Partial Differential Equations I

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Abstract. This is part one of a two semester course on partial differential equations. The course was offered in Fall 2011 at the Courant Institute for Mathematical Sciences, a division of New York University. The primary reference will be *Partial Differential Equations* by L. Evans (Ed. 2). Office hours will be Monday 4-6 pm in 711 WWH. Grading will be 70% final and 30% weekly homework.

Contents

CHAPTER 0

Introduction

Lecture 1, 9/6/11

First some notation. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function. Throughout the course we'll write

$$\partial_i f = \frac{\partial}{\partial x_i} f(x_1, \ldots, x_d) = \frac{d}{ds} f(x_1, \ldots, x_i + s, \ldots, x_d)$$

to mean the ith partial of f.

1. What is a PDE?

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with $d \geq 2$.[1] The most general form for a PDE is a relation

$$F\left(x, f, (\partial_i f)_{i=1,\ldots,n}, (\partial_i \partial_j f)_{i,j=1\ldots,n}, \ldots, (\partial_{i_1} \cdots \partial_{i_n} f)\right) = 0.$$

Example 1.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Laplace's equation is

$$\partial_x^2 f(x,y) + \partial_y^2 f(x,y) = 0.$$

The relation

$$\sin(x)\, \partial_x f(x,y) + (\partial_y f(x,y))^2 = 0$$

is also a PDE.

The *order* of a PDE is the highest number of derivatives which occurs. In the example above, the first is order two and the second order one. A PDE is *linear* if any linear combination of solutions is a solution. The first example above is linear; the second is not. Very often, we think of the variable $x = (x_1, \ldots, x_d)$ as $(t, x_1, \ldots, x_d) \in \mathbb{R} \times \mathbb{R}^d$. Here, t denotes time and $x_1, \ldots, x_d$ space. In such a situation, we speak of an *evolution equation.*

Example 1.2. The equation

$$\partial_t f(t,x,y) + \partial_x^2 f(t,x,y) + \partial_y^2 f(t,x,y) = 0$$

is an evolution equation.

Physics has always been a source of good PDE problems: PDE are the language of physics. In the 18th and 19th century, PDE began with continuum mechanics, e.g. Navier-Stokes. In the 20th century, quantum mechanics (Schrodinger) and general relativity (Einstein) arose. Typically, one considers a physical setting and prescribes data (e.g. the system at time $t = 0$, or system on the boundary), and then asks for a solution to the PDE which describes the physics. The first question is: What is the correct notion of a solution? Consider the PDE

$$\partial_x f + \partial_y f = 0$$

[1] $d = 1$ is the realm of ODEs.

for $f : \mathbb{R}^2 \to \mathbb{R}$. One notion of a solution is when $f \in C^1$, so that $\partial_x f$ and $\partial_y f$ are defined pointwise. This is called a *classical solution*, but there are limitations to this approach. In particular, physically interesting solutions may not be C^1(e.g. shocks); moreover, for certain data there may not exist any solutions in C^1.

Instead, consider the map from the data to the solution. A PDE is said to be *well-posed* if

- for any data, there exists a solution[2]
- this solution is unique
- the map from data to solution is continuous.

This definition of well-posedness is due to Hadamard.

2. Four Important PDE

Consider the problem of classification. The simplest case is: $d = 2$, linear, constant coefficients, and with order ≤ 2. If the order is 1, then the PDE must be of the form

$$a\partial_x f(x, y) + b\partial_y f(x, y) + cf(x, y) = 0$$

for (real) constants a, b, c. If $c = 0$ then this is called the "transport equation". If the order is 2, there are three possibilities (up to an affine change of variable):

$$\begin{aligned} \partial_x^2 f + \partial_y^2 f &= 0, \\ \partial_x^2 f - \partial_y^2 f &= 0, \\ \partial_x f - \partial_y^2 f &= 0. \end{aligned}$$

The first is known as "Laplace's equation", the second as the "wave equation", and the third as the "heat equation". The first is the prototype for an "elliptic" equation, the second for a "hyperbolic" equation, and the third for a "parabolic" equation.

In higher d the classification becomes too complicated, but the prototypes remain important. The transport equation in $\mathbb{R}^d$ is

$$\partial_t f + b \cdot \nabla f = 0,$$

the Laplace equation is

$$\Delta f = \partial_1^2 f + \partial_2^2 f + \cdots + \partial_d^2 f = 0,$$

the wave equation in $\mathbb{R} \times \mathbb{R}^d \ni (t, x)$ is

$$\partial_t^2 f - \Delta f = 0,$$

and the heat equation is

$$\partial_t f - \Delta f = 0.$$

We will spend a good amount of time studying these four equations. The most important results we'll see are included in table 1 on the next page.

In this course, we'll first cover the standard linear equations (Chapter 2 of the text). Then we'll cover first order nonlinear equations (Chapter 3). Finally, we'll discuss Fourier analysis towards PDE.

[2] Here the notion of solution may not always be classical.

	Smoothing	Finite Speed	Max Principle	Energy Methods
Transport	N	Y	N	Y
Laplace	Y	N	Y	Y
Heat	Y	N	Y	Y
Wave	N	Y	N	Y

Table 1. Important results on four linear PDE.

CHAPTER 1

The Transport Equation

1. The Homogeneous Problem

Consider an unknown function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$. As usual we write

$$\nabla u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_d u \end{pmatrix}$$

to mean the gradient of u. Given a vector $b \in \mathbb{R}^d$, the *transport equation* is

$$\partial_t u + b \cdot \nabla u = 0$$

for $u \in C^1$. Another way of writing this is

$$\partial_t u + \sum_{i=1}^{d} b_i \partial_i u = 0.$$

Proposition 1.1. *The general solution reads*

$$u(t, x) = f(x - bt)$$

where $f \in C^1\left(\mathbb{R}^d, \mathbb{R}\right)$.

Proof. Observe that the equation can be seen as a certain directional derivative of u, as

$$\partial_t u(t + \lambda, x + \lambda b) + b \cdot \nabla u(t + \lambda, x + \lambda b) = \frac{d}{d\lambda}\left[u\left((t, x) + \lambda(1, b)\right)\right],$$

and so the equation reads

$$\frac{d}{d\lambda}\left[u(t + \lambda, x + \lambda b)\right] = 0.$$

Thus

$$u(t + \lambda, x + \lambda b) = C(t, x)$$

for C independent of λ. But we are free to pick any λ we want, so pick $\lambda = -t$ and $\lambda = 0$ in particular. Then we arrive at

$$u(0, x - tb) = u(t, x)$$

and hence

$$u(t, x) = f(x - tb)$$

once we define

$$f(y) = u(0, y).$$

Conversely, let us check that $f(x - tb)$, for $f \in C^1$, is a solution of the transport equation:

$$\partial_t f(x - tb) = -\sum_i b_i \partial_i f(x - tb)$$

by the chain rule, and so

$$\partial_t f(x - tb) + b \cdot \nabla f(x - tb) = 0.$$

□

Now we pose a general problem, known as an *Initial Value Problem* or *Cauchy Problem* for the linear transport equation. The problem is to solve

$$\begin{cases} \partial_t u(t,x) + b \cdot \nabla u(t,x) = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = g(x) & x \in \mathbb{R}^d \end{cases}$$

with $u, g \in C^1$. Here g is the initial data.

THEOREM 1.2. *The unique solution to this IVP is*

$$u(t,x) = g(x - bt).$$

PROOF. Clearly $g(x - tb)$ solves the IVP. Conversely, if u solves the IVP, it has to read

$$u(t,x) = f(x - tb)$$

by the proposition. Setting $t = 0$ we get

$$u(0,x) = f(x)$$

and hence $f = g$. So the solution is unique. □

REMARKS.

(1) Why is this called the transport equation? From the solution, we see that the initial data g is transported in time. This is most easily seen with $d = 1$ and $b = 1$.
(2) The claim and the theorem give a means of defining a notion of solution for non-differentiable u, g. We saw that if $f \in C^1$, then $f(x - bt)$ solves

$$\partial_t u + b \cdot \nabla u = 0.$$

Now we may decide that any function f gives rise to a (generalized) solution $f(x - bt)$. This type of thinking will be important later on (e.g. it gives rise to the notion of a distributional solution).

2. The Inhomogeneous Problem

Consider the problem

$$\begin{cases} \partial_t u(t,x) + b \cdot \nabla u(t,x) = h(t,x) & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = g(x) & x \in \mathbb{R}^d \end{cases}$$

with $u \in C^1\left(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}\right), h \in C^1\left(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}\right), g \in C^1\left(\mathbb{R}^d, \mathbb{R}\right)$ and for $b \in \mathbb{R}^d$.

THEOREM 2.1. *The solution reads*

$$u(t,x) = g(x - tb) + \int_0^t h(s, x + (s-t)b)\, ds.$$

Proof. Suppose u solves the inhomogenous problem. Then we have

$$\begin{aligned}\frac{d}{d\lambda}u\left(t+\lambda,x+\lambda b\right)&=\left(\partial_t u+b\cdot\nabla\right)u\left(t+\lambda,x+\lambda b\right)\\&=h\left(t+\lambda,x+\lambda b\right).\end{aligned}$$

Integrating from $-t$ to 0 gives

$$\begin{aligned}u\left(t,x\right)-u\left(0,x-tb\right)&=\int_{-t}^{0}h\left(t+\lambda,x+\lambda b\right)\,d\lambda\\&=\int_0^t h\left(s,x+\left(s-t\right)b\right)\,ds\end{aligned}$$

after the change of variable $s=t+\lambda$. And $u\left(0,x-tb\right)=g\left(x-tb\right)$ by assumption.

Conversely,

$$g\left(x-tb\right)+\int_0^t h\left(s,x+\left(t-s\right)b\right)\,ds$$

is a solution. □

CHAPTER 2

The Laplace Equation

Let $u : \mathbb{R}^d \to \mathbb{R}$, $x = (x_1, \ldots, x_d) \mapsto u(x)$ be a C^2 function. The equation

$$\Delta u = 0$$

is called the *Laplace equation*. Similarly

$$-\Delta u = f$$

is called the *Poisson equation*. If u solves the Laplace equation, it is called a *harmonic* function.

Remark. The Laplace equation arises from electrostatics. Suppose we have a charge distribution $\rho(x)$ with $x \in \mathbb{R}^3$. Then it generates an electric field

$$E(x) = -\nabla \phi(x)$$

and ϕ solves

$$\Delta \phi = \rho.$$

Lecture 2, 9/13/11

1. The Fundamental Solution

We begin by looking for a solution which depends only on the radial coordinate, $r = |x|$. To do so, we need to write the Laplacian in polar coordinates.

Claim. If $u(x) = u(r)$ then

$$\Delta u = \left(\frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} \right) u(r).$$

Proof. Write $r = |x| = \sqrt{x_1^2 + \cdots + x_d^2}$ and apply the chain rule. $\square$

So we look to solve

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} \right) u(r) = 0. \tag{1.1}$$

Write

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} \right) = r^{1-d} \frac{d}{dr} \left(r^{d-1} \frac{d}{dr} u(r) \right),$$

then u solves (1.1) iff

$$\begin{aligned} r^{d-1} \frac{d}{dr} u(r) &= c_0 \\ \iff u'(r) &= c_0 r^{1-d} \end{aligned}$$

for a constant c_0. So u solves (1.1) iff

$$u(r) = \begin{cases} c_1 r^{2-d} + c_2 & d \geq 3 \\ c_1 \log r + c_2 & d = 2 \end{cases},$$

where c_1, c_2 are constants.

Definition 1.1. The *fundamental solution of the Laplace equation* is

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & d=2 \\ \frac{1}{d(d-2)\alpha(d)}\frac{1}{|x|^{d-2}} & d \geq 3 \end{cases}.$$

Here, $\alpha(d)$ is the volume of the unit ball in $\mathbb{R}^d$.

Theorem 1.2. *Consider the equation*

$$-\Delta u = f \tag{1.2}$$

in $\mathbb{R}^d$ *with* $f \in C_c^3(\mathbb{R}^d, \mathbb{R})$. *The map* $u : \mathbb{R}^d \to \mathbb{R}$ *given by*

$$u(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy$$

is in $C^2(\mathbb{R}^d, \mathbb{R})$, *and is a solution to* (1.2).

Remarks.

(1) $\int \Phi(x-y) f(y)\, dy$ is called the *convolution* of Φ and f and is denoted $\Phi \star f$. Note that $\Phi \star f = f \star \Phi$, i.e.

$$\int \Phi(x-y) f(y)\, dy = \int \Phi(y) f(x-y)\, dy.$$

(2) Formally we compute

$$\begin{aligned} \Delta_x \int \Phi(x-y) f(y)\, dy &= \int \Delta_x \Phi(x-y) f(y)\, dy \\ &= -\int \delta_x \cdot f(y)\, dy \\ &= -f(x). \end{aligned}$$

Of course to make this rigorous we need the theory of distributions. The crucial idea is that

$$\Delta_x \Phi = -\delta_0.$$

We'll justify this later.

(3) Even though f is compactly supported, $\Phi \star f$ will not be. We can interpret this to mean that the Δ operator is not localized.

Recall the Divergence (Gauss) theorem:

Theorem 1.3 (Divergence/Gauss). *Consider a domain* D *with boundary* ∂D *and exterior normal* N. *Then*

$$\int_D div\ F\, dx = \int_{\partial D} F \cdot N\, dS.$$

Corollary 1.4 (Integration by parts). *In particular,*

$$\int_D f(\partial_i g)\, dx = -\int_D (\partial_i f)\, g\, dx + \int_{\partial D} (fg)\, N \cdot e_i\, dS.$$

Now we'll prove theorem 1.2.

PROOF. First we'll prove $u \in C^2$. Write

$$\frac{u(x+he_i)-u(x)}{h} = \int \Phi(y)\,\frac{f(x-y+he_i)-f(x-y)}{h}\,dy.$$

Since $f \in C^3$, $\frac{f(x-y+he_i)-f(x-y)}{h} \rightrightarrows \partial_i f(x-y)$ uniformly in y as $h \to 0$. So

$$\int \Phi(y)\,\frac{f(x-y+he_i)-f(x-y)}{h}\,dy \to \int \Phi(y)\,\partial_i f(x-y)\,dy$$

as $h \to 0$. It's not hard to show the right hand side is continuous in x, and hence $u \in C^1$ and

$$\partial_i u = \int \Phi(y)\,\partial_i f(x-y)\,dy.$$

Repeating the argument gives $u \in C^2$ with

$$\partial_i\partial_j u = \int \Phi(y)\,\partial_i\partial_j f(x-y)\,dy.$$

Now we'll compute Δu. For simplicity take $d \geq 3$ (the $d = 2$ case follows a similar argument). By the computation above,

$$\Delta u(x) = \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Phi(y)\,\Delta f(x-y)\,dy + \int_{B(0,\epsilon)} \Phi(y)\,\Delta f(x-y)\,dy.$$

First note

$$\int_{B(0,\epsilon)} \Phi(y)\,\Delta f(x-y)\,dy \to 0$$

as $\epsilon \to 0$. The first reason is because the integrand is integrable on a shrinking domain. More explicitly,

$$\Big|\int_{B(0,\epsilon)} \Phi(y)\,\Delta f(x-y)\,dy\Big| \leq M\int_{B(0,\epsilon)} \frac{1}{|y|^{d-2}}\,dy \leq M'\epsilon.$$

Next, observe that

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Phi(y)\,\Delta f(x-y)\,dy &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Phi(y)\sum_i \partial_i^2 f(x-y)\,dy\\ &= -\int_{\mathbb{R}^d\setminus B(0,\epsilon)} \sum_i \partial_i\Phi(y)\,\partial_i f(x-y)\,dy\\ &\quad+\int_{\partial B(0,\epsilon)} \sum_i \Phi(y)\,\partial_i f(x-y)\,\frac{y^i}{|y|}\,dS.\end{aligned}$$

The second integral goes to zero as $\epsilon \to 0$, for $\Phi \sim C|x|^{-(d-2)}$ and so

$$\Big|\int_{\partial B(0,\epsilon)} \sum_i \Phi(y)\,\partial_i f(x-y)\,\frac{y^i}{|y|}\,dS\Big| \leq M\epsilon^{2-d}\epsilon^{d-1} = M\epsilon.$$

Now integrate by parts again to get

$$\begin{aligned}-\int_{\mathbb{R}^d\setminus B(0,\epsilon)} \sum_i \partial_i\Phi(y)\,\partial_i f(x-y)\,dy &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Delta\Phi(y)\,f(x-y)\,dy\\ &\quad-\int_{\partial B(0,\epsilon)} \sum_i \partial_i\Phi(y)\,f(x-y)\,\frac{y^i}{|y|}\,dS.\end{aligned}$$

Again the first integral goes to zero. We compute

$$\partial_i \Phi(x) = -\frac{1}{d\alpha(d)} \frac{x^i}{|x|^d},$$

and hence

$$\begin{aligned}
-\int_{\partial B(0,\epsilon)} \sum_i \partial_i \Phi(y) f(x-y) \frac{y^i}{|y|} \, dS &= -\int_{\partial B(0,\epsilon)} \frac{1}{d\alpha(d)} \frac{1}{\epsilon^{d-1}} f(x-y) \, dS \\
&= -\frac{1}{|\partial B(0,\epsilon)|} \int_{\partial B(0,\epsilon)} f(x-y) \, dS \\
&\to -f(x)
\end{aligned}$$

as $\epsilon \to 0$. Putting it all together, we get

$$\Delta u = -f(x)$$

after letting $\epsilon \to 0$. □

2. Properties of Harmonic Functions

We've shown the existence of solutions $u \in C^2(\mathbb{R}^d, \mathbb{R})$ to the Poisson equation in $\mathbb{R}^d$. To arrive at uniqueness, we will first prove the "mean value formula" for Laplace's equation.

THEOREM 2.1 (Mean value formula). *If $u \in C^2(\mathbb{R}^d, \mathbb{R})$ satisfies $\Delta u = 0$, then*

$$u(x) = \fint_{\partial B(x,r)} u(y) \, dS = \fint_{B(x,r)} u(y) \, dy$$

for any $r > 0$.

There are numerous corollaries. For example:

(1) u cannot have isolated zeros. (Suppose $u(x_0) = 0$, then by the mean value formula there exists $x \in \partial B(x_0, r)$ with $u(x) = 0$ for all $r > 0$.)
(2) u cannot have local maxima/minima.

PROOF OF THEOREM. Another integration by parts. We'll get the first part of the theorem now. Set

$$F(r) = \fint_{\partial B(x,r)} u(y) \, dS(y).$$

It suffices to show $F' \equiv 0$, as $\lim_{r \to 0} F(r) = u(x)$. Change variables to get

$$F(r) = \fint_{\partial B(0,1)} u(x + rz) \, dS(z).$$

Then

$$\begin{aligned}
F'(r) &= \fint_{\partial B(0,1)} \sum_i z^i \partial_i u(x + rz) \, dS(z) \\
&= \frac{1}{|\partial B(0,1)|} \int_{B(0,1)} r \Delta u(x + rz) \, dz \\
&= 0
\end{aligned}$$

by assumption.

To prove that

$$u(x) = \fint_{B(0,r)} u(y)\ dy,$$

just write

$$\int_{B(0,r)} u\, dy = \int \int_{\partial B(0,r)} u\, dS_r dr.$$

□

The mean value formula has a converse.

Theorem 2.2. *Conversely, if $u \in C^2$ satisfies the mean value formula, then it solves the Laplace equation.*

Proof. Argue by contradiction. Assume that u satisfies the mean value formula but $\Delta u \neq 0$. Then in a neighborhood of some point, $\Delta u > 0$. But then there exists x, r such that $F'(r) \neq 0$ in the previous proof. □

Remark. As a consequence one might try to declare that solutions to the Laplace equation are (integrable) functions which satisfy the mean value formula. In fact this does not weaken the problem: we'll see that any solution of the Laplace equation must in fact be smooth.

Now let U be a bounded, open subset of $\mathbb{R}^d$ and suppose $u \in C^2(U, \mathbb{R})$.

Theorem 2.3 (Maximum Principle). *Suppose $u \in C^2(U)$, $u \in C^0\left(\overline{U}\right)$, and $\Delta u = 0$ in U. Then,*

(1) $\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x)$.
(2) *If u acheives $M = \max_{x\in\overline{U}} u(x)$ at a point $x \in U$ and U is connected, then $u \equiv M$.*

Proof. It's easy to check that (2) $\Longrightarrow$ (1). (Argue by contradiction.) So we'll only prove (2). By contradiction, suppose there exists $x_0 \in U$ such that $u(x_0) = M$. Consider the set $E = \{x : u(x) = M\}$. E is

- not empty ($x_0 \in E$)
- closed (u is in particular continuous)
- open (the mean value formula).

To see the third, take $x \in E$ and r such that $B(x, r) \subset U$. Then by the mean value formula,

$$M = u(x) = \fint_{B(x,r)} u(y)\ dy$$

which implies that $u = M$ on $B(x, r)$, and hence that E is open. Since U is connected, $E = U$ and we are done.

□

Remark. There is also a "minimum principle". Replace "max" by "min" throughout the previous theorem.

Now we can prove that solutions to the Laplace equation are unique, at least in bounded domains.

THEOREM 2.4. *Consider $f \in C(U,\mathbb{R})$, $g \in C(\partial U,\mathbb{R})$. There is at most one solution of*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

for $C^2(U) \cap C^0(\overline{U})$.

PROOF. Argue by contradiction. Suppose there are two solutions, u^1 and u^2, and set $w = u^1 - u^2$. Then w solves

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}.$$

By the maximum principle, we find $w \leq 0$ in U. By the minimum principle, we find $w \geq 0$ in U. Hence $w \equiv 0$ and hence the proof. $\square$

The previous proof used the following consequence of the minimum principle: If

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

and $g \geq 0$, then $u \geq 0$ throughout U. The max/min principles give us much control over solutions to Laplace's equation.

THEOREM 2.5. *Suppose $u \in C^0(U)$ and satisfies the mean value formula. Then $u \in C^\infty(U)$.*

COROLLARY 2.6. *If u is harmonic, then it is C^∞.*

This says that the Laplace operator is a smoothing operator. Note this is not at all the same for the transport equation, where the solutions have the same regularity as the boundary data. The Poisson equation is not smoothing either.

Lecture 3, 9/20/11

PROOF OF THEOREM. Step 1. Convolution with a smooth function regularizes. Let $\eta \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ be a smooth function with compact support. Let $f \in C^0(\mathbb{R}^d \to \mathbb{R})$. Then $\eta \star f \in C^\infty$. Indeed we have

$$(\eta \star f)(x) = \int \eta(x-y) f(y)\, dy,$$

and differentiation with respect to x commutes with the integral sign so that $\partial_i(\eta \star f) = \partial_i \eta \star f$. Since η is smooth we can repeat the argument as many times as we please.

Step 2. u can be represented as the convolution of itself with a smooth function. To motivate this, note that the mean value formula can be written as

$$\begin{aligned} u(x) &= \fint_{B(x,r)} u(y)\, dy \\ &= \frac{1}{r^n C} \int_{|x-r|<r} u(y)\, dy \\ &= \frac{1}{r^n C} \int \chi_{B(0,r)}(x-y)\, u(y)\, dy. \end{aligned}$$

But $\chi_{B(0,r)}$ is not smooth, so we'll have to do some work. In what follows we take for granted the existence of $\eta \in C_0^\infty(B(0,1),\mathbb{R})$ which is radial and has $\int \eta = 1$.

Consider the set $U_\epsilon = \{x : d(x, \partial U) > \epsilon\}$ for fixed $\epsilon > 0$. Given $x \in U_\epsilon$, we claim $u(x) = u \star \frac{1}{\epsilon^d}\eta\left(\frac{\cdot}{\epsilon}\right)$. From this the theorem follows. Towards the claim, write

$$\begin{aligned}
\left(u \star \frac{1}{\epsilon^d}\eta\left(\frac{\cdot}{\epsilon}\right)\right)(x) &= \int \frac{1}{\epsilon^d}\eta\left(\frac{|y|}{\epsilon}\right) u(x-y)\ dy \\
&= \int \frac{1}{\epsilon^d}\eta\left(\frac{r}{\epsilon}\right) \int_{S(0,r)} u(x-y)\ dS_r(y)\, dr \\
&= \int \frac{1}{\epsilon^d}\eta\left(\frac{r}{\epsilon}\right) |S(0,r)| u(x)\ dr \\
&= u(x) \int \frac{1}{\epsilon^d}\eta\left(\frac{r}{\epsilon}\right) |S(0,r)|\, dr \\
&= u(x) \int \frac{1}{\epsilon^d}\eta\left(\frac{y}{\epsilon}\right)\ dy \\
&= u(x).
\end{aligned}$$

This completes the proof. □

At this point we know for fixed n that $|D^n u| \leq C(\epsilon)$ on U_ϵ. But as we get close to the boundary, the bound may blow up. This next theorem gives us a quantitative control on the bound. First some notation. We'll call $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ a *multi-index* and write $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. The size of α will be $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

THEOREM 2.7. *Suppose u is harmonic in U and $B(x,r) \subset U$. Then for all α there exists C_α so that*

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{r^{|\alpha|}} ||u||_{L^\infty(B(x,r))}$$

and

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{r^{n+|\alpha|}} ||u||_{L^1(B(x,r))}$$

hold for all n.

NOTE. Evans gives a complete expression for C_α.

How can we understand this theorem? Consider the problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

for $g \in C(\partial U)$. Suppose there exists a bounded solution u, then by the theorem there exists C so that

$$|\partial_i^k u(x)| \leq \frac{C}{|d(x, \partial U)|^k}.$$

PROOF. Take η as in the previous proof and define $U_r = \{x : d(x, \partial U) > r\} = \{x : B(x,r) \subset U\}$. By the previous proof, we know $u(x) = \left(u \star \frac{1}{r^d}\eta\left(\frac{\cdot}{r}\right)\right)(x)$ if $x \in U_r$. Now differentiate. Explicitly, we have

$$\begin{aligned}
D^\alpha u(x) &= \int u(y) \frac{1}{r^{d+|\alpha|}} (D^\alpha \eta)\left(\frac{x-y}{r}\right) dy \\
\implies |D^\alpha u(x)| &\leq \frac{C_\alpha}{r^{d+|\alpha|}} \int_{B(x,r)} |u(y)|\, dy
\end{aligned}$$

where $|D^\alpha \eta| < C_\alpha$. This implies the second estimate, from which follows the first. □

This has as consequence the next theorem.

THEOREM 2.8 (Liouville). *Suppose* $u \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ *is harmonic and bounded. Then* u *is constant.*

PROOF. We saw that for any x,

$$|Du| \le \frac{C}{r^{n+1}} \int_{B(x,r)} |u| \, dy.$$

By assumption, $|u| < M$ so

$$|Du| \le \frac{C'}{r^n}.$$

Now let $r \to \infty$ to conclude the result. $\square$

As a corollory we get uniqueness on $\mathbb{R}^d$.

COROLLARY 2.9. *If* $f \in C_0^2\left(\mathbb{R}^d, \mathbb{R}\right)$*, then any bounded solution of* $-\Delta u = f$ *equals* $f \star \Phi + c$ *for some constant* c*.*

PROOF. First, we show $f \star \Phi$ is bounded. Suppose $\operatorname{supp}(f) \subset B(0, R)$, then

$$\begin{aligned} |f \star \Phi(x)| &\le M \int_{B(0,R)} |\Phi(x-y)| \, dy \\ &\le M \int_{B(x,R)} |\Phi(y)| \, dy \\ &\le M \int_{B(0,R)} |\Phi(y)| \, dy \\ &= M'. \end{aligned}$$

Now suppose w is bounded and satisfies $-\Delta w = f$. Then $w - \Phi \star f$ solves $\Delta(w - \Phi \star f) = 0$ and $w - \Phi \star f$ must be bounded. So by Liouville $w - \Phi \star f$ must be constant. $\square$

But c can be arbitrary. To restore complete uniqueness, we need to reduce our scope.

DEFINITION 2.10. u *vanishes at* ∞ if $\sup_{|x|=R} |u(x)| \to 0$ as $R \to \infty$.

COROLLARY 2.11. *Suppose* $f \in C_0^2$ *and* $n \ge 3$*. The unique solution of* $-\Delta u = f$ *in* $\mathbb{R}^d$ *which vanishes at* ∞ *is* $\Phi \star f$*.*

PROOF SKETCH. To show $f \star \Phi$ vanishes at ∞, write

$$f \star \Phi(x) = \int f(y) \, \Phi(x-y) \, dy$$

and argue about the size of Φ when x is far from y. To get uniqueness, suppose w solves $-\Delta w = f$ and vanishes at ∞. Then $w = f \star \Phi + c$, but as $f \star \Phi \to 0$ as $x \to \infty$ we must have $c = 0$. $\square$

EXERCISE 2.12. Work out the details in the previous proof.

Recall that a function $u : \mathbb{R}^d \to \mathbb{R}$ is *analytic* if

$$u(x+h) = \sum_{\alpha \in \mathbb{N}^k, k \in \mathbb{N}} \frac{1}{|\alpha|!} |D^\alpha u(x)| h^\alpha.$$

THEOREM 2.13. *If u is harmonic in $U \subset \mathbb{R}^d$, then u is analytic in U.*

PROOF SKETCH. Write

$$u(x+h) = \sum_{\alpha \in \mathbb{N}^k, k \leq N} \frac{1}{|\alpha|!} D^\alpha u(x) h^\alpha + \sum_{|\alpha| = N+1} \frac{1}{|\alpha|!} D^\alpha u(x+th) h^\alpha$$

for some $0 < t < 1$. We want to show the error goes to zero as $N \to \infty$. To do this we need a stronger version of theorem 2.7. See Evans for the complete proof. □

THEOREM 2.14 (Harnack's Inequality). *Suppose U is an open, bounded set and V is a connected, open set with $\overline{V} \subset U$. Also suppose u is harmonic in U. Then there exists a constant C (independent of u) such that if $u \geq 0$ on U, then also*

$$\sup_V u \leq C \inf_V u.$$

PROOF. We'll show for any $x, y \in V$, $u(x) \leq Cu(y)$ for some C. Let $r = \frac{1}{2} d(V, \partial U)$ so that $B(x, 2r) \subset U$ if $x \in V$. Assume $|x - y| \leq r$, then by the mean value formula

$$\begin{aligned} u(x) &= \frac{1}{|B(x,2r)|} \int_{B(x,2r)} u(z)\, dz \\ &\geq \frac{1}{|B(x,2r)|} \int_{B(y,r)} u(z)\, dz \\ &= \frac{|B(y,r)|}{|B(x,2r)|} u(y). \end{aligned}$$

So the claim holds for $|x - y| \leq r$.

Now suppose x, y are arbitrary. It is possible to find N balls $B(x, r)$ (N uniformly bounded) which cover V and have $B(x_i, r) \cap B(x_{i+1}, r) = \emptyset$. Then $u(x) \leq C^N u(y)$ and since N is bounded we are done. □

Lecture 4, 9/27/11

3. Green's Function

Now we generalize the notion of fundamental solution to general open sets. Recall the fundamental solution

$$\Phi(x) = \begin{cases} c_2 \log|x| & d = 2 \\ \frac{c_d}{|x|^{d-2}} & d \geq 3 \end{cases},$$

and the uniqueness result, that for $f \in C_0^2(\mathbb{R}^d)$ the problem

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^d \\ u \to 0 & \text{as } |x| \to \infty \end{cases}$$

has the unique solution

$$u(x) = \Phi \star f(x) = \int \Phi(x-y) f(y)\, dy.$$

Now consider the problem

$$\begin{cases} \Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}.$$

Again we want to write $u = \Phi \star f$, but for some other Φ. At the very least, we'll need to be able to integrate over the boundary of U. So we have

Definition 3.1. Suppose $U \subset \mathbb{R}^d$ is an open set. We say the boundary ∂U is C^n if, given any $x_0 \in \partial U$, there exists $r > 0$ and an invertible $\phi \in C^n\left(\mathbb{R}^d, \mathbb{R}^d\right)$ with $\phi^{-1} \in C^n\left(\mathbb{R}^d, \mathbb{R}^d\right)$ so that $U \cap B(x_0, r) = \phi\left(\mathbb{R}^d_- \cap B(x_0, r)\right)$ (equivalently, such that $\partial U \cap B(x_0, r) = \phi\left(\partial \mathbb{R}^d_-\right) \cap B(x_0, r)$).

We'll need to correct Φ to match the prescribed boundary condition.

Definition 3.2. The *corrector* ϕ^x is the solution of

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } U \\ \phi^x(y) = \Phi(x-y) & \text{on } \partial U \end{cases}.$$

Of course, we don't know yet if such a solution exists. But grant it for now.

Definition 3.3. The *Green's function* for Δ is

$$G(x,y) = \Phi(x-y) - \phi^x(y).$$

Remark. $G = 0$ on ∂U and $\Delta_y G = 0$ on U except at x, where (formally) $\Delta_y G = \delta_x$.

Theorem 3.4. *G is symmetric.*

To show this, we recall

Proposition 3.5 (Green's Identity). *For $\phi, \psi : U \to \mathbb{R}$,*

$$\int_U \phi \Delta \psi - \psi \Delta \phi = \int_{\partial U} \phi \frac{\partial \psi}{\partial N} - \psi \frac{\partial \phi}{\partial N}$$

where $\frac{\partial \phi}{\partial N} = N \cdot \nabla \phi$.

Proof of theorem. Apply Green's identity on $U \setminus [B(x,\epsilon) \cup B(y,\epsilon)]$ to $w(z) = G(x,z)$ and $v(z) = G(y,z)$. Note that $\Delta w = \Delta v = 0$ and $v|_{\partial U} = w|_{\partial U} = 0$, so we get

$$\int_{\partial B(x,\epsilon)} w \frac{\partial v}{\partial N} - v \frac{\partial w}{\partial N} = \int_{\partial B(y,\epsilon)} -w \frac{\partial v}{\partial N} + v \frac{\partial w}{\partial N}.$$

Taking $\epsilon \to 0$ yields $v(x) = w(y)$ and hence the result. $\square$

Exercise 3.6. Fill in the details in the above proof.

Theorem 3.7. *Suppose $U \subset \mathbb{R}^d$ is a bounded, open set with $\partial U \in C^2$. Given $f \in C^2(U)$ and $g \in C^2(\partial U)$, the unique solution in C^2 to*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

is

$$u(x) = \int_U G(x,y) f(y)\, dy - \int_{\partial U} \frac{\partial G}{\partial N}(x,y) g(y)\, dy.$$

Proof. Step 1. This is the only possible solution. Suppose u is a solution of the problem. Then Green's identity on $U \setminus B(x,\epsilon)$ yields

$$\int_{U \setminus B(x,\epsilon)} u(y) \Delta \Phi(x-y) - \Phi \Delta u = \int_{\partial U} u \frac{\partial \Phi}{\partial N} - \Phi \frac{\partial u}{\partial N} - \int_{\partial B(x,\epsilon)} u \frac{\partial \Phi}{\partial N} - \Phi \frac{\partial u}{\partial N}.$$

Notice that $\Delta\Phi \equiv 0$ on $U\backslash B(x,\epsilon)$, and that

$$\int_{\partial B(x,\epsilon)} u\frac{\partial\Phi}{\partial N} - \Phi\frac{\partial u}{\partial N} \to -u(x)$$

as $\epsilon \to 0$. Applying Green's identity on U for u and ϕ^x yields

$$\int_U u(y)\,\Delta\phi^x(y) - \phi^x\Delta u = \int_{\partial U} u\frac{\partial\phi^x}{\partial N} - \phi^x\frac{\partial u}{\partial N}.$$

And notice that $\Delta\phi^x \equiv 0$ in U. Taking $\epsilon \to 0$ and subtracting yields the desired formula.

Step 2. The formula solves the equation. We have

$$\Delta u = \Delta\int_U \Phi(x-y)\,f(y)\,dy + \Delta\int \phi^x(y)\,f(y)\,dy - \Delta\int_{\partial U}\frac{\partial G}{\partial N}g(y)\,dy.$$

Observe

$$\Delta\int_U \Phi(x-y)\,f(y)\,dy = -f(x)$$

and

$$\Delta\int \phi^x(y)\,f(y)\,dy = \int \Delta_x\phi^x(y)\,f(y)\,dy = 0$$

since $\Delta_x\phi^y(x) = 0$. And the third term is zero as well, so $\Delta u = -f$.

Now we check that $u = g$ on ∂U. Set $x_0 \in \partial U$. Note that $G(x_0, y) = 0$ for all y, so

$$u(x_0) = -\int_{\partial U}\frac{\partial G}{\partial N}(x_0, y)\,g(y)\,dy.$$

Take the following steps:

(1) Show u has continuous extension around x_0.
(2) Extend g inside U.
(3) Apply the divergence theorem on $U\backslash B(x,\epsilon)$.
(4) Let $\epsilon \to 0$.

This will prove the theorem.

□

Exercise 3.8. Fill in the details of the proof.

So we have reduced the problem to showing there exists ϕ^x which solves

$$\begin{cases}\Delta_y\phi^x(y) = 0 & \text{in } U\\ \phi^x(y) = \Phi(x-y) & \text{on } \partial U\end{cases}.$$

Before we entertain an existence theorem for generic open sets, we'll try to build intuition with a classical example – the upper half-plane.

3.1. The Upper Half-Plane. Now we set $U = \mathbb{R}^d_+$, i.e. we consider the problem

$$\begin{cases}\Delta u = 0 & \text{on } \mathbb{R}^d_+\\ u = g & \text{on } \partial\mathbb{R}^d_+\end{cases}.$$

Note that the previous theorem does not apply, for $\mathbb{R}^d_+$ is not bounded. Still, we proceed by solving

$$\begin{cases}\Delta_y\phi^x(y) = 0 & \text{in } \mathbb{R}^d_+\\ \phi^x(y) = \Phi(x-y) & \text{on } \partial\mathbb{R}^d_+\end{cases}$$

and then writing down the Green's function.

We'll produce a corrector ϕ^x explicitly. Given $x = (x^1, \dots, x^d) \in \mathbb{R}^d_+$, set $\overline{x} = (x^1, \dots, x^{d-1}, -x^d)$. We claim that a solution to the corrector problem is given by $\Phi(\overline{x} - y)$. Indeed, $\Delta_y \Phi(\overline{x} - y) = 0$ if $y \in \mathbb{R}^d_+$, and if $y \in \partial\mathbb{R}^d_+$ then $\Phi(x - y) = \Phi(\overline{x} - y)$. ($\Phi(x)$ depends only on $|x|$, and $|x - y| = |\overline{x - y}| = |\overline{x} - y|$.) Thus the Green's function for the upper half-plane is

$$G(x, y) = \Phi(x - y) - \Phi(\overline{x} - y).$$

A quick calculation, and then we'll write down the full solution in the upper-half plane. Motivated by the previous theorem, we note $\Phi(z) = \frac{c_d}{|z|^{d-2}}$ and compute $\partial_{z^d}\Phi(z) = (d-2)\, c_d \frac{z^d}{|z|^d}$. Thus,

$$\partial_{y^d}\Phi(x - y) - \partial_{y^d}\Phi(\overline{x} - y) = -2c_d(d-2)\frac{x^d}{|x - y|^d}.$$

THEOREM 3.9. *A solution of*

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^d_+ \\ u = g & \text{on } \partial\mathbb{R}^d_+ \end{cases}$$

for $g \in C^2_0$ is given by

$$u(x) = \int_{\partial\mathbb{R}^d_+} \frac{\partial}{\partial y^d} G(x, y)\, g(y)\, dy = \frac{2x^d}{d \cdot \alpha(d)} \int \frac{g(y)}{|x - y|^d}\, dy$$

for $x \in \mathbb{R}^d_+$, extended by continuity to $\partial\mathbb{R}^d_+$.

PROOF. Step 1. Show $\Delta u = 0$ in $\mathbb{R}^d_+$. We have

$$u(x) = \int_{\partial\mathbb{R}^d_+} \partial_{y^d}(\Phi(x - y) - \Phi(\overline{x} - y))\, g(y)\, dy$$

for $x \in \mathbb{R}^d_+$, and both $\Phi(x - y)$ and $\Phi(\overline{x} - y)$ are harmonic in x as long as $x \neq y$, $\overline{x} \neq y$.

Step 2. Show u can be extended by continuity, and that its boundary value is g. Write $x' = (x^1, \dots, x^{d-1})$ and similarly y', then

$$u(x', h) = c_0 \int \frac{h}{\left((x' - y')^2 + h^2\right)^{d/2}} g(y')\, dy'.$$

Write

$$c_0 \frac{h}{\left((x' - y')^2 + h^2\right)^{d/2}} = c_0 \frac{1}{h^{d-1}} \frac{1}{\left(\left(\frac{x'-y'}{h}\right)^2 + 1\right)^{d/2}} = c_0 \frac{1}{h^{d-1}} Z\left(\frac{x' - y'}{h}\right)$$

where $Z(z) = \frac{1}{(1+z^2)^{d/2}} : \mathbb{R}^{d-1} \to \mathbb{R}$. We have thus

$$u(x', h) = \left(\frac{c_0}{h^{d-1}} Z\left(\frac{\cdot}{h}\right) \star g\right)(x').$$

Now $Z > 0$, $\int_{\mathbb{R}^{d-1}} Z < \infty$, and in fact $c_0 \int Z = 1$. We want to show $u(x', h) \to g(x')$ as $h \to 0$.

CLAIM. $\int \frac{c_0}{h^{d-1}} Z\left(\frac{x'-y}{h}\right) g(y')\, dy' \to g(x')$ as $h \to 0$, provided $Z > 0$, $\int Z < \infty$, $c_0 \int Z = 1$, and g is continuous.

PROOF. Take $x' = 0$ and fix $\epsilon > 0$. Take R so that $\int_{|x|>R} Z(x)\, dx < \epsilon$ and take δ so that $|g(x) - g(y)| < \epsilon$ if $|x - y| < \delta$. Then

$$\begin{aligned}\int \frac{c_0}{h^{d-1}} Z\left(\frac{-y'}{h}\right) g(y')\, dy' = & \int_{|y'|>Rh} \frac{c_0}{h^{d-1}} Z\left(\frac{-y'}{h}\right) g(y')\, dy' \\ & + \int_{|y'|<Rh} \frac{c_0}{h^{d-1}} Z\left(\frac{-y'}{h}\right) g(0)\, dy \\ & + \int_{|y'|<Rh} \frac{c_0}{h^{d-1}} Z\left(\frac{-y'}{h}\right) (g(y') - g(0))\, dy.\end{aligned}$$

The first integral is $< \epsilon \sup|g|$, the second is within ϵ of $g(0)$, and the third is $< \epsilon$. □

EXERCISE 3.10. Check the details above.

This completes the proof. □

We have found a solution

$$u(x) = \frac{2x^d}{d \cdot \alpha(d)} \int \frac{g(y)}{|x-y|^d}\, dy$$

to the problem

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^d_+ \\ u = g & \text{on } \partial\mathbb{R}^d_+ \end{cases}$$

with $g \in C^2_0$. Suppose $d \geq 3$. Then we claim this is the unique solution in the class of C^2 solutions which go to zero at infinity, i.e. which satisfy

$$\lim_{R\to\infty} \sup_{|x|=R} |u(x)| = 0.$$

On $\mathbb{R}^d$, we used Liouville's theorem to conclude this. But Liouville's theorem fails in the half-plane. Indeed, suppose $x_0 \in \mathbb{R}^d_-$, then $\Phi(x_0 - y)$ is harmonic for $y \in \mathbb{R}^d_+$ but

$$\Phi(x_0 - y) = \frac{1}{|x_0 - y|^d} < \frac{1}{|x_0|^d}.$$

So how should be proceed? The only other tool we have is the maximum principle. To demonstrate uniqueness, it suffices to prove that the unique solution to

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^d_+ \\ u = 0 & \text{on } \partial\mathbb{R}^d_+ \\ u \to 0 & \text{at } \infty \end{cases}$$

is $u \equiv 0$. Fix $\epsilon > 0$ and take R so that $|u(x)| < \epsilon$ if $|x| = R$. Consider $U = \mathbb{R}^d_+ \cap B(0, R)$. By the maximum principle, $|u| < \epsilon$ in U. Let $\epsilon \to 0$ and $R \to \infty$ to get the result. Lecture 5, 10/4/11

3.2. The Ball. Now take $U = B(0,1) \subset \mathbb{R}^d$. Again, to find the Green's function one has to find the corrector, i.e. the solution of

$$\begin{cases} -\Delta\phi^x = 0 & \text{in } B \\ \phi^x(y) = \Phi(x - y) & \text{on } \partial B \end{cases}.$$

EXERCISE 3.11. Check that the corrector for the unit ball is

$$\phi^x(y) = \Phi\left(|x|\left(y - \frac{x}{|x|^2}\right)\right).$$

(Hint: Harmonicity is translation invariant and the fundamental solution is homogeneous.)

So the Green's function for the unit ball is

$$G(x,y) = \Phi(x-y) - \Phi\left(|x|\left(y - \frac{x}{|x|^2}\right)\right).$$

Hence the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = g & \text{on } \partial B \end{cases}$$

is

$$u(x) = \int_{\partial B(0,1)} \frac{1}{d\alpha(d)} \frac{1-|x|^2}{|x-y|^d} g(y)\, dS(y).$$

The kernel

$$K(x,y) = \frac{1}{d\alpha(d)} \frac{1-|x|^2}{|x-y|^d}$$

is called the *Poisson kernel for* $B(0,1)$.

EXERCISE 3.12. Check that u extends to the boundary ∂B and satisfies $u = g$ there.

4. Energy Methods

Recall the maximum principle, which gave control to the L^∞-norm of harmonic u. Another approach is to study the L^2- and H^1-norms

$$||u||_{L^2}^2 = \int |u|^2$$

$$||\nabla u||_{H^1}^2 = \int |\nabla u|^2.$$

These are the so-called "energy methods". The first result we'll see is uniqueness.

THEOREM 4.1. *Let U be a bounded, open set with smooth boundary. There exists at most one solution $u \in C^2\left(\overline{U}\right)$ of*

$$\begin{cases} -\Delta u = f & \textit{in } U \\ u = g & \textit{on } \partial U \end{cases}$$

where $f, g \in C^2$.

We already proved a version of this with the maximum principle. Now we'll prove the result with energy methods.

PROOF. Assume there are two solutions u^1, u^2. Set $w = u^1 - u^2$, then w solves

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}.$$

We have $-\Delta w = 0$, so

$$\int_U w(-\Delta w)\, dx = 0.$$

Integrating by parts yields

$$\int |\nabla w|^2 \, dx = 0,$$

hence $\Delta w \equiv 0$. But then $w \equiv 0$ and so we're done. □

Now define the *energy functional*

$$E(u) = \int \left(\frac{1}{2} |\nabla u|^2 - fu \right) dx$$

and the *admissible functions*

$$\mathcal{A} = \left\{ u \in C^2\left(\overline{U}\right), \; u = g \text{ on } \partial U \right\}.$$

THEOREM 4.2 (Dirichlet's Principle). *u solves the problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

with $f, g \in C^2$ iff u minimizes $E(u)$ over $\mathcal{A}$.

PROOF. Assume that u solves the problem. Take $w \in C^2\left(\overline{U}\right)$, then $-\Delta u - f = 0$ so

$$\int_U (-\Delta u - f)(u - w) \, dx = 0.$$

Integrate by parts to get

$$\int_U |\nabla u|^2 - \nabla u \cdot \nabla w - fu + fw \, dx = 0$$

or equivalently

$$\int \left(|\nabla u|^2 - uf \right) dx = \int (\nabla u \cdot \nabla w - fw) \, dx.$$

Cauchy-Schwarz gives

$$\nabla u \cdot \nabla w \leq \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla w|^2$$

and so

$$\int \left(|\nabla u|^2 - uf \right) dx \leq \int \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla w|^2 - fw \right) dx$$

which says $E(u) \leq E(w)$. (Note this also implies u is the unique minimizer – consider Cauchy-Schwarz more carefully.)

Now assume u is a minimizer. Take $\phi \in C^2\left(\overline{U}\right)$ with $\operatorname{supp}(\phi) \subset U$, then

$$\begin{aligned} E(u + \phi) &= E(u) + \epsilon \int (\nabla u \cdot \nabla \phi - \phi f) \, dx + \epsilon^2 \frac{1}{2} \int |\nabla \phi|^2 \, dx \\ &= E(u) + \epsilon \int \phi (-\Delta u - f) \, dx + \epsilon^2 \frac{1}{2} \int |\nabla \phi|^2 \, dx \end{aligned}$$

after integration by parts. For small ϵ, only the linear term matters; since we can take ϵ positive or negative, it must be that

$$\int \phi (-\Delta u - f) \, dx = 0.$$

But ϕ was arbitrary, so it must be that

$$-\Delta u - f = 0.$$

□

This gives an interesting way to look for solutions to the problem. It takes some functional analysis to make it work.

CHAPTER 3

The Heat Equation

We begin with Fourier's law. Consider a material and assume that the energy is

$$\text{energy} = c \cdot T$$

for some constant c. (We'll take $c = 1$ implicitly from here on.) Then Fourier's law says that the energy flow is

$$q = -k\nabla T.$$

(Again, we'll take $k = 1$.) If we have a domain Ω inside the material, then

$$\frac{d}{dt}\text{energy inside } \Omega = -\text{flow of energy across } \partial\Omega$$

or

$$\frac{d}{dt}\int_{\Omega} T(x,t)\,dx = \int_{\partial\Omega} q \cdot dS = \int_{\partial\Omega} \nabla T \cdot dS.$$

By the divergence theorem,

$$\frac{d}{dt}\int_{\Omega} T\,dx = \int_{\Omega} \triangle T\,dx$$

and passing the derivative under the integral gives

$$\int_{\Omega} (\partial_t T - \triangle T)\,dx = 0.$$

As Ω is arbitrary, we get

$$\partial_t T - \triangle T = 0.$$

So now the problem is to find $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, $(x,t) \mapsto u(x,t)$ which solves the *heat equation*

$$\partial_t u - \Delta u = 0.$$

Fourier invented Fourier series to solve this equation. If time permits, we'll cover this towards the end of the course. Instead, we'll now try to find a fundamental solution.

1. The Fundamental Solution

Look for a solution of the form

$$u(x,t) = \frac{1}{t^\alpha}\phi\left(\frac{x}{t^\beta}\right).$$

We have

$$\begin{aligned}(\partial_t - \Delta)\,u(x,t) &= -\alpha t^{-\alpha-1}\phi\left(\frac{x}{t^\beta}\right) - \beta t^{-\alpha-\beta-1}x \cdot \nabla\phi\left(\frac{x}{t^\beta}\right) + t^{-2\beta-\alpha}\triangle\phi\left(\frac{x}{t^\beta}\right) \\ &= -\alpha t^{-\alpha-1}\phi(y) - \beta t^{-\alpha-1}y \cdot \nabla\phi(y) + t^{-2\beta-\alpha}\triangle\phi(y)\end{aligned}$$

after setting $y = xt^{-\beta}$. We want this to be zero for all $t > 0$, so it will be convenient to require the exponents on t to all be the same; thus take $\beta = 1/2$. Then the equation to solve is

$$-\alpha\phi - \frac{1}{2}y \cdot \nabla\phi(y) + \triangle\phi(y) = 0.$$

We'll see soon that it is best to take $\alpha = d/2$. Now if we further assume $\phi = \phi(r)$, we get the ODE

$$-\frac{n}{2}\phi - \frac{1}{2}r\phi' + \phi'' + \frac{n-1}{2}\phi' = 0$$

which is solved by

$$\phi(r) = Ce^{-r^2/4}.$$

DEFINITION 1.1. The *fundamental solution for the heat equation* is

$$\Phi(t,x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t < 0 \end{cases}.$$

CLAIM. For all $t > 0$, $\int \Phi(t,x)\ dx = 1$.

PROOF. Note $\int \Phi(t,x)\ dx$ does not depend on t. So it suffices to show $\int \Phi(1/4, x)\ dx = 1$. But it is a standard result that

$$\int_{\mathbb{R}} e^{-x^2}\ dx = \sqrt{\pi}$$

so after integrating iteratively we're done. □

Now we are equipped to consider the standard initial value problem.

THEOREM 1.2. *A solution of*

$$\begin{cases} \partial_t u - \triangle u = 0 & t > 0 \\ u(t=0) = g \end{cases}$$

with $g \in C^2$ is given by

$$u(x,t) = (\Phi(t) \star g)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y)\ dy.$$

Recall the solution to Laplace's equation in the upper-half plane. Now we can think of $\mathbb{R}^d \times \mathbb{R}^+ = \mathbb{R}^{d+1}_+$ to intuitively understand the solution above. In the upper-half plane, the boundary data was matched since the closer we got to the boundary the more concentrated the kernel became. It's the same here – the smaller t gets, the more concetrated the kernel becomes. But as it's integral is always one, $u(t,x) \to g(x)$ as $t \to 0$. Here are the details.

PROOF SKETCH. First show that u solves $(\partial_t - \Delta)u = 0$. Write

$$u(t,x) = \int \Phi(x-y,t)\, g(y)\ dy$$

then apply $\partial_t - \Delta$ and commute with $\int$. For y fixed, $(\partial_t - \Delta_x)\Phi(x-y,t) = 0$ so we're done.

Next, show that u has the right initial data. More precisely, extend u by continuity up to $t = 0$ and check that $u(0,x) = g(x)$. Observe that

$$u(t,x) = \left(\frac{1}{t^{d/2}}\Psi\left(\frac{\cdot}{\sqrt{t}}\right) \star g\right)(x)$$

where $\Phi(x,t) = \frac{1}{t^{d/2}}\Psi\left(x/\sqrt{t}\right)$. Since $\Psi > 0$ and $\int \Psi = 1$ we see $u(t,x) \to g(x)$ as $t \to 0$. $\square$

REMARKS.

(1) $u \in C^\infty$ for $t > 0$. This follows because convolving with a smooth function regularizes. In light of this, we say the heat equation homogenizes the data. Since the heat equation models a diffusive process, this is physically intuitive.

(2) The equation exhibits infinite propagation speed. Consider compactly supported $g = u(t=0)$. Note that for $t > 0$, the kernel is non-zero for any x. So as soon as t becomes positive, u will be supported everywhere. Then information will have traveled at infinite speed.

2. Duhamel's Principle

We want now to solve the non-homogeneous problem. Consider first an ODE

$$\begin{cases} \frac{d}{dt}u - Lu = 0 \\ u(t=0) = g \end{cases}.$$

This has the solution

$$u(t) = e^{tL}g.$$

Now consider the problem

$$\begin{cases} \frac{d}{dt}u - Lu = f \\ u(t=0) = g \end{cases}.$$

This has solution

$$u(t) = e^{tL}g + \int_0^t e^{(t-s)L}f(s)\, ds.$$

To check, write

$$\left(\frac{d}{dt} - L\right)u = e^{(t-t)L}f(t) = f(t)$$

after commuting the derivative and hitting the upper bound t.

This has an analogue for PDE. But we have to be careful. The key fact above was that $e^{0L} = I$. But now, the fundamental solution is

$$\Phi(x,t) = \frac{C}{t^{d/2}}e^{-\frac{|x-y|^2}{4t}}$$

which does not make sense for $t = 0$. We'll still be able to translate the Duhamel formula for ODE into a sucessful formula for the heat equation. For us, $L = \Delta$ and $e^{t\Delta}f = \Phi(t) \star f$.

THEOREM 2.1. *A solution of*

$$\begin{cases} \partial_t u - \Delta u = f & t > 0 \\ u(t=0) = g \end{cases}$$

with bounded $f, g \in C^2$ is

$$u(t,x) = \Phi(t) \star g + \int_0^t \Phi(t-s) \star f\, ds.$$

REMARK. This is easy with distribution theory, since then $(\partial_t - \Delta)\Phi = \delta$. But we don't want to take the approach.

PROOF SKETCH. First $u(t=0)=g$. Note that

$$\left|\int_0^t (\Phi(t-s)\star f)(x)\ ds\right| \le t\sup|f| \to 0$$

as $t\to 0$ for

$$\begin{aligned}\left|\int \Phi(t-s,x-y)\,f(y)\ dy\right| &\le \int |\Phi(t-s,x-y)||f(y)|\,dy\\ &\le \sup|f|\int \Phi(t-s,x-y)\ dy\\ &= \sup|f|.\end{aligned}$$

Now $(\partial_t-\Delta)\,u=f$. We need to show

$$(\partial_t-\Delta)\int_0^t\int \Phi(t-s,x-y)\,f(y,s)\ dyds = f.$$

Write

$$\int_0^t\int \Phi(t-s,x-y)\,f(y,s)\ dyds = \int_0^t\int \Phi(s,y)\,f(x-y,t-s)\ dyds$$

by the change of variables $s'=t-s$, $y'=x-y$. Then

$$\begin{aligned}(\partial_t-\Delta)\int_0^t\int \Phi(s,y)\,f(x-y,t-s)\ dyds &= \int_0^t\int \Phi(s,y)\,(\partial_t-\Delta_x)\,f(x-y,t-s)\ dyds\\ &\quad+\int \Phi(t,y)\,f(x-y,0)\ dy. \qquad (2.1)\end{aligned}$$

Now

$$\begin{aligned}\int_0^t\int \Phi(s,y)\,(\partial_t-\Delta_x)\,f(x-y,t-s)\ dyds &= \int_0^t\int \Phi(s,y)\,(-\partial_s-\Delta_y)\,f(x-y,t-s)\ dyds\\ &= \int_0^\epsilon\int \Phi(s,y)\,(-\partial_s-\Delta_y)\,f\,dyds\\ &\quad+\int_\epsilon^t\int \Phi(s,y)\,(-\partial_s-\Delta_y)\,f\,dyds.\end{aligned}$$

We have

$$\int_0^\epsilon\int \Phi(s,y)\,(-\partial_s-\Delta_y)\,f\,dyds \to 0$$

as $\epsilon\to 0$, and after integration by parts

$$\begin{aligned}\int_\epsilon^t\int \Phi(s,y)\,(-\partial_s-\Delta_y)\,f\,dyds &= \int_\epsilon^t\int (-\partial_s-\Delta_y)\,\Phi(s,y)\,f\,dy\,ds\\ &\quad-\int \Phi(t,y)\,f(x-y,0)\ dy\\ &\quad+\int \Phi(\epsilon,y)\,f(x-y,t-\epsilon)\ dy\end{aligned}$$

The first integral is zero, and the second cancels out the term $\int \Phi(t,y)\,f(x-y,0)\ dy$ in (2.1). The third goes to $f(x,t)$ as $\epsilon\to 0$, so we're done. $\square$

Lecture 6, 10/18/11

3. Properties of Solutions

We'll follow a similar program to what we saw with the Laplace equation. First we'll prove a mean value formula. From this we'll derive a maximum principle and then uniqueness; we'll also conclude a result about smoothness. The domain we'll consider is a so-called "parabolic cylinder":

DEFINITION 3.1. Given an open, bounded set $U \subset \mathbb{R}^d$, it generates the *parabolic cylinder* $U_T = U \times (0, T]$. We'll say it's *boundary* is $\Gamma_T = U \times \{t = 0\} \cup \partial U \times [0, T]$.

NOTE. The definition of boundary above is not the same as usual. It is tailored to our study of the heat equation.

As with the Laplace equation, we might want to consider an open ball $B(x, r) \subset U_T$ to derive a mean value formula. However, t and x no longer play symmetric roles. Also there is now directionality in time; solutions become more smooth as time goes forward, and not the other way around. The right thing to do is to replace $B(x, r)$ by level sets of the fundamental solution,

$$E(x_0, t_0, r) = \left\{(s, y) : \Phi(x_o - y, t_o - s) > \frac{1}{r^d}\right\}.$$

THEOREM 3.2 (Mean value formula). *If $u \in C^2(U_T)$ satisfies $\partial_t u - \Delta u = 0$, then*

$$u(x, t) = \frac{1}{4r^d} \int_{E(x,t,r)} u(y, s) \frac{|x - y|^2}{|t - s|^2} \, dyds$$

whenever $E(x, t, r) \subset U_T$.

REMARK. This is a weighted average (in contrast to the mean value formula for Laplace's equation).

PROOF. The equation is translation-invariant, so take $x = t = 0$. Consider the function

$$\phi(r) = \frac{1}{4r^d} \int_{E(0,0,r)} u(y, s) \frac{|y|^2}{s^2} \, dyds.$$

We need to show $\phi' \equiv 0$ and $\phi(r) \to u(0, 0)$ as $r \to 0$. We'll only do the first here. The second is straightforward. By the change of variables $\left(ry', r^2 s'\right) = (y, s)$ we have

$$\phi(r) = \int_{E(0,0,1)} u\left(ry, r^2 s\right) \frac{|y|^2}{s^2} \, dyds.$$

So then

$$\begin{aligned}
\phi'(r) &= \int_{E(0,0,1)} \left(y^i \partial_i u + 2rs\partial_t u\right) \frac{|y|^2}{s^2} \, dyds \\
&= \frac{1}{r^{d+1}} \int_{E(0,0,r)} y^i u_i \frac{|y|^2}{s^2} \, dyds + \frac{1}{r^{d+1}} \int_{E(0,0,r)} 2u_t \frac{|y|^2}{s} \, dyds \\
&= I + II.
\end{aligned}$$

(We've introduced Einstein summation.) Set

$$\Psi = \log \Phi(y, s) + \log r^d = -\frac{d}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + d \log r$$

with $\Psi \equiv 0$ on $\partial E(0,0,r)$. Then write

$$\begin{aligned} II &= \frac{1}{r^{d+1}} \int_{E(0,0,r)} 2u_t y^i \Psi_i \, dyds \\ &= \frac{1}{r^{d+1}} \int_{E(0,0,r)} \left(4u_{si} y^i \Psi + 4du_t \Psi\right) dyds \\ &= \frac{1}{r^{d+1}} \int_{E(0,0,r)} \left[-4du_s \Psi + 4u_i y^i \left(-\frac{d}{2s} - \frac{|y|^2}{4s^2}\right)\right] dyds \end{aligned}$$

where we've integrated by parts in y_i twice. So we've obtained

$$II = \frac{1}{r^{d+1}} \int_{E(0,0,r)} \left(-4du_s \Psi - \frac{2d}{s} u_i y^i\right) dyds - I.$$

Now $u_s = \Delta u$ by assumption, and after integrating by parts in y_i we find

$$\int_{E(0,0,r)} \left(-4du_s \Psi - \frac{2d}{s} u_i y^i\right) dyds = 0.$$

Hence $I + II = 0$ so $\phi' \equiv 0$. □

Exercise 3.3. Show $\phi(r) \to u(0,0)$ as $r \to 0$ to complete the proof.

Theorem 3.4 (Maximum principle). *Suppose U has smooth boundary. If $u \in C^2\left(\overline{U_T}\right)$ satisfies $\partial_t u - \Delta u = 0$, then*

(1) $\max_{\overline{U_T}} u = \max_{\Gamma_T} u$,
(2) If U is connected and u acheives its maximum on $\overline{U_T} \backslash \Gamma_T$, then u is constant.

Remarks.

(1) This fits with our interpretation of the heat equation as averaging. This intuition is confirmed by the maximum principle.
(2) There is also a minimum principle.

Proof sketch. If suffices to prove (2). Suppose u is maximized at $(x_0, t_0) \notin \Gamma_T$ with $u(x_0, t_0) = M$, and find r so that $E(x_0, t_0, r) \subset U_T$. Then by the mean value formula

$$u(x_0, t_0) = \frac{1}{4r^d} \int_{E(x_0,t_0,r)} u(y,s) \frac{|x_0 - y|^2}{|t_0 - s|^2} dyds,$$

and since this is a weighted average we conclude $u \equiv M$ on $E(x_0, t_0, r)$. This implies $u \equiv M$ on $U \times [0, t_0]$, and since this holds for all t_0 the claim follows. □

Exercise 3.5. Fill in the details in the proof above. Note the clopen argument used in the proof of the corresponding result for Laplace's equation does not work because (x_0, t_0) is not in the interior of $E(x_0, t_0, r)$.

As a result we get uniqueness.

Corollary 3.6. *The problem*

$$\begin{cases} \partial_t u - \Delta u = g & \text{in } U_T \\ u(t=0) = f & \text{on } U \\ u(t,x) = h & \text{on } \partial U \times [0,T] \end{cases}$$

has at most one solution $u \in C^2(U_T)$.

PROOF. Suppose $u, \tilde{u}$ are two solutions, then define $w = u - \tilde{u}$. By linearity, $\partial_t w - \Delta w = 0$ in U_T, and $w = 0$ on Γ_T. By the maximum principle, $w \leq 0$ on U_T. By the minimum principle, $w \geq 0$ on U_T. Hence $w \equiv 0$. $\square$

THEOREM 3.7 (Maximum principle in $\mathbb{R}^d$). *Let $u \in C^2$ solve*

$$\begin{cases} \partial_t u - \Delta u = g & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ u\,(t=0) = f \end{cases}$$

with bounded $f \in C$, and suppose

$$|u\,(x,t)| \leq Ae^{a|x|^2}$$

for constants $A, a > 0$. Then u is bounded and $\sup u \leq \sup f$.

As a result, we get uniqueness in a class which allows for growth at infinity.

COROLLARY 3.8. *The only solution $u \in C^2$ to*

$$\begin{cases} \partial_t u - \Delta u = g & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ u\,(t=0) = f \end{cases}$$

with bounded $f \in C$ and with

$$|u\,(x,t)| \leq Ae^{a|x|^2}$$

is given by $u\,(t) = \Phi\,(t) \star f$.

The proof of the corollary goes as usual. So we'll only discuss the theorem here.

PROOF SKETCH OF THEOREM. It suffices to take $\max f > 0$. Set T so that $4aT < 1$. We'll prove first that $\sup_{t\in[0,T]} u \leq \max f$. Find $\epsilon > 0$ so that $4a\,(T+\epsilon) < 1$, and define

$$v\,(x,t) = u\,(x,t) - \frac{\delta}{4\,(T+\epsilon-t)^{d/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}}$$

for $\delta > 0$. By the same computation which shows the fundamental solution solves the heat equation, one can prove the extra term on the right solves the heat equation. So v solves the heat equation.

We want to apply the maximum principle on $B\,(y,r)\times[0,T]$ for a certain choice of r. Observe that on $\partial B\,(y,r) \times [0,T]$,

$$v\,(x,t) \leq Ae^{a(|y|+r)^2} - \frac{\delta}{4\,(T+\epsilon-t)^{d/2}} e^{\frac{r^2}{4(T+\epsilon)}} \to -\infty$$

as $r \to \infty$. So for big enough r, $v \leq 0$ on $\partial B\,(y,r) \times [0,T]$. Given such an r, the maximum principle on $B\,(y,r) \times [0,T]$ says

$$\begin{aligned} v\,(y,t) &\leq \sup_{B(0,r)} \left(f + \delta \cdot (\text{something})\right) \\ &\leq \sup_{\mathbb{R}^d} \left(f + \delta \cdot (\text{something})\right) \\ &\to \sup_{\mathbb{R}^d} f \end{aligned}$$

as $\delta \to 0$. $\square$

EXERCISE 3.9. Fill in the details in the proof above.

Finally we have a result on smoothness.

THEOREM 3.10. *If u solves $\partial_t u - \Delta u = 0$ on U_T, then it is smooth on U_T.*

REMARK. The proof is similar to the corresponding result for Laplace's equation. The idea is to use the mean value formula to write u as convolution with a smooth kernel.

There is a quantitative version of this result.

THEOREM 3.11. *If $\partial_t u - \Delta u = 0$ on $A = \left\{|y-x| < r,\ t - r^2 < s < t\right\}$, then*

$$\sup_A \left| D_x^k D_t^l u \right| < \frac{C_{kl}}{r^{k+u+n+2}} ||u||_{L^1(A)}$$

where C_{kl} depends only on k, l.

REMARK. Again, the proof is similar to the corresponding result for Laplace's equation.

4. Energy Methods

The maximum principle concerns the L^∞-norm of u, i.e. $\sup|u|$. As we've seen before, considering the L^2-norm of u, i.e. $\int u^2$, can lead to new proofs. Here is a uniqueness result.

THEOREM 4.1. *There exists at most one $u \in C^2(U_T)$ which solves*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

with $f, g \in C$.

PROOF. Suppose $u, \tilde{u}$ are two solutions. Then $w = u - \tilde{u}$ solves

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \end{cases}.$$

As usual, we multiply $\partial_t w - \Delta w = 0$ by w and integrate to get

$$\int \partial_t w(t) \cdot w(t)\, dx - \int \Delta w \cdot w\, dx = 0.$$

Integrating by parts yields

$$\frac{1}{2}\partial_t \int w^2(t)\, dx + \int |\Delta w|^2\, dx = 0,$$

and thus

$$\frac{1}{2}\partial_t \int w^2(t)\, dx \leq 0.$$

Since $\int w^2 = 0$ at $t = 0$, $\int w^2 = 0$ for all $t > 0$. This completes the proof. □

CHAPTER 4

The Wave Equation

Lecture 7, 10/31/11

So far we've seen the (linear) transport, Laplace, and heat equations. Now we'll study a more complicated transport equation – the wave equation. The *wave equation* for a map $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$ is

$$\partial_t^2 u - \Delta u = 0.$$

Since the equation is second order in time, to determine a solution we need to specify $u(t = 0)$ and $u_t(t = 0)$. (This is a general phenomenon in PDE – the order in time of the equation determines the amount of initial data needed to prescribe a solution.) The initial value problem corresponding to the wave equation is thus

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(t = 0) = f \\ u_t(t = 0) = g \end{cases}.$$

The wave equation describes the physics of light and sound. Consider plucking a string and try to describe the resulting oscillations. The equation that governs the displacement of the string is non-linear, but linearizing yields the wave equation. More generally, consider an elastic material and let $u(t, x)$ be the displacement of x. For small u, the force which the material exerts (to first order) is given by Hooke's law: $F = -\nabla u$. (We've normalized the spring coefficient.) Newton's law in a domain Ω says

$$\partial_t \int_\Omega \partial_t u(x, t)\, dx = \int_{\partial\Omega} \nabla u \cdot dS.$$

(We've taken density to be one everywhere.) After applying the divergence theorem and rearranging terms,

$$\int_\Omega \left(\partial_t^2 u - \Delta u\right) dx = 0.$$

This holds on all domains Ω, hence

$$\partial_t^2 - \Delta u = 0.$$

1. One-Dimensional Wave Equation

Take $d = 1$ and consider

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t = 0) = f \\ u_t(t = 0) = g \end{cases}.$$

The trick is to notice $\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x)$, so if $\partial_t^2 - \partial_x^2 u = 0$ then upon setting $v = (\partial_t - \partial_x) u$ we see $(\partial_t + \partial_x) v = 0$. So v satisfies the homogeneous transport equation, and hence $v(x,t) = a(x-t)$. Now u solves $(\partial_t - \partial_x) u = a(x-t)$, the non-homogeneous transport equation, and hence

$$u(x,t) = b(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(z)\, dz.$$

We have yet to identify the functions a, b. To do so, we use the data. From the formula, we see $u(t=0) = b(x)$ and $\partial_t u(t=0) = b'(x) + a(x)$. So the data determines $b = f$ and $a = g - f'$. So we get

$$u(x,t) = f(x+t) + \frac{1}{2}\int_{x-t}^{x+t} g(z) - f'(z)\, dz$$

and hence the following theorem.

Theorem 1.1 (d'Alembert). *The solution of*

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t=0) = f \\ u_t(t=0) = g \end{cases}$$

is

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(z)\, dz.$$

A proof would follow the discussion above, but would also demonstrate the formula solves the given problem. This is not a hard calculation.

Remarks.

(1) The equation is not smoothing, but we gain a little reguarity. If $f \in C^m$ and $g \in C^{m-1}$, then $u \in C^m$. This behavior is closer to the transport equation then to the Laplace or heat equation.
(2) The equation exhibits a finite propagation speed. Let $\phi = u(t=0)$, then if supp$(\phi) \subset$B$(0,1/2)$, supp$(u) \subset B(0, t+1/2)$. In fact, supp$(u) \subset B(t,1/2) \cup B(-t,1/2)$. So signals propagate at speed one. This is an example of Huygen's principle.

As with the transport equation, we could define a class of solutions for $f \in L^1, g \in L^1$. These are the so-called "weak solutions", which we will not study here.

We want to solve the wave equation in higher dimensions. The next theorem concerning reflection in $d = 1$ will prove useful.

Theorem 1.2. *Let $f, g \in C^2$. Then the unique solution of*

$$\begin{cases} u_{tt} - u_{xx} = 0 & t \geq 0,\ x \geq 0 \\ u(t=0) = f & x \geq 0 \\ u_t(t=0) = g & x \geq 0 \\ u(x=0) = 0 & t \geq 0 \end{cases}$$

is

$$u(x,t) = \begin{cases} \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(z)\, dz & x \geq t \geq 0 \\ \frac{1}{2}\left(f(x+t) - f(t-x)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(z)\, dz & t \geq x \geq 0 \end{cases}.$$

Proof sketch. Let u be a solution, then extend it for all x as an odd function to v. Then v solves the wave equation on $\mathbb{R}^2$. So we have a formula for v. □

2. Three-Dimensional Wave Equation

First consider any $d \geq 2$. We want to solve

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u\,(t=0) = g \\ u_t\,(t=0) = h \end{cases}.$$

Define the *spherical means*

$$U\,(x,r,t) = ⨍_{\partial B(x,r)} u\,(y,t)\; dS(y)$$
$$G\,(x,r) = ⨍_{\partial B(x,r)} g\,(y)\; dS(y)$$
$$H\,(x,r) = ⨍_{\partial B(x,r)} h\,(y)\; dS(y).$$

Lemma 2.1 (Euler-Poisson-Darboux). *Fix x, then U solves*

$$U_{tt} - U_{rr} - \frac{d-1}{r} U_r = 0$$

so long as u solves the wave equation.

Proof. It's easy to differentiate U in time to get

$$U_{tt} = ⨍_{\partial B} u_{tt}\,(y,t)\; dS(y).$$

Differentiating w.r.t. r is more challenging. We have

$$\begin{aligned} U\,(r,t) &= ⨍_{\partial B(x,r)} u\,(y,t)\; dS(y) \\ &= ⨍_{\partial B(0,1)} u\,(x+rz,t)\; dS(z) \end{aligned}$$

after a change of variables. Differentiating, changing variables back, and applying the divergence theorem yields

$$\begin{aligned} \partial_r U &= ⨍_{\partial B(0,1)} z \cdot \nabla u\,(x+rz,t)\; dS(z) \\ &= ⨍_{\partial B(0,r)} n \cdot \nabla u\,(x+y,t)\; dS(y) \\ &= \frac{1}{|\partial B\,(0,1)| r^{d-1}} \int_{B(x,r)} \Delta u\, dy. \end{aligned}$$

Now observe

$$U_{rr} - \frac{d-1}{r} U_r = r^{1-d} \frac{d}{dr} \left(r^{d-1} U_r\right),$$

so we multiply U_r above by r^{d-1} and differentiate:

$$r^{d-1} U_r = \frac{1}{|\partial B\,(0,1)|} \int_{B(x,r)} \Delta u\, dy$$

so

$$\frac{d}{dr} \left(r^{d-1} U_r\right) = \frac{1}{|\partial B\,(0,1)|} \int_{\partial B(x,r)} \Delta u\, dS(y).$$

Now multiplying by r^{1-d} again gives

$$r^{1-d}\frac{d}{dr}\left(r^{d-1}U_r\right) = \fint_{\partial B(x,r)} \Delta u \, dS(y).$$

Hence

$$U_{tt} - U_{rr} - \frac{d-1}{r}U_r = \fint_{\partial B(x,r)} \left(u_{tt} - \Delta u\right) \, dS(y) = 0$$

so long as u solves the wave equation. $\square$

Now we specialize to $d = 3$. Then the equation in the previous lemma becomes

$$U_{tt} - u_{rr} - \frac{2}{r}U_r = 0.$$

As $t \to 0$, U and $\partial_t U$ are given by the data; as $r \to 0$, $U \to u\,(x, t)$. The idea is to reduce to $d = 1$ by considering $\tilde{U}\,(r, t) = rU\,(r, t)$, since

$$U_{tt} - U_{rr} - \frac{2}{r}U_r = r\left(\tilde{U}_{tt} - \tilde{U}_{rr}\right).$$

So if u solves the wave equation, then by the lemma we see

$$0 = \tilde{U}_{tt} - \tilde{U}_{rr}.$$

As $t \to 0$, $\tilde{U} \to \tilde{G}$ and $\partial_t\tilde{U} \to \tilde{H}$ where $\tilde{G} = rG$ and $\tilde{H} = rH$. And as $r \to 0$, $\tilde{U} \to 0$. Putting it all together, we have arrived at the one-dimensional problem

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & r, t \geq 0 \\ \tilde{U}\,(t = 0) = \tilde{G} & r \geq 0 \\ \partial_t\tilde{U}\,(t = 0) = \tilde{H} & r \geq 0 \\ \tilde{U}\,(r = 0) = 0 & t \geq 0 \end{cases}.$$

This is the one-dimensional wave equation on the half-line, for which we have a formula (see theorem 1.2). As $r \to 0$, $U \to u$, and so we end up with a formula for u solving the three-dimensional wave equation.

Theorem 2.2 (Kirchhoff formula). *In $d = 3$, the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u\,(t = 0) = g \\ u_t\,(t = 0) = h \end{cases}$$

is given by

$$u\,(x, t) = \fint_{\partial B(x,t)} \left[t \cdot h\,(y) + g\,(y) + \nabla g\,(y) \cdot (y - x)\right] \, dS\,(y).$$

Remarks.

(1) The wave equation in $d = 3$ is not regularizing. A more detailed description requires Sobolev spaces.
(2) The wave equation in $d = 3$ exhibits finite speed of propagation, and obeys Huygen's principle. Given compact data, u is supported in the "light cone" corresponding to the data. (D1)

3. Two-Dimensional Wave Equation

For $d = 2$, the same strategy does not apply. Instead, the idea is to look at $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$ and apply the results of the previous section. Explicitly, if $u(t, x_1, x_2)$ solves $u_{tt} - \Delta u = 0$ in $\mathbb{R} \times \mathbb{R}^2$, then $v(t, x_1, x_2, x_3) = u(t, x_1, x_2)$ solves $v_{tt} - \Delta v = 0$ in $\mathbb{R} \times \mathbb{R}^3$. This is the so-called "method of descent".

Let us be more explicit. Suppose u solves

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^2 \\ u(t=0) = g & \text{on } \mathbb{R}^2 \\ u_t(t=0) = h & \text{on } \mathbb{R}^2 \end{cases}.$$

Define $\overline{u}(t, x_1, x_2, x_3) = u(t, x_1, x_2)$, then $\overline{u}$ solves

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \overline{u}(t=0) = \overline{g} & \text{on } \mathbb{R}^3 \\ \overline{u}_t(t=0) = \overline{h} & \text{on } \mathbb{R}^3 \end{cases}.$$

So $\overline{u}$ is given by a formula, and then we can extract u. The idea is that

$$u(t, x_1, x_2) = \overline{u}(t, x_1, x_2, x_3) = \int_{\partial B_{\mathbb{R}^3}((x_1, x_2, 0), t)} f(t, y_1, y_2)\, dy$$

where f does not depend on y_3. So we can smash down the y_3-coordinate and go back to $\mathbb{R}^2$.

THEOREM 3.1 (Poisson formula). *In $d = 2$, the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^2 \\ u(t=0) = g & \text{on } \mathbb{R}^2 \\ u_t(t=0) = h & \text{on } \mathbb{R}^2 \end{cases}$$

is

$$u(x, t) = \frac{1}{2} \fint_{B(x,t)} \frac{tg(y) + t^2 h(y) + t(y - x) \cdot \nabla g(y)}{\sqrt{t^2 - |y - x|^2}}\, dy.$$

REMARK. There is still a finite propagation speed. But u is no longer supported only on a light cone. Physically, this says that if sound obeyed the wave equation in $\mathbb{R}^2$, then once sound was observed it would never go away. Huygen's principle is certanily violated here.

4. Energy Methods

Although we no longer have a maximum principle, we can still acheive uniqueness through energy methods.

THEOREM 4.1. *There exists at most one solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U \times [0, T] \\ u(t=0) = g & \text{on } U \times \{t = 0\} \\ \partial_t u(t=0) = h & \text{on } U \times \{t = 0\} \\ u = i & \text{on } \partial U \times [0, T] \end{cases}.$$

Proof. It suffices to consider the homogenous case. Multiply $u_{tt}-\Delta u=0$ by u_t, and integrate over U to get

$$\int_U u_{tt}u_t-\int_U u_t\Delta u=0.$$

Rewriting the first term and integrating by parts on the second yields

$$\frac{1}{2}\partial_t\int_U (u_t)^2+\int_U \nabla u_t\nabla u=0.$$

Integrating by parts again,

$$\frac{1}{2}\partial_t\left[\int_U (u_t)^2+|\nabla u|^2\right]=0$$

which holds for all t. Since $\int_U (u_t)^2+|\nabla u|^2=0$ at $t=0$, we're done. $\square$

Next we have an energy version of the light cone. This is a localized version of the previous theorem.

Theorem 4.2. *Let u,v solve*

$$\begin{cases}u_{tt}-\Delta u=0\\ u\,(t=0)=g_1\\ \partial_t u\,(t=0)=h_1\end{cases}$$

and

$$\begin{cases}v_{tt}-\Delta v=0\\ v\,(t=0)=g_2\\ \partial_t v\,(t=0)=h_2\end{cases}.$$

Suppose $g_1\equiv g_2, h_1\equiv h_2$ on $B\,(x,t)\subset\mathbb{R}^d$. Then $u\equiv v$ in the light cone.

Proof. Assume u solves

$$\begin{cases}u_{tt}-\Delta u=0\\ u\,(t=0)=0 & \text{on } B\,(x,t)\\ \partial_t u\,(t=0)=0 & \text{on } B\,(x,t)\end{cases}$$

and define

$$e\,(s)=\int_{B(x+s,t-s)}\left(|u_t|^2+|\nabla u|^2\right)\,dy$$

Differentiating in s gives

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}e\,(s)&=\int_B u_tu_{tt}+\nabla u\nabla u_t-\frac{1}{2}\int_{\partial B}|u_t|^2+|\nabla u|^2\\ &=\int_B u_tu_{tt}-u_t\Delta u+\int_{\partial B}u_tn\cdot\nabla u\,dS-\frac{1}{2}\int_{\partial B}|u_t|^2+|\nabla u|^2\end{aligned}$$

after integrating by parts. The first term is zero, but the second is not of one sign. But we can use Cauchy-Schwarz to guarantee the second and third together are of one sign. Applying Cauchy-Schwarz yields

$$\left|\int_{\partial B}u_tn\cdot\nabla u\right|\le\frac{1}{2}\int_{\partial B}\left(|u_t|^2+|\nabla u|^2\right)\,dS$$

and hence $\frac{d}{ds}e\,(s)\le 0$. Since $e\,(0)=0$, $e\equiv 0$ for all s.

$\square$

CHAPTER 5

First Order PDE

Lecture 8, 11/8/11

Now we study nonlinear pde. "First order" here indicates that the only derivatives which show up in the equation are first order derivatives. Previously, we derived explicit solutions as much as possible and then derive properties of the equations via those solutions. With nonlinear pde, this is rarely possible.

The setting is as follows. We have an unknown map $u : \mathbb{R}^d \to \mathbb{R}$ and a function $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. The pde is

$$F\left(\nabla u\left(x\right), u\left(x\right), x\right) = 0$$

on $\mathbb{R}^d$. A more general formulation is with $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^d$ is an open set and $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Given $\Gamma \subset U$ a subset, the boundary (initial) value problem is

$$\begin{cases} F\left(\nabla u, u, x\right) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}.$$

Where do such pde arise? The equations of continuum mechanics (e.g. fluid mechanics) are conservation laws in divergence form. A typical law looks like

$$\partial_t f + \operatorname{div} g = 0.$$

This is a first order pde. Wave propagation (Hamilton Jacobi) is also a source of first order pde.

Examples 0.3.

(1) The Eikonal equation is

$$||\nabla u|| = 1.$$

(2) The Hamilton Jacobi equation is

$$\partial_t u + H\left(\nabla u\right) = 0.$$

(3) The Bergers equation is

$$\partial_t u + u \partial_x u = 0.$$

1. Characteristics

Let $u : \mathbb{R}^d \to \mathbb{R}$ and $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Usually we indicate the arguments by writing $F = F\left(p, z, x\right)$. The pde we are trying to solve is

$$F\left(\nabla u\left(x\right), z\left(x\right), x\right) = 0.$$

The idea is to somehow replace the pde by an ode. Then it will be easier to solve.

Let us parametrize x as $x(s)$, then the unknowns are

$$\begin{cases} x(s) \\ z(s) = u(x(s)) \\ p(s) = \nabla u(x(s)) \end{cases}.$$

Supposing u is a solution, we look for a path x in $\mathbb{R}^d$ so that we can find an ode involving only $x(s), z(s), p(s)$:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{p} \end{pmatrix} = G \begin{pmatrix} x \\ z \\ p \end{pmatrix}.$$

By the chain rule,

$$\dot{z}(s) = \sum_{i=1}^{d} \dot{x}^i(s) u_i(x(s)) = \dot{x}^i u_i$$

with summation implied. Also, $p^i(s) = \partial_i u(x(s))$ so that

$$\dot{p}^i(s) = \dot{x}^j u_{ij}.$$

Now if we differentiate the pde w.r.t. the kth coordinate, we find

$$u_{ik} F_{p_i} + u_k F_z + F_{x_k} = 0.$$

Choosing $\dot{x} = \nabla_p F$, we arrive at

$$\begin{aligned} \dot{z} &= p \cdot \nabla_p F \\ \dot{p} &= -pF_z - \nabla_x F. \end{aligned}$$

Thus we have found a closed ode involving only x, z, p. Note that everything is parametrized by s, i.e. $x = x(s)$, $z = z(s)$, $p = p(s)$, and $F = F(p(s), z(s), x(s))$. We have proved the following theorem.

THEOREM 1.1 (Method of characteristics). *Suppose u solves*

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}.$$

If X solves

$$\dot{X}(s) = \nabla_p F(\nabla u(X(s)), u(X(s)), X(s)),$$

then $p(s) = \nabla u(X(s))$ and $z(s) = u(X(s))$ solve

$$\begin{cases} \dot{z} = p \cdot F_p \\ \dot{p} = -F_z p - F_x \end{cases}.$$

REMARK. Roughly, the theorem says that if you have prescribed data on $\Gamma \subset U$ then the solution u is determined locally by an ode. That is, if we know $u, \nabla u$ at some point x_0, then we can compute $u, \nabla u$ on the whole characteristic $\{X(s), s \in \mathbb{R}\}$ simply by solving an ode. It can be shown this ode is nice enough to have a unique solution (locally).

Before we see some examples, consider the possibility of using characteristics to build up a solution to the pde, starting from $\Gamma \subset U$. If all the characteristics emanating from Γ do not cross, this sounds like a plausible way to discover solutions. But if characteristics cross, we'll only get local solutions. Worse yet, if a characteristic lies tangent to Γ, then the boundary data is inherently restricted. The worst possible situation would be when Γ itself is a characteristic.

1.1. Linear Equations. The easiest case is the linear case. Let $F(p,z,x) = b(x)\cdot p + c(x)\,z$, then the pde is

$$b(x)\cdot\nabla u(x) + c(x)\,u(x) = 0.$$

Once we assume $F(p(x(s)), z(x(s)), x(s)) = 0$ on the path $x(s)$ then the ode given above becomes

$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = p\cdot b(x) = -c(x(s))\,z(s) \end{cases}.$$

Note we don't need the evolution equation for p as the ode above is closed. This is due to our assumption on F (which is justified since we are interested in solutions, afterall).

Example 1.2. Let's specialize to the transport equation,

$$\partial_t u + a\cdot\nabla_y u = 0$$

where we've set $x = (t,y)$ and $u = u(t,y)$. The ode becomes

$$\begin{cases} \dot{t}(s) = 1 \\ \dot{y}(s) = a \\ \dot{z}(s) = 0 \end{cases},$$

so u is constant on the line $\{(s, y_0 + as)\}$ for any y_0. Hence the solution is

$$u(t,y) = u(t=0, y-at).$$

This process shows us something deeper than just how to solve the transport equation. Indeed, the characteristics are the lines on which data propagates. This constrains the boundary data which can be prescribed – a well-posed problem will have all characteristics intersecting the boundary exactly once. Also notice that all the information is propagating at finite speed. And there is no smoothing.

Example 1.3. Let $U = \left\{(x_1,x_2)\;\middle|\; x_1 > 0,\ x_2 \geq 0\right\}$ and let $\Gamma = \{x_2 = 0\}\cap U$. Solve

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in U} \\ u = g & \text{on } \Gamma \end{cases}.$$

Now, $F_p = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ and so the ode is

$$\begin{cases} \dot{x}_1(s) = -x_2 \\ \dot{x_2}(s) = x_1 \\ \dot{z}(s) = z \end{cases}.$$

Integrating gives

$$\begin{cases} x_1(s) = x_1^0\cos s \\ x_2(s) = x_1^0\sin s \\ z(s) = g\left(x_1^0\right)e^s \end{cases}$$

where we've imposed the initial conditions

$$\begin{cases} x_1(0) = x_1^0 \\ x_2(0) = 0 \\ z(0) = g\left(x_1^0\right) \end{cases}.$$

Now the characteristics are arcs of circles.

Pick $(X, Y) \in U$ then the solution of the ode reaches X, Y if

$$\begin{cases} x_1^0 \cos s = X \\ x_1^0 \sin s = Y \end{cases}$$

and so with $x_1^0 = \sqrt{X^2 + Y^2}$ and $s = \arctan(Y/X)$. So the solution is

$$u(X, Y) = z(s) = g\left(x_1^0\right) e^s = g\left(\sqrt{X^2 + Y^2}\right) e^{\arctan(Y/X)},$$

which fills out the whole space.

1.2. Quasilinear Equations. The pde is *quasilinear* if $F(p, z, x)$ is linear in p, i.e. if $F(p, z, x) = b(x, z) \cdot p + c(x, z)$. Then $\nabla_p F = b(x, z)$ so the ode becomes

$$\begin{cases} \dot{x} = b(x, z) \\ \dot{z} = -c(x, z) \end{cases}.$$

As before, we'll assume $F(p, z, x) = 0$ on the path $x(s)$.

Example 1.4. Bergers equation is

$$\partial_t u + u \partial_x u = 0$$

with $t, x, u \in \mathbb{R}$. Given data g at $t = 0$ we have the initial value problem

$$\begin{cases} \partial_t u + u \partial_x u = 0 \\ u(t = 0, x) = g(x) \end{cases}.$$

(This is a simplification of the Euler equation:

$$\partial_t u + u \cdot \nabla u = -\nabla p$$

for $t \in \mathbb{R}$ and $x, u \in \mathbb{R}^d$.) Here, (t, x) plays the role of x, $b(x, z) = \begin{pmatrix} 1 \\ z \end{pmatrix}$, and $c(x, z) = 0$. So the characteristic equation is

$$\begin{cases} \dot{t}(s) = 1 \\ \dot{x}(s) = z(x(s)) \\ \dot{z}(s) = 0 \end{cases}.$$

The data is

$$\begin{cases} t(0) = 0 \\ x(0) = x_0 \\ z(0) = g(x_0) \end{cases}.$$

Integrating yields

$$\begin{cases} t(s) = s \\ x(s) = x_0 + s z(x_0) \\ z(s) = g(x_0) \end{cases}.$$

Do the characteristics fill up the whole space? Pick $(T,X) \in \mathbb{R}^2$, then

$$\begin{cases} T = s \\ X = x_0 + sz(x_0) \end{cases}$$

iff

$$X = x_0 + Tg(x_0).$$

So for general T, g the Berger's equation does not have a unique solution – the characteristics may cross! And at a crossing point, u becomes multivalued (indeed no longer C^2) so the whole framework of characteristics breaks down. But observe that if g is a decreasing function, then there exist unique global solutions. Crossing of characteristics is linked to the formation of shocks (as in fluid dynamics).

2. Local Existence of Solutions

We have seen two objections to the existence of general global solutions to first order pde. The first example had to do with choosing a bad surface Γ on which to impose initial conditions (i.e. in the transport equation). The second had to do with crossing of characteristics (i.e. in Bergers equation). But we can acheive local existence. Recall we aim to solve

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{on } U \\ u = g & \text{on } \Gamma \end{cases}$$

with $u : U \subset \mathbb{R}^d \to \mathbb{R}$, $F : \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R}$, and $\Gamma \subset U$ a smooth boundary.

Claim. In order to solve close to $x_0 \in \Gamma$, it is possible to consider that Γ is flat and equal (locally) to $\partial\mathbb{R}^d_+$.

Proof sketch. It is possible to find neighborhoods V, W of x_0, x and a smooth, bijective map $\Phi : V \ni x_0 \to W \ni x$ so that $\Phi(\Gamma \cap V) = \mathbb{R}^{d-1} \cap W$. Then set $v = u \circ \Phi^{-1}$, which solves an equation of the form

$$\begin{cases} G(\nabla v, v, x) = 0 & \text{in } W \\ u = g \circ \Phi^{-1} & \text{on } \mathbb{R}^{d-1} \cap W \end{cases}$$

iff u solves

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{on } U \\ u = g & \text{on } \Gamma \end{cases}.$$

Hence the claim. □

Now we have $x_0 \in \mathbb{R}^{d-1}$ which is naturally identified as a subset of $\mathbb{R}^d$. We want to find a neighborhood W of x_0 in $\mathbb{R}^d$ so that

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } W \\ u = g & \text{on } W \cap \mathbb{R}^{d-1} \end{cases}$$

has a solution. This is called "solving locally". And we hope to use the method of characteristics. Observe that, locally, characteristics are smooth and do not cross (an ode result). But they might lie tangent to $\mathbb{R}^{d-1}$, in which case it would be hard to prescribe arbitrary data on Γ. A most degenerate case would be if a characteristic were to lie completely in $\mathbb{R}^{d-1}$.

Definition 2.1. We say $x_0 \in \mathbb{R}^{d-1}$ is a *characteristic point* if the characteristic through x_0 is tangent to $\mathbb{R}^{d-1}$.

We want to solve $F(\nabla u, u, x) = 0$ by the method of characteristics, i.e. we want to solve

$$\begin{cases} \dot{x} = \nabla_p F \\ \dot{z} = p \cdot \nabla_p F \\ \dot{p} = -pF_z - \nabla_x F \end{cases}$$

with the data $x(0) = x_0, z(0) = u(x_0), p(0) = \nabla u(x_0)$ on $\mathbb{R}^{d-1}$. But a priori, we have no way of saying what $\partial_d u(x_0)$ is; we only know that $u = g$ on $\mathbb{R}^{d-1}$. But still we can say $\partial_i u = \partial_i g$ on $\mathbb{R}^{d-1}$ if $i \leq d-1$. For the dth derivative $\alpha = \partial_d u$, we resort to the pde and demand

$$0 = F\left((\partial_1 g)(x_0), (\partial_2 g)(x_0), \ldots, (\partial_{d-1} g)(x_0), \alpha, g(x_0), x_0\right).$$

Definition 2.2. The data $X = (\partial_1 g(x_0), \ldots, \partial_{d-1} g(x_0), \alpha, g(x_0), x_0)$ are called *admissible* if $F(X) = 0$.

Now suppose we have x_0, α so that X is admissible. Is it possible to find $\beta(x)$ so that

$$\begin{cases} F(\partial_1 g(x_0), \ldots, \partial_{d-1} g(x_0), \beta(x), g(x), x) = 0 \\ \beta(x_0) = \alpha \end{cases}$$

close to x_0? By the implicit function theorem, this is possible so long as $\frac{\partial F}{\partial p_d}(x_0) \neq 0$. Then we can extend the data locally away from x_0 and carry out the method of characteristics. is the analytic expression of the non-tangency condition from before.

Lecture 9, 11/15/11

Theorem 2.3 (Implicit function theorem). *Let $f : \mathbb{R}^{d-1}_\alpha \times \mathbb{R}_\beta \to \mathbb{R}$ be smooth and suppose $f(0,0) = 0$. If $\partial_\beta f(0,0) \neq 0$ then there exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ close to $(0,0)$ so that $f(\alpha, \beta) = 0 \iff \beta = \phi(\alpha)$.*

Now apply the theorem to our problem with

$$f(\alpha, \beta) = F(\partial_1 g(\alpha), \ldots, \partial_{d-1} g(\alpha), \beta, g(\alpha), \alpha)$$

and at $(x_0, 0)$. Then the condition $\partial_\beta f(0,0) \neq 0$ becomes

$$\partial_{p_d} F(\partial_1 g(x_0), \ldots, \partial_{d-1} g(x_0), p_d(x_0), g(x_0), x_0) \neq 0.$$

Call $(x_0, 0) \in \Gamma$ *non-characteristic* if this condition is satisfied. Then by the implicit function theorem, we find admissible data $(\partial_1 g(x), \ldots, \partial_{d-1} g(x), p_d(x), g(x), x)$ for $x \in \Gamma$ close to x_0. The geometric interpretation of x_0 being non-characteristic is that the characteristic line through x_0 is not tangent to Γ. Now we are ready to solve the pde.

Theorem 2.4. *Consider the bvp*

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } \mathbb{R}^d \\ u = g & \text{on } \Gamma = \mathbb{R}^{d-1} \subset \mathbb{R}^d \end{cases}$$

and assume Γ is non-characteristic at x_0. Assume

$$(\partial_1 g(x_0), \ldots, \partial_{d-1} g(x_0), p_d(x_0), g(x_0), x_0)$$

is admissible, so we can find admissible data for x in a neighborhood of x_0. Then there exists an open interval I of 0 and an open neighborhood $W \subset \Gamma$ of x_0 so that the solution $(X(s,x'), z(s,x'), p(s,x'))$ of

$$\begin{cases} \dot{X} = F_p \\ \dot{z} = p \cdot F_p \\ \dot{p} = -F_z p - F_x \end{cases}$$

with initial data $X(s=0,x') = x'$, $z(s=0,x') = g(x')$, $p(s=0,x') = p(x')$ is well-defined for $(s,x') \in I \times W$. Moreover, there exists a neighborhood $V \subset \mathbb{R}^d$ of x_0 so that the mapping $I \times W \to V$, $(s,x') \mapsto X(s,x')$ is bijective. Finally, for any $\overline{x} \in V$ let (s,x') solve $X(s,x') = \overline{x}$ and define $u(\overline{x}) = z(s,x')$. Then u solves the bvp in V.

Remark. This says that we can build up a solution locally by the method of characteristics. Recall we always had $z(s,x') = u(X(s,x'))$ in the method of characteristics. So the definition of u above quite natural.

Proof. Step 1. Solve the ode. Invoke a general ode result.

Step 2. $X : I \times W \to V$ is bijective. Apply the inverse function theorem. Recall for $F : \mathbb{R}^d \to \mathbb{R}^d$, if dF_0 is invertible, then F is a local diffeomorphism. So we compute $dX(s=0, x'=x_0)$:

$$\begin{aligned} \partial_s X(s,x') &= \nabla_p F(s,x') \\ \partial_{x_i} X^k(0,x') &= \delta_{ik}, \end{aligned}$$

and so

$$dX(0,x_0) = \begin{pmatrix} \partial_{p_1} F & \partial_{p_2} F & \cdots & & \partial_{p_d} F \\ 1 & & & & 0 \\ & 1 & & & \vdots \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}.$$

This is invertible as Γ is non-characteristic at x_0.

Step 3. Solve the pde. Prove that

$$F(p(s,x'), z(s,x'), X(s,x')) = 0$$

for all s, x'. It is true for $s = 0$, and

$$\begin{aligned} \partial_s F(p,z,X) &= \dot{p}\partial_p F + \dot{z}\partial_z F + \dot{X}\partial_x F \\ &= (-F_z p - F_x) F_p + (p \cdot F_p) F_z + F_p F_z \\ &= 0. \end{aligned}$$

This is not suprising, given how we derived the characteristic equations.

Step 4. Show $p(s,x') = \nabla u(X(s,x'))$. Combined with step 3, this gives the theorem. Indeed, given $\overline{x} \in V$ pick (s,x') so that $X(s,x') = \overline{x}$. Then by the previous step,

$$0 = F(p(s,x'), z(s,x'), X(s,x')) = F(\nabla u(\overline{x}), u(\overline{x}), \overline{x})$$

and so u is a solution. We admit the following claim.

Claim. The following relations hold:

$$\dot{z}(x',s)=p^j\dot{X}^j=\sum_{i=1}^{d}p^j(x',s)\,\dot{X}^j(x',s)$$
$$z_i(x',s)=p^jX_i^j.$$

Note that (s,x') can be viewed as functions of $\overline{x}$ due to the relation $X(s,x')=\overline{x}\iff(s,x')=X^{-1}(\overline{x})$. Now differentiate to get

$$\begin{aligned}\partial_{\overline{x}_i}[u(\overline{x})]&=\partial_{\overline{x}_i}[z(s(\overline{x}),x'(\overline{x}))]\\&=s_{\overline{x}_i}\dot{z}+x'^{j}_{\overline{x}_i}z_j\\&=s_{\overline{x}_i}p^k\dot{X}^k+x'^{j}_{\overline{x}_i}p^kX_j^k\end{aligned}$$

by the claim. So

$$\begin{aligned}\partial_{\overline{x}_i}[u(\overline{x})]&=p^k\left(s_{\overline{x}_i}\dot{X}^k+x'^{j}_{\overline{x}_i}X_j^k\right)\\&=p^k\partial_{\overline{x}_i}\left(X^k(s,x')\right)\\&=p^k\partial_{\overline{x}_i}\overline{x}^k\\&=p^i\end{aligned}$$

which finishes the proof.

□

This completes our discussion of local existence. Uniqueness comes for free by the method of characteristics (modulo choice of admissible data). In practice, there is usually only one choice of admissible data.

3. The Hamilton-Jacobi Equations

Let us now consider an important first order pde. The *Hamilton-Jacobi equations* are

$$\begin{cases}u_t+H(\nabla_xu,x)=0\\u(t=0,x)=g(x)\end{cases}$$

where $u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$. As usual, it is possible to restrict u to an open subset $U\subset\mathbb{R}^d$. We would like to solve these equations. The first thing to try is the characteristic method.

The characteristic equations are

$$\begin{cases}\dot{X}=F_p\\\dot{z}=p\cdot F_p\\\dot{p}=-F_zp-F_x\end{cases}.$$

We have a time variable, so make the replacements

$$\begin{aligned}X&\to(X_0,X)\\z&\to z\\p&\to(p_0,p).\end{aligned}$$

We have

$$F(p_0,p,z,X_0,X)=p_0+H(p,X)$$

and hence

$$\begin{cases} \dot{X}_0 = 1 \\ \dot{X} = \nabla_p H\left(X, p\right) \\ \dot{z} = p_0 + p \cdot \nabla_p H \\ \dot{p}_0 = 0 \\ \dot{p} = -\nabla_x H\left(X, p\right) \end{cases} .$$

Notice we can take the second and last ode together to form a closed system.

DEFINITION 3.1. If $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, then the ode

$$\begin{cases} \dot{X} = \nabla_p H \\ \dot{p} = -\nabla_x H \end{cases}$$

is called *Hamilton's equation*. The function H is called the *Hamiltonian*.

Next we introduce some basic variational principles, with the aim of studying Hamilton's equation.

3.1. Calculus of Variations. We start with a function $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, to be thought of as dual to the Hamiltonian H. (L is called the *Lagrangian*.) Label the entries as $L = L\left(x, p\right)$.

DEFINITION 3.2. The *action functional* I_t on paths $w : [0, t] \to \mathbb{R}^d$ is given by

$$I_t\left[w\right] = \int_0^t L\left(w\left(s\right), w'\left(s\right)\right)\, ds.$$

THEOREM 3.3. *Fix points $x, y \in \mathbb{R}^d$. A minimizer of I_t in the class*

$$\mathcal{A}_{x,y} = \left\{w \in C^2\left(\left[0,t\right], \mathbb{R}^d\right),\ w\left(0\right) = x,\ w\left(t\right) = y\right\}$$

satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial s}\nabla_p L\left(w\left(s\right), w'\left(s\right)\right) = \nabla_x L\left(w\left(s\right), w'\left(s\right)\right). \tag{3.1}$$

REMARK. This is called the *least-action principle*. Fermat observed in the 17th century that light obeys a form of this principle; it has since then been realized that physics as a whole obeys this principle.

PROOF. Suppose that w is a minimizer. Take $\phi \in C^\infty\left(\left[0,t\right], \mathbb{R}^d\right)$ with $\operatorname{supp}\left(\phi\right) \subset (0, t)$. Then

$$\begin{aligned} I\left(w + \epsilon\phi\right) &= \int_0^t L\left(w + \epsilon\phi, w' + \epsilon\phi'\right)\, ds \\ &= I\left(w\right) + \epsilon\left[\int_0^t \phi \cdot \nabla_x L\left(w, w'\right) + \phi' \cdot \nabla_p L\left(w, w'\right)\, ds\right] + O\left(\epsilon^2\right). \end{aligned}$$

For $I\left(w\right)$ to be minimal, we need

$$\int_0^t \phi \cdot \nabla_x L\left(w, w'\right) + \phi' \cdot \nabla_p L\left(w, w'\right)\, ds = 0.$$

Integrating by parts yields

$$\int_0^t \phi \cdot \left[\nabla_x L\left(w, w'\right) - \frac{d}{ds}\nabla_p L\left(w, w'\right)\right]\, ds = 0.$$

This holds for all ϕ, so it must be that

$$\nabla_x L(w, w') - \frac{d}{ds} \nabla_p L(w, w') = 0$$

which is Euler-Lagrange. $\square$

Lecture 10
11/22/11

EXERCISE 3.4. Check the last step of the proof.

Here is a classical application of this theorem.

EXAMPLE 3.5. Motion of a particle in a field. Let $L(x, v) = \frac{1}{2} m||v||^2 - \phi(x)$. Then $\nabla_v L = mv$ and $\nabla_x L = -\nabla \phi$, so Euler-Lagrange reads

$$m\ddot{x} = -\nabla \phi(x).$$

This is Newton's equation. Note that the Lagrangian L does not have a clear physical meaning, e.g. it is not the energy. But somehow this process results in the physics we know.

The Euler-Lagrange equation is a second order pde on $\mathbb{R}^d$. Hamilton's equations, on the other hand, constitute a first order system on $\mathbb{R}^d \times \mathbb{R}^d$. We'll introduce a new variable to make the transition from second order to first order.

DEFINITION 3.6. The *generalized momentum* is given by

$$p(s) = \nabla_v L(x(s), \dot{x}(s)).$$

Now we make an assumption. If x and p are given, we assume there is a unique v so that

$$p = \nabla_v L(x, v).$$

Furthermore, we assume the map $(x, p) \mapsto v(x, p)$ is smooth. Then we can freely switch between $(x, \dot{x})$ and (x, p).

DEFINITION 3.7. The *Hamiltonian H associated to the Lagrangian L* is given by

$$H(x, p) = p \cdot v(x, p) - L(x, v(x, p)).$$

EXAMPLE 3.8. Continuing the last example, we find $\nabla_v L(x, v) = mv$ so that $p = mv$. So then $H(x, p) = \frac{p^2}{2m} + \phi(x)$. Notice H has a physical meaning here – it is exactly the energy. The assumption that p and v are in smooth, one-to-one correspondence means here that $m \neq 0$.

All of this is interesting because of the following theorem.

THEOREM 3.9. *If x solves the Euler-Lagrange equation* (3.1) *for a given Lagrangian L, then $(x(s), p(s))$ with $p(s) = \nabla_v L(x(s), \dot{x}(s))$ solves Hamilton's equations*

$$\begin{cases} \dot{x} = \nabla_p H(x, p) \\ \dot{p} = -\nabla_x H(x, p) \end{cases}$$

where H is the Hamiltonian associated to L.

PROOF. Recall that $\dot{x}(s) = v(x(s), p(s))$. So

$$\begin{aligned} \frac{\partial}{\partial x_i} H(p, x) &= \frac{\partial}{\partial x_i} (p \cdot v(p, x) - L(v(p, x), x)) \\ &= p \cdot v_{x_i} - L_{v^k} v^k_{x_i} - L_{x_i} \\ &= -L_{x_i}(x, \dot{x}) \end{aligned}$$

since $p = \nabla_v L(x, \dot{x})$. But x solves Euler-Lagrange, and thus

$$\nabla_x H(p, x) = -\frac{d}{ds} \nabla_v L(x, \dot{x}) = -\dot{p}.$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial p_i} H(p, x) &= \frac{\partial}{\partial p_i} \left(p \cdot v(p, x) - L(v(p, x), x)\right) \\ &= v^i(p, x) + p \cdot v_{p_i} - L_{v^k} v^k_{p_i} \\ &= v^i(p, x) \end{aligned}$$

and thus

$$\nabla_p H(p, x) = v(p, x) = \dot{x}.$$

Hence the result. □

EXAMPLE 3.10. Again we go back to the classical example. In the Lagrangian picture, $L(x, v) = \frac{m}{2} v^2 - \phi(x)$ gave the Euler-Lagrange equations

$$m\ddot{x} = -\nabla \phi(x).$$

To switch to the Hamiltonian picture, we set $p = mv$. Then the Hamiltonian becomes $H(x, p) = \frac{1}{2m} p^2 + \phi(x)$ and Hamilton's equations read

$$\begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = -\nabla \phi(x) \end{cases}.$$

PROPOSITION 3.11. *$H(x(s), p(s))$ is constant along trajectories of*

$$\begin{cases} \dot{x} = \nabla_p H \\ \dot{p} = -\nabla_x H \end{cases}.$$

PROOF. Compute

$$\begin{aligned} \frac{d}{ds} H(x(s), p(s)) &= \dot{x} \cdot \nabla_x H + \dot{p} \cdot \nabla_p H \\ &= \nabla_p H \cdot \nabla_x H - \nabla_x H \cdot \nabla_p H \\ &= 0. \end{aligned}$$

Thus H is contant along trajectories. □

The relation between the Lagrangian picture and the Hamiltonian picture is best explained with convex analysis. Recall the following definitions.

DEFINITIONS 3.12. A set S is called *convex* if for all $x, y \in S$ we have that $\lambda x + (1 - \lambda) y \in S$ for $\lambda \in [0, 1]$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *convex* if the set $\{(x, y), y > f(x)\}$ is convex, or equivalently if $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$ for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$.

Now consider $H = H(p)$ and $L = L(v)$ the Hamiltonian and Lagrangian functions. Let us assume that the mapping $v \mapsto L(v)$ is convex. This is very much related to our previous assumption that the equation $p = \nabla_v L(v)$ uniquely determines v from p, and holds for all examples of interest (e.g. the classical example from before). Let us also assume that $L(v) / ||v|| \to \infty$ as $||v|| \to \infty$. This means that the graph of L grows faster than linearly.

Definition 3.13. If L is convex, its *Legendre transform* is given by

$$L^*(p) = \sup_v \{v \cdot p - L(v)\}.$$

Recall we associated a Hamiltonian H with a given Lagrangian L, via

$$H(p) = p \cdot v(p) - L(v(p)).$$

Proposition 3.14. *Suppose we are given a smooth Lagrangian L for which $v \mapsto \nabla_v L(v)$ is bijective. Then the Legendre transform of L is its associated Hamiltonian H, i.e. $L^* = H$.*

Proof. Consider the map $v \mapsto p \cdot v - L(v)$. It reaches a unique maximum, for it is a concave function which goes to $-\infty$ as the input goes to ∞. To find the maximum, differentiate in v, so demand

$$\nabla_v (p \cdot v - L(v)) = p - \nabla_v L(v) = 0.$$

This holds iff $p = \nabla_v L(v)$, so iff $v = v(p)$. Hence

$$\sup_v \{p \cdot v - L(v)\} = p \cdot v(p) - L(v(p))$$

and hence the claim. □

Theorem 3.15 (Convex duality). *Set $H = L^*$. Then,*

(1) H is convex,
(2) $\frac{H(p)}{||p||} \to \infty$ as $||p|| \to \infty$,
(3) $H^* = L$.

Proof. Proof of (1) is as follows. We have

$$H(p) = \sup_v \{p \cdot v - L(v)\}$$

which is a supremum of linear functions. So it is convex. Explicitly,

$$\begin{aligned} H(\lambda p + (1-\lambda) p') &= \sup_v \{\lambda p \cdot v + (1-\lambda) p' \cdot v - L(v)\} \\ &\leq \lambda \sup_v \{p \cdot v - L(v)\} + (1-\lambda) \sup_v \{p' \cdot v - L(v)\} \\ &= \lambda H(p) + (1-\lambda) H(p'), \end{aligned}$$

and so H is convex.

Proof of (2). Fix $R > 0$ and take $v = Rp/||p||$. Then

$$\begin{aligned} H(p) &= \sup_v \{p \cdot v - L(v)\} \\ &\geq p \cdot R\frac{p}{||p||} - L\left(R\frac{p}{||p||}\right) \\ &\geq R||p|| - \sup_{\partial B(0,R)} L \end{aligned}$$

which implies

$$\frac{H(p)}{||p||} \geq R - \frac{\sup_{\partial B(0,R)} L}{||p||}.$$

Taking $p \to \infty$ and then $R \to \infty$ yields the result.

Proof of (3). Write

$$H(p) = \sup_v \{p \cdot v - L(v)\} \geq p \cdot v - L(v)$$

for all v. So then,

$$L(v) \geq p \cdot v - H(p)$$

for all v, p and hence

$$L(v) \geq \sup_p \{p \cdot v - H(p)\} = H^*(v).$$

In the other direction,

$$\begin{aligned} H^*(v) &= \sup_p \left\{ p \cdot v - \sup_r \{p \cdot r - L(r)\} \right\} \\ &= \sup_p \inf_r \{p \cdot (v - r) + L(r)\}. \end{aligned}$$

Fix v, then as L is convex, there exists $s \in \mathbb{R}^d$ so that

$$L(r) \geq L(v) + s \cdot (r - v).$$

Taking $p = s$,

$$H^*(v) \geq \inf_r \{s \cdot (v - r) + L(r)\} \geq L(v)$$

and hence the result. $\square$

3.2. The Hopf-Lax Formula. Now we solve the Hamilton-Jacobi equations

$$\begin{cases} \partial_t u - H(\nabla u) = 0 \\ u(t = 0) = g \end{cases}$$

via the characteristic equations

$$\begin{cases} \dot{X} = \nabla_p H \\ \dot{p} = 0 \\ \dot{z} = \nabla_p H(p) \cdot p - H(p) \end{cases}.$$

Recall the full characteristics take the form (X_0, X, z, p_0, p), but the ode above is enough (it is a closed system). The initial conditions are $X(0) = x$, $p(0) = \nabla g(x)$, and $z(0) = g(x)$. It is possible to integrate:

$$\begin{cases} X(s) = x + s\nabla_p H(\nabla g(x)) \\ p(s) = \nabla g(x) \\ z(s) = \int_0^s [\nabla_p H(p) \cdot p - H(p)] \, dt \end{cases}.$$

But the mapping $(x, s) \mapsto X(s) = x + s\nabla_p H(\nabla_g(x))$ is only bijective for small s. (Characteristics may cross.) Can we work around this?

We aim to prolong u beyond where $(x, s) \mapsto X(s)$ ceases to be bijective. Observe that the characteristic equation can be written as

$$\begin{cases} \dot{X} = \nabla_p H(p) \\ \dot{p} = 0 \\ \dot{z} = L\left(\dot{X}\right) \end{cases}.$$

The idea is to recast these in a variational manner. Before characteristics cross, u will satisfy

$$u(x, t) = \inf_{X(t) = x} \left\{ \int_0^t L\left(\dot{X}(s)\right) ds + g(X(0)) \right\}.$$

The characteristic equations show that $\dot{X}$ is constant for the minimizing path. Hence the minimizing path looks like $\gamma(s) = y + \frac{s}{t}(x-y)$, where $y = X(0)$. So the variational problem becomes

$$u(x,t) = \inf_{X(t)=x} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

This is called the *Hopf-Lax formula*. One can prove that this formula admits a solution valid for any time. In what sense will the extended solution be a "solution" to the pde? Although it cannot be a classical solution, one can prove the following:

- u will be Lipschitz continuous.
- Hence u will be a.e. differentiable.
- At the points where a derivative exists, u satisfies $u_t + H(\nabla u) = 0$.
- $u(t=0) = g$.

Hence u solves the pde a.e. and satisfies the initial data.

Lecture 11
11/29/11

4. Conservation Laws

Let $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$ and $F : \mathbb{R} \to \mathbb{R}$ be given. A *conservation law* is any pde of the form

$$\begin{cases} \partial_t u + \nabla_x (F(u)) = 0 \\ u(t=0) = g \end{cases}.$$

There are conservation laws for vector-valued u, but they are not understood in general. In fact, the conservation laws we consider here are very close to the best that we can say about such equations.

First we apply the method of characteristics. The characteristics (X, z, p) are of the form (T, X, z, p_T, p) where $F = p_T + F'(z)\,p$. The characteristic equations are

$$\begin{cases} \dot{T} = 1 \\ \dot{X} = F'(z) \\ \dot{z} = 0 \end{cases}.$$

Thus if u is a smooth solution, then it must be constant along lines of the type $(s, x_0 + sF'(u))$. But as we saw in Burger's equation

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0$$

that the solution does not remain smooth for all time. Instead, shocks form in finite time. So we need a more general notion of solution.

4.1. Integral Solutions.

A *classical solution* of a pde is a map u which is everywhere differentiable and satisfies the pde pointwise. Assume now that u is a classical solution of

$$\begin{cases} \partial_t u + \nabla_x (F(u)) = 0 \\ u(t=0) = g \end{cases}.$$

Take $v \in C_0^\infty\left([0,\infty] \times \mathbb{R}^d\right)$, multiply the pde by v, and integrate on $\mathbb{R}_x \times \mathbb{R}_t^+$. So

$$0 = \int_0^\infty \int (\partial_t u + \nabla_x F(u))\, v \, dx dt.$$

Now we integrate by parts in both t and x to get

$$\begin{aligned} 0 &= -\int u(t=0)\,v(t=0)\,dx - \int_0^\infty\int uv_t\,dxdt - \int_0^\infty\int F(u)\,v_x\,dxdt \\ &= -\int g\cdot v(t=0)\,dx - \int_0^\infty\int uv_t\,dxdt - \int_0^\infty\int F(u)\,v_x\,dxdt \end{aligned}$$

where we have used the initial data. So we make the definition

DEFINITION 4.1. u is said to be an *integral (weak) solution* if it satisfies

$$\int g\cdot v(t=0)\,dx + \int_0^\infty\int [uv_t + F(u)\,v_x]\,dxdt = 0$$

for any $v \in C_0^\infty\left([0,\infty]\times\mathbb{R}^d\right)$. Such solutions are also said to be *solutions in the sense of distributions*.

PROPOSITION 4.2. *If $u \in C^1$, then it is a classical solution iff it is an integral solution.*

If u is piecewise C^1 and C^0, then it may be continuous across some smooth hypersurface $\Gamma \subset \mathbb{R}^d$ while not being C^1 across Γ. But we still have

PROPOSITION 4.3. *If $u \in C^0$ is piecewise C^1, then it is an integral solution iff it satisfies the equation pointwise away from Γ.*

PROOF SKETCH. Take v with support away from $t = 0$. Then integrating by parts away from Γ we find

$$\int_0^\infty\int [uv_t + F(u)\,v_x]\,dxdt = -\int_0^\infty\int [u_t + \nabla_x F(u)]\,v\,dxdt$$

since the boundary terms cancel by continuity. But $u_t + \nabla_x F(u) \equiv 0$ away from Γ so the RHS is zero. □

Suppose now u is piecewise C^1 but not continuous across Γ. We specialize to the case $d = 1$, so $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. (D1) Suppose u is C^1 away from $\Gamma = \{x = s(t)\}$, then if u is an integral solution u will also be a classical solution away from Γ. Let u_l and u_r be the restrictions of u to the domain on the left of Γ, Γ_l, and to the domain on the right of Γ, Γ_r. We ask: when will u be an integral solution?

Suppose u is an integral solution, then for all v supported away from $t = 0$ we find

$$\begin{aligned} 0 &= \int\int [v_t u + F(u)\,v_x]\,dxdt \\ &= \int\int_{\Gamma_l} + \int\int_{\Gamma_r} \\ &= -\int\int_{\Gamma_l} [u_t + \nabla_x F(u)]\,v\,dxdt - \int\int_{\Gamma_r} [u_t + \nabla_x F(u)]\,v\,dxdt \\ &\quad + \int_\Gamma \left[u_l\nu^2 + F(u_l)\,\nu^1\right] v\,dl - \int\int \left[u_r\nu^2 + F(u_r)\,\nu^1\right] v\,dl \end{aligned}$$

where ν is the normal to Γ. Since the first two integrals are zero (u is a classical solution away from Γ) we find

$$0 = \int_\Gamma \left[\nu^2(u_l - u_r) + \nu^1(F(u_l) - F(u_r))\right] v\,dl = 0$$

for all v. Therefore,

$$\nu^2 (u_l - u_r) + \nu^1 (F(u_l) - F(u_r)) = 0$$

itself. Since we have parametrized Γ by $x = s(t)$, we find $\nu^2/\nu^1 = -\dot{s}(t)$ and hence

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r)\,. \tag{4.1}$$

This last relation is called the *Rankine-Hugoniot condition*.

Theorem 4.4. *u is an integral solution iff u is a classical solution away from Γ and* (4.1) *holds along* Γ.

Remark. In situations where the pde comes from physics, it turns out that condition (4.1) represents a physically correct requirement on solutions.

4.2. Burger's Equation. We only look at $d = 1$. Recall Burger's equation is

$$\begin{cases} u_t + \frac{1}{2}\left(u^2\right)_x = 0 \\ u(t=0) = g \end{cases}$$

Solutions u are constant along lines of with slope u, i.e. of the form $\{(s, x_0 + su)\}$. So the characteristics have slope $g(x_0)$ where $g = u(t=0)$, and thus can cross. (D2)

Consider u given by $u_l \equiv a$ and $u_r \equiv b$. (D3) Then (4.1) becomes

$$\frac{1}{2}\left(a^2 - b^2\right) = \dot{s}(a-b)$$

and so we require $\dot{s} = (a+b)/2$. Therefore, u will be an integral solution iff $\Gamma = \left\{x = \frac{1}{2}(a+b)t + x_0\right\}$. So

$$u(x,t) = \begin{cases} a & x < t\left(\frac{a+b}{2}\right) \\ b & x > t\left(\frac{a+b}{2}\right) \end{cases}$$

is a solution.

Now consider the initial data

$$g(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}.$$

The characteristics look like (D4). The solution above works to the right of $x = 1$. Working out the details shows that

$$u(x,t) = \begin{cases} 1 & x \le t \\ \frac{1-x}{1-t} & t \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

is a solution as well. (D5) We can use this last solution on $t \le 1$, and then glue to the solution above to find a global solution (this requires a gluing lemma).

Have we finished the whole story? There is a big problemhere: non-uniqueness. Consider Berger's equation with the initial data

$$g(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 0 \end{cases}.$$

Here are two possible solutions. We have the *shock wave*

$$u(x,t) = \begin{cases} 0 & x < \frac{t}{2} \\ 1 & x > \frac{t}{2} \end{cases},$$

and the *rarefaction wave*

$$u(x,t) = \begin{cases} 1 & x > t \\ \frac{x}{t} & 0 < x < t \\ 0 & x < 0 \end{cases}.$$

Another possibility would be that for some time there is a shock wave, and then it becomes a rarefaction wave.

Can we find a condition to exclude (non-physical) solutions? (D6) Our criterion is as follows: An *entropy solution* is a solution such that at any shock, characteristics points towards the shock. This will resolve the problem is non-uniqueness. There are other reasons that we choose entropy solutions:

- Information propagates along characteristics. Solution which are not entropy solutions have information propagating away from a shock. This is non-physical.
- Smooth solutions are better than discontinuous solutions.
- The entropy criterion ensures that the solution is stable (w.r.t. the data and the equation).

In general, a shock will be entropic iff $F'(u_l) > \dot{s} > F'(u_r)$, i.e. iff

$$F'(u_l) > \frac{F(u_l) - F(u_r)}{u_l - u_r} > F'(u_r).$$

This is known as the *Lax condition*.

Example 4.5. Let

$$g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

be the initial data. We look for the entropy solution. For Burger's equation, the Lax condition becomes

$$u_l > \frac{1}{2}(u_l + u_r) > u_r,$$

so the only entropic shocks are those such that $u_l > u_r$. Gluing the rarefaction wave and the shock gives

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & x < t \\ 1 & t < x < 1 + \frac{t}{2} \\ 0 & x > 1 + \frac{t}{2} \end{cases}.$$

The two meet at $t = 2$. Past $t = 2$, we look for a solution of the type

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < s(t) \\ 0 & x > s(t) \end{cases}.$$

The Rankine-Hugoniot condition becomes

$$\begin{cases} \dot{s}(t) = \frac{1}{2}\frac{s(t)}{t} \\ s(2) = 2 \end{cases}$$

and so $s(t) = \sqrt{2t}$. Therefore an entropic solution past $t = 2$ is

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < \sqrt{2t} \\ 0 & x > \sqrt{2t} \end{cases}.$$

Lecture 12
12/6/11

Observe that for Burger's equation, the Lax condition simply becomes $u_l > u_r$. This is true in a more general setting.

Proposition 4.6. *If F is strictly convex, then a shock Γ is entropic iff $u_l > u_r$ across Γ.*

Next, we go back to the Hamilton-Jacobi equations in an effort to understand why the entropy condition gives uniqueness.

4.3. The Lax-Oleinik Formula. Integrate the relations

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}$$

from 0 to x to get

$$\begin{cases} U_t + F(U_x) = F(U_x)(0) \\ U(t=0) = h \end{cases}$$

where $U = \int_0^x u\,dx$. This is almost Hamilton-Jacobi, expect for the undertermined integration constant. Alternatively, we can start with Hamilton-Jacobi

$$\begin{cases} U_t + F(U_x) = 0 \\ U(t=0) = h \end{cases}$$

and then differentiate w.r.t. x to get

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}$$

where $u = \frac{d}{dx}U$, $g = \frac{d}{dx}h$.

The steps above suggest that we can go freely between the conservation law and the Hamilton-Jacobi equations. Formally, at least, the above implies the *Lax-Oleinik formula*

$$u(x,t) = \frac{\partial}{\partial x}\left(\min_y \left[tL\left(\frac{x-y}{t}\right) + h(y)\right]\right)$$

with $L = F^*$ (for convex F). The next theorem justifies all of this.

Theorem 4.7. *Let F be strictly convex and $g \in L^\infty$.*

(1) The Lax-Oleinik formula gives

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right)$$

for almost every x, where $G = \left(\frac{d}{dx}F\right)^{-1}$, and where $y(x,t)$ is the point at which

$$Z_{x,t} : y \mapsto tL\left(\frac{x-y}{t}\right) + h(y)$$

reaches its minimum (which is unique for almost every x).

(2) *u is an integral solution of*

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}.$$

(3) *u satisfies the estimate*

$$u(x+z,t) - u(x,t) \leq \frac{C}{t}z \tag{4.2}$$

for $t, z > 0$, $x \in \mathbb{R}$.

(4) *u is unique amongst solutions satisfying* (4.2)*.*

Remark. (3) says in particular that if u contains a shock, then $u_r < u_l$ and hence the shock is entropic. To see this, send $z \to 0$ in relation (4.2) and argue that the sign of equality cannot hold otherwise there would be no shock.

Example 4.8. Let us see what the theorem says for Burger's equation, where $F(u) = u^2/2$. Then $u(x,t) = (x - y(x,t))/t$ where $y(x,t)$ minimizes

$$y \mapsto \frac{(x-y)^2}{2t} + h(y).$$

We will sketch a proof in the case of Burger's equation, where $F(u) = u^2/2$.

Proof sketch. Step 1. Fix $x_1 < x_2$, then there exists y_1 so that $Z_{x,t}(y)$ is minimized at y_1.

Claim. If $y < y_1$, then

$$\frac{(x_2-y_1)^2}{2t} + h(y_1) < \frac{(x_2-y)^2}{2t} + h(y).$$

The claim is easy to show. Indeed, if $x_2 = x_1$ then the statement holds as an inequality. Differentiating both sides w.r.t. x_2 gives that $\frac{d}{dx_2}LHS < \frac{d}{dx_2}RHS$.

Step 2. Set $y(x,t) = \inf\left\{y \,\middle|\, Z_{x,t}(y) \text{ is minimal}\right\}$. The claim implies that $y(x,t)$ is increasing in x. So the map $x \mapsto y(x,t)$ has at most a countable number of points of discontinuity. Where continuous, the minimum is reached at a single point. So $y(x,t)$ is unique except for a countable number of points, and so u is well-defined almost everywhere.

Step 3. The formula for Hamilton-Jacobi reads

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial x}\left(\min_y \left[\frac{(x-y)^2}{2t} + h(y)\right]\right) \\ &= \frac{\partial}{\partial x} Z_{x,t}(y(x,t)) \\ &= Z'_{x,t}(y(x,t))\frac{\partial y}{\partial x} + \frac{\partial Z}{\partial x}(y(x,t)) \\ &= \frac{x - y(x,t)}{t} \end{aligned}$$

as $Z'_{x,t}(y(x,t))\frac{\partial y}{\partial x}=0$.

Step 4. u is an integral solution. This is proved via Hamilton-Jacobi.

Step 5. For $z>0$ write

$$\begin{aligned} u(x,t) &= \frac{x-y(x,t)}{t} \\ &\geq \frac{x-y(x+z,t)}{t} \\ &= \frac{x+z-y(x+z,t)}{t}-\frac{z}{t} \\ &= u(x+z,t)-\frac{z}{t} \end{aligned}$$

which proves relation (4.2).

Step 6. Uniqueness. This is not trivial.

□

We end with two applications of the Lax-Oleinik formula.

4.3.1. *Riemann's Problem.* Consider solving

$$\begin{cases} u_t+\nabla_x F(u)=0 \\ u(t=0)=g \end{cases}$$

where

$$g(x)=\begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}.$$

Assume that F is strictly convex. The following theorem holds.

THEOREM 4.9. *Given the above,*

(1) If $u_l>u_r$, the unique entropy solution is

$$u(x,t)=\begin{cases} u_l & \frac{x}{t}<\frac{F(u_l)-F(u_r)}{u_l-u_r} \\ u_r & \frac{x}{t}>\frac{F(u_l)-F(u_r)}{u_l-u_r} \end{cases}.$$

(2) If $u_l<u_r$, the unique entropy solution is

$$u(x,t)=\begin{cases} u_l & \frac{x}{t}<F'(u_l) \\ G\left(\frac{x}{t}\right) & F'(u_l)<\frac{x}{t}<F'(u_r) \\ u_r & \frac{x}{t}>F'(u_r) \end{cases}$$

where $G=\left(\frac{d}{dx}F\right)^{-1}$.

REMARK. This says that the only entropy solutions are (1) shocks and (2) rarefaction waves.

We will sketch a proof in the case of Burger's equation.

PROOF SKETCH. Proof of (1). Suppose we have a shock solution so that $u_l>u_r$. Then (4.2) holds, so (1) is implied by the previous theorem.

Proof of (2). For Burgers,

$$u(x,t)=\begin{cases} u_l & \frac{x}{t}<u_l \\ \frac{x}{t} & u_l<\frac{x}{t}<u_r \\ u_r & \frac{x}{t}>u_r \end{cases}.$$

Check that u is continuous, u is a classical solution where smooth, u is an integral solution, and u satisfies (4.2). Then the previous theorem implies u is the unique entropy solution. $\square$

4.3.2. *Large Time Behavior.* Consider

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}$$

with F strictly convex. We claim that if $g \in C_0^\infty(\mathbb{R})$, then $\max_{x\in\mathbb{R}} \{u(\cdot,t)\}$ tends towards zero as $t \to \infty$. So u dies at $t = \infty$. However,

EXERCISE 4.10. Show that $\int u(x,t)\, dx$ is constant for $t > 0$.

What is happening here? It turns out that the mechanism behind the decay of u is shocks. Consider that without shocks, the maximum remains constant as u is constant along characteristics.

THEOREM 4.11. *Let u be the entropy solution of*

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}$$

where F is strictly convex and $g \in L^1 \cap L^\infty$. Then

$$|u(x,t)| \lesssim \frac{1}{\sqrt{t}}.$$

We will prove the result for Burger's equation.

PROOF. The Lax-Obeiniek formula gives

$$u(x,t) = \frac{x - y(x,t)}{t}$$

where $y(x,t)$ satisfies

$$Z_{x,t}(y(x,t)) = \min_y \{Z_{x,t}(y)\} = \frac{(x-y)^2}{2t} + h(y).$$

Observe that

$$Z_{x,t}(x) \le \sup|h| = M < \infty,$$

so then $Z_{x,t}(y(x,t)) \le M$ and here

$$\frac{(x - y(x,t))^2}{2t} \le 2M.$$

Hence

$$|x - y(x,t)| \le 2\sqrt{M}\sqrt{t}$$

and in conclusion

$$|u(x,t)| \le \frac{|x - y(x,t)|}{t} \le \frac{2\sqrt{M}}{\sqrt{t}}.$$

The proof in general is not that much different from above.

$\square$

To understand the asymptotic behavior, we introduce the notion of an "N-wave".

Definition 4.12. The *N-wave* with parameters p, q, d, σ is given by

$$N(x,t) = \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & -\sqrt{pdt} < x - \sigma t < \sqrt{qdt} \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 4.13. *Let u be the entropic solution of*

$$\begin{cases} u_t + \nabla_x F(u) = 0 \\ u(t=0) = g \end{cases}.$$

Let N be the N-wave with parameters $p = -2\min_y \left\{\int_{-\infty}^{y} g\right\}$, $q = 2\max_y \left\{\int_{y}^{\infty} g\right\}$, $\sigma = F'(0)$, and $d = F''(0)$. Then

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|\, dx \lesssim \frac{1}{t}.$$