spaces, and $f : Y \to X_{,x}$ is a covering map. Recall we defined the *degree* of f as $\deg(f) = \#f^{-1}(p)$ for any $p \in X$. This was independent of p. Finally, let any $y \in f^{-1}(x)$ be the basepoint for Y.

We now investigate the so-called "lifting problem". Notice that for a small enough set $A \subset X$, we get exactly $\deg(f)$ copies of A upstairs in Y. For example, if $f : \mathbb{R} \to S^1$ is the infinite-cover, then there are an infinite number of copies of a small neighborhood of any point $s \in S^1$. But there is no copy of the entire S^1 in $\mathbb{R}^1$. More formally, we have the lifting problem: Given a map $g : A_{,a} \to X_{,x}$, does there exist a map$_+$ $\hat{g} : A_{,a} \to Y_{,y}$ so that $f \circ \hat{g} = g$? (This is existence and uniqueness, combined.) Such a map $\hat{g}$ is called a *lift* of g (to the cover Y). The lift preserving the base-point is unique.

THEOREM 3.1 (Lifting Theorem). *Let X, Y be path connected, let $f : Y_{,y} \to X_{,x}$ be a covering map$_+$, and let $g : A_{,a} \to X_{,x}$ be a map$_+$. There exists a lift$_+$ iff $Im\, g_* \subset Im\, f_*$. Moreover, if a lift$_+$ exists, then it is unique.*

PROOF OF $\implies$. Formally, if the diagram of spaces (D6) exists, then the diagram of groups (D7) exists. That is, if there exists a map $\hat{g}$ so that $f \circ \hat{g} = g$, then the induced maps must satisfy $f_* \circ \hat{g}_* = g_*$. Hence $\operatorname{Im} g_* \subset \operatorname{Im} f_*$.

The direction $\impliedby$ is much harder, and we won't see it now. □

COROLLARY 3.2. *If A is simply connected, then there exists a unique lift$_+$.*

PROOF. The condition is that $\operatorname{Im} g_* \subset \operatorname{Im} f_*$ where $g_* : \pi_1(A) \to \pi_1(X)$. But $\pi_1(A) = 0$ so the condition is vacuous. □

COROLLARY 3.3. *If $f : X \to Y$ is a covering map, then $f_* : \pi_k(Y) \to \pi_k(X)$ is an isomorphism for $k > 1$.*

EXAMPLE 3.4. $\pi_k\left(S^1\right) \simeq \pi_k\left(\mathbb{R}^1\right) = 0$ for $k > 1$ by the corollary. Also $\pi_k\left(S^1 \times \cdots \times S^1\right) = 0$ either since $\pi_k\left(S^1\right) = 0$ or because there is a covering map wrapping $\mathbb{R}^n$ around T^n.

PROOF SKETCH. Given $\omega : S^k \to X$ representing $[\omega] \in \pi_k(X)$, we can lift this to Y as S^k is simply connected for $k > 1$. Hence $f \circ \hat{\omega} = \omega$ for some $\hat{\omega}$, and hence $f_*([\hat{\omega}]) = [\omega]$ in $\pi_k(X)$. This is surjectivity. The proof that f_* is injective is similar, by lifting maps of D^{k+1}. □

EXERCISE 3.5. Fill in the details above.

EXAMPLE 3.6. As we observed before, there is a map $q : S^n \to \mathbb{R}P^n = S^n / v \sim (-v)$ which is the quotient map. This is a covering map of degree two. Thus

$$\pi_k(\mathbb{R}P^n) \simeq \pi_k(S^n) = \begin{cases} 0 & 1 < k < n \\ ? & \end{cases}.$$

The cases $k \geq n$ are unknown in general.

Let us go in another direction. Consider $S^0 \subset S^1 \subset S^2 \subset \cdots$, then define $S^\infty = \cup_k S^k$. This space is contractible (for the spheres are nested to begin with). Similarly, $\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots$ and $\mathbb{R}P^\infty = \cup_k \mathbb{R}P^k$. One can show that $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}_2$ (we'll prove it later). And $\pi_k(\mathbb{R}P^\infty) \simeq \pi_k(S^\infty) = 0$ for S^∞ is contractible. So $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$.

Here is a nice fact about covering spaces.

PROPOSITION 3.7. *Let X, Y be path connected, and $f : Y \to X$ a covering map. Then $f_* : \pi_1(Y) \to \pi_1(X)$ is injective.*

REMARK. So we can view $\pi_1(X)$ as being isomorphic to $\operatorname{Im} f_*$.

EXERCISE 3.8. Prove the proposition via the technique used in the proof of corollary 3.3.

4. Comparing Covering Maps

Suppose $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ are covering maps. Is there a map between Y_1 and Y_2 so that the diagram commutes?

EXAMPLE 4.1. Take $X = S^1$, $Y_1 = \mathbb{R}^1$ and $f_1 : x \mapsto e^{2\pi i x}$ the infinite cover, $Y_2 = S^1$ and f_2 the k degree cover $f_2 : z \mapsto z^k$. The comparison map is $h : \mathbb{R}^1 \to S^1$ is given by $h(x) = e^{2\pi i x/k}$. Then $f_2 \circ h = f_1$.

DEFINITION 4.2. Given covers $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$, a *map of covering spaces* is a map $h : Y_1 \to Y_2$ so that $f_2 \circ h = f_1$.

PROPOSITION 4.3. *There exists such a map h iff Im $(f_1)_* \subset$ Im $(f_2)_*$.*

PROOF. Apply the lifting theorem. □

DEFINITION 4.4. Two covers Y_1, Y_2 are said to be *equivalent* if there are maps of covers h, k in both directions so that $k \circ h = \mathrm{id}_{Y_1}$ and $h \circ k = \mathrm{id}_{Y_2}$.

PROPOSITION 4.5. *Covers Y_1, Y_2 are equivalent iff Im $(f_1)_* =$ Im $(f_2)_*$.*

PROOF. Use the previous proposition to get inclusions both ways. □

Now consider the mapping from covers of X to subgroups of $\pi_1(X)$. We have proved that this mapping must be injective. Thus covers of X are determined by their fundamental groups, regarded as subgroups of $\pi_1(X)$.

EXAMPLE 4.6. We have $\pi_1\left(S^1\right) = \mathbb{Z}$. The subgroups are $(k\mathbb{Z}) \simeq \mathbb{Z}$ $(k > 0)$ and $\{0\}$. We've shown this must be the whole story when it comes to covering maps. The covering space corresponding to the subgroup $\{0\}$ is $\mathbb{R}^1$, and the cover corresponding to $k\mathbb{Z}$ is the map $f_k : S^1 \to S^1$, $z \mapsto z^k$. These are all the connected covering spaces of S^1.

COROLLARY 4.7. *If X is simply connected, then the only connected cover is $\mathrm{id}_X : X \to X$.*

PROOF. $\pi_1(X) = \{0\}$. □

As it turns out, the mapping from covers of X to subgroups of $\pi_1(X)$ is also surjective. So it is bijective. We'll prove this next time via a construction.

Here is a special case of the above.

PROPOSITION 4.8. *If X is a connected space, then X has at most one simply connected covering space. Moreover, this covering space maps to all the other covers.*

We'll prove a simply connected covering space always exists. And $\pi_k(X) = \pi_k\left(\tilde{X}\right)$. (??)

DEFINITION 4.9. Given a connected space X, its simply connected covering space is called the *universal cover* of X, and is denoted $\tilde{X}$.

COROLLARY 4.10. *If X is simply connected, then $\tilde{X} = X$.*

EXERCISE 4.11. $\widetilde{X \times Y} = \tilde{X} \times \tilde{Y}$.

EXAMPLES 4.12.

(1) $\widetilde{S^1 \times S^n} = \mathbb{R} \times S^n$ with $n > 1$.

(2) $\widetilde{\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}} = \mathbb{R}^n$
(3) $\widetilde{\mathbb{R}P^n} = S^n$ with $n > 1$.
(4) $\widetilde{\text{Klein}} = \mathbb{R}^2$. (D9)
(5) If X is simply connected, then $\widetilde{S^1 \times X} = \mathbb{R} \times X$.

EXERCISE 4.13. Work out the details in the previous examples.

EXERCISE 4.14. What is the universal cover of $S^1 \vee S^2$?