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The stabilization and K-theory of pointed derivators

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by

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ABSTRACT OF THE DISSERTATION

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This thesis is concerned with two disparate results in the field of abstract homotopy theory, treated through the lens of derivators. In Chapter 2, we recall essential results in the theory of derivators, mostly drawn from [Gro13], but provide new proofs in many cases. We do this in order to expand results on derivators to results on half derivators, a weaker notion that is required for many examples of interest. In Chapter 3, we revisit and improve Alex Heller's results on the stabilization of derivators in [Hel97], recovering his results entirely. Along the way we give some details of the localization theory of derivators and prove some new results in that vein. We are able to give a 2-categorical statement of stabilization that should allow for future applications to derivator K-theory, which was the impetus for the study. In Chapter 4, we answer questions in derivator K-theory asked by Muro and Raptis in [MR17]. We define a new class of pointed derivators suitable for K-theory and prove some interesting properties about these per se. We then prove that derivator K-theory satisfies additivity and has the structure of an infinite loop space. We anticipate being able to prove properties of derivator K-theory using the theorems herein.

The dissertation of Ian Alexander Coley is approved.

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To Love's Eternal Glory

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CHAPTER 1

Introduction

Algebraic K-theory is a general tool for understanding complicated mathematical objects arising in homotopy theory, algebraic geometry, differential topology, representation theory, and other fields. Since Quillen's seminal work [Qui73], the field of algebraic K-theory has enjoyed incredible popularity and expansion beyond abelian or exact categories. Waldhausen in [Wal85] set the tone for how K-theory would be constructed for more and more general objects. A common philosophy is that, if we expand the class of objects on which K-theory is defined, we should make sure that our new definitions agree with the old. Waldhausen made sure this was the case when he defined his K-theory of categories with cofibrations and weak equivalences.

A stumbling block, however, was including triangulated categories into Waldhausen's framework. Defined first by Verdier in his doctoral thesis [Ver96] in 1963, triangulated categories are invaluable in the study of homological algebra and homotopy theory. The bounded derived category associated to an exact or abelian category has a natural triangulated structure. The homotopy categories of both ordinary and G-equivariant spectra are similarly naturally triangulated.

Neeman in the 1990's published a series of papers on the K-theory of triangulated categories starting with [Nee97a] and [Nee97b]. There are a number of interesting properties of his construction, but we do not mention them here. In the years following his publications, various authors proved that a satisfactory functorial construction would never be possible, in the following sense. Starting from an exact category, we can take its K-theory via Quillen's or Waldhausen's definition, or pass to its bounded derived category and take its triangulated K-theory. If triangulated K-theory were to extend Quillen's K-theory, these two constructions would give the same K-groups.

Schlichting in [Sch02] gives a general argument showing that we should not expect the triangulated category to retain all K-theoretic information. Specifically, he constructs two Waldhausen categories $\mathcal{W}_1$ and $\mathcal{W}_2$, arising from abelian categories of modules over a commutative ring, such that their homotopy (triangulated) categories are equivalent but their Waldhausen K_4 differs. However, on the triangulated side, each homotopy category $\mathrm{Ho}(\mathcal{W}_i)$ appears as a Verdier localization of the same category with equivalent localising subcategories. This leads to a contradiction between two desirable properties for K-theory: agreement and localization.

Schlichting's result points to the need for a richer structure for homotopy theory than triangulated categories alone. This is not a new idea; from our first homological algebra class, we learn that the cone construction in a triangulated category is non-functorial. The slogan 'unique up to unique isomorphism', central to how we approach category theory, abandons us. There are a few different ways to give ourselves more data to work with.

In some senses, the best replacement for triangulated category is *stable* $(\infty,1)$*-categories*, i.e. higher-categorical triangulated categories, particularly for algebraic K-theory. Recent work of Blumberg-Gepner-Tabuada in [BGT13] proves that algebraic K-theory is the universal additive invariant of a stable (small) ∞-category. This extends the origins of K-theory precisely; the Grothendieck group of a commutative monoid is the universal abelian group such that any additive invariant on the monoid must factor through it. However, there are reasons to mistrust $(\infty,1)$-categories: the literature is daunting and there are competing (though equivalent) models. Though the theory of $(\infty,1)$-categories is ideal for universal constructions, it is often difficult to know concretely what has been constructed. In the example of algebraic K-theory, there is an $(\infty,1)$-category of 'non-commutative motives' in which algebraic K-theory is found – but the rest of the category is quite mysterious.

There are lower-categorical tools that work well and do not have these drawbacks. An early tool in studying triangulated categories, developed by Quillen in [Qui67] before his work on algebraic K-theory, is that of *model categories.* A model category is the data of a category we wish to treat homotopically and extra information allowing us to pass from the rigid structure to the homotopy category. A solution to the non-functoriality of the cone can be solved in such a framework. For nice enough model categories $\mathcal{M}$, the category of arrows $\operatorname{Ar}\mathcal{M}$ inherits a compatible model category structure. We can define the cone of a morphism before passing to the homotopy categories, i.e. $\operatorname{Ho}(\operatorname{Ar}\mathcal{M}) \to \operatorname{Ho}(\mathcal{M})$ rather than $\operatorname{Ar}\operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(\mathcal{M})$. If we knew only the category $\operatorname{Ho}(\mathcal{M})$, this first approach will not be possible, so in this sense we have given ourselves more to work with.

Let us interpret this in the triangulated setting. Let $\mathcal{A}$ be an abelian category. Then the arrow category $\operatorname{Ar}\mathcal{A}$ is still abelian, so we can take its bounded derived category $D^b(\operatorname{Ar}\mathcal{A})$, which we can think of as homotopy classes of maps of chain complexes. Then we can define the cone construction as an exact functor of abelian categories $C^b(\operatorname{Ar}\mathcal{A}) \to C^b(\mathcal{A})$ *before* we invert quasi-isomorphisms. We still have a functor upon passing to the derived category, and so have a functorial cone construction with a new domain. However, there is a forgetful functor $D^b(\operatorname{Ar}\mathcal{A}) \to \operatorname{Ar}D^b(\mathcal{A})$ which takes a homotopy class of a map to a map of homotopy classes. While $D^b(\operatorname{Ar}\mathcal{A})$ is triangulated, $\operatorname{Ar}D^b(\mathcal{A})$ is not, but this functor can be shown to be full and essentially surjective (and almost never faithful). We have constructed a cone *functor* because we had access to $\mathcal{A}$ itself and not just the triangulated category $D^b(\mathcal{A})$ and thus were able to build an auxiliary diagram category $\operatorname{Ar}\mathcal{A}$ to fill in the gaps in information.

This is our slogan: we would like to study not only a triangulated category, but a whole system of triangulated diagram categories. An equivalence of homotopy categories as in [Sch02] does not necessarily give rise to an equivalence of systems, and thus we are able to better distinguish distinct homotopy theories. Grothendieck in [Gro90] coined the term *derivator* for a system of derived categories, and this is the framework in which we will work to address questions about the K-theory of triangulated categories. At this point, we can give an overview of this thesis and state our main results.

In Chapter 2, we address the theory of derivators, developed initially (under different names) by Heller in [Hel88], Grothendieck in [Gro90], and (in the triangulated setting) Franke in [Fra96]. In brief, a derivator represents an abstract bicomplete homotopy theory; we attach the adjective *triangulated* to a derivator when it represents a stable (bicomplete) homotopy theory. We detail the fundamental proof techniques used in the theory of derivators, and in particular demonstrate the diagrammatic flavor which is unique to this field. Such proofs have been well-articulated recently by Moritz Rahn (né Groth) in [Gro13]. We give new proofs in many cases in order to develop a more robust theory of *half derivators*, i.e. ones representing homotopy theories that may not be bicomplete, but still admit many limits or colimits. Our results are used significantly throughout the course of this thesis, but are less interesting to reiterate without context.

The results in Chapter 2 allow us to proceed with our investigation of derivator K-theory. The K-theory of triangulated derivators was defined by Maltsiniotis in [Mal07] and Garkusha in [Gar06] and [Gar05], and revisited by Muro and Raptis in [MR17]. We will give a more complete picture of the development of this field in Chapter 4. Muro-Raptis proved that the definition of K-theory still makes sense for derivators which are not triangulated, and form a class which we call *left pointed derivators*. Not every morphism of left pointed derivators will induce a map on K-theory, but we prove that it is easy to understand those that do. We can summarize the work of §4.2 as follows:

Main Result 1. Let $\mathbb{D}, \mathbb{E}$ be left pointed derivators on $\mathbf{Dir_f}$, the 2-category of finite direct categories. Then a morphism of derivators $\Phi\colon \mathbb{D} \to \mathbb{E}$ induces a map on derivator K-theory if and only if it is cocontinuous. Moreover, every morphism of derivators arising from a functor in $\mathbf{Dir_f}$, i.e. a pullback, left Kan extension, or right Kan extension by zero, is cocontinuous.

Having established the morphisms at our disposal, we proceed with the proof of the additivity of derivator K-theory. Cisinki and Neeman proved that the K-theory of triangulated derivators satisfies a form of additivity in [CN08], but their proof involves Neeman's theory of regions and, in particular, does not admit an obvious analogy in the non-triangulated situation.

Main Result 2 (Theorems 4.4.5 and 4.4.19)**.** Let $\mathbb{D}$ and $\mathbb{E}$ be left pointed derivators. Then the following are equivalent:

(1) The map

$$\mathbb{D}_{\text{cof}} \xrightarrow{(0,0)^* \times (1,1)^*} \mathbb{D} \times \mathbb{D}$$

induces a homotopy equivalence on derivator K-theory, where $\mathbb{D}_{\mathrm{cof}}$ is the left pointed derivator of cofiber sequences in $\mathbb{D}$.

(2) If $\Xi\colon \mathbb{D} \to \mathbb{E}_{\mathrm{cof}}$ is a cofibration morphism of derivators, then there exists a homotopy between the target of the cofibration morphism and the source plus quotient. Specifically,

$$K(T) \simeq K(S) \sqcup K(Q) \ (\cong K(S \sqcup Q))$$

where $\sqcup$ denotes the coproduct.

The first statement is the form of additivity proven by Cisinski-Neeman and conjectured by Maltsiniotis. The second statement is the one proven in Theorem 4.4.19 and uses techniques of Grayson in [Gra11] that have a more diagrammatic flavor appropriate for general derivator theory. We are able to obtain as a corollary to additivity the delooping of the K-theory space $K(\mathbb{D})$, and so conclude that $K(\mathbb{D})$ is an infinite loop space for a general left pointed derivator $\mathbb{D}$, which was not known in any cases before. This answers two questions of Muro-Raptis posed in [MR17].

The third chapter of this thesis is motivated by another question of Muro-Raptis in that same paper: Is derivator K-theory of a left pointed derivator invariant under an appropriately defined notion of stabilization which would produce a triangulated derivator? The stabilization of derivators was developed by Heller in [Hel97], though he does not use the term ‘derivator’. However, Heller rarely gives more than a cursory proof of his main theorems and propositions. Therefore it was impossible to begin approaching the technical question asked by Muro-Raptis without revisiting (and revising) Heller’s work and translating it into modern language. This culminated in [Col19]. We have placed these results in Chapter 3 because, though motivated by the work of Chapter 4, they have relatively little to do with K-theory.

The first main result in Chapter 3 is a general statement about the localization of pointed derivators. We first describe localizations of derivators in a way different from [Lag17]; rather than inverting a class of maps, we construct adjoints to fully faithful inclusions of prederivators. In order to construct a particularly useful (co)localization of derivators for our purposes, we develop the following useful result:

Main Result 3 (Construction 3.3.5)**.** For any functor $u\colon J \to K$, there is a morphism of derivators $\mathbb{D}^J \to \mathbb{D}^{J\times[1]}$ given by

$$X \mapsto (u_! u^* X \to X).$$

That is, the counit of the $(u_!, u^*)$ adjunction may be constructed coherently. The unit of the adjunction (u^*, u_*) can also be built coherently in an analogous way.

Since coherence is of the utmost importance in derivator constructions, this is a broadly desirable result. We use it to construct the derivator of prespectrum objects associated to a pointed derivator $\mathbb{D}$. From there, we prove that we can localize onto the stable Ω-spectrum objects, which recovers Heller's result. In the event that our original derivator $\mathbb{D}$ admits a mild hypothesis about the commuting of certain colimits with limits, which we call regular, we obtain the following theorem.

Main Result 4 (Theorem 3.6.15)**.** There is a pseudofunctor $\mathrm{St}\colon \mathbf{Der}_! \to \mathbf{StDer}_!$ associating to any regular pointed derivator $\mathbb{D}$ a stable derivator $\mathrm{St}\,\mathbb{D}$. This pseudofunctor is a localization of 2-categories, i.e. it admits a fully faithful right adjoint. In particular, for any stable derivator $\mathbb{S}$, there is an equivalence of categories of cocontinuous morphisms

$$\mathrm{stab}^*\colon\ \mathrm{Hom}_!(\mathrm{St}\,\mathbb{D}, \mathbb{S}) \to \mathrm{Hom}_!(\mathbb{D}, \mathbb{S})$$

induced by precomposition with a cocontinuous morphism $\mathrm{stab}\colon \mathbb{D} \to \mathrm{St}\,\mathbb{D}$.

Not every stable derivator is triangulated, but if our original pointed derivator $\mathbb{D}$ arose from a model category or abelian category as above, the resulting $\mathrm{St}\,\mathbb{D}$ will be triangulated. We will explain this distinction more in Chapter 2.

Our statement of the stabilization of derivators recalls similar results in $(\infty, 1)$-category theory. Lurie in [Lur16, §1.4] describes stabilization for a huge class of ∞-categories, with stronger results for ∞-categories admitting nicer properties. His result that most resembles ours is the following:

Theorem (Corollary 1.4.4.5, [Lur16])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable. Then precomposition with $\Sigma^\infty_+ \colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ induces an equivalence of ∞-categories of left adjoint functors

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

Here, Σ^∞_+ is Lurie's stabilization functor.

Lurie relies on an adjoint functor theorem in order to prove this equivalence, something we both do not have access to and do not require. Moreover, any $(\infty, 1)$-category gives rise to a (pre)derivator, considered most thoroughly in recent work by Carlson. Our stabilization result subsumes Lurie's, at least for small quasicategories, which embed 2-fully faithfully into the category of prederivators by [Car18, Theorem 2.1].

Unfortunately we are unable to give a positive or negative answer to Muro-Raptis' original question about the invariance of K-theory under stabilization, but were able to (re)prove that there is a stabilization of the sort the question requires. We hope to address this in future post-doctoral work.

CHAPTER 2

The theory of derivators

2.1 2-categorical notions

Before giving the definition of a derivator, we will need to set up the categorical context in which we are working. We will assume the reader is familiar with basic definitions in category theory, and if not is directed to [Mac71] for a good introduction.

For a word on set-theoretic concerns before continuing: recall that a *class* is a collection that may not be a set. It is sometimes beneficial in higher category theory to fix a Grothendieck universe $\mathscr{U}$ and thereby formalize what it means to be 'not a set'. This was first developed in [AGV71, Exposé I], and a more modern discussion can be found in, e.g., [Shu08]. Because we will not have to address seriously any issues of size in this thesis, we fix the following definitions:

Definition 2.1.1. A category is called *small* if its class of objects is actually a set. A category is *essentially small* if it is equivalent to a small category. Equivalently, a category is essentially small if the collection of isomorphism classes of its objects is a set.

This being settled, we can define a 2-category and fix some notation.

Definition 2.1.2. A *2-category* $\mathbf{C}$ consists of a class of objects, and for any two objects $J, K \in \mathbf{C}$, a category of morphisms $\mathbf{C}(J, K)$. Additionally, for any $I, J, K \in \mathbf{C}$, the composition map

$$\mathbf{C}(J, K) \times \mathbf{C}(I, J) \to \mathbf{C}(I, K)$$

must be a functor. We will use different notation for the category of morphisms depending on the 2-category in question.

One can think of a 2-category as a 1-category enriched in 1-categories, so that the morphism sets in $\mathbf{C}$ actually have the structure of a (small) category. We have three collections in a 2-category: objects, morphisms, and morphisms-between-morphisms, which we will call *2-morphisms.* The canonical example of a 2-category is the 2-category of 1-categories.

Example 2.1.3. Define the 2-category $\mathbf{Cat}$ as follows. The objects are small categories; the morphisms are functors; and the 2-morphisms are natural transformations. By way of general notation, we will denote small categories by J, K, or other capital Roman letters. Functors between small categories will be $u, v \colon J \to K$, or other lowercase Roman letters. Natural transformations will be denoted $\alpha \colon u \Rightarrow v$ or other lowercase Greek letters. We will try to reserve a, b, c, x, y for objects of a particular category $K \in \mathbf{Cat}$ and $f \colon x \to y$ for maps in J, which we will continue to call 'maps' rather than 'morphisms'. We will unfortunately make an exception to each of these conventions in the course of this thesis.

We will also consider the 2-category $\mathbf{CAT}$, defined analogously to above but with objects all categories, which may or may not be small. There is a slight set-theoretic issue in that the functor-categories here might fail to be small, but we will we ignore this inconvenience.

Having defined a 2-category, we can now define maps between 2-categories. These come in a variety of flavors, but we will give the one required for our purposes.

Definition 2.1.4. Let $\mathbf{C}, \mathbf{D}$ be 2-categories. A *strict 2-functor* $\mathbb{F} \colon \mathbf{C} \to \mathbf{D}$ consists of the following data: a functor $\mathbb{F} \colon \mathbf{C} \to \mathbf{D}$ on the underlying 1-categories, and for any $J, K \in \mathbf{C}$, a functor $\mathbb{F}_{J,K} \colon \mathbf{C}(J, K) \to \mathbf{D}(\mathbb{F}(J), \mathbb{F}(K))$. That is, $\mathbb{F}$ respects composition of both 1- and 2-morphisms strictly, not just up to natural isomorphism or some weaker notion.

We have many competing notions of subcategory in the theory of 2-categories, but the one we will need is closest to the case of 1-categories.

Definition 2.1.5. A *full 2-subcategory* $\mathbf{C}$ of a 2-category $\mathbf{D}$ is a subclass of objects $\mathbf{C} \subset \mathbf{D}$ with the choice of morphism categories $\mathbf{C}(J, K) = \mathbf{D}(J, K)$ for all $J, K \in \mathbf{C}$.

Example 2.1.6. As described above, $\mathbf{Cat}$ is a full 2-subcategory of $\mathbf{CAT}$ on the subclass of small categories.

2.2 Particulars in CAT

We will need a few constructions which occur inside of **CAT** for the axioms of a derivator. In fact, the same constructions may be made within an arbitrary 2-category, but the majority of 2-categories in this thesis will be 2-subcategories of **CAT**, so there is no need to strain one's intuition too much. The generalizations will be left to the interested reader with the help of [Bor94, §7].

A commutative square in the 2-category **CAT** has more data than a usual commutative square. We write such squares

(2.2.1)
$$\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "u_2"'] & B \arrow[d, "v_1"] \arrow[dl, Rightarrow, shorten <=10pt, shorten >=10pt, "\alpha"'] \\
C \arrow[r, "v_2"'] & D
\end{tikzcd}$$

The data is as follows: four categories A, B, C, D; four functors u_1, v_1, u_2, v_2; and a natural transformation $\alpha\colon v_1 u_1 \Rightarrow v_2 u_2$. Thus the square does not commute per se, but there is a natural map $\alpha_a\colon v_1(u_1(a)) \to v_2(u_2(a))$ in D for any $a \in A$. Of course, if α is $\mathrm{id}_{v_1 u_1}$ or a natural isomorphism, then this square commutes in the 1-categorical sense as well as we can expect.

We could also have the natural transformation point the other way, in which case we write

$$\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "u_2"'] & B \arrow[d, "v_1"] \\
C \arrow[r, "v_2"'] \arrow[ur, Rightarrow, shorten <=10pt, shorten >=10pt, "\beta"] & D
\end{tikzcd}$$

and have for every $a \in A$ a natural map $\beta_a\colon v_2(u_2(a)) \to v_1(u_1(a))$ in D. Of course this is the same as the above up to flipping the square over the line $\overline{AD}$, but in the case that we have fixed the data of the outside of the square but are varying the natural transformation, we will write it in this way.

There is a particular type of square that will arise repeatedly in this thesis, so we describe it now for future reference.

Definition 2.2.2. Let $u\colon J \to K$ be any functor, and let $k \in K$ be any object. We define the *comma category* (u/k) as follows: its objects are pairs $j \in J$ with a map $f\colon u(j) \to k$, and a map $(j, f) \to (j', f')$ is a map $g\colon j \to j'$ in J making the obvious diagram commute:

(2.2.3)
$$\begin{tikzcd}
u(j) \arrow[rr, "u(g)"] \arrow[dr, "f"'] & & u(j') \arrow[dl, "f'"] \\
& k &
\end{tikzcd}$$

One should read the notation (u/k) as 'u over k', which reminds the reader that maps in (u/k) take place over the identity on k.

There is an analogous category (k/u) ('u under k') where the objects are instead pairs $j \in J$ with $f\colon k \to u(j)$.

These categories fit into a canonical commutative square

(2.2.4)
$$\begin{tikzcd}
(u/k) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(u/k)}"'] & J \arrow[d, "u"] \arrow[dl, Rightarrow, shorten=12pt, "\alpha"'] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

Here, e denotes the final category with one object and one (identity) morphism. Thus $k\colon e \to K$ is a functor which classifies the object $k \in K$. The functor $\pi_{(u/k)}\colon (u/k) \to e$ is the unique functor which sends all objects to the only object in e and all maps to the only map in e. Finally, $\mathrm{pr}\colon (u/k) \to J$ (read: 'projection') is the forgetful functor $(j, f\colon u(j) \to k) \mapsto j$.

Let $(j, f) \in (u/k)$. The composition around the top and right gives $u(\mathrm{pr}(j, f)) = u(j)$, and the composition around the left and bottom gives $k(\pi_{(u/k)}(j, f)) = k$. Thus the natural transformation $\alpha_{(j,f)}\colon u(j) \to k$ has an obvious candidate: the map $f\colon u(j) \to k$ that was part of the original data of (j, f). If we make this a definition, then α is indeed a natural transformation and we have a commutative square in **CAT**.

There is an analogous construction for the comma category (k/u):

$$\begin{tikzcd}
(k/u) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(u/k)}"'] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] \arrow[ur, Rightarrow, shorten=12pt, "\beta"] & K
\end{tikzcd}$$

Such categories satisfy a universal property in the 2-category **CAT**: they are a kind of pullback, in the sense that any choice of $T \in \mathbf{CAT}$ with two functors and a natural transformation making the square below commute

$$\begin{tikzcd}
T \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

has a unique factorization through the comma category which respects both the functors and the natural transformations. The terminology for this universal object is a bit disputed, but the square can be accurately called a *lax pullback square.*

We can also generalize the above construction to obtain the lax pullback of any cospan of categories

$$\begin{tikzcd}
 & J_1 \arrow[d, "u_1"] \\
J_2 \arrow[r, "u_2"'] & K
\end{tikzcd}$$

to obtain a comma category (u_1/u_2) or (u_2/u_1). The objects in (u_1/u_2) are triples $j_1 \in J_1$, $j_2 \in J_2$, and a map $f\colon u_1(j_1) \to u_2(j_2)$. The maps in (u_1/u_2) come from maps $g_1\colon j_1 \to j_1'$ and $g_2\colon j_2 \to j_2'$ in J_1 and J_2 respectively making the requisite square commute:

$$\begin{tikzcd}
u_1(j_1) \arrow[r, "u_1(g_1)"] \arrow[d, "f"'] & u_1(j_1') \arrow[d, "f'"] \\
u_2(j_2) \arrow[r, "u_2(g_2)"'] & u_2(j_2')
\end{tikzcd}$$

We will need this generality in §2.6 but not immediately.

To give some intuition for comma categories, we present a specific case which will use repeatedly throughout this thesis. Recall that a poset may be considered as a category in the following way: if K is a poset and $k, k' \in K$, then we will let $\mathrm{Hom}(k, k') = \{*\}$ if $k \leq k'$ and $\mathrm{Hom}(k, k') = \varnothing$ otherwise. There is not really such a thing as 'composition' in this category, or rather any two maps with compatible codomain and domain compose uniquely.

Suppose that K is a poset, and $u\colon J \to K$ is a functor. Consider any comma category (u/k). Then for any $j \in J$, there is at most one map $u(j) \to k$, so the 'projection' map $\mathrm{pr}\colon (u/k) \to J$ is injective on objects. Similarly, for any $(j, f), (j', f') \in (u/k)$, suppose there is a map $j \to j'$. Then Diagram 2.2.3 automatically commutes, so we also have a map

$(j, f) \to (j', f')$. We also know that every map $(j, f) \to (j', f')$ comes from a map $j \to j'$ in J by the construction of (u/k), so we conclude the following:

Lemma 2.2.5. If $u\colon J \to K$ is a functor and K is a poset, then for any $k \in K$, the forgetful functor $\mathrm{pr}\colon (u/k) \to J$ is injective on objects and fully faithful. Thus we can view (u/k) as the full subcategory of J on those objects j such that $u(j)$ admits any map to k. The dual statement is true for $\mathrm{pr}\colon (k/u) \to J$.

We will be dealing with a lot of comma categories because they play a role in one of the axioms of a derivator as a means of computing arbitrary Kan extensions. In the course of many of the proofs in this thesis, they will arise from maps into a poset, and thus this identification lemma is highly useful.

2.3 The calculus of mates

There is one more phenomenon to explore before defining a derivator. A thorough but older (i.e. pre-LaTeX) reference for calculus of mates can be found in [KS74].

Suppose that we are given the following commutative square in **CAT**:

(2.3.1)
$$\begin{tikzcd}
J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J_2 \arrow[d, "u_2"] \\
K_1 \arrow[r, "w"'] & K_2
\end{tikzcd}$$

such that u_1 and u_2 admit right adjoints r_1 and r_2, respectively. Then we can extend the above picture:

$$\begin{tikzcd}
K_1 \arrow[r, "r_1"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "\overset{\varepsilon_1}{\swarrow}"] & J_1 \arrow[r, "v"] \arrow[d, "u_1"] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J_2 \arrow[d, "u_2"] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\overset{\eta_2}{\swarrow}"] & \\
 & K_1 \arrow[r, "w"'] & K_2 \arrow[r, "r_2"'] & J_2
\end{tikzcd}$$

where ε_1 and η_2 are the counit and unit of the respective adjunctions. In total, this gives us a natural transformation $v \circ r_1 \Rightarrow r_2 \circ w$ which we call the *right mate of* α and denote α_*.

Similarly, if we have the other flavor of commutative square and u_1, u_2 admit left adjoints

ℓ_1, ℓ_2, we obtain

$$\begin{tikzcd}
K_1 \arrow[r, "\ell_1"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, shift left=3, "\overset{\eta_1}{\Nearrow}"] & J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\overset{\beta}{\Nearrow}"] & J_2 \arrow[d, "u_2"'] \arrow[dr, bend left, "="] \arrow[dr, phantom, shift right=3, "\overset{\varepsilon_2}{\Nearrow}"] & \\
& K_1 \arrow[r, "w"'] & K_2 \arrow[r, "\ell_2"'] & J_2
\end{tikzcd}$$

to construct $\beta_! \colon \ell_2 \circ w \Rightarrow v \circ \ell_1$.

In the situation of Diagram 2.3.1, if u_1, u_2 admit right adjoints and v, w admit left adjoints, then it makes sense to talk about both $\alpha_!$ and α_*. A fair question is the relationship between these two mates, which we will answer shortly.

For one more piece of setup: consider a commutative diagram comprised of two squares

$$\begin{tikzcd}
J_1 \arrow[r, "v_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\overset{\alpha_1}{\Swarrow}"] & J_2 \arrow[r, "v_2"] \arrow[d, "u_2"'] \arrow[dr, phantom, "\overset{\alpha_2}{\Swarrow}"] & J_3 \arrow[d, "u_3"] \\
K_1 \arrow[r, "w_1"'] & K_2 \arrow[r, "w_2"'] & K_3
\end{tikzcd}$$

We can take composite of these natural transformations (called their *pasting*)

$$\alpha_2 \odot \alpha_1 \colon u_3 \circ v_2 \circ v_1 \Rightarrow w_2 \circ w_1 \circ u_1$$

and take its left/right mate or look at the mates one at a time.

Proposition 2.3.2 (Lemma 1.14, [Gro13])**.**

(1) The calculus of mates is compatible with pasting. That is, $(\alpha_2 \odot \alpha_1)_! = (\alpha_2)_! \odot (\alpha_1)_!$ and similar for the right mates.

(2) The different formations of mates are inverse to each other. That is $\alpha = (\alpha_!)_* = (\alpha_*)_!$ when α admits both a left and right mate.

(3) In the case that α admits both a left and a right mate, then $\alpha_!$ is a natural isomorphism if and only if α_* is a natural isomorphism.

Once we define the axioms of a derivator, we will use all of these properties in order to study some first consequences of those axioms. As a remark on notation, henceforth we will usually not use $\circ$ when writing a composition of functors and just write vu for $v \circ u$.

2.4 Prederivators and derivators

A good first reading on derivators is [Gro13], to which we will refer repeatedly throughout. A more thorough reference is [Gro19], but as of the completion of this thesis this remains a work in progress. However, in an effort to be self-contained, we will provide as much detail for these known results as we will need to expand upon them in Chapters 3 and 4.

Definition 2.4.1. A *prederivator* is a strict 2-functor $\mathbb{D}\colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$, where op reverses only the 1-morphisms and leaves the 2-morphisms as usual.

By way of notation, for a morphism $u\colon J \to K$ in $\mathbf{Cat}$ we denote by u^* the functor $\mathbb{D}(u)\colon \mathbb{D}(K) \to \mathbb{D}(J)$ in $\mathbf{CAT}$, and for $\alpha\colon u \Rightarrow v$ in $\mathbf{Cat}$ we denote by α^* the natural transformation $\mathbb{D}(\alpha)\colon u^* \Rightarrow v^*$. Composition is respected strictly, so that $(vu)^* = u^*v^*$ and $(\alpha \odot \beta)^* = \alpha^* \odot \beta^*$. Identities are also preserved, so that $(\mathrm{id}_J)^* = \mathrm{id}_{\mathbb{D}(J)}$ and $(\mathrm{id}_u)^* = \mathrm{id}_{u^*}$.

Example 2.4.2. For the following examples, let $u\colon J \to K$ be a functor in $\mathbf{Cat}$.

(1) Let $\mathcal{C} \in \mathbf{CAT}$ be any category. Define the prederivator $\mathbb{D}_{\mathcal{C}}$ via the Yoneda embedding:

$$K \mapsto \mathrm{Fun}(K,\mathcal{C}), \quad u\colon J \to K \;\mapsto u^*\colon \mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(J,\mathcal{C})$$

with the action on natural transformations obvious. Here the map u^* is precomposition with u, and for this reason we usually refer to u^* as a *pullback functor* even when $\mathbb{D}$ is not this specific prederivator.

(2) Let $\mathcal{A}$ be a Grothendieck abelian category, e.g. the category of R-modules for any commutative ring R. We can then form the (unbounded) derived category $D(\mathcal{A})$ without any set-theoretic issues. For any small category K, the category $\mathrm{Fun}(K,\mathcal{A})$ is still a small abelian category, so we may take its derived category as well. Define the prederivator $\mathbb{D}_{\mathcal{A}}$ by $\mathbb{D}_{\mathcal{A}}(K) = D(\mathrm{Fun}(K,\mathcal{A}))$ with u^* induced by precomposition.

(3) Let $\mathcal{M}$ be a combinatorial model category, e.g. the category of simplicial sets with either the standard or Joyal model structure. The first model structure makes $\mathbf{sSet}$ a model for topological spaces and the second makes it a model for $(\infty, 1)$-categories.

In any case, for any small category K the category $\mathrm{Fun}(K, \mathcal{M})$ is still a combinatorial model category (using either the injective or projective model structure) where the weak equivalences are defined pointwise. Then define the prederivator $\mathbb{D}_{\mathcal{M}}$ by $\mathbb{D}_{\mathcal{M}}(K) = \mathrm{Ho}(\mathrm{Fun}(K, \mathcal{M}))$ with u^* again induced by precomposition.

(4) Let $\mathbb{D}$ be any prederivator. Then for any $I \in \mathbf{Cat}$ we may obtain another prederivator $\mathbb{D}^I$ defined by $\mathbb{D}^I(K) = \mathbb{D}(I \times K)$ and $\mathbb{D}^I(u) = \mathbb{D}(\mathrm{id}_I \times u)$. We usually call $\mathbb{D}^I$ a *shifted prederivator*. This is a useful way of constructing new prederivators from old ones and is a key technique in the theory of derivators.

Recall we are working under the slogan ‘system of diagram categories’, so we would like to define some axioms to ensure that $\mathbb{D}(K)$ looks like K-shaped diagrams in some category.

We can immediately identify what category that should be. Let $\mathbb{D}$ be a prederivator, K a small category, and $k \in K$ be any object. Recall that we have a functor that classifies the object k which we denote $k \colon e \to K$. Then for any $X \in \mathbb{D}(K)$, we have an object $k^*X \in \mathbb{D}(e)$. Suppose that $f \colon k_1 \to k_2$ is a map in K. Then we have a corresponding natural transformation $f^* \colon k_1^* \Rightarrow k_2^*$ and thus a map $f^*X \colon k_1^*X \to k_2^*X$ in $\mathbb{D}(e)$. Repeating this process for all objects and maps in K, we obtain a functor

$$\mathrm{dia}_K \colon \mathbb{D}(K) \to \mathrm{Fun}(K, \mathbb{D}(e))$$

which sends $X \in \mathbb{D}(K)$ to the functor which assembles all the above data. We call this an *underlying diagram functor*, and its existence implies that the prederivator $\mathbb{D}$ should be modeling K-shaped diagrams in $\mathbb{D}(e)$, which we call the *underlying category* or the *base* of the prederivator. We will refer to the categories $\mathbb{D}(K)$ as *coherent* diagrams, as opposed to the *incoherent* diagrams $\mathrm{Fun}(K, \mathbb{D}(e))$.

This motivates the following definition:

Definition 2.4.3. A *semiderivator* is a prederivator $\mathbb{D}$ satisfying the following two axioms:

(Der1) Coproducts are sent to products. Explicitly, consider any set $\{K_a\}_{a\in A}$ of small categories, and let $i_b \colon K_b \to \coprod_{a\in A} K_a$ be the inclusion for any $b \in A$. Pulling back along this inclusion gives a functor

$$i_b^*\colon \mathbb{D}\left(\coprod_{a\in A} K_a\right) \to \mathbb{D}(K_b)$$

which induces a map to the product

$$\prod_{b\in A} i_b^*\colon \mathbb{D}\left(\coprod_{a\in A} K_a\right) \to \prod_{b\in A}\mathbb{D}(K_b)$$

We require this map to be an equivalence of categories for any collection $\{K_a\}_{a\in A}$.

(Der2) Isomorphisms are detected pointwise. That is, for any $K \in \mathbf{Cat}$, the underlying diagram functor dia_K is conservative. More specifically, a map $f\colon X \to Y$ is an isomorphism in $\mathbb{D}(K)$ if and only if the map $k^*f\colon k^*X \to k^*Y$ is an isomorphism for all $k \in K$.

This is the bare minimum such that $\mathbb{D}$ acts like a system of diagram categories. Der1 means that choosing two disconnected diagrams is the same (up to equivalence) as choosing a diagram on each component, and Der2 says that $\mathbb{D}(e)$ has control over isomorphisms of diagrams. Each of the prederivators in Example 2.4.2 is a semiderivator, as they are precisely constructed to be systems of diagram categories.

Note that both Der1 and Der2 are *properties* of a prederivator $\mathbb{D}$. The functors in these axioms always exist for any prederivator; we are investigating whether they are equivalences or conservative (respectively). Thus we are not placing additional structures on a prederivator to obtain a semiderivator, but examining how nice the 2-functor $\mathbb{D}$ happens to be.

A derivator is a semiderivator that admits homotopy limits and colimits, as well as more general homotopy Kan extensions that may be computed pointwise as limits and colimits. We will again motivate these as best as possible. Recall that in ordinary category theory, limits and colimits can be thought of as adjoint functors. Specifically, if we let K be a diagram shape and $\mathcal{C}$ a category, then we can consider the functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ which sends $c \in \mathcal{C}$ to the constant diagram of shape K. Assuming that $\mathcal{C}$ admits limits and colimits of shape K, we have the following adjunction:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\Delta"'] \\
\mathrm{Fun}(K,\mathcal{C}) \arrow[u, bend left=70, "\mathrm{colim}_K"] \arrow[u, bend right=70, "\lim_K"']
\end{tikzcd}$$

The colimit functor is the left adjoint of Δ, and the limit functor is the right adjoint.

In derivators, we will replace $\mathcal{C}$ by $\mathbb{D}(e)$ and $\mathrm{Fun}(K,\mathcal{C})$ by $\mathbb{D}(K)$. The analogue of Δ in this case is the following functor: Consider the projection $\pi_K \colon K \to e$ of K to the final category. Then for any $x \in \mathbb{D}(e)$, we have a coherent diagram $\pi_K^* x \in \mathbb{D}(K)$. To get a handle on this object, we can restrict to $k \in K$. We then notice that

$$k^* \pi_K^* x = (\pi_K \circ k)^* x$$

by strict 2-functoriality. The map $\pi_K \circ k \colon e \to e$ must be the identity functor, and thus $(\pi_K \circ k)^* = \mathrm{id}_{\mathbb{D}(e)}^*$ so $k^* \pi_K^* x = x$ for any $x \in \mathbb{D}(e)$. Using the same sort of reasoning we can see that for any $k \to k'$ in K, the map $k^* X \to k'^* X$ is the identity on x. Therefore π_K^* is a coherent constant diagram functor. Having established this, we can now give the axioms of a derivator. Because it will be necessary in the long run, we will give this definition in halves.

Definition 2.4.4. A semiderivator $\mathbb{D}$ is a *left derivator* if it satisfies the following two axioms:

(Der3L) The base of the semiderivator $\mathbb{D}(e)$ is (homotopically) cocomplete. Specifically, for every functor $u \colon J \to K$, the pullback u^* admits a left adjoint, which we denote $u_! \colon \mathbb{D}(J) \to \mathbb{D}(K)$ and call the *(homotopy) left Kan extension along u*. As a special case, this includes $\pi_K \colon K \to e$ and thus $\mathbb{D}(e)$ admits all (coherent) colimits.

(Der4L) Left Kan extensions can be computed pointwise. Let $u \colon J \to K$ and $k \in K$. Then recall that we have the following lax pullback square in **Cat** from Diagram 2.2.4

$$\begin{tikzcd}
(u/k) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\alpha}{\swArrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

where we let $\pi = \pi_{(u/k)}$ for brevity. Applying the semiderivator $\mathbb{D}$ to this square, we obtain the following square in **CAT**, remembering that functors are reversed and natural transformations are not:

$$\begin{tikzcd}
\mathbb{D}((u/k)) & \mathbb{D}(J) \arrow[l, "\mathrm{pr}^*"'] \\
\mathbb{D}(e) \arrow[u, "\pi^*"] & \mathbb{D}(K) \arrow[l, "k^*"] \arrow[u, "u^*"']
\arrow[from=1-2, to=2-1, phantom, "\alpha^* \Swarrow"]
\end{tikzcd}$$

We would like to use techniques in the calculus of mates from §2.3, but our square does not have the correct orientation. If we flip it around so that the functors point to the bottom-right, we obtain

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "k^*"] \arrow[d, "u^*"'] & \mathbb{D}(e) \arrow[d, "\pi^*"] \\
\mathbb{D}(J) \arrow[r, "\mathrm{pr}^*"'] & \mathbb{D}((u/k))
\arrow[from=2-1, to=1-2, phantom, "\alpha^* \Nearrow"]
\end{tikzcd}$$

By Der3L, both vertical functors admit left adjoints, so we may construct the left mate of α^*, which we denote by $\alpha_!$ rather than $(\alpha^*)_!$:

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "u_!"] \arrow[dr, bend right, "="'] & \mathbb{D}(K) \arrow[r, "k^*"] \arrow[d, "u^*"'] & \mathbb{D}(e) \arrow[d, "\pi^*"] \arrow[dr, bend left, "="] & \\
 & \mathbb{D}(J) \arrow[r, "\mathrm{pr}^*"'] & \mathbb{D}((u/k)) \arrow[r, "\pi_!"'] & \mathbb{D}(e)
\arrow[from=1-1, to=1-2, phantom, shift right=6, "\Nearrow"]
\arrow[from=2-2, to=1-3, phantom, "\alpha^* \Nearrow"]
\arrow[from=2-3, to=2-4, phantom, shift left=6, "\Nearrow"]
\end{tikzcd}$$

In total we have the natural transformation $\alpha_! \colon \pi_! \mathrm{pr}^* \Rightarrow k^* u_!$. We require this map to be a natural isomorphism.

Once again, note that these are properties of a semiderivator $\mathbb{D}$, not additional structures.

To analyze Der4L a bit more, for $X \in \mathbb{D}(J)$, $k^* u_! X$ is the value of the coherent diagram $u_! X$ at the point $k \in K$. The lefthand side of $\alpha_!$ is the colimit of shape (u/k) of the diagram obtained by pulling back X to the comma category. This is particularly useful in situations like that of Lemma 2.2.5, where we have better knowledge of the comma category and the projection functor than the functor $u \colon J \to K$.

For the sake of completeness, we give the dual definition explicitly.

Definition 2.4.5. A semiderivator $\mathbb{D}$ is a *right derivator* if it satisfies the following two axioms:

(Der3R) The base of the semiderivator $\mathbb{D}(e)$ is (homotopically) complete. Specifically, for every functor $u\colon J \to K$, the pullback u^* admits a right adjoint, which we denote $u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ and call the *(homotopy) right Kan extension along* u. As a special case, this includes $\pi_K\colon K \to e$ and thus $\mathbb{D}(e)$ admits all (coherent) limits.

(Der4R) Right Kan extensions can be computed pointwise. Let $u\colon J \to K$ and $k \in K$. Then recall that we have the following lax pullback square in **Cat**

$$\begin{tikzcd}
(k/u) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\beta}{\Nearrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

where we let $\pi = \pi_{(k/u)}$ for brevity. Applying the semiderivator $\mathbb{D}$ to this square and taking the right mate $\beta_* = (\beta^*)_*$ we obtain a natural transformation $\beta_*\colon k^*u_* \Rightarrow \pi_*\mathrm{pr}^*$. We require this map to be a natural isomorphism.

Remark 2.4.6. Historically in derivator literature, a derivator which admits all colimits and in which left Kan extensions may be computed pointwise is called a right derivator. This is due to an analogy with right exact functors that we feel does not justify the confusing terminology. Moreover, this thesis contains (to the author's knowledge) the most in-depth work on 'half derivators', so we reserve the right to rechristen these objects.

Definition 2.4.7. A *derivator* is a semiderivator that is both a left and a right derivator. That is, it is a prederivator satisfying Der1, Der2, Der3L, Der3R, Der4L, and Der4R.

Example 2.4.8.

(1) For a category $\mathcal{C}$, the prederivator $\mathbb{D}_{\mathcal{C}}$ is a left (resp. right) derivator if and only if $\mathcal{C}$ is cocomplete (resp. complete).

(2) For a Grothendieck abelian category $\mathcal{A}$ (which is in particular bicomplete), the prederivator $\mathbb{D}_{\mathcal{A}}$ is a derivator.

(3) For a combinatorial model category $\mathcal{M}$, working under the assumption that model categories are bicomplete, the prederivator $\mathbb{D}_{\mathcal{M}}$ is a derivator.

We can also generate new derivators from old:

Proposition 2.4.9 (Theorem 1.25, [Gro13])**.** Let $\mathbb{D}$ be a left (resp. right) derivator and I a small category. Then the shifted prederivator $\mathbb{D}^I$ is also a left (resp. right) derivator.

Remark 2.4.10. There is a 'fifth axiom' for derivators which is not necessary in all contexts, ours included. We mention it, however, for context and because it relates to the development of derivators as an enhancement of triangulated categories.

Consider the ordinal $[1]$ as a category, that is, the category with two objects $0, 1$ and one non-identity map $0 \to 1$. For a (pre)derivator $\mathbb{D}$, the category $\mathbb{D}([1])$ is 'coherent arrows' in $\mathbb{D}(e)$. Not only can we look at $\operatorname{dia}_{[1]} \colon \mathbb{D}([1]) \to \operatorname{Fun}([1], \mathbb{D}(e))$, but also we could do the same for the (pre)derivator $\mathbb{D}^K$ for any small category K. This functor is defined to be

$$\operatorname{dia}_{[1],K} \colon \mathbb{D}^K([1]) = \mathbb{D}(K \times [1]) \to \operatorname{Fun}([1], \mathbb{D}(K)) = \operatorname{Fun}([1], \mathbb{D}^K(e))$$

We call this a *partial underlying diagram functor*; we forget the $[1]$-dimension of coherence but leave the K-dimension of the diagram coherent.

Definition 2.4.11. A prederivator $\mathbb{D}$ is *strong* if it satisfies the following axiom:

(Der5) For any category $K \in \mathbf{Cat}$, the functor $\operatorname{dia}_{[1],K}$ is full and essentially surjective.

Essential surjectivity means that whenever we have a map in $\mathbb{D}(K)$, we can lift it to a coherent arrow between K-shaped diagrams, i.e. an element of $\mathbb{D}(K \times [1])$. Fullness means that whenever we have a commutative square in $\mathbb{D}(K)$, we can lift it to a map in $\mathbb{D}(K \times [1])$.

This is a condition which usually holds when a prederivator comes from some sort of explicit model; all the derivators of Example 2.4.8 are strong. Constructions made inside the theory of derivators (of which there will be many in this thesis) will usually preserve the property of being a left/right derivator, but not necessarily of being strong. Lagkas in [Lag17] gives a heuristic for constructing non-strong derivators arising in the context of monads over a triangulated category translated into derivator theory.

Looking at the bases of the derivators in Example 2.4.8, we find a limitation: what if we have some category that does not admit all limits and colimits, but still admits some?

What we wanted to produce a derivator modelling diagrams in the bounded derived category $D^b(\mathcal{A})$ of an abelian category or the model category of finite CW-complexes? We could easily produce a semiderivator for these cases, but the homotopical bicompleteness on all of **Cat** throws us off.

The solution is to restrict our attention at times to a full 2-subcategory $\mathbf{Dia} \subset \mathbf{Cat}$ that contains only those diagrams over which we may take a limit or colimit, and replace **Cat** by **Dia** in all the axioms above. However, not every 2-subcategory is appropriate; for example, given any functor $u\colon J \to K$ in **Dia** and any $k \in K$, we should also have $(u/k) \in \mathbf{Dia}$ in order to make sense of Der3L. The following axioms were first defined in [Mal07, p.3] and refined in [Gro13, Definition 1.12]

Definition 2.4.12. A full 2-subcategory $\mathbf{Dia} \subset \mathbf{Cat}$ is a *diagram 2-category* if it satisfies the following axioms:

(Dia1) **Dia** contains all finite posets.

(Dia2) **Dia** is closed under coproducts and (1-categorical) pullbacks.

(Dia3) For every $u\colon J \to K$ in **Dia** and every $k \in K$, $(u/k), (k/u) \in \mathbf{Dia}$.

(Dia4) If $K \in \mathbf{Dia}$, then $K^{\mathrm{op}} \in \mathbf{Dia}$.

(Dia5) For every Grothendieck fibration $u\colon J \to K$ in **Cat**, if for all $k \in K$ the fiber J_k is in **Dia** and $K \in \mathbf{Dia}$, then $J \in \mathbf{Dia}$ as well.

In the case that our diagram 2-category contains only finite categories, we will replace Dia2 by the following:

($\mathrm{Dia2_f}$) **Dia** is closed under *finite* coproducts and (1-categorical) pullbacks.

To motivate the above axioms: Dia1 makes sure that we have something in **Dia** to work with; Dia2 makes sure we can check Der1; Dia3 makes sure we can check Der3L and Der3R; Dia4 is to preserve some semblance of duality; and Dia5 is related to the idea that, in this situation, we should be able to 'build' J out of the fibres J_k and the base K, so it should be a valid diagram shape as well. The smallest choice of diagram 2-category is $\mathbf{Pos_f}$ the

2-category of all finite posets (in this case we take the modified Dia2). The largest choice is, of course, **Cat** itself.

Remark 2.4.13. A common choice for **Dia** is $\mathbf{Dir_f}$, the 2-category of finite direct categories, i.e. categories whose nerve has finitely many nondegenerate simplices. Keller in [Kel07] gives a construction of a derivator $\mathbb{D}_{\mathcal{E}}$ with domain $\mathbf{Dir_f}$ for any exact category $\mathcal{E}$ with $\mathbb{D}_{\mathcal{E}}(e) = D^b(\mathcal{E})$. Cisinski in [Cis10, Théorème 2.21] proves that a Waldhausen category $\mathcal{W}$ whose weak equivalences satisfy some mild properties gives rise to a left derivator $\mathbb{D}_{\mathcal{W}}$ with domain $\mathbf{Dir_f}$ such that $\mathbb{D}_{\mathcal{W}}(e) = \operatorname{Ho}\mathcal{W}$. This latter example will be extremely important when we study derivator K-theory in Chapter 4.

When we begin proving theorems in this thesis, we will be clear on which **Dia** we are considering our derivators. For the moment, we will allow **Dia** arbitrary.

2.5 The 2-category of (pre)derivators

Having set up the objects of our study, we can now describe the morphisms between them and (as it turns out) the 2-morphisms between those.

Definition 2.5.1. Let $\mathbb{D}, \mathbb{E}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$ be prederivators. A *morphism of prederivators* $\Phi\colon \mathbb{D} \to \mathbb{E}$ is a pseudonatural transformation of the associated 2-functors. This consists of the following data: for each $K \in \mathbf{Dia}$ we have a functor $\Phi_K\colon \mathbb{D}(K) \to \mathbb{E}(K)$ and for every $u\colon J \to K$ we have a natural isomorphism $\gamma_u^\Phi\colon u^*\Phi_K \Rightarrow \Phi_J u^*$

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\gamma_u^\Phi}{\swarrow\!\!\!\swarrow}"] & \mathbb{E}(K) \arrow[d, "u^*"] \\
\mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J)
\end{tikzcd}$$

where we have slightly abused notation by writing u^* for both $\mathbb{D}(u)$ and $\mathbb{E}(u)$.

The family $\{\gamma_u^\Phi\}$ is subject to coherence conditions. Foremost, for two composable functors $u\colon J \to K$ and $v\colon I \to J$ we require that the pasting on the left be equal to the square on the right:

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\gamma_u^\Phi}{\swarrow}"] & \mathbb{E}(K) \arrow[d, "u^*"] \\
\mathbb{D}(J) \arrow[r, "\Phi_J"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\overset{\gamma_v^\Phi}{\swarrow}"] & \mathbb{E}(J) \arrow[d, "v^*"] \\
\mathbb{D}(I) \arrow[r, "\Phi_I"'] & \mathbb{E}(I)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "(uv)^*"'] \arrow[dr, phantom, "\overset{\gamma_{uv}^\Phi}{\swarrow}"] & \mathbb{E}(K) \arrow[d, "(uv)^*"] \\
\mathbb{D}(I) \arrow[r, "\Phi_I"'] & \mathbb{E}(I)
\end{tikzcd}$$

In addition, we require $\gamma^\Phi_{\mathrm{id}_J} = \mathrm{id}_{\mathbb{D}(J)}$. There is also required compatibility with natural transformations in **Dia**. For two functors $u, v\colon J \to K$ and a natural transformation $\alpha\colon u \Rightarrow v$, we require the below pastings to be equal:

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "u^*"] \arrow[d, bend right=70, "v^*"'] \arrow[d, phantom, shift right=5ex, "\overset{\alpha^*}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\gamma_u^\Phi}{\swarrow}"] & \mathbb{E}(K) \arrow[d, "u^*"] \\
\mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\overset{\gamma_v^\Phi}{\swarrow}"] & \mathbb{E}(K) \arrow[d, "v^*"'] \arrow[d, bend left=70, "u^*"] \arrow[d, phantom, shift left=5ex, "\overset{\alpha^*}{\Leftarrow}"] \\
\mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J)
\end{tikzcd}$$

The general definition of a pseudonatural transformation can be found at [Bor94, Definition 7.5.2].

A *morphism of (left/right) derivators* is a just a morphism of prederivators; there is no additional condition. A morphism $\Phi\colon \mathbb{D} \to \mathbb{E}$ is an *equivalence of (pre)derivators* if Φ_K is an equivalence of categories for every $K \in$ **Dia**.

Having claimed that we are assembling a 2-category, we now define the 2-morphisms.

Definition 2.5.2. If $\Phi, \Psi\colon \mathbb{D} \to \mathbb{E}$ are two morphisms of (pre)derivators, a 2-morphism $\mu\colon \Phi \to \Psi$ is given by a *modification* of pseudonatural transformations of 2-functors. This is a natural transformation $\mu_K\colon \Phi_K \Rightarrow \Psi_K$ for every $K \in$ **Dia** satisfying the following coherence condition: if $u, v\colon J \to K$ are two functors and $\alpha\colon u \Rightarrow v$ is a natural transformation, then we have an equality of pastings

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\alpha^*\Downarrow"] & \mathbb{D}(J) \arrow[r, bend left, "\Phi_J"] \arrow[r, bend right, "\Psi_J"'] \arrow[r, phantom, "\mu_J\Downarrow"] & \mathbb{E}(J)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{D}(K) \arrow[r, bend left, "\Phi_K"] \arrow[r, bend right, "\Psi_K"'] \arrow[r, phantom, "\mu_K\Downarrow"] & \mathbb{E}(K) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\alpha^*\Downarrow"] & \mathbb{E}(J)
\end{tikzcd}$$

See also [Bor94, Definition 7.5.3]. A modification μ is called an *isomodification* if μ_K is a natural isomorphism for every $K \in$ **Dia**.

This gives us a 2-category **PDer** of prederivators and full 2-subcategories left derivators, right derivators, and derivators. We only name this last one **Der**. By way of notation, we will reserve uppercase Greek letters Φ, Ψ for morphisms of derivators, lowercase Greek letters μ, ν for modifications. Moreover, if we are working with arbitrary diagrams $K \in \mathbf{Dia}$ and want to examine $\Phi_K(X)$ for arbitrary $X \in \mathbb{D}(K)$, we will usually just write $X \in \mathbb{D}$ and $\Phi(X)$ or sometimes ΦX, leaving 'at any $K \in \mathbf{Dia}$' implicit.

On the surface, these definitions require a ton of compatible information, and it seems unlikely that we would ever be able to construct morphisms of derivators. There is a ready source of morphisms, however, which arise from functors in **Dia**. Consider a functor $u\colon J \to K$ and the associated pullback $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$. Then we can consider $u^*\colon \mathbb{D}^K \to \mathbb{D}^J$ as a morphism between the associated shifted derivators. Let us be explicit about where the structure isomorphisms γ^{u^*} come from in this case. Let $v\colon A \to B$ be a functor in **Dia**. Then we need to populate the below square with a natural isomorphism:

$$\begin{tikzcd}
\mathbb{D}^K(B) \arrow[r, "u_B^*"] \arrow[d, "v^*"'] & \mathbb{D}^J(B) \arrow[d, "v^*"] \arrow[dl, phantom, "\overset{\gamma_v^{u^*}}{\swarrow}"] \\
\mathbb{D}^K(A) \arrow[r, "u_A^*"'] & \mathbb{D}^J(A)
\end{tikzcd}$$

If we make more explicit what all these maps are in terms of the derivator $\mathbb{D}$, the above is equal to

$$\begin{tikzcd}
\mathbb{D}(K \times B) \arrow[r, "(u\times \mathrm{id}_B)^*"] \arrow[d, "(\mathrm{id}_K \times v)^*"'] & \mathbb{D}(J \times B) \arrow[d, "(\mathrm{id}_J \times v)^*"] \arrow[dl, phantom, "\overset{\gamma_v^{u^*}}{\swarrow}"] \\
\mathbb{D}(K \times A) \arrow[r, "(u\times \mathrm{id}_A)^*"'] & \mathbb{D}(J \times A)
\end{tikzcd} \tag{2.5.3}$$

So we are looking for a transformation $(\mathrm{id}_J \times v)^*(u \times \mathrm{id}_B)^* \Rightarrow (u \times \mathrm{id}_A)^*(\mathrm{id}_K \times v)^*$. But by strict 2-functoriality of $\mathbb{D}$, these are the same functor $(u \times v)^*$, so the choice $\gamma_v^{u^*} = \mathrm{id}_{(u\times v)^*}$ fits the bill. This choice happily satisfies all compatibility conditions. In fact, we have terminology for just this situation.

Definition 2.5.4. A morphism of derivators $\Phi\colon \mathbb{D} \to \mathbb{E}$ is called *strict* if for any $u\colon J \to K$ in **Dia**, the corresponding structure natural isomorphism γ_u^Φ is the identity.

Another class of morphisms of derivators arise from left and right Kan extensions, but these are not strict. For any $u\colon J \to K$ in **Dia**, if $\mathbb{D}$ is a left derivator then we have a morphism $u_!\colon \mathbb{D}^J \to \mathbb{D}^K$. For $v\colon A \to B$, the structure isomorphisms in this case need to populate the square

$$\begin{tikzcd}
\mathbb{D}(J \times B) \arrow[r, "(u\times \mathrm{id}_B)_!"] \arrow[d, "(\mathrm{id}_J \times v)^*"'] \arrow[dr, phantom, "\overset{\gamma_v^{u_!}}{\Swarrow}"] & \mathbb{D}(K \times B) \arrow[d, "(\mathrm{id}_K \times v)^*"] \\
\mathbb{D}(J \times A) \arrow[r, "(u\times \mathrm{id}_A)_!"'] & \mathbb{D}(K \times A)
\end{tikzcd}$$

The natural transformation above is the left mate of Diagram 2.5.3, and it is not hard to show that this mate is a natural isomorphism. Instead of showing it directly, we will show shortly how to make this conclusion. Unfortunately, there is no reason for this mate to be the identity, which corresponds to the fact that colimits are unique up to unique isomorphism, but not strictly unique.

For the purposes of both stabilization and K-theory, we will care to focus on a particular subclass of morphisms of derivators.

Definition 2.5.5. Let $\mathbb{D}, \mathbb{E}$ be left derivators and $u\colon J \to K$ in **Dia**. We say that a morphism $\Phi\colon \mathbb{D} \to \mathbb{E}$ *preserves left Kan extensions along* u if the left mate of $(\gamma_u^\Phi)^{-1}$ is a natural isomorphism. Specifically, we have the pasting

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "u_!"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{(\gamma_u^\Phi)^{-1}}{\Nearrow}"] & \mathbb{E}(K) \arrow[d, "u^*"] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\Nearrow"] & \\
 & \mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J) \arrow[r, "u_!"'] & \mathbb{E}(K)
\end{tikzcd}$$

giving us a natural transformation $(\gamma_u^\Phi)_!^{-1}\colon u_!\Phi_J \Rightarrow \Phi_K u_!$ which we demand is an isomorphism, where again we slightly abuse notation by writing $u_!$ for the left adjoint to both $\mathbb{D}(u)$ and $\mathbb{E}(u)$. If the morphism Φ preserves left Kan extensions along all $u\colon J \to K$ in **Dia**, we say that Φ is *cocontinuous.*

Cocontinuity means that we can compute the left Kan extension along u in $\mathbb{D}$ then apply Φ, or apply Φ and compute the left Kan extension along u in $\mathbb{E}$ and we obtain isomorphic objects. There is an analogous definition of a *continuous* morphism that we will spell out explicitly for reference.

Definition 2.5.6. Let $\mathbb{D}, \mathbb{E}$ be right derivators and $u\colon J \to K$ in **Dia**. We say that a morphism $\Phi\colon \mathbb{D} \to \mathbb{E}$ *preserves right Kan extensions along* u if the right mate of γ_u^Φ is a natural isomorphism. Specifically, we have the pasting

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "u_*"] \arrow[dr, bend right, "="'] & \mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[d, "u^*"'] \arrow[dl, phantom, "\swarrow"] \arrow[dr, phantom, "\overset{\gamma_u^\Phi}{\swarrow}"] & \mathbb{E}(K) \arrow[d, "u^*"] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\swarrow"'] & \\
 & \mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J) \arrow[r, "u_*"'] & \mathbb{E}(K)
\end{tikzcd}$$

giving us a natural transformation $(\gamma_u^\Phi)_*\colon \Phi_K u_* \Rightarrow u_* \Phi_J$. If the morphism Φ preserves right Kan extensions along all $u\colon J \to K$ in **Dia**, we say that Φ is *continuous.*

Remark 2.5.7. It might seem like Proposition 2.3.2(3) says that a morphism of derivators Φ is cocontinuous if and only if it is continuous. However, cocontinuity uses $(\gamma_\Phi^u)^{-1}$ while continuity uses γ_Φ^u, and the calculus of mates says nothing about the relationship between the mates of α and α^{-1}.

We will now focus on cocontinuity, leaving the dual formulations to the reader. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a morphism between left derivators. We will be able to determine if Φ is cocontinuous in two key ways. For the first, we make the following series of observation. Der2 tells us that isomorphisms may be checked pointwise, so that for a particular $X \in \mathbb{D}(J)$, we have that

$$(\gamma_u^\Phi)_{!,X}^{-1}\colon u_! \Phi_J X \to \Phi_K u_! X$$

if an isomorphism if and only if

$$k^*(\gamma_u^\Phi)_{!,X}^{-1}\colon k^* u_! \Phi_J X \to k^* \Phi_K u_! X$$

is an isomorphism for all $k \in K$. This tactic will be common, so we give it a name: precomposing or postcomposing a natural transformation by a functor is generally called *whiskering.* We will now modify the domain and codomain of this map (up to isomorphism) to make it easier to study.

We may postcompose with the structure isomorphism $\gamma_k^\Phi\colon k^*\Phi_K \Rightarrow \Phi_e k^*$ to obtain

$$k^* u_! \Phi_J X \xrightarrow{k^*(\gamma_u^\Phi)_{!,X}^{-1}} k^* \Phi_K u_! X \xrightarrow[\cong]{(\gamma_k^\Phi)_{u_! X}} \Phi_e k^* u_! X$$

Now using Der4L, we can precompose both the domain and codomain by the natural isomorphism $\pi_! \mathrm{pr}^* \Rightarrow k^* u_!$, recalling the notation from Definition 2.4.4. This gives us

$$\begin{tikzcd}
k^* u_! \Phi_J X \arrow[r, "k^*(\gamma_u^\Phi)^{-1}_{!,X}"] & k^* \Phi_K u_! X \arrow[r, "(\gamma_k^\Phi)_{u_! X}", "\cong"'] & \Phi_e k^* u_! X \\
\pi_! \mathrm{pr}^* \Phi_J X \arrow[u, "\cong"] & & \Phi_e \pi_! \mathrm{pr}^* X \arrow[u, "\cong"']
\end{tikzcd}$$

As one final step, we can commute pr^* and Φ using $\gamma^\Phi_{\mathrm{pr}}$ and complete the above to a commutative square

$$\begin{tikzcd}
k^* u_! \Phi_J X \arrow[r, "k^*(\gamma_u^\Phi)^{-1}_{!,X}"] & k^* \Phi_K u_! X \arrow[r, "(\gamma_k^\Phi)_{u_! X}", "\cong"'] & \Phi_e k^* u_! X \\
\pi_! \mathrm{pr}^* \Phi_J X \arrow[u, "\cong"] \arrow[r, "\cong", "\pi_!(\gamma^\Phi_{\mathrm{pr}})_X"'] & \pi_! \Phi_{(u/k)} \mathrm{pr}^* X \arrow[r, "(\gamma_\pi^\Phi)^{-1}_{!,\mathrm{pr}^* X}"'] & \Phi_e \pi_! \mathrm{pr}^* X \arrow[u, "\cong"']
\end{tikzcd}$$

We are able to fill in the bottom-right map because of the functoriality of the calculus of mates and the coherence conditions imposed on the structure isomorphisms $\{\gamma^\Phi\}$. We have thus reduced the question of all $(\gamma_u^\Phi)_!^{-1}$ being natural isomorphisms to the specific case of $(\gamma_\pi^\Phi)_!^{-1}$ being a natural isomorphism for all maps $\pi \colon (u/k) \to e$, at least on objects of the form $\mathrm{pr}^* X$. This is a key technique in derivator proofs: we have reduced a general problem to more specific one in the base of the derivator. We have the following conclusion:

Lemma 2.5.8 (Proposition 2.3, [Gro13])**.** A morphism $\Phi \colon \mathbb{D} \to \mathbb{E}$ between left derivators is cocontinuous if and only if Φ preserves left Kan extensions along all maps $\pi \colon K \to e$ for $K \in \mathbf{Dia}$, i.e. if Φ preserves (homotopy) colimits.

For the second technique, we recall the following from 1-category theory: left adjoint functors preserve colimits and right adjoint functors preserve limits. Since preserving colimits is enough to preserve all left Kan extensions, we need to figure out what an adjunction of morphisms of derivators should be. There is a general definition in any 2-category:

Definition 2.5.9. Let $\Phi \colon \mathbb{D} \to \mathbb{E}$ and $\Psi \colon \mathbb{E} \to \mathbb{D}$ be two morphisms of (pre)derivators. We say that Φ *is left adjoint to* Ψ (equivalently, Ψ *is right adjoint to* Φ) if there exist two modifications $\eta \colon \mathrm{id}_{\mathbb{D}} \Rightarrow \Psi\Phi$ and $\varepsilon \colon \Phi\Psi \Rightarrow \mathrm{id}_{\mathbb{E}}$ satisfying the usual triangle identities.

In particular, an adjunction (Φ, Ψ) gives rise to an adjunction of functors (Φ_K, Ψ_K) for each $K \in \mathbf{Dia}$. However, this condition is not sufficient. A morphism of derivators $\Phi\colon \mathbb{D} \to \mathbb{E}$ may admit a right adjoint to $\Phi_K\colon \mathbb{D}(K) \to \mathbb{E}(K)$ for all $K \in \mathbf{Dia}$, but part of the data of a right adjoint morphism of derivators is the structure isomorphisms, which we have no way of recovering in this general situation.

Let us name these levelwise right adjoints Ψ_K, hoping to assemble them to a morphism of derivators $\Psi\colon \mathbb{E} \to \mathbb{D}$. For all $u\colon J \to K$, we need to come up with a structure isomorphism in the following square:

$$\begin{tikzcd}
\mathbb{E}(K) \arrow[r, "\Psi_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\gamma_u^\Psi}{\Swarrow}"] & \mathbb{D}(K) \arrow[d, "u^*"] \\
\mathbb{E}(J) \arrow[r, "\Psi_J"'] & \mathbb{D}(J)
\end{tikzcd}$$

Using the fact that (Φ_K, Ψ_K) and (Φ_J, Ψ_J) are adjunctions, we may take the left mate of this square (after flipping it for convenience):

$$\begin{tikzcd}
\mathbb{D}(K) \arrow[r, "\Phi_K"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{E}(K) \arrow[r, "u^*"] \arrow[d, "\Psi_K"] \arrow[dr, phantom, "\overset{\gamma_u^\Psi}{\Nearrow}"] & \mathbb{E}(J) \arrow[d, "\Psi_J"] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\Nearrow"] & \\
& \mathbb{D}(K) \arrow[r, "u^*"'] & \mathbb{D}(J) \arrow[r, "\Phi_J"'] & \mathbb{E}(J)
\end{tikzcd}$$

This gives us a transformation $\Phi_J u^* \Rightarrow u^* \Phi_K$. We are already equipped with a candidate transformation here, namely $(\gamma_u^\Phi)^{-1}$. Thus we have the notion that $(\gamma_u^\Psi)_! = (\gamma_u^\Phi)^{-1}$. Using Proposition 2.3.2(2), we may take the right mate of both these transformations and conclude that the natural choice for structure isomorphisms is $\gamma_u^\Psi = ((\gamma_u^\Psi)_!)_* = (\gamma_u^\Phi)_*^{-1}$.

Unfortunately, we do not know that γ_u^Ψ defined this way is an isomorphism. The left and right mates of γ_u^Φ have no particular properties for an arbitrary morphism Φ. However, as discussed above, if Φ is cocontinuous then $(\gamma_u^\Phi)_!^{-1}$ are natural isomorphisms for all $u\colon J \to K$ in $\mathbf{Dia}$. By Proposition 2.3.2(3), this implies that $(\gamma_u^\Phi)_*^{-1}$ are natural isomorphisms as well, which allows us to furnish the collection $\{\Psi_K\}$ with structure isomorphisms $\{\gamma_u^\Psi\}$. We conclude:

Lemma 2.5.10 (Proposition 2.9, [Gro13])**.** Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a morphism of left derivators such that each Φ_K admits a right adjoint Ψ_K. Then the collection $\{\Psi_K\}$ assemble to a morphism of derivators $\Psi\colon \mathbb{E} \to \mathbb{D}$ which is right adjoint to Φ if and only if Φ is cocontinuous.

Using these two lemmas, we can give a number of cocontinuous morphisms of derivators.

Example 2.5.11. Let $u\colon J \to K$ be a functor in **Dia**.

(1) If u admits a categorical right adjoint $v\colon K \to J$, then $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ is right adjoint to $v^*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ because (strict) 2-functors send adjunctions to adjunctions, though in our case which is left and which is right swaps (see Proposition 2.6.2 below). We can upgrade this to, for any prederivator $\mathbb{D}$, a left adjoint morphism $v^*\colon \mathbb{D}^K \to \mathbb{D}^J$ which preserves any left Kan extensions that $\mathbb{D}^K$ happens to have.

(2) If $\mathbb{D}$ is a left derivator, then the left adjoint functor $u_!\colon \mathbb{D}(J) \to \mathbb{D}(K)$ lifts to a left adjoint morphism of derivators $u_!\colon \mathbb{D}^J \to \mathbb{D}^K$ with right adjoint u^*. Similarly, if $\mathbb{D}$ is a right derivator, $u^*\colon \mathbb{D}^K \to \mathbb{D}^J$ is a left adjoint morphism of derivators.

2.6 Homotopy exact squares

At this point, we will begin to use more seriously the calculus of mates to prove a few statements that apply broadly to any derivator $\mathbb{D}$ on any diagram 2-category **Dia**. The idea is the following: the axiom Der4 has given us a class of squares in **Dia** whose left or right mates will always be natural isomorphisms. On the other hand, Proposition 2.3.2 gives us a way to compare the mates of pastings with pastings of mates. Therefore we might be able to conclude that some other squares automatically have this same property. To get specific:

Definition 2.6.1. Let $\mathbb{D}$ be a derivator. Consider the following square in **Dia**:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "p"'] & B \arrow[d, "q"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha"'] \\
C \arrow[r, "w"'] & D
\end{tikzcd}$$

We call such a square $\mathbb{D}$-*exact* if the right mate of $\mathbb{D}(\alpha) = \alpha^*$, namely $\alpha_*\colon w^*q_* \Rightarrow p_*v^*$, is a natural isomorphism. Equivalently, by Proposition 2.3.2(3), we could ask that the left mate $\alpha_!\colon w_!q^* \Rightarrow p^*v_!$ be a natural isomorphism. If a square is $\mathbb{D}$-exact for all derivators $\mathbb{D}$, we say it is *homotopy exact.*

By the axioms we have put on derivators, we already know that the following squares are homotopy exact for all $u\colon J \to K$ and all $k \in K$:

$$\begin{tikzcd}
(u/k) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
(k/u) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\beta}{\Nearrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

We have a second class of examples coming from adjunctions in **Dia**.

Proposition 2.6.2 (Proposition 1.18, [Gro13])**.** Let $\ell\colon A \to B$ be a left adjoint functor. Then the following commutative square is homotopy exact:

$$\begin{tikzcd}
A \arrow[r, "\ell"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Nearrow}"] & B \arrow[d, "\pi_B"] \\
e \arrow[r, "\mathrm{id}_e"'] & e
\end{tikzcd}$$

Dually, for any right adjoint functor $r\colon A \to B$, the following square is homotopy exact:

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Swarrow}"] & B \arrow[d, "\pi_B"] \\
e \arrow[r, "\mathrm{id}_e"'] & e
\end{tikzcd}$$

Proof. We will prove the second case, with the modifications for the first obvious. We make use of the following fact: any strict 2-functor sends adjunctions to adjunctions. The data of an adjunction is a pair of functors with a unit and counit satisfying the triangle identities. Specifically for our case: $r\colon A \to B$, $\ell\colon B \to A$, and $\eta\colon \mathrm{id}_A \Rightarrow r\ell$, $\varepsilon\colon \ell r \Rightarrow \mathrm{id}_B$ such that $\varepsilon_\ell \ell\eta = \mathrm{id}_\ell$ and $r\varepsilon\eta_r = \mathrm{id}_r$. Applying any (pre)derivator $\mathbb{D}$ gives us $r^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$, $\ell^*\colon \mathbb{D}(A) \to \mathbb{D}(B)$, and

$$\eta^*\colon \mathrm{id}_{\mathbb{D}(A)} = (\mathrm{id}_A)^* \Rightarrow (r\ell)^* = \ell^* r^*, \quad \varepsilon^*\colon r^*\ell^* = (\ell r)^* \Rightarrow (\mathrm{id}_B)^* = \mathrm{id}_{\mathbb{D}(B)}$$

The equalities above are a consequence of the strict 2-functoriality of $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$, and that the triangle equalities still hold is another consequence. However, the composition of the left and right adjoint now seem to be backwards, but this is because our domain was $\mathbf{Dia}^{\mathrm{op}}$ to begin with; the order of the adjunction is reversed. Thus the adjunction (ℓ, r) in **Dia** yields an adjunction (r^*, ℓ^*) in **CAT** for any prederivator $\mathbb{D}$.

Now, consider the image of our square after applying a derivator $\mathbb{D}$:

$$\begin{tikzcd}
\mathbb{D}(A) \arrow[dr, phantom, "\overset{\mathrm{id}^*}{\Swarrow}"] & \mathbb{D}(B) \arrow[l, "r^*"'] \\
\mathbb{D}(e) \arrow[u, "\pi_A^*"] & \mathbb{D}(e) \arrow[l, "\mathrm{id}_e^*"] \arrow[u, "\pi_B^*"']
\end{tikzcd}$$

Because r^* is now a left adjoint, we can take the right mate of this square to obtain $\mathrm{id}_*\colon \mathrm{id}_{e,*}\,\pi_B^* \Rightarrow \ell^*\pi_A^*$, where we write $\mathrm{id}_{e,*}$ for illustration but note that it is just the identity on $\mathbb{D}(e)$ because id_e^* is. But now, $\ell^*\pi_A^* = (\pi_A\ell)^*$ by strict 2-functoriality, and $\pi_A\ell\colon B \to e$ must be equal to $\pi_B\colon B \to e$ as e is the final category. The properties of the calculus of mates expressed in Proposition 2.3.2 implies that id_* expresses this equality, and thus we conclude that the square is homotopy exact. □

Corollary 2.6.3 (Lemma 1.19(1), [Gro13])**.** Suppose that B is a category with a final object $b_1 \in B$. Then for any $X \in \mathbb{D}(B)$, there is a natural isomorphism $b_1^*X \xrightarrow{\sim} \pi_{B,!}X$. Similarly, if B is a category with an initial object $b_0 \in B$, then there is a natural isomorphism $\pi_{B,*}X \xrightarrow{\sim} b_0^*X$.

The one-line proof is that the functor $b_1\colon e \to B$ is a right adjoint and $b_0\colon e \to B$ is a left adjoint.

There is a particular derivator technique in the calculus of mates that we have not used yet, but will need to use throughout this section and the remainder of this thesis. We state it below as a general lemma.

Lemma 2.6.4. Consider the following square in **Dia**:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "q"] \\
C \arrow[r, "w"'] & D
\end{tikzcd}$$

We may paste onto this square the comma category associated to any $c \in C$:

$$\begin{tikzcd}
(p/c) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(p/c)}"'] \arrow[dr, phantom, "\overset{\gamma}{\Swarrow}"] & A \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "q"] \\
e \arrow[r, "c"'] & C \arrow[r, "w"'] & D
\end{tikzcd} \tag{2.6.5}$$

Then our original square is homotopy exact if and only if the pasting

$$\begin{tikzcd}
(p/c) \arrow[r, "v\,\mathrm{pr}"] \arrow[d, "\pi_{(p/c)}"'] \arrow[dr, phantom, "\Swarrow"] & B \arrow[d, "q"] \\
e \arrow[r, "w(c)"'] & D
\end{tikzcd}$$

(with natural transformation $\alpha \odot \gamma$) is homotopy exact for every $c \in C$.

Proof. There is a small technical point that we will see proven shortly: though the pasting in **Dia** is $\alpha \odot \gamma$, after applying $\mathbb{D}$ we obtain a pasting $\gamma^* \odot \alpha^*$ because of the contravariance with respect to functors and covariance with respect to natural transformations. In our notation, we have $(\alpha \odot \gamma)_! = \gamma_! \odot \alpha_!$ by Proposition 2.3.2(1). For both directions of the proof, by Der4L we know that $\gamma_!$ is a natural isomorphism.

For the forward direction, assume our original square was homotopy exact, so that $\alpha_!$ is a natural isomorphism. Then by the reasoning above, the natural transformation $(\alpha \odot \gamma)_!$ is the pasting of two natural isomorphisms, thus is itself a natural isomorphism which proves that the pasting is homotopy exact.

We now turn to the converse, and assume that the pasting is homotopy exact. We will look at the natural transformation $(\gamma \odot \alpha)_! = \alpha_! \odot \gamma_!$ applied to some $X \in \mathbb{D}(B)$ one step at a time. Applying any derivator $\mathbb{D}$ to Diagram 2.6.5 we obtain (after rotating)

$$\begin{tikzcd}
\mathbb{D}(D) \arrow[r, "w^*"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\alpha^*\Nearrow"] & \mathbb{D}(C) \arrow[r, "c^*"] \arrow[d, "p^*"'] \arrow[dr, phantom, "\gamma^*\Nearrow"] & \mathbb{D}(e) \arrow[d, "\pi^*_{(p/c)}"] \\
\mathbb{D}(B) \arrow[r, "v^*"'] & \mathbb{D}(A) \arrow[r, "\mathrm{pr}^*"'] & \mathbb{D}((p/c))
\end{tikzcd}$$

Taking left mates, we obtain

$$\begin{tikzcd}
\mathbb{D}(B) \arrow[r, "q_!"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(D) \arrow[r, "w^*"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\alpha^*\Nearrow"] & \mathbb{D}(C) \arrow[d, "p^*"'] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\Nearrow"] & & & \\
& \mathbb{D}(B) \arrow[r, "v^*"'] & \mathbb{D}(A) \arrow[r, "p_!"'] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(C) \arrow[r, "c^*"] \arrow[d, "p^*"'] \arrow[dr, phantom, "\gamma^*\Nearrow"] & \mathbb{D}(e) \arrow[d, "\pi^*_{(p/c)}"'] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\Nearrow"] & \\
& & & \mathbb{D}(A) \arrow[r, "\mathrm{pr}^*"'] & \mathbb{D}((p/c)) \arrow[r, "\pi_{(p/c),!}"'] & \mathbb{D}(e)
\end{tikzcd}$$

The top horizontal pasting is $\alpha_!$ and the bottom is $\gamma_!$. Moreover, the transformation in the middle $p^* \Rightarrow p^* p_! p^* \Rightarrow p^*$ is just the identity by the triangle identities of the adjunction $(p_!, p^*)$, so we have not introduced anything aberrant in this process. This is the key ingredient in the proof of the compatibility of the calculus of mates with pasting.

Rewriting this diagram with $\alpha_!$ and $\gamma_!$ and rotating it to our preferred orientation we obtain

$$\begin{tikzcd}
\mathbb{D}(B) \arrow[r, "v^*"] \arrow[d, "q_!"'] \arrow[dr, phantom, "\overset{\alpha_!}{\Swarrow}"] & \mathbb{D}(A) \arrow[r, "\mathrm{pr}^*"] \arrow[d, "p_!"'] \arrow[dr, phantom, "\overset{\gamma_!}{\Swarrow}"] & \mathbb{D}((p/c)) \arrow[d, "\pi_{(p/c),!}"] \\
\mathbb{D}(D) \arrow[r, "w^*"'] & \mathbb{D}(C) \arrow[r, "c^*"'] & \mathbb{D}(e)
\end{tikzcd}$$

Starting with any object $X \in \mathbb{D}(B)$, we obtain the following commutative diagram:

$$\begin{tikzcd}
c^* w^* q_! X \arrow[r, "c^*\alpha_{!,X}"] \arrow[d, "="'] & c^* p_! v^* X \arrow[r, "\gamma_{!,v^*X}", "\cong"'] & \pi_{(p/c),!}\mathrm{pr}^* v^* X \arrow[d, "="] \\
w(c)^* q_! X \arrow[rr, "(\gamma\odot\alpha)_{!,X}", "\cong"'] & & \pi_{(p/c),!}(v\,\mathrm{pr})^* X
\end{tikzcd}$$

We have isomorphisms where indicated because we know homotopy exactness in these situations, and the vertical equalities follow from the strict 2-functoriality of any (pre)derivator. Thus $c^*\alpha_{!,X}$ must be an isomorphism for any $c \in C$ and $X \in \mathbb{D}(B)$.

In some other context, we might be stuck here, but we have at our disposal the axiom Der2. Rephrasing the axiom slightly for our purposes here, it states that a map $f\colon X_1 \to X_2$ in $\mathbb{D}(C)$ is an isomorphism if and only if $c^* f\colon c^* X_1 \to c^* X_2$ is an isomorphism in $\mathbb{D}(e)$ for all $c \in C$. We apply this axiom to $f = \alpha_{!,X}$, $X_1 = w^* q_! X$, and $X_2 = p_! v^* X$ to conclude that $\alpha_{!,X}$ is actually an isomorphism in $\mathbb{D}(C)$. Therefore $\alpha_!$ itself is a natural isomorphism without whiskering with c^* and thus the corresponding square is homotopy exact. □

Lemma 2.6.6. In the situation of Lemma 2.6.4, we might have pasted vertically instead of horizontally to obtain for any $b \in B$:

$$\begin{tikzcd}
(b/v) \arrow[r, "\pi_{(b/v)}"] \arrow[d, "\mathrm{pr}"'] & e \arrow[d, "b"] \\
A \arrow[r, "v"] \arrow[d, "p"'] & B \arrow[d, "q"] \\
C \arrow[r, "w"'] & D
\arrow[Rightarrow, from=1-2, to=2-1, shorten=6mm, "\beta"']
\arrow[Rightarrow, from=2-2, to=3-1, shorten=6mm, "\alpha"']
\end{tikzcd}$$

Then the same conclusion holds: the total pasting is homotopy exact if and only if our original square is homotopy exact.

Proof. The proof here involves instead the right mates of α and β and that the top square is homotopy exact by Der4R. As such, we rotate our vertical pasting to become a horizontal one:

$$\begin{tikzcd}
(b/v) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(b/v)}"'] & A \arrow[r, "p"] \arrow[d, "v"'] & C \arrow[d, "w"] \\
e \arrow[r, "b"'] & B \arrow[r, "q"'] & D
\arrow[Rightarrow, from=2-1, to=1-2, shorten=6mm, "\beta"]
\arrow[Rightarrow, from=2-2, to=1-3, shorten=6mm, "\alpha"]
\end{tikzcd}$$

Following a sort of reasoning dual to the proof of the lemma, one obtains a commutative diagram for any $X \in \mathbb{D}(C)$,

$$\begin{tikzcd}
b^*q^*w_*X \arrow[r, "b^*\alpha_{*,X}"] \arrow[d, "="'] & b^*v_*p^*X \arrow[r, "\beta_{*,p^*X}", "\cong"'] & \pi_{(b/v),*}\mathrm{pr}^*p^*X \arrow[d, "="] \\
q(b)^*w_*X \arrow[rr, "(\beta\odot\alpha)_{*,X}", "\cong"'] & & \pi_{(b/v),*}(p\,\mathrm{pr})^*X
\end{tikzcd}$$

This proves that $\alpha_{*,X}$ is a pointwise isomorphism, thus by Der2, α_* is a natural isomorphism.

□

Remark 2.6.7. It might have made sense to state the preceding two lemmas as one, or to present the second lemma as a corollary of the first. But there is an important difference: Lemma 2.6.4 is proven using *left* mates and Lemma 2.6.6 with *right* mates. As we noted around Remark 2.4.6, we will be focusing on half derivators for a good portion of this thesis. Since left derivators do not necessarily admit right mates (and vice versa), this is cause for concern.

Fortunately, this concern is easily allayed. We have the following proposition which tells us that there is less of a distinction between Der4L and Der4R than previous described.

Proposition 2.6.8 (Proposition 1.26, [Gro13])**.** Let $\mathbb{D}$ be a semiderivator satisfying Der3L. Then $\mathbb{D}$ satisfies Der4L if and only if for an arbitrary comma squares

$$\begin{tikzcd}
(u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J_1 \arrow[d, "u_1"] \\
J_2 \arrow[r, "u_2"'] & K
\end{tikzcd}$$

the left mate $\alpha_!$ is an isomorphism.

Proof. The backwards direction of this proof is automatic, as Der4L concerns a specific instance of the general phenomenon. Thus we assume that $\mathbb{D}$ satisfies Der4L, i.e. $\mathbb{D}$ is a left derivator. We wlll begin by using Lemma 2.6.4 and paste horizontally for some $j_2 \in J_2$:

$$\begin{tikzcd}
(\mathrm{pr}_2/j_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\swarrow}"] & (u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J_1 \arrow[d, "u_1"] \\
e \arrow[r, "j_2"'] & J_2 \arrow[r, "u_2"'] & K
\end{tikzcd}$$

We do not know that the outside square is (left) homotopy exact yet, so we will paste again on the left the comma square that would make the total pasting homotopy exact by Der4L:

$$\begin{tikzcd}
(u_1/u_2(j_2)) \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & (\mathrm{pr}_2/j_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\swarrow}"] & (u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J_1 \arrow[d, "u_1"] \\
e \arrow[r] & e \arrow[r, "j_2"'] & J_2 \arrow[r, "u_2"'] & K
\end{tikzcd} \tag{2.6.9}$$

Now the total pasting is homotopy exact by Der4L, up to describing the map r. The codomain of r consists of an object $(j_1', j_2', f\colon u_1(j_1') \to u_2(j_2'))$ of (u_1/u_2) along with a map $g\colon \mathrm{pr}_2(j_1', j_2', f) = j_2' \to j_2$. The domain of r consists of an object $j_1'' \in J_1$ and a map $h\colon u_1(j_1'') \to u_2(j_2)$ (where j_2 is fixed). We therefore define r by

$$r(j_1'', h\colon u_1(j_1'') \to u_2(j_2)) = (j_1'', j_2, h\colon u_1(j_1'') \to u_2(j_2), j_2 = j_2)$$

We claim that this map is a right adjoint. The left adjoint is given by

$$\ell(j_1', j_2', f\colon u_1(j_1') \to u_2(j_2'), g\colon j_2' \to j_2) = (j_1', u_2(g) \circ f\colon u_1(j_1') \to u_2(j_2') \to u_2(j_2))$$

To sketch the bijection on hom-sets, let us describe $\mathrm{Hom}_{(u_1/u_2(j_2))}(\ell(j_1', j_2', f, g), (j_1'', h))$. The maps are maps $a\colon j_1' \to j_1''$ in J_1 such that the following diagram commutes:

$$\begin{tikzcd}
u_1(j_1') \arrow[r, "u_1(a)"] \arrow[d, "f"'] & u_1(j_1'') \arrow[d, "h"] \\
u_2(j_2') \arrow[r, "u_2(g)"'] & u_2(j_2)
\end{tikzcd}$$

Meanwhile, maps in $\mathrm{Hom}_{(\mathrm{pr}_2/j_2)}((j_1', j_2', f, g), r(j_1'', h))$ are maps $b\colon (j_1', j_2', f) \to (j_1'', j_2, h)$ in (u_1/u_2), which themselves are certain pairs $b_1\colon j_1' \to j_1''$ and $b_2\colon j_2' \to j_2$ in J_1 and J_2 respectively, all making the following diagrams commute:

$$\begin{tikzcd}
u_1(j_1') \arrow[r, "f"] \arrow[d, "u_1(b_1)"'] & u_2(j_2') \arrow[d, "u_2(b_2)"] \\
u_1(j_1'') \arrow[r, "h"'] & u_2(j_2)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
j_2' \arrow[dr, "g"] \arrow[dd, "b_2"'] & \\
& j_2 \\
j_2 \arrow[ur, "="'] &
\end{tikzcd}$$

Commutativity of the second diagram means that $b_2 = g$ is forced, thus the data is $b_1\colon j_1' \to j_1''$ making the square commute. But this is exactly the same as the other hom-set, proving a bijection. That this bijection is natural in both arguments is easy to verify.

Since r is a right adjoint, by Proposition 2.6.2 we know that the lefthand square of Diagram 2.6.9 is homotopy exact. Thus in that diagram, the total pasting is exact, as are the middle and lefthand squares, which implies that the righthand square is homotopy exact as well, completing the proof. □

The dual statement is also true: for any right derivator $\mathbb{D}$, the right mate α_* as in the preceding proposition will be an isomorphism, which will use in the proof Lemma 2.6.6 instead. While Lemmas 2.6.4 and 2.6.6 seemed to be left- and right-dependent (respectively), the pasted square in both cases is $\mathbb{D}$-exact for any half derivator $\mathbb{D}$, so these techniques apply equally well to half derivators.

Thus for the rest of this section, 'homotopy exact' should be read as '$\mathbb{D}$-exact for any half derivator $\mathbb{D}$'. We do not wish to redefine the actual term, as there might be some strange edge cases that we have not thought up, but want to emphasize that the homotopy exact squares that we deal with below are also $\mathbb{D}$-exact for all half derivators $\mathbb{D}$.

We now prove a general recognition principle for homotopy exactness and subsequently apply to a case of particular interest. Our statements come from [GPS14], but we caution the reader that our notation differs; it is chosen to be harmonious with the rest of this section. Further, we have expanded the proofs from that paper to show more precisely the derivator technology in action.

Definition 2.6.10 (Definition 3.5, [GPS14])**.** Let $\mathbb{D}$ be a (half) derivator. A category $K \in \mathbf{Dia}$ is $\mathbb{D}$-*contractible* if the following square is $\mathbb{D}$-exact:

$$\begin{tikzcd}
K \arrow[r, "\pi_K"] \arrow[d, "\pi_K"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\swarrow}"] & e \arrow[d] \\
e \arrow[r] & e
\end{tikzcd}$$

We can check this by verifying either that the left mate $\pi_{K,!}\pi_K^* \Rightarrow \mathrm{id}_{e,!}\,\mathrm{id}_e^* \cong \mathrm{id}_{\mathbb{D}(e)}$, the counit of the $(\pi_!, \pi^*)$ adjunction, is a natural isomorphism or that the right mate $\mathrm{id}_{\mathbb{D}(e)} \cong \mathrm{id}_e^*\,\mathrm{id}_{e,*} \Rightarrow \pi_{K,*}\pi_K^*$, the unit of the (π_*, π^*) adjunction, is a natural isomorphism. We say K is *homotopy contractible* if it is $\mathbb{D}$-contractible for all (half) derivators $\mathbb{D}$.

As we have shown above, categories admitting an initial or final object are homotopy contractible. When the question of homotopy contractibility arises, our usual tactic will be to try to prove the existence of one of these objects.

Definition 2.6.11 (Definition 3.7, [GPS14])**.** Consider the following square in **Dia**:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & B \arrow[d, "q"] \\
C \arrow[r, "w"'] & D
\end{tikzcd}$$

Suppose that $b \in B$ and $c \in C$ admit a map $\delta\colon q(b) \to w(c)$ in D. In this case, we define the double comma category $(b/A/c)_\delta$ as follows: its objects are triples $a \in A$ with two maps $f\colon b \to v(a)$ and $g\colon p(a) \to c$ in B and C respectively such that the following square commutes:

$$\begin{tikzcd}
q(b) \arrow[r, "\delta"] \arrow[d, "q(f)"'] & w(c) \\
q(v(a)) \arrow[r, "\alpha_a"'] & w(p(a)) \arrow[u, "w(p)"']
\end{tikzcd}$$

A map $(a, f, g) \to (a', f', g')$ is a map $a \to a'$ in A compatible with f, f' and g, g' in the obvious sense.

Proposition 2.6.12 (Theorem 3.8, [GPS14])**.** In the notation of Definition 2.6.11, the square is homotopy exact if the category $(b/A/c)_\delta$ is homotopy contractible for every possible triple $b \in B$, $c \in C$, and $\delta\colon q(b) \to w(c)$.

Proof. The idea is that these categories are measuring the possible failures of homotopy exactness, so if they are all homotopy contractible we have no problem. To construct the double comma category in relation to our starting square, we first paste on the left by the homotopy exact comma square associated to some $c \in C$:

$$\begin{tikzcd}
(p/c) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi_{(p/c)}"'] \arrow[dr, phantom, "\overset{\beta}{\Swarrow}"] & A \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "q"] \\
e \arrow[r, "c"'] & C \arrow[r, "w"'] & D
\end{tikzcd}$$

By Lemma 2.6.4, to prove that α is homotopy exact it suffices to show that the outside pasting is homotopy exact.

To do so, we now consider the composite $(p/c) \to B$ of the top arrow, and paste vertically by the homotopy exact comma square associated to some $b \in B$:

$$\begin{tikzcd}
(b/A/c) \arrow[rr, "\pi_{(b/A/c)}"] \arrow[d, "\mathrm{pr}_2"'] & \overset{\gamma}{\Swarrow} & e \arrow[d, "b"] \\
(p/c) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi_{(p/c)}"'] \arrow[dr, phantom, "\overset{\beta}{\Swarrow}"] & A \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "q"] \\
e \arrow[r, "c"'] & C \arrow[r, "w"'] & D
\end{tikzcd} \tag{2.6.13}$$

By Lemma 2.6.6, the bottom rectangle (i.e. the pasting $\beta \odot \alpha$) is homotopy exact if and only if the outside pasting is exact.

We have written $(b/A/c)$ for the comma category $(b/v\,\mathrm{pr}_1)$, but should justify that claim before moving on. An object in $(b/v\,\mathrm{pr}_1)$ is a pair $(a, f\colon p(a) \to c) \in (p/c)$ and a map $g\colon b \to v(\mathrm{pr}(a, f)) = v(a)$ in B. Maps in this comma category are maps under b that come from (p/c), i.e. maps in A that are both under b and over c. We have baptised this category $(b/A/c)$ accordingly, noting the similarity to the comma categories $(b/A/c)_\delta$ under consideration.

So we now look at the total pasting of the diagram:

$$\begin{tikzcd}
(b/A/c) \arrow[r, "\pi_{(b/A/c)}"] \arrow[d, "\pi_{(b/A/c)}"'] \arrow[dr, phantom, "\overset{\delta}{\Swarrow}"] & e \arrow[d, "q(b)"] \\
e \arrow[r, "w(c)"'] & D
\end{tikzcd} \tag{2.6.14}$$

The natural transformation $\delta = \beta \odot \alpha \odot \gamma$ evaluated at an object $(a, f, g) \in (b/A/c)$ is the map in D given by

$$q(b) \xrightarrow{q(f)} q(v(a)) \xrightarrow{\alpha_a} w(p(a)) \xrightarrow{w(g)} w(c) \tag{2.6.15}$$

Putting everything above together, the proposition holds if we can show Diagram 2.6.14 is homotopy exact, which we can achieve by proving $\delta_!$ is a natural isomorphism.

We will analyse Diagram 2.6.14 from another angle, and for now we will name fewer of our natural transformations. Consider the comma category $(q(b)/w(c))$ which sits in the homotopy exact square

$$\begin{tikzcd}
(q(b)/w(c)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\text{ex}}{\Swarrow}"] & e \arrow[d, "q(b)"] \\
e \arrow[r, "w(c)"'] & D
\end{tikzcd}$$

We note that this category is discrete, equal to the set $\mathrm{Hom}_D(q(b), w(c))$, as any maps in the comma category are induced by maps in e, which has no nontrivial maps. The category $(b/A/c)$ has a natural forgetful functor to $(q(b), w(c))$ which remembers only the composed map of Equation 2.6.15. Call this functor $\mathrm{pr}_{b,c}$. We may therefore split Diagram 2.6.14 as the composite

$$\begin{tikzcd}
(b/A/c) \arrow[r, "\pi_{(b/A/c)}"] \arrow[d, "\pi_{(b/A/c)}"'] \arrow[dr, phantom, "\overset{\delta}{\Swarrow}"] & e \arrow[d, "q(b)"] \\
e \arrow[r, "w(c)"'] & D
\end{tikzcd}
=
\begin{tikzcd}
(b/A/c) \arrow[r, "\pi_{(b/A/c)}"] \arrow[d, "\mathrm{pr}_{b,c}"'] \arrow[dr, phantom, "\overset{\epsilon}{\Swarrow}"] & e \arrow[d] \\
(q(b)/w(c)) \arrow[r, "\pi_{(q(b)/w(c))}"] \arrow[d, "\pi_{(q(b)/w(c))}"'] \arrow[dr, phantom, "\overset{\text{ex}}{\Swarrow}"] & e \arrow[d, "q(b)"] \\
e \arrow[r, "w(c)"'] & D
\end{tikzcd} \tag{2.6.16}$$

The last step in our proof is to prove that the top square with the natural transformation ϵ is homotopy exact. Once we know this, by Proposition 2.3.2(1), Diagram 2.6.14 is the pasting of two homotopy exact squares, therefore is itself homotopy exact, which we showed above implies the proposition.

We will now paste one more time so that we may (finally) use our hypothesis. Consider any map $\delta\colon q(b) \to w(c)$ in D. This is an object of $(q(b)/w(c))$, so there is a classifying functor $\delta\colon e \to (q(b)/w(c))$. If we take the comma category corresponding to $\mathrm{pr}_{b,c}$ over δ we may paste in our final homotopy exact square

$$\begin{tikzcd}
(b/A/c)_\delta \arrow[r, "\mathrm{pr}_3"] \arrow[d, "\pi_{(b/A/c)_\delta}"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\Swarrow}"] & (b/A/c) \arrow[r, "\pi_{(b/A/c)}"] \arrow[d, "\mathrm{pr}_{b,c}"'] \arrow[dr, phantom, "\overset{\epsilon}{\Swarrow}"] & e \arrow[d] \\
e \arrow[r, "\delta"'] & (q(b)/w(c)) \arrow[r, "\pi_{(q(b)/w(c))}"'] & e
\end{tikzcd}$$

Why can we identify $(b/A/c)_\delta = (\mathrm{pr}_{b,c}/\delta)$? The data of an object in the comma category is a triple $(a, f, g) \in (b/A/c)$ and a map in $(q(b)/w(c))$ from the composite of Equation 2.6.15 to δ. But $(q(b)/w(c))$ is a discrete category, which implies that the composite must actually be equal to δ, which was the precise definition of the objects in $(b/A/c)_\delta$. The maps in both $(b/A/c)_\delta$ and $(\mathrm{pr}_{b,c}/\delta)$ must be over δ, which shows that we have identical categories.

We apply Lemma 2.6.4 one more time: the righthand square is homotopy exact if and only if the total pasting is homotopy exact for any $\delta \in (q(b), w(c))$:

$$\begin{tikzcd}
(b/A/c)_\delta \arrow[r, "\pi_{(b/A/c)_\delta}"] \arrow[d, "\pi_{(b/A/c)_\delta}"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Swarrow}"] & e \arrow[d] \\
e \arrow[r] & e
\end{tikzcd}$$

This is precisely the condition that $(b/A/c)_\delta$ is homotopy contractible. Now all the squares we had hoped were homotopy exact are proven to be so, and we are done. $\square$

We will now work on applications of Proposition 2.6.12 that will arise in the rest of this thesis.

Definition 2.6.17. Let $v\colon A \to B$ be a functor. We say that v is *homotopy final* if for every $b \in B$, the comma category (b/v) is homotopy contractible. We say that v is *homotopy initial* if for every $b \in B$, the comma category (v/b) is homotopy contractible.

The usual context of (homotopy) final and initial functors is that pulling back along them preserves colimits and limits (respectively). We will now see that this is the case in derivators as well.

Proposition 2.6.18. Let $v\colon A \to B$ be a homotopy final functor. Then the following square is homotopy exact:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "\pi_A"'] & B \arrow[d, "\pi_B"] \arrow[dl, Rightarrow, shorten <=2ex, shorten >=2ex, "\mathrm{id}"'] \\
e \arrow[r] & e
\end{tikzcd}$$

Dually, if $v\colon A \to B$ is homotopy initial, then the following square is homotopy exact:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "\pi_A"'] & B \arrow[d, "\pi_B"] \\
e \arrow[r] \arrow[ur, Rightarrow, shorten <=2ex, shorten >=2ex, "\mathrm{id}"] & e
\end{tikzcd}$$

Proof. In the first case, it suffices by Proposition 2.6.12 to show that $(b/A/*)_\delta$ is homotopy contractible for any $b \in B$, $* \in e$, and $\delta\colon \pi_B(b) \to \mathrm{id}_e(*)$ in e. An object in $(b/A/*)_\delta$ will be a triple (a, f, g) with $a \in A$, $f\colon b \to v(a)$ in B, and $g\colon \pi_A(a) \to *$ in e such that the composite

$$\pi_B(b) \xrightarrow{\pi_B(f)} \pi_B(v(a)) \xrightarrow{=} \mathrm{id}_e(\pi_A(a)) \xrightarrow{\mathrm{id}_e(g)} \mathrm{id}_e(*)$$

is the map δ. All this is much ado about nothing: all the objects in the above diagram are the sole object $* \in e$ and all maps must be the sole map $\mathrm{id}_*\colon * \to *$ in e. Thus the only piece of data that matters in $(b/A/*)_\delta$ is the map $f\colon b \to v(a)$, as the map g must also be id_*. We therefore identify $(b/A/*)_\delta = (b/v)$ and obtain the proposition. The proof in the other case is dual. ☐

Reading off literally what it means for the above squares to be homotopy exact, for any homotopy final $v\colon A \to B$ and any $X \in \mathbb{D}(B)$, we have an isomorphism

$$\mathrm{id}_{!,X}\colon \pi_{A,!}v^*X \xrightarrow{\cong} \pi_{B,!}X$$

which gives us the appropriate derivator extension of the 1-categorical definition.

There is a third class of homotopy exact squares that will conclude our discussion.

Proposition 2.6.19 (Proposition 1.20, [Gro13])**.** Let $u\colon J \to K$ be a fully faithful functor. Then the following commutative square in **Dia** is homotopy exact:

$$\begin{tikzcd}
J \arrow[r, "\mathrm{id}_J"] \arrow[d, "\mathrm{id}_J"'] \arrow[dr, phantom, "\overset{\mathrm{id}_u}{\Swarrow}"] & J \arrow[d, "u"] \\
J \arrow[r, "u"'] & K
\end{tikzcd}$$

Specifically, the left mate is the counit of the (u^*, u_*) adjunction and right mate is the unit of the (u^*, u_*) adjunction (as we noted in Definition 2.6.10). Since these are natural isomorphisms, we have that $u_*, u_!\colon \mathbb{D}(J) \to \mathbb{D}(K)$ are both fully faithful.

Proof. We will use a similar proof to that of Proposition 2.6.12. We start by pasting the homotopy exact comma square on the left associated to some $j \in J$:

$$\begin{tikzcd}
(\mathrm{id}_J/j) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(\mathrm{id}_J/j)}"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\Swarrow}"] & J \arrow[r, "\mathrm{id}_J"] \arrow[d, "\mathrm{id}_J"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Swarrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "j"'] & J \arrow[r, "u"'] & K
\end{tikzcd}$$

We now want to think about the category (id_J/j). Its objects are $j' \in J$ along with a map $f\colon j' \to j$, and its maps are morphisms over j. The functor $u\colon J \to K$ induces a functor $(\mathrm{id}_J/j) \to (u/u(j))$, where (j', f) gets mapped to $(j', u(f))$. This map is an isomorphism of categories because u is fully faithful: any map $u(j') \to u(j)$ in K must come from a map $j' \to j$ in J and uniquely so. Let $v\colon (u/u(j)) \to (\mathrm{id}_J/j)$ be the inverse of this functor. Pasting again, we have

$$\begin{tikzcd}
(u/u(j)) \arrow[r, "v", "\cong"'] \arrow[d, "\pi_{(u/u(j))}"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\Swarrow}"] & (\mathrm{id}_J/j) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(\mathrm{id}_J/j)}"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\Swarrow}"] & J \arrow[r, "\mathrm{id}_J"] \arrow[d, "\mathrm{id}_J"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Swarrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "\mathrm{id}_e"'] & e \arrow[r, "j"'] & J \arrow[r, "u"'] & K
\end{tikzcd} \tag{2.6.20}$$

The square we have pasted is homotopy exact because v is an isomorphism, so in particular a right adjoint functor and Proposition 2.6.2 applies. Using two applications of Lemma 2.6.4, our original square is homotopy exact if and only if the total pasting is homotopy exact.

What is this total horizontal pasting? It is

$$\begin{tikzcd}
(u/u(j)) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\Swarrow}"] & J \arrow[d, "u"] \\
e \arrow[r, "u(j)"'] & K
\end{tikzcd}$$

which we identify as the Der4L comma square associated to $u(j) \in K$. This square is therefore homotopy exact, and we are done. $\square$

Remark 2.6.21. We emphasize one last time that this proof works just as well for half derivators, so any left or right Kan extensions along fully faithful functors that exist for a half derivator $\mathbb{D}$ will also be fully faithful.

2.7 Pointed derivators

The derivators we study in this thesis will satisfy an additional axiom. For now, we restrict our attention to full derivators $\mathbb{D}$ on an arbitrary diagram 2-category **Dia**.

Definition 2.7.1. A derivator $\mathbb{D}$ is *pointed* if its underlying category $\mathbb{D}(e)$ is pointed, i.e. the unique morphism from the initial to the final object is an isomorphism. We will write $0 \in \mathbb{D}(e)$ for its zero object.

Example 2.7.2.

(1) For a bicomplete category $\mathcal{C}$, the represented derivator $\mathbb{D}_{\mathcal{C}}$ is a pointed derivator if $\mathcal{C}$ is pointed as a category.

(2) For a small Grothendieck abelian category $\mathcal{A}$, the derivator $\mathbb{D}_{\mathcal{A}}$ is a pointed derivator.

(3) For a combinatorial model category $\mathcal{M}$, the derivator $\mathbb{D}_{\mathcal{M}}$ is pointed if $\mathcal{M}$ is pointed.

Note that $\mathbb{D}(e)$ admits an initial and a final object in any derivator $\mathbb{D}$. Let $\varnothing$ denote the empty category. By Der1, we have an equivalence of categories

$$\mathbb{D}(\varnothing) = \mathbb{D}(\varnothing \sqcup \varnothing) \xrightarrow{\sim} \mathbb{D}(\varnothing) \times \mathbb{D}(\varnothing)$$

which implies that $\mathbb{D}(\varnothing)$ is equivalent to e. Let $\pi_\varnothing \colon \varnothing \to e$ be the unique functor in **Dia**. Then the left or right Kan extension along $\pi_\varnothing$ picks out a single object in $\mathbb{D}(e)$, which are the final and initial objects (respectively). Whether these are isomorphic is again a property of the derivator $\mathbb{D}$ (specifically, a property of its underlying category).

This property used to be called *weakly pointed*. There is an obvious way to strengthen this axiom: we ask that for all $K \in \mathbf{Dia}$, $\mathbb{D}(K)$ is a pointed category, and that $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ is a pointed morphism for any $u\colon J \to K$. If $\mathbb{D}$ is a derivator, this is automatic. Let $\pi_K\colon K \to e$ be the projection to the point. Then the pullback π_K^* is both a left and right adjoint in any derivator $\mathbb{D}$, so $0_K := \pi_K^*(0)$ should be both an initial and final object in $\mathbb{D}(K)$, meaning that $\mathbb{D}(K)$ is also pointed. Similarly, $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ is both a left and right adjoint, so it preserves initial and final objects, so it sends 0_K to 0_J. In fact, $u_!, u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ are also pointed because each is an adjoint functor.

There is also a notion of *strongly pointed*, which was originally called Der6 in [Mal07]. In order to state it, we need to recall two particular classes of functors in **Dia**.

Definition 2.7.3. Let $u\colon J \to K$ be a fully faithful functor that is injective on objects.

(1) The functor u is a *sieve* if for any morphism $k \to u(j)$ in K, k lies in the image of u.

(2) The functor u is a *cosieve* if for any morphism $u(j) \to k$ in K, k lies in the image of u.

Definition 2.7.4. [Mal07, p.6] A derivator $\mathbb{D}$ is *strongly pointed* if for every sieve (resp. cosieve) $u\colon J \to K$ in **Dia**, u_* (resp. $u_!$) admits a right adjoint $u^!$ (resp. left adjoint $u^?$).

Asking for these exceptional adjoints is confusing until we know more about the right Kan extension along a sieve or the left Kan extension along a cosieve. We will examine the first case, since it occurs more often in this thesis.

Let $\mathbb{D}$ be a pointed derivator, $u\colon J \to K$ a sieve, and $X \in \mathbb{D}(J)$. Then we can examine u_*X pointwise using Der4R, which involves understanding the comma category (k/u) for all $k \in K$. Recall that its objects are pairs $j \in J$ with a morphism $k \to u(j)$ and its maps are maps in J under k.

Suppose that k is not in the image of u. Then because u is a sieve, there cannot be any maps $k \to u(j)$ for any $j \in J$, so the comma category (k/u) is empty. Hence

$$k^*u_*X \cong \pi_{\varnothing,*}\mathrm{pr}^*X \cong 0$$

because the right Kan extension along $\pi_\varnothing\colon \varnothing \to e$ we have established gives the zero object.

Now if k is in the image of u, write $k = u(j')$ and consider any object $(j, f\colon k \to u(j))$ in the comma category (k/u). Because u is fully faithful, any map $f\colon k = u(j') \to u(j)$ in K must be the image of a map $f'\colon j' \to j$ in J. Thus we can consider (k/u) to have objects $(j, f'\colon j' \to j)$ and maps in J under j'. This category admits the initial object $(j', \mathrm{id}_{j'})$. For any $(j, f'\colon j' \to j)$ we have the unique map from $(j', \mathrm{id}_{j'})$ given by

$$\begin{tikzcd} & j' \arrow[dl, "="'] \arrow[dr, "f'"] & \\ j' \arrow[rr, "f'"'] & & j \end{tikzcd}$$

We can therefore apply Corollary 2.6.3. This gives us the following chain of isomorphisms:

$$k^*u_*X \xrightarrow{\cong} \pi_{(k/u),*}\mathrm{pr}^*X \xrightarrow{\cong} \pi_{e,*}(j', \mathrm{id}_{j'})^*\mathrm{pr}^*X$$

We have no clear sense of what pr^*X looks like, but we know the composite $\mathrm{pr}(j', \mathrm{id}_{j'})\colon e \to J$ is the inclusion of the object j'. Hence $(j', \mathrm{id}_{j'})^*\mathrm{pr}^*X = j'^*X$. Finally, since $\pi_e\colon e \to e$ is just a fancier way of writing id_e, we have

$$k^*u_*X \cong \pi_{e,*}(j', \mathrm{id}_{j'})^*\mathrm{pr}^*X = \mathrm{id}_{e,*}\, j'^*X \cong j'^*X$$

Thus when $k \in K$ is in the image of u, the value of u_*X at k is exactly what it was in $\mathbb{D}(J)$ under the fully faithful inclusion $u\colon J \to K$.

Combining the above with Proposition 2.6.19, we can sum up the above work (similar to [Gro13, Proposition 1.23]).

Proposition 2.7.5. Let $\mathbb{D}$ be a pointed derivator, and let $u\colon J \to K$ be a sieve (resp. cosieve). Then $u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ (resp. $u_!$) is fully faithful, with essential image $X \in \mathbb{D}(K)$ such that $k^*X \cong 0$ for all $k \in K \setminus u(J)$.

We call these *extension by zero morphisms*. We have the following surprising result:

Proposition 2.7.6 (Corollaries 3.5 and 3.8,[Gro13])**.** A derivator is pointed if and only if it is strongly pointed.

The backwards direction is easy: using the exceptional adjoints to the (co)sieve $\varnothing \to e$, we can show that the initial object is also final. The forwards direction is difficult. We will need details of the construction later on in Chapter 3, and will address them there.

2.8 Stable derivators

We now need to define what we mean by stable in the theory of derivators. To begin, any pointed derivator admits an intrinsic notion of suspension and loop endofunctors.

Notation 2.8.1. Let $\square$ be the category

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

Let $i_{\ulcorner} \colon \ulcorner \to \square$ be the full subcategory lacking the element $(1,1)$ and $i_{\lrcorner} \colon \lrcorner \to \square$ the full subcategory lacking $(0,0)$.

Definition 2.8.2. Let $\mathbb{D}$ be a pointed derivator and $X \in \mathbb{D}(\square)$. We say that X is *cocartesian* (i.e. a pushout square) if X is in the essential image of $i_{\ulcorner,!} \colon \mathbb{D}(\ulcorner) \to \mathbb{D}(\square)$. We say that X is *cartesian* (i.e. a pullback square) if X is in the essential image of $i_{\lrcorner,*}$.

Definition 2.8.3. Let $\mathbb{D}$ be a pointed derivator. Define the *suspension* endomorphism $\Sigma \colon \mathbb{D} \to \mathbb{D}$ by the composition

$$\mathbb{D} \xrightarrow{(0,0)_*} \mathbb{D}^{\ulcorner} \xrightarrow{i_{\ulcorner,!}} \mathbb{D}^{\square} \xrightarrow{i_{\lrcorner}^*} \mathbb{D}^{\lrcorner} \xrightarrow{(1,1)^*} \mathbb{D}.$$

Define the *loop* endomorphism $\Omega \colon \mathbb{D} \to \mathbb{D}$ by the composition

$$\mathbb{D} \xrightarrow{(1,1)_!} \mathbb{D}^{\lrcorner} \xrightarrow{i_{\lrcorner,*}} \mathbb{D}^{\square} \xrightarrow{i_{\ulcorner}^*} \mathbb{D}^{\ulcorner} \xrightarrow{(0,0)^*} \mathbb{D}.$$

Note that $(0,0) \colon e \to \ulcorner$ is a sieve, so the right Kan extension along this morphism is extension by zero (see Proposition 2.7.5). Suspension of an object $x \in \mathbb{D}$ is thus defined to be

$$\begin{tikzcd}
 & x \arrow[r] \arrow[dd] & 0 & & x \arrow[r] \arrow[dd] & 0 \arrow[dd] & & & 0 \arrow[dd] & \\
x \mapsto & & & \mapsto & & & \mapsto & & & \mapsto \Sigma x \\
 & 0 & & & 0 \arrow[r] & \Sigma x & & 0 \arrow[r] & \Sigma x &
\end{tikzcd}$$

where the square in the middle is cocartesian. Loop is defined analogously. We write the restriction in two steps to motivate the following proposition:

Proposition 2.8.4 (Proposition 3.17, [Gro13])**.** Let $\mathbb{D}$ be a pointed derivator. Then (Σ, Ω) is an adjunction of endomorphisms of $\mathbb{D}$.

Remark 2.8.5. Technically we can define suspension and loop for any derivator, not just pointed ones. However, if the initial and final object do not coincide, then the 'extension by zero' morphisms $(0,0)_*$ and $(1,1)_!$ behave a bit more strangely.

Definition 2.8.6. A pointed derivator $\mathbb{D}$ is *stable* if (Σ, Ω) is an adjoint equivalence of derivators.

There are other perspectives on stability in derivators. Groth in [Gro16a] has explored some equivalent definitions of stability, and the interested reader is directed there. One that is familiar from other contexts is the following:

Proposition 2.8.7 (Theorem 3.1, [Gro16a])**.** A pointed derivator $\mathbb{D}$ is stable if and only if cocartesian and cartesian squares coincide.

That is, we only have bicartesian squares in a stable derivator.

Remark 2.8.8. Why should we care about stable derivators? If a derivator is both stable and strong, we call it *triangulated* for the following reason:

Proposition 2.8.9 (Theorem 4.16, Corollary 4.19, [Gro13])**.** Let $\mathbb{D}$ be a triangulated derivator. For each $K \in \mathbf{Dia}$, the category $\mathbb{D}(K)$ has a canonical triangulated structure using the suspension functor Σ_K. Moreover, all functors $u_!, u^*, u_*$ respect the triangulated structure.

Derivators as an enhancement of triangulated categories is an important part of the history of the subject. The derivators associated to a stable model category $\mathbb{D}_{\mathcal{M}} \colon K \mapsto \mathrm{Ho}(\mathrm{Fun}(K, \mathcal{M}))$ and to an abelian category $\mathbb{D}_{\mathcal{A}} \colon K \mapsto D^b(\mathrm{Fun}(K, \mathcal{A}))$ are triangulated. For our purposes, however, we care less about the triangulated structure and more about stability per se.

In Chapter 3, we will prove how pointed derivators satisfying a relatively mild hypothesis admit a canonical stabilization.

CHAPTER 3

Stabilization

3.1 Agenda

The stabilization of derivators was first addressed by Alex Heller in [Hel97], where derivators are considered with slightly different axioms under the name ‘homotopy theories’. As we remarked in the introduction, Heller’s paper is essentially correct, but has major gaps that we repaired in [Col19]. The results of that paper is the content of this chapter.

The stabilization of a pointed derivator occurs in a few steps. First, we will need to establish some results in the localization theory of derivators and prove a general construction of a new class of pointed derivators. We then define the prederivator of prespectrum objects correspond for a pointed derivator $\mathbb{D}$, which falls under the framework of our preceding general construction and thus is a pointed derivator. Inside the derivator of prespectrum objects there is a subprederivator of (stable) spectrum objects, and we show that it arises as a localization of the prespectrum derivator (using the mild assumption of *regularity*). At last we show that the above work assembles into a statement at the level of 2-categories of derivators.

3.2 Localization of derivators

There is no one resource for this topic at the moment, but we will try to summarize the theory as it relates to our ultimate goal of stabilization. For a general reference on localizations of categories, see [GZ67, §1] or [Kra10].

Let $\mathcal{C}$ be an ordinary category. If we have some class of morphisms $S \subset \mathcal{C}^{[1]}$ that we would like to invert, we can ask whether there is a category $\mathcal{C}[S^{-1}]$ and functor $L_S \colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ inverting S and admitting a fully faithful left or right adjoint. The localization functor L_S and the category $\mathcal{C}[S^{-1}]$ are essentially unique and satisfy a universal property in **CAT**. There are more specialized localization theories in, for example, model categories or triangulated categories.

However, we are going to work with a broad class of derivators so will not necessarily have more refined machinery. Moreover, we will care less about starting with a class of morphisms in $\mathbb{D}(e)$, but will have some subcategory of $\mathbb{D}(e)$ which we would like to be reflective or coreflective. If this is the case, we obtain a (co)localization of $\mathbb{D}(e)$ onto that subcategory and can compute which morphisms have been inverted.

Let $\mathbb{D}$ be a derivator. A prederivator $\mathbb{E}$ is called a *full subprederivator* of $\mathbb{D}$ if there is a morphism $\iota \colon \mathbb{E} \to \mathbb{D}$ which is levelwise fully faithful. There is no reason for $\mathbb{D}$ to reflect any of its properties onto $\mathbb{E}$, but there is a straightforward criterion.

Lemma 3.2.1. [Cis08, Lemme 4.2] Let $\mathbb{E}$ be a full subprederivator of a derivator $\mathbb{D}$. Assume that the morphism $\iota \colon \mathbb{E} \to \mathbb{D}$ admits either a left adjoint L or right adjoint R. Then $\mathbb{E}$ is also a derivator.

If ι admits a left adjoint, we call $\mathbb{E}$ a *localization* of $\mathbb{D}$, and if ι admits a right adjoint we call it a *colocalization*. We give the proof of this lemma because Cisinski's result uses terminology different from ours, contains several typographical errors, and is in French.

Proof. Let us prove Der1 and Der2 first, which do not depend on which side the adjoint is. For Der1, consider the following commutative diagram for $K_i \in \mathbf{Dia}$:

$$\begin{tikzcd}
\mathbb{E}\left(\coprod_{i\in I} K_i\right) \arrow[r, "\iota_{\coprod K_i}"] \arrow[d, "\prod j_i^*"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}\left(\coprod_{i\in I} K_i\right) \arrow[r, "A_{\coprod K_i}"] \arrow[d, "\prod j_i^*"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{E}\left(\coprod_{i\in I} K_i\right) \arrow[d, "\prod j_i^*"] \\
\prod_{i\in I} \mathbb{E}(K_i) \arrow[r, "\prod \iota_{K_i}"'] & \prod_{i\in I} \mathbb{D}(K_i) \arrow[r, "\prod A_{K_i}"'] & \prod_{i\in I} \mathbb{E}(K_i)
\end{tikzcd}$$

where A is the adjoint to ι. Because $\mathbb{D}$ satisfies Der1, the middle vertical functor is an equivalence of categories. We would like to prove that the other vertical functor is also an equivalence, so we will show it is fully faithful and essentially surjective.

First we examine the lefthand square. The functor $\iota_{\coprod K_i}$ is fully faithful by assumption, hence the top composition $\prod j_i^* \circ \iota_{\coprod K_i}$ is fully faithful. Thus the bottom composition $\prod \iota_{K_i} \circ \prod j_i^*$ is also fully faithful, and by assumption $\prod \iota_{K_i}$ is fully faithful. For any $X, Y \in \mathbb{E}(\coprod K_i)$, we have isomorphisms

$$\begin{tikzcd}
 & \mathrm{Hom}_{\prod \mathbb{E}(K_i)}(\prod j_i^* X, \prod j_i^* Y) \arrow[dr, "\prod \iota_{K_i}", "\cong"'] & \\
\mathrm{Hom}_{\mathbb{E}(\coprod K_i)}(X, Y) \arrow[ur, "\prod j_i^*"] \arrow[rr, "\cong"'] & & \mathrm{Hom}_{\prod \mathbb{D}(K_i)}(\prod \iota_{K_i}(\prod j_i^* X), \prod \iota_{K_i}(\prod j_i^* Y))
\end{tikzcd}$$

which proves that $\prod j_i^*$ induces an isomorphism on hom-sets, i.e. is fully faithful.

Next we examine the righthand square. Whether A is the left or right adjoint to ι, it is levelwise essentially surjective. For $A = L$, because ι_K is fully faithful, the counit of the adjunction $L_K \iota_K \Rightarrow \mathrm{id}_{\mathbb{E}(K)}$ is an isomorphism. Hence for any $X \in \mathbb{E}(K)$, we have $X \cong L_K(\iota_K X)$, so X is in the essential image of L_K. For $A = R$, the unit of the adjunction is an isomorphism and the same result follows.

The bottom composition $\prod A_{K_i} \circ \prod j_i^*$ is thus essentially surjective, which implies the top composition $\prod j_i^* \circ A_{\coprod K_i}$ is too. Thus for any $Z \in \prod \mathbb{E}(K_i)$, there exists some $X \in \mathbb{D}(\coprod K_i)$ such that $\prod j_i^*(A_{\coprod K_i} X) \cong Z$. In particular, the object $A_{\coprod K_i} X = Y \in \mathbb{E}(\coprod K_i)$ satisfies $\prod j_i^* Y \cong Z$, so that $\prod j_i^*$ is essentially surjective as well. Having shown that $\prod j_i^*$ is both fully faithful and essentially surjective, it is an equivalence of categories and thus $\mathbb{E}$ satisfies Der1.

For Der2, suppose that $f \colon X \to Y$ is a map in $\mathbb{E}(K)$. We need to show that f is an isomorphism if and only if $k^* f \colon k^* X \to k^* Y$ is an isomorphism for all $k \in K$. The image $\iota_K f$ in $\mathbb{D}(K)$ is an isomorphism if and only if f is an isomorphism as ι_K is fully faithful. Further, $k^* \iota_K f \cong \iota_e k^* f$ as ι is a morphism of (pre)derivators, and $\iota_e k^* f$ is an isomorphism if and only if $k^* f$ is an isomorphism. Putting all these implications together, we obtain Der2 for $\mathbb{E}$.

Now we prove that $\mathbb{E}$ admits left and right Kan extensions, and at this point we assume

$A = L$ is the left adjoint, the dual case being similar but not identical. Let $u\colon J \to K$ be a functor in **Dia**, $X \in \mathbb{E}(J)$ and $Y \in \mathbb{E}(K)$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{E}(J)}(X, u^*Y) &\cong \mathrm{Hom}_{\mathbb{D}(J)}(\iota_J X, \iota_J u^* Y) \\
&\cong \mathrm{Hom}_{\mathbb{D}(J)}(\iota_J X, u^* \iota_K Y) \\
&\cong \mathrm{Hom}_{\mathbb{D}(K)}(u_! \iota_J X, \iota_K Y) \\
&\cong \mathrm{Hom}_{\mathbb{E}(K)}(L_K u_! \iota_J X, Y)
\end{aligned}$$

The first isomorphism is due to ι_J being fully faithful, the second because ι is a morphism of prederivators, the third because $\mathbb{D}$ admits left Kan extensions, and the fourth because L_K is left adjoint to ι_K. Hence we have constructed a left adjoint in $\mathbb{E}$ to u^*.

For the right Kan extension, we have the following chain of isomorphisms. Using the same notation as above,

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{E}(J)}(u^*Y, X) &\cong \mathrm{Hom}_{\mathbb{D}(J)}(\iota_J u^* Y, \iota_J X) \\
&\cong \mathrm{Hom}_{\mathbb{D}(J)}(u^* \iota_K Y, \iota_J X) \\
&\cong \mathrm{Hom}_{\mathbb{D}(K)}(\iota_K Y, u_* \iota_J X) \qquad (3.2.2) \\
&\longrightarrow \mathrm{Hom}_{\mathbb{D}(K)}(\iota_K Y, \iota_K L_K u_* \iota_J X) \\
&\cong \mathrm{Hom}_{\mathbb{E}(K)}(Y, L_K u_* \iota_J X)
\end{aligned}$$

We have included here one arrow, because the argument differs here. That map is induced by composition with the unit $\mathrm{id}_{\mathbb{D}(K)} \Rightarrow \iota_K L_K$ of the adjunction, and we claim that this is an isomorphism in this case. Specifically, we claim that $u_* \iota_J X$ is an L_K-local object, so that the unit of the adjunction is an isomorphism.

To see this, suppose that $f\colon a \to b$ is an arrow in $\mathbb{D}(K)$ such that $L_K f$ is an isomorphism in $\mathbb{E}(K)$. Then we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{D}(K)}(b, u_*\iota_J X) \arrow[r, "-\circ f"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{D}(K)}(a, u_*\iota_J X) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbb{D}(J)}(u^*b, \iota_J X) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{D}(J)}(u^*a, \iota_J X) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbb{E}(J)}(L_J u^*b, X) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{E}(J)}(L_J u^*a, X) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbb{E}(J)}(u^*L_K b, X) \arrow[r, "\cong", "-\circ u^*L_K f"'] & \mathrm{Hom}_{\mathbb{E}(J)}(u^*L_K a, X)
\end{tikzcd}$$

The first two vertical maps are isomorphisms using the adjunctions (u^*, u_*) and (L_J, ι_J), and the last vertical isomorphism is because L is a morphism of (pre)derivators. The bottom horizontal arrow is an isomorphism because $L_K f$ is an isomorphism, hence every map in the above diagram is an isomorphism. Therefore $u_*\iota_J X$ is an L_K-local object.

Therefore the lone arrow in Equation 3.2.2 is also an isomorphism, proving that $L_K u_* \iota_J$ is right adjoint to u^* in $\mathbb{E}$. This proves Der3R and Der3L for $\mathbb{E}$.

To show Der4, we need to verify that for any $u\colon J \to K$, $k \in K$, and $X \in \mathbb{E}(J)$, the canonical map

$$L_e \pi_! \iota_{(u/k)} \operatorname{pr}^* X \to k^* L_K u_! \iota_J X$$

is an isomorphism, where $\pi\colon (u/k) \to e$ is the projection to the point and $\operatorname{pr}\colon (u/k) \to J$ the forgetful functor. Because ι and k are morphisms of derivators, we have that $\iota_{(u/k)} \operatorname{pr}^* \cong \operatorname{pr}^* \iota_J$ and $k^* L_K \cong L_e k^*$. Hence the above map factors as

$$\begin{tikzcd}
L_e \pi_! \iota_{(u/k)} \operatorname{pr}^* X \arrow[r] & k^* L_K u_! \iota_J X \arrow[d, "\cong"] \\
L_e \pi_! \operatorname{pr}^* \iota_J X \arrow[r] \arrow[u, "\cong"] & L_e k^* u_! \iota_J X
\end{tikzcd}$$

The bottom map is an isomorphism, as it is L_e applied to the isomorphism guaranteed by Der4 for $\mathbb{D}$ at $\iota_J X \in \mathbb{D}(J)$. Therefore the top map is an isomorphism as well. There is no difference for the right Kan extensions, as the above argument only relied on commuting pullback functors with L and ι. This proves that $\mathbb{E}$ is a derivator. □

In order to determine when such an adjoint as in Lemma 3.2.1 might exist, we should compare the derivator situation to the analogous situation in ordinary category theory.

Proposition 3.2.3 (Proposition 2.4.1, [Kra10])**.** Let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor and $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow L$ be a natural transformation. Then the following are equivalent:

(1) $\eta_L\colon L \Rightarrow L^2$ and $L\eta\colon L \Rightarrow L^2$ are natural isomorphisms.

(2) There exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ with fully faithful right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ such that $L = GF$ and $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is the unit of the adjunction.

Such a functor L is called a *localization functor*.

Case (1) is equivalent to the seemingly stronger condition that $L\eta = \eta_L$ and η_L is a natural isomorphism. The proof of this equivalence can be found, for example, at [BD, Remark 2.3]. Case (2) is the categorical version of Lemma 3.2.1, but we have not given the derivator version of case (1). We need one more proposition to begin that discussion.

Proposition 3.2.4 (Proposition 2.6.1, [Kra10])**.** Let $\mathcal{C}$ be a category and $\mathcal{E} \subset \mathcal{C}$ a replete, full subcategory. Then the following are equivalent:

(1) There exists a localization functor $L\colon \mathcal{C} \to \mathcal{C}$ with essential image $\mathcal{E}$.

(2) The inclusion functor $\mathcal{E} \to \mathcal{C}$ admits a left adjoint.

In the case that we already have a localization functor, the left adjoint to $\mathcal{E} \to \mathcal{C}$ is given by L (with restricted codomain).

We now want to generalize the two propositions above, and will temporarily halt our notational conventions for morphisms of derivators for the sake of analogy. Suppose that $\mathbb{E} \subset \mathbb{D}$ is a full subprederivator of a derivator $\mathbb{D}$ obtained by taking full and replete subcategories levelwise, and let $G\colon \mathbb{E} \to \mathbb{D}$ be the inclusion. If G admits a left adjoint $F\colon \mathbb{D} \to \mathbb{E}$, then we can define $L = GF\colon \mathbb{D} \to \mathbb{D}$ and $\eta\colon \mathrm{id}_{\mathbb{D}} \Rightarrow GF$ the unit of the adjunction. The essential image of L is $\mathbb{E}$; for $Y \in \mathbb{E}(J)$, we have $L(G(Y)) = GFG(Y) \cong G(Y)$ as the counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{E}}$ of the (F, G) adjunction is an isomodification.

Finally, using the triangle identities for the (F, G) adjunction we have

$$L\eta = GF\eta = (G\varepsilon_F)^{-1} = \eta_{GF} = \eta_L. \tag{3.2.5}$$

The natural transformation $(G\varepsilon_F)^{-1}$ only makes sense if $G\varepsilon_F$ is an isomodification, but this is clear as ε is an isomodification. Therefore $L\eta = \eta_L$ is an isomodification and L is a localization morphism.

Now, suppose we start with an endomorphism $L\colon \mathbb{D} \to \mathbb{D}$ of a derivator $\mathbb{D}$ with a modification $\eta\colon \mathrm{id}_{\mathbb{D}} \Rightarrow L$ such that η_L and $L\eta$ are isomodifications. Without loss of generality, we may assume $\eta_L = L\eta$, as the proof in the categorical case translates immediately to the derivator context.

Let $\mathcal{E}_K \subset \mathbb{D}(K)$ denote the essential image of L_K for every $K \in \mathbf{Dia}$. We claim these categories assemble into a prederivator. For $X \in \mathbb{D}(K)$ and $u\colon J \to K$, we have that $u^*L_KX \cong L_Ju^*X$ because L is a morphism of derivators. If $Y \in \mathcal{E}_K$, we have $u^*Y \cong u^*L_KX$ for some $X \in \mathbb{D}(K)$, hence $u^*Y \cong L_Ju^*X$, so $u^*Y \in \mathcal{E}_J$. Because the pullbacks u^* restrict to these subcategories, we get a full subprederivator $\mathbb{E} \subset \mathbb{D}$.

Define a morphism $F\colon \mathbb{D} \to \mathbb{E}$ by $F_J(X) = L_J(X)$ and $\gamma_u^F = \gamma_u^L$ and $G\colon \mathbb{E} \to \mathbb{D}$ by the inclusion. Then the unit of the adjunction $\eta\colon \mathrm{id}_{\mathbb{D}} \Rightarrow GF$ should be defined by the same η as above. For the counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{E}}$, we know that η_Y is an isomorphism for any $Y \in \mathbb{E}$, so we let ε be the inverse of η. For the triangle identities, we have

$$F\eta = L\eta = \eta_L = (\varepsilon_L)^{-1} = (\varepsilon_F)^{-1}$$

so $\varepsilon_F F\eta = \mathrm{id}_F$ as required. Similarly, as G is just the inclusion,

$$\eta_G = G\eta = (G\varepsilon)^{-1}$$

so $G\varepsilon\eta_G = \mathrm{id}_G$. We obtain an adjunction, and thus have proved the following:

Proposition 3.2.6. Let $\mathbb{D}$ be a derivator and $\mathbb{E} \subset \mathbb{D}$ a replete, full subprederivator (i.e. levelwise replete, full subcategories). Then the following are equivalent:

(1) There exists a morphism of derivators $L\colon \mathbb{D} \to \mathbb{D}$ with essential image $\mathbb{E}$ and a modification $\eta\colon \mathrm{id}_{\mathbb{D}} \Rightarrow L$ such that η_L and $L\eta$ are an isomodifications.

(2) The levelwise inclusion morphism $\mathbb{E} \to \mathbb{D}$ admits a left adjoint.

The following general context will be found in the case of the localization of Theorem 3.3.4. Suppose that $F\colon \mathbb{D} \to \mathbb{D}$ is a left adjoint endomorphism of a (pre)derivator $\mathbb{D}$ such that its right adjoint G is fully faithful on the (essential) image of F. Then the assignment $L = GF$ and $\eta\colon \operatorname{id}_{\mathbb{D}} \Rightarrow GF$ the unit of the adjunction makes L a localization morphism and the essential image of F is a reflective subprederivator of $\mathbb{D}$.

3.3 Vanishing subderivators

Our goal is the following: for any pointed derivator $\mathbb{D}$, there is a certain diagram category V such that objects in $\mathbb{D}(V)$ will be a good starting point for prespectrum objects. However, we will have a certain vanishing criterion for the objects that we really care about, that is, there is a collection $\{v_i\}$ of objects of V such that we want $v_i^* X = 0 \in \mathbb{D}(e)$. We would then want these objects to underlie not only a subprederivator, but a full (pointed) subderivator of $\mathbb{D}^V$.

We approach this situation generally, starting with notation for the above situation.

Notation 3.3.1. Let $\mathbb{D}$ be a pointed derivator, let $B \in \mathbf{Dia}$, and let $i\colon A \to B$ be a full subcategory. Define $\mathbb{D}(B, A)$ to be the full subcategory of $\mathbb{D}(B)$ given by

$$\mathbb{D}(B, A) = \{X \in \mathbb{D}(B) : i^* X = 0 \in \mathbb{D}(A)\}$$

This is the same as asking for pointwise vanishing at each object of A, because all the maps in the full subcategory on those objects must be zero maps.

Lemma 3.3.2. As above, $\mathbb{D}(B, A)$ is the base of a prederivator.

Proof. We let $\mathbb{D}^{B,A}$ denote the purported prederivator. We define $\mathbb{D}^{B,A}(K)$ to be the subcategory of $X \in \mathbb{D}(B \times K)$ such that $(i \times \operatorname{id}_K)^* X = 0 \in \mathbb{D}(A \times K)$. Let $u\colon J \to K$ be any morphism in **Dia**. Then we have the following commutative square:

(3.3.3)
$$\begin{tikzcd}
A \times J \arrow[r, "\mathrm{id}_A \times u"] \arrow[d, "i\times \mathrm{id}_J"'] & A \times K \arrow[d, "i\times \mathrm{id}_K"] \\
B \times J \arrow[r, "\mathrm{id}_B \times u"'] & B \times K
\end{tikzcd}$$

Applying $\mathbb{D}$ to this square, we obtain an equality

$$(i \times \mathrm{id}_J)^*(\mathrm{id}_B \times u)^* = (\mathrm{id}_A \times u)^*(i \times \mathrm{id}_K)^*$$

by strict 2-functoriality. Therefore consider some $X \in \mathbb{D}^{B,A}(K)$. By definition, we have $(i \times \mathrm{id}_K)^*X = 0$, so $(\mathrm{id}_A \times u)^*(i \times \mathrm{id}_K)^*X = 0$ as well (as restriction functors are pointed). By the above equality, this shows that the object $Y = (\mathrm{id}_B \times u)^*X$, which a priori is an element of $\mathbb{D}(B \times J)$ but not necessarily $\mathbb{D}^{B,A}(J)$, satisfies $(i \times \mathrm{id}_J)^*Y = 0$, so it is indeed in the vanishing subcategory. □

The prederivator $\mathbb{D}^{B,A}$ is easy to understand in the case that $i\colon A \to B$ is a sieve (resp. cosieve). The complementary inclusion $j\colon B \setminus A \to A$ is a cosieve (resp. sieve). We can then identify $\mathbb{D}^{B,A}$ with the essential image of $j_!\colon \mathbb{D}^{B\setminus A} \to \mathbb{D}^B$ (resp. j_*). These morphisms are (levelwise) fully faithful, so $j_!\colon \mathbb{D}^{B\setminus A} \to \mathbb{D}^{B,A}$ is an equivalence of derivators, and thus $\mathbb{D}^{B,A}$ shares all nice properties of $\mathbb{D}$. In the case that $i\colon A \to B$ is not a (co)sieve, the result is identical but the proof significantly harder. The following theorem is [Hel97, Proposition 7.4], which was given without proof.

Theorem 3.3.4. Let $\mathbb{D}$ be a pointed derivator and $i\colon A \to B$ be a full subcategory in **Dia**. Then there exists a localization of $\mathbb{D}^B$ with essential image $\mathbb{D}^{B,A}$.

Heller claims that the localization morphism is given by the cofiber of the counit of the $(i_!, i^*)$ adjunction. We shall see that this is the case, under a sufficiently sophisticated interpretation of the claim. In the case of usual homotopy theory, the construction of cofibers is not functorial. This is one of the main weaknesses of triangulated categories, but one of the strengths of derivators is that we can obtain functorial (and even coherent, as we will shortly show) cofibers.

Proof. Our overall goal is the following: there is an adjunction $(L, R)\colon \mathbb{D}^B \to \mathbb{D}^B$ such that $\mathbb{D}^{B,A}$ is the essential image of L and R is fully faithful on $\mathbb{D}^{B,A}$. As we noted after Proposition 3.2.6, this gives us a localization morphism $RL\colon \mathbb{D}^B \to \mathbb{D}^B$.

Construction 3.3.5. For any functor $u\colon J \to K$, there is a morphism of derivators $\mathbb{D}^J \to \mathbb{D}^{J\times[1]}$ given by

$$X \mapsto (u_! u^* X \to X).$$

That is, the counit of the $(u_!, u^*)$ adjunction may be constructed coherently.

The idea for this construction originates from personal correspondence with Kevin Carlson and the proof is joint with Ioannis Lagkas.

Consider the category $\mathrm{Cyl}(u)$ constructed as follows: its objects are the disjoint union of J and K, and its morphisms are defined as follows:

$$\mathrm{Hom}_{\mathrm{Cyl}(u)}(x, y) = \begin{cases} \mathrm{Hom}_J(x, y) & x, y \in J \\ \mathrm{Hom}_K(x, y) & x, y \in K \\ * & x \in J, y \in K \text{ and } u(x) = y \\ \varnothing & \text{otherwise} \end{cases}$$

In other words, the category is J glued to K along the map u, so we will refer to this as a mapping cylinder. We will abuse notation and refer to $J, K \subset \mathrm{Cyl}(u)$ as full subcategories. Let $\overline{u}\colon\ \mathrm{Cyl}(u) \to K \times [1]$ be defined by

$$\overline{u}(x) = \begin{cases} (u(x), 0) & x \in J \\ (x, 1) & x \in K \end{cases}$$

with the action of $\overline{u}$ on the maps in J, K obvious and sending the unique 'gluing' map $j \to k$ in $\mathrm{Cyl}(u)$ to the vertical map $(u(j), 0) \to (k, 1)$ as $u(j) = k$. Let $p\colon K \times [1] \to K$ be the projection.

We claim that $\overline{u}_! \overline{u}^* p^*\colon \mathbb{D}^K \to \mathbb{D}^{K\times[1]}$ gives coherently the counit of the $(u_!, u^*)$ adjunction.

To verify this, we use the following diagram:

$$\begin{tikzcd}
J \arrow[r, "i_0"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & \mathrm{Cyl}(u) \arrow[r, "q"] \arrow[d, "\overline{u}"] & K \arrow[dd, "\mathrm{id}_K"] \\
K \arrow[r, "s"] \arrow[d, "\mathrm{id}_K"'] \arrow[dr, phantom, "\overset{\beta}{\Swarrow}"] & K \times [1] \arrow[d, "\mathrm{id}_{K\times[1]}"] \arrow[ur, phantom, "\gamma\Swarrow"] & \\
K \arrow[r, "t"'] & K \times [1] \arrow[r, "p"'] & K
\end{tikzcd} \tag{3.3.6}$$

In the above, we write $q = p\overline{u}$; $s, t\colon K \to K \times [1]$ are the inclusion into the source $K \times \{0\}$ and target $K \times \{1\}$ of the coherent arrow, respectively; and $i_0\colon J \to \mathrm{Cyl}(u)$ is the inclusion of the full subcategory. The transformation α is the identity, β is the transformation $s \Rightarrow t$ as in the underlying diagram functor $\mathrm{dia}_{[1],K}$, and γ is also the identity.

The map we must study is, for $X \in \mathbb{D}^K$,

$$\mathrm{dia}_{[1],K}(\overline{u}_!\overline{u}^*p^*X)\colon s^*\overline{u}_!\overline{u}^*p^*X \to t^*\overline{u}_!\overline{u}^*p^*X$$

which is encoded by the natural tranformation β. To be more specific, the left mate of β^* is the natural transformation

$$\beta_!\colon \mathrm{id}_{K,!}\, s^* \Rightarrow t^*\, \mathrm{id}_{K\times[1],!}$$

Since the left Kan extension along an identity is an isomorphism, in essence we have a transformation $\beta_!\colon s^* \Rightarrow t^*$. We then need to whisker this transformation with $\overline{u}_!q^* = \overline{u}_!\overline{u}^*p^*$ to obtain the desired map. We pick the left mate (rather than using β^* itself) because we need to whisker with a left Kan extension.

We would like to understand how the transformation $\beta_{!,\overline{u}_!q^*}$ fits into the left mate of total pasting of the diagram. To that end, we take the left mates of α, β, γ and paste them as follows:

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[d, "u_!"'] \arrow[dr, phantom, "\overset{\alpha_!}{\Swarrow}"] & \mathbb{D}(\mathrm{Cyl}(u)) \arrow[l, "i_0^*"'] \arrow[d, "\overline{u}_!"] & \mathbb{D}(K) \arrow[l, "q^*"'] \arrow[dd, "\mathrm{id}_{K,!}"] \\
\mathbb{D}(K) \arrow[d, "\mathrm{id}_{K,!}"'] \arrow[dr, phantom, "\overset{\beta_!}{\Swarrow}"] & \mathbb{D}(K \times [1]) \arrow[l, "s^*"'] \arrow[d, "\mathrm{id}_{K\times[1],!}"] \arrow[ur, phantom, "\gamma_!\Swarrow"] & \\
\mathbb{D}(K) & \mathbb{D}(K \times [1]) \arrow[l, "t^*"] & \mathbb{D}(K) \arrow[l, "p^*"]
\end{tikzcd}$$

Working through these left mates one at a time, we obtain the composite transformation

$$\mathrm{id}_{K,!}\, u_! i_0^* q^* \xrightarrow{\mathrm{id}_{K,!}\, \alpha_{!,p^*}} \mathrm{id}_{K,!}\, s^* \overline{u}_! p^* \xrightarrow{\beta_{!,\overline{u}_! q^*}} t^*\, \mathrm{id}_{K\times[1],!}\, \overline{u}_! q^* \xrightarrow{t^*\gamma_!} t^* p^*\, \mathrm{id}_{K,!}.$$

The total pasting of the diagram is

$$\begin{tikzcd}
J \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\mathrm{id}\Swarrow"] & K \arrow[d, "\mathrm{id}_K"] \\
K \arrow[r, "\mathrm{id}_K"'] & K
\end{tikzcd}$$

and its left mate is $\varepsilon\colon u_! u^* \Rightarrow \mathrm{id}_K^*\, \mathrm{id}_{K,!} \cong \mathrm{id}_{\mathbb{D}^K}$, i.e. the counit of the $(u_!, u^*)$ adjunction. Thus the above composite is the counit of the adjunction by Proposition 2.3.2(1). Therefore it suffices to show that $\alpha_{!,p^*}$ and $t^*\gamma_!$ are natural isomorphisms, thereby obtaining

$$\begin{tikzcd}
\mathrm{id}_{K,!}\, i_! i_0^* q^* \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{K,!}\, \alpha_{!,p^*}"', "\cong"] & u_! u^* \arrow[r, "\varepsilon"] & \mathrm{id}_{\mathbb{D}^K} \arrow[r, "\cong"] & t^* p^*\, \mathrm{id}_{K,!} \\
\mathrm{id}_{K,!}\, s^* \overline{u}_! p^* \arrow[rrr, "\beta_{!,\overline{u}_! q^*}"'] & & & t^*\, \mathrm{id}_{K\times[1],!}\, \overline{u}_! q^* \arrow[u, "\cong", "t^*\gamma_!"']
\end{tikzcd} \tag{3.3.7}$$

so that $\beta_{!,\overline{u}_! q^*}$ is (up to isomorphism) the counit of the adjunction.

We begin with $\alpha_!$. We claim that $\alpha_!$ is actually a natural isomorphism even without whiskering with p^*. We use [Gro13, Proposition 1.24]: if that square is a (1-categorical) pullback and the bottom horizontal functor is a Grothendieck fibration or the right vertical functor is a Grothendieck opfibration, then $\alpha_!$ is an isomorphism. For us, the square is a pullback, which requires the transformation α to be the identity (and so it is). Moreover, s is a sieve, which in particular is a discrete Grothendieck fibration. This proves that $\alpha_!$, hence $\alpha_{!,p^*}$, is an isomorphism.

For $\gamma_!$, it is not true that this transformation is an isomorphism in general, but it is after applying t^*. To prove this, we give another pasting

$$\begin{tikzcd}
K \arrow[r, "i_1"] \arrow[d, "\mathrm{id}_K"'] \arrow[dr, phantom, "\theta\Swarrow"] & \mathrm{Cyl}(u) \arrow[r, "q"] \arrow[d, "\overline{u}"'] \arrow[dr, phantom, "\gamma\Swarrow"] & K \arrow[d, "\mathrm{id}_K"] \\
K \arrow[r, "t"'] & K \times [1] \arrow[r, "p"'] & K
\end{tikzcd} \tag{3.3.8}$$

where $i_1\colon K \to \mathrm{Cyl}(u)$ is the inclusion into the bottom of the mapping cylinder and θ is just the identity. Taking left mates and composing, we obtain

$$\mathrm{id}_{K,!}\, i_1^* q^* \xrightarrow{\theta_{!,q^*}} t^* \overline{u}_! q^* \xrightarrow{t^*\gamma_!} t^* p^*\, \mathrm{id}_{K,!}.$$

But the total pasting is just

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}_B"] \arrow[d, "\mathrm{id}_B"'] \arrow[dr, phantom, "\mathrm{id}\Swarrow"] & B \arrow[d, "\mathrm{id}_B"] \\
B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd}$$

whose left mate is clearly an isomorphism. Therefore we just need to show that $\theta_{!,q^*}$ is an isomorphism to complete this argument.

The functor $\overline{u}$ is a cosieve, and a fortiori a discrete Grothendieck opfibration. Moreover, the lefthand square in Diagram 3.3.8 is clearly a pullback square, and has the identity natural transformation, hence by [Gro13, Proposition 1.24] again the transformation $\theta_!$ is an isomorphism, even without whiskering with q^*. We thus obtain

$$\begin{tikzcd}
\mathrm{id}_{K,!}\, i_1^* q^* \arrow[r, "\theta_{!,q^*}", "\cong"'] \arrow[rr, bend right, "\cong"'] & t^* \overline{u}_! q^* \arrow[r, "t^*\gamma_!"] & t^* p^* \,\mathrm{id}_{K,!},
\end{tikzcd}$$

proving that $t^*\gamma_!$ is an isomorphism. This proves that Diagram 3.3.7 has isomorphisms where claimed and so the functor $\overline{u}_!\overline{u}^*p^* \colon \mathbb{D}^K \to \mathbb{D}^{K\times[1]}$ is a coherent counit morphism

$$X \mapsto (u_! u^* X \to X).$$

We will use this construction for the inclusion $i \colon A \to B$. Having constructed the counit coherently, we now take the cone of the counit via a canonical morphism $C \colon \mathbb{D}^{[1]} \to \mathbb{D}$. In brief, for $(f \colon x \to y)$ in $\mathbb{D}^{[1]}$, $C(f)$ is computed as the pushout along zero

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\
0 \arrow[r] & C(f)
\end{tikzcd}$$

We will give the details of this construction for the second half of this proof.

In total this gives us a morphism

$$L \colon \mathbb{D}^B \xrightarrow{\overline{i}_!\overline{i}^*p^*} \mathbb{D}^{B\times[1]} \xrightarrow{C} \mathbb{D}^B.$$

We now need to show that the image of L is contained in $\mathbb{D}^{B,A}$. To that end, we consider $Z = i^*L(X) \in \mathbb{D}^A$ for any $X \in \mathbb{D}^B$. Because i^* is cocontinuous, it commutes with C, so we can compute Z (up to canonical isomorphism) as the pushout

$$\begin{tikzcd}
i^*i_!i^*X \arrow[r, "i^*\varepsilon_X"] \arrow[d] & i^*X \arrow[d] \\
0 \arrow[r] & Z
\end{tikzcd}$$

We claim this top map is an isomorphism. Because $i_!\colon \mathbb{D}^A \to \mathbb{D}^B$ is fully faithful by Proposition 2.6.19, the unit $\eta\colon \mathrm{id}_{\mathbb{D}^A} \Rightarrow i^*i_!$ is an isomorphism, and one of the triangle identities gives us

$$\begin{tikzcd}
i^*X \arrow[r, "\eta_{i^*X}", "\cong"'] \arrow[dr, bend right, "="'] & i^*i_!i^*X \arrow[d, "i^*\varepsilon_X"] \\
 & i^*X
\end{tikzcd}$$

so $i^*\varepsilon_X$ is also an isomorphism by two out of three. By [Gro13, Proposition 3.12], the pushout of an isomorphism is an isomorphism, hence $Z = 0$. Therefore $L(X) \in \mathbb{D}^{B,A}$ by definition.

Suppose that $Y \in \mathbb{D}^{B,A}$. Then to compute $L(Y)$, we have the pushout

$$\begin{tikzcd}
i_!i^*Y \arrow[r] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & L(Y)
\end{tikzcd}$$

Because $Y \in \mathbb{D}^{B,A}$, $i^*Y = 0$, so the lefthand vertical map is an isomorphism. Hence the map $Y \to L(Y)$ is also an isomorphism. Therefore the essential image of L is precisely $\mathbb{D}^{B,A}$.

Every part of the construction of L is a left adjoint morphism of derivators, so L has a right adjoint $R\colon \mathbb{D}^B \to \mathbb{D}^B$. We now prove that R is fully faithful on $\mathbb{D}^{B,A} \subset \mathbb{D}^B$. To that end, again let $Y \in \mathbb{D}^{B,A}$.

The cone morphism $C\colon \mathbb{D}^{[1]} \to \mathbb{D}$ is a composition of three morphisms. Recall from Notation 2.8.1 that $\square$ denotes the category

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

Let $i_\ulcorner\colon \ulcorner \to \square$ be the inclusion away from $(1,1)$ as before and let $i_{[1]}\colon [1] \to \ulcorner$ be the inclusion of the horizontal arrow.

The first step of C is the extension by zero morphism $i_{[1],*}\colon \mathbb{D}^{[1]} \to \mathbb{D}^\ulcorner$. The second step is the left Kan extension $i_{\ulcorner,!}\colon \mathbb{D}^\ulcorner \to \mathbb{D}^\square$ which computes the pushout. The final step is the restriction $(1,1)^*\colon \mathbb{D}^{\square} \to \mathbb{D}$ to the terminal object of $\square$, which we define to be the cone of the coherent morphism we began with.

The right adjoint to $(1,1)^*$ is $(1,1)_*$. By Corollary 2.6.3, $(1,1)_*$ is isomorphic to the constant diagram functor $\pi_{\square}^*$, so that

$$(1,1)_*Y \cong \begin{tikzcd} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] & Y \end{tikzcd}$$

with all morphisms identities. The right adjoint to $i_{\ulcorner,!}$ is $i_{\ulcorner}^*$, which restricts the constant diagram to the upper-left corner. The right adjoint to $i_{[1],*}$ is $i_{[1]}^!$, an exceptional right adjoint. The construction of this adjoint as a composition of three morphisms can be found following [Gro13, Corollary 3.8]. The first step is the extension by zero

$$\begin{tikzcd} & Y \arrow[r] \arrow[dl] & Y \\ Y & & \end{tikzcd} \mapsto \begin{tikzcd} 0 \arrow[dd] & & \\ & Y \arrow[r] \arrow[dl] & Y \\ Y & & \end{tikzcd}$$

Call the resulting diagram shape $\ulcorner'$ and the resulting object $Y' \in \mathbb{D}^{B\times\ulcorner'}$. The second step is to compute the right Kan extension along the inclusion $s\colon \ulcorner' \to \ulcorner \times [1]$. The last step will be to restrict ourselves to $(0,0,0) \to (1,0,0)$ in $\ulcorner \times [1]$. It will suffice to compute $(0,0,0)^*s_*\ulcorner'$ and $(1,0,0)^*s_*Y'$, which we can do using Der4. We have that

$$(1,0,0)^*s_*Y' \cong \pi_{((1,0,0)/s),*}\operatorname{pr}^* Y',$$

Examining the comma category $((1,0,0)/s)$ using Lemma 2.2.5, we note that $(1,0,0)$ maps only to $(1,0,1) \in \ulcorner \times [1]$. Therefore the comma category consists only of the element $(1,0,1)$ and the unique map $(1,0,0) \to (1,0,1)$, so that $\pi_{(1,0,0)/s)} \cong \mathrm{id}_e$ and

$$\pi_{(1,0,0)/s),*}\operatorname{pr}^* Y' \cong \mathrm{id}_{e,*}(1,0,1)^*Y' \cong Y.$$

The object at $(0,0,0)$ is a little more complicated. The comma category $((0,0,0)/s)$ is actually all of $\ulcorner'$ because $(0,0,0)$ is the initial element of $\ulcorner \times [1]$. Therefore

$$(0,0,0)^*s_*Y' \cong \pi_{\ulcorner',*}Y'.$$

Fortunately, $\ulcorner'$ contains a homotopy initial subcategory. Using Proposition 2.6.18, if $\ell\colon I \to \ulcorner'$ is a left adjoint functor, then we may compute

$$\pi_{\ulcorner',*}Y' \cong \pi_{I,*}\ell^*Y'.$$

Let $\ell\colon I \to \ulcorner'$ be the inclusion of the full subcategory excluding the element $(1,0,1)$. The right adjoint $r\colon \ulcorner' \to I$ is the identity on I but takes $(1,0,1) \mapsto (0,0,1)$. Because $(1,0,1)$ is only the codomain of one map in $\ulcorner'$, we just need to verify that

$$\mathrm{Hom}_{\ulcorner'}(\ell(0,0,1),(1,0,1)) \cong \mathrm{Hom}_I((0,0,1), r(1,0,1)).$$

But each of these are one-point sets, so we indeed have an adjunction. Naturality of the hom-set bijection follows because we are working in posets. Therefore we need to compute the limit of

$$\ell^*Y' = \begin{tikzcd} 0 \arrow[dd] & \\ & Y \arrow[dl] \\ Y & \end{tikzcd}$$

which is just $(0,0)^*$ of the pullback of that diagram. Because the map $Y \to Y$ is an isomorphism and the pullback of an isomorphism is still an isomorphism, we have $\pi_{I,*}\ell^*Y' \cong 0$.

Therefore the underlying diagram of $i^*_{[1]}s_*Y'$ is $(0 \to Y) \in \mathbb{D}^{B\times[1]}$. The underlying morphism must be the zero morphism, so we do not need to identify it using the calculus of mates as we did in Construction 3.3.5. The remaining right adjoints are the restriction along $\bar{i}\colon \mathrm{Cyl}(i) \to B \times [1]$ followed by the right Kan extension along the composition

$$q\colon \mathrm{Cyl}(i) \to B \times [1] \to B$$

First, the object $\bar{i}^*(0 \to Y)$ we could write as $(0_A \to Y)$, so that we have forgotten the elements in $B \setminus A$ in the domain of the coherent arrow.

Now we finally use the assumption that $Y \in \mathbb{D}^{B,A}$. We know that $i^*Y = 0_A$, so $(0_A \to Y) = q^*Y \in \mathbb{D}^{\mathrm{Cyl}(i)}$. Therefore the right adjoint to L evaluated at $Y \in \mathbb{D}^{B,A}$ is exactly q_*q^*Y. We now claim that the unit $\mathrm{id}_{\mathbb{D}^B} \Rightarrow q_*q^*$ is an isomorphism, so that the right adjoint to L is isomorphic to the identity on elements of $\mathbb{D}^{B,A}$.

By usual category theory, the unit of (q^*, q_*) is an isomorphism if and only if q^* is fully faithful. We now show that this is the case. The functor $q\colon \mathrm{Cyl}(i) \to B$ itself is left adjoint to the functor $i_1\colon B \to \mathrm{Cyl}(i)$ defined by $b \mapsto (b, 1)$. To illustrate this, let $(a, n) \in \mathrm{Cyl}(i)$ and $b \in B$. Then

$$\mathrm{Hom}_B(q(a,n), b) = \mathrm{Hom}_B(a, b) \cong \mathrm{Hom}_{\mathrm{Cyl}(i)}((a,n),(b,1)) = \mathrm{Hom}_{\mathrm{Cyl}(i)}((a,n), i_1(b))$$

where we here use the identification $q(a, n) = i(a) = a \in B$ because i is fully faithful. The adjunction (q, i_1) in **Dia** under $\mathbb{D}$ becomes an adjunction (i_1^*, q^*) from $\mathbb{D}^B$ to $\mathbb{D}^{\overline{A}}$. Because adjoints are essentially unique, we have that $q^* \cong i_{1,*}$. Because i_1 is fully faithful, $i_{1,*}$ is also fully faithful by Proposition 2.6.19, which proves q^* is as well.

Hence we finally may conclude that the morphism of prederivators $L\colon \mathbb{D}^B \to \mathbb{D}^B$ has a right adjoint which is fully faithful on the essential image of L. We have thus accomplished the plan at the beginning of this proof, and so obtain a localization of the derivator $\mathbb{D}^B$ with essential image $\mathbb{D}^{B,A}$. □

Corollary 3.3.9. The prederivator $\mathbb{D}^{B,A}$ is a pointed derivator.

Proof. Because $\mathbb{D}^{B,A}$ is a localization of a derivator, Lemma 3.2.1 applies, so $\mathbb{D}^{B,A}$ is a derivator. Clearly $0 \in \mathbb{D}(B)$ is in the subcategory $\mathbb{D}(B, A)$, so $\mathbb{D}^{B,A}$ is pointed as well. □

Remark 3.3.10. The left and right Kan extensions of the localized derivator $\mathbb{D}^{B,A}$ have explicit formulas based on those of $\mathbb{D}^B$. For a functor $u\colon J \to K$, write $u'_!$ and u'_* for the left and right Kan extensions of u in $\mathbb{D}^{B,A}$. Then the proof of Lemma 3.2.1 provides

$$u'_! = L_K u_! R_J \quad \text{and} \quad u'_* = L_K u_* R_J.$$

However, the restriction of $u_!$ and u_* to $\mathbb{D}^{B,A}$ have their images in $\mathbb{D}^{B,A}$ as well. Because the morphism of derivators $i^*\colon \mathbb{D}^B \to \mathbb{D}^A$ is continuous and cocontinuous and u_* is pointed, we have for any $X \in \mathbb{D}^B(J)$,

$$i^* u_* X \cong u_* i^* X \cong u_* 0 \cong 0.$$

Therefore $u_*X \in \mathbb{D}^{B,A}(K)$ already. Similarly, $u_!X \in \mathbb{D}^{B,A}(K)$. This is a much quicker way to prove the above corollary, but does not give us the localization morphism which will prove instrumental in the next section.

The situation of vanishing subderivators is exceptional among localizations of derivators. In general, the localized derivator $\mathbb{E} \subset \mathbb{D}$ need not be closed under both left and right Kan extensions in $\mathbb{D}$, but will obtain its own left and right Kan extensions as above.

Remark 3.3.11. There is also a construction of $\mathbb{D}^{B,A}$ as a colocalization of $\mathbb{D}^B$, i.e. the inclusion $\mathbb{D}^{B,A} \to \mathbb{D}^B$ admits a right adjoint. This is the case if and only if the dual situation of Proposition 3.2.6(1) holds. We obtain the colocalization morphism $\mathbb{D}^B \to \mathbb{D}^B$ by composing an adjunction similar to the one above.

For any functor $u\colon J \to K$, we can construct the coherent unit of the (u^*, u_*) adjunction using a right mate version of Construction 3.3.5. Considering our inclusion $i\colon A \to B$, we can take the fiber of the unit and obtain a morphism $R'\colon \mathbb{D}^B \to \mathbb{D}^B$. We can see that the image of R' is contained in $\mathbb{D}^{B,A}$ because $Z' = i^*R'(X)$ for $X \in \mathbb{D}^B$ is computed as the pullback

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & i^*X \arrow[d, "i^*\eta_X"] \\
0 \arrow[r] & i^*i_*i^*X
\end{tikzcd}$$

where the righthand map is an isomorphism for reasons dual to the above. Hence $Z' = 0$ so $R'(X) \in \mathbb{D}^{B,A}$. The rest of the proof proceeds similarly.

3.4 Prespectra in (pointed) derivators

We come now to the specific vanishing subderivator that is the starting point for stabilization. Unfortunately, we need to make a change in our notational convention for $\square$ at this point.

Notation 3.4.1. Henceforth let $\square$ be the category

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[u]
\end{tikzcd}$$

Let $i_{\llcorner} \colon \llcorner \to \square$ be the full subcategory lacking the element $(1,1)$ and $i_{\urcorner} \colon \urcorner \to \square$ the full subcategory lacking $(0,0)$.

We want to see $\square$ as a subcategory of $\mathbb{Z}^2$ considered as a poset, so this orientation makes more sense.

Our aim for stabilization is to force the (Σ, Ω) adjunction on a pointed derivator $\mathbb{D}$ to be an equivalence. We cannot do this solely in $\mathbb{D}(e)$, but will use the higher structure of the derivator $\mathbb{D}$ in a way recalling [Hov01, Definition 1.1].

Definition 3.4.2. Consider the poset $\mathbb{Z}^2$ viewed as a category. Let $V \subset \mathbb{Z}^2$ be the full subcategory of those (i,j) such that $|i-j| \leq 1$. Let ∂V be the full subcategory of V on (i,j) such that $|i-j| = 1$. We define the *pointed derivator of prespectrum objects on* $\mathbb{D}$ to be $\operatorname{Sp}\mathbb{D} := \mathbb{D}^{V,\partial V}$.

We will often write X_i for $(i,i)^*X$ for brevity. This should cause no confusion as $(i,j)^*X = 0$ if $i \neq j$ by the vanishing criterion. A prespectrum $X \in \operatorname{Sp}\mathbb{D}$ then has underlying diagram

$$\begin{tikzcd}
 & & & & {} \\
 & & 0 \arrow[r] & X_1 \arrow[ur, dotted] & \\
 & 0 \arrow[r] & X_0 \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & X_{-1} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

extending infinitely in both directions.

To justify the definition, we need to describe in what sense a prespectrum contains information about the (Σ, Ω) adjunction on $\mathbb{D}$. On the surface, a prespectrum $X \in \operatorname{Sp}\mathbb{D}$ is equivalent to a discrete collection of objects $\{X_i\}$ for $i \in \mathbb{Z}$. However, the category V and the vanishing of ∂V naturally encode the structure maps of the prespectrum.

Let $X \in \operatorname{Sp}\mathbb{D}$, and let $i_n \colon \square \to V$ be the functor defined by $(a,b) \mapsto (a+n, b+n)$. Then we have

$$i_n^* X = \begin{tikzcd}
0 \arrow[r] & X_{n+1} \\
X_n \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd} \in \mathbb{D}^{\square}.$$

We follow [Gro16b, §2] for the following notation. The category $\square$ is the cone on the category $\urcorner$, that is, the category $\urcorner$ with an initial object freely added. Recall from Definition 2.8.3 (and bearing in mind our new Notation 3.4.1) that the loop of an object $Y \in \mathbb{D}$ is $(0,0)^* i_{\urcorner,*}$ applied to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y \\
& 0 \arrow[u]
\end{tikzcd}$$

From the square $i_n^* X$, we would like to ignore the corner at $(0,0)$ and apply $i_{\urcorner,*}$ to the rest of the diagram. This should give us the object ΩX_{n+1}, and we can now describe how close X_n is to ΩX_{n+1}. This will be the nth structure map of X.

This occurs formally as follows: let A be the cone on the category $\square$, i.e. the double cone on $\urcorner$. Denote by $\varnothing$ the object that is added to $\square$ to form A. Let $s_\square \colon \square \to A$ be the functor sending $(0,0)$ to $\varnothing$ and which is the identity elsewhere. Then it is a straightforward computation by Der4 that

$$s_{\square,*} i_n^* X = \begin{tikzcd}
& 0 \arrow[r] & X_{n+1} \\
& \Omega X_{n+1} \arrow[u] \arrow[r] & 0 \arrow[u] \\
X_n \arrow[ur] \arrow[uur, bend left] \arrow[urr, bend right] & &
\end{tikzcd} \in \mathbb{D}^A. \tag{3.4.3}$$

If we restrict to the map $\varnothing \to (0,0)$ in A, this gives us a coherent map $(X_n \to \Omega X_{n+1})$ in $\mathbb{D}^{[1]}$. We could have gotten *incoherent* structure maps using the universal property of the pullback in $\mathbb{D}$, but we require coherence for what follows.

Repeating this construction for all $n \in \mathbb{Z}$ gives us the structure maps of a prespectrum $X \in \operatorname{Sp}\mathbb{D}$. We will make a formal definition later, but we can anticipate that a (stable) spectrum is a prespectrum such that all of these structure maps are isomorphisms.

We will first describe how to pass from $\mathbb{D}$ to $\operatorname{Sp}\mathbb{D}$ in a canonical way, then in the following section discuss stable spectra and the stabilization morphism.

Proposition 3.4.4. The morphism $(0,0)^* \colon \operatorname{Sp}\mathbb{D} \to \mathbb{D}$ sending $X \mapsto X_0$ admits a left adjoint.

Remark 3.4.5. For any prederivator $\mathbb{E}$, we can define morphisms $\operatorname{Sp}\mathbb{D} \to \mathbb{E}$ by using the composition $\operatorname{Sp}\mathbb{D} \subset \mathbb{D}^V \to \mathbb{E}$, but we cannot necessarily define morphisms $\mathbb{E} \to \operatorname{Sp}\mathbb{D}$ in the same way. Without more information there is no reason to suspect that an arbitrary morphism $\mathbb{E} \to \mathbb{D}^V$ will vanish on ∂V.

We know that the morphism $(0,0)_!$ is left adjoint to $(0,0)^* \colon \mathbb{D}^V \to \mathbb{D}$, but $(0,0)_! \colon \mathbb{D} \to \mathbb{D}^V$ is not the adjoint we are looking for. More specifically, let us calculate $(1,0)^*(0,0)_! x$ for some $x \in \mathbb{D}$. By Der4, we have a canonical isomorphism

$$\pi_{((0,0)/(1,0)),!}\, \mathrm{pr}^* x \to (1,0)^*(0,0)_! x,$$

But since V is a poset, using Lemma 2.2.5 we see that $((0,0)/(1,0))$ contains only one element, namely $(e, (0,0) \to (1,0))$. Therefore pr and $\pi_{((0,0)/(1,0))}$ are both isomorphisms between e, giving $\pi_{((0,0)/(1,0)),!}\, \mathrm{pr}^* x \cong x$. Therefore the object $(0,0)_! x$ does not vanish on ∂V. Hence $(0,0)_!$ does not have its image in $\mathbb{D}^{V,\partial V} \subset \mathbb{D}^V$.

We can generalise the above argument to other points in V. For any (i,j) with $i, j > 0$, $((0,0)/(i,j))$ is still isomorphic to the terminal category, so $\pi_{((0,0)/(i,j)),!}\, \mathrm{pr}^* x \cong x$. For $i, j < 0$, we have $(i,j)^*(0,0)_! x \cong 0$ because the comma category $((0,0)/(i,j))$ is empty. Hence $(0,0)_! x$ has the form

$$\begin{tikzcd}
& & & & {} \\
& & x \arrow[r] & x \arrow[ur, dotted] & \\
& 0 \arrow[r] & x \arrow[r] \arrow[u] & x \arrow[u] & \\
& 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

However, if we compose $(0,0)_!$ with the localization $\mathbb{D}^V \to \mathbb{D}^{V,\partial V}$ we will necessarily obtain a left adjoint to $(0,0)^*$, as $(0,0)^*$ composed with the inclusion $\mathbb{D}^{V,\partial V} \to \mathbb{D}^V$ is still $(0,0)^*$. This means that something must happen on the ∂V part of the diagram, but it is unclear from this description (slick as it is) what is actually going on at the level of objects. In order to understand better the left adjoint, the below proof is constructive.

Proof. We construct the left adjoint in three stages. Let $V^{\leq 0}$ be the full subcategory of V on those (i,j) such that $i, j \leq 0$. Then consider the functor $(0,0) \colon e \to V^{\leq 0}$. This functor is a cosieve, so the left Kan extension $(0,0)_! \colon \mathbb{D} \to \mathbb{D}^{V^{\leq 0}}$ is extension by zero.

Next, let V' be the full subcategory of V which contains both $V^{\leq 0}$ and ∂V. That is, V' contains all (i, j) except for (i, i) with $i > 0$. Let $\iota_{\leq 0}\colon V^{\leq 0} \to V'$ be the inclusion. This map is a sieve, so the right Kan extension $\iota_{\leq 0,*}$ is extension by zero. By Proposition 2.7.6, $\iota_{\leq 0,*}$ admits a right adjoint which we name $\iota^!_{\leq 0}$. Finally, we can consider the inclusion $\iota\colon V' \to V$. Therefore we obtain the following picture

$$\begin{tikzcd}
\mathbb{D} \arrow[d, bend right, "(0,0)_!"'] \arrow[ddd, bend right=80, "L"'] \\
\mathbb{D}^{V^{\leq 0}} \arrow[u, bend right, "(0,0)^*"'] \arrow[d, bend right, "\iota_{\leq 0,*}"'] \\
\mathbb{D}^{V'} \arrow[u, bend right, "\iota^!_{\leq 0}"'] \arrow[d, bend right, "\iota_!"'] \\
\mathbb{D}^{V} \arrow[u, bend right, "\iota^*"']
\end{tikzcd}$$

Let us call this left adjoint L. We will compute L explicitly at the end of this section, and prove that for $x \in \mathbb{D}$, Lx is the connective suspension prespectrum with $(Lx)_0 \cong x$. This is then a derivator version of Σ^∞. We now need to check that on $\mathbb{D}^{V,\partial V}$, the right adjoint to L is $(0, 0)^*\colon \mathbb{D}^{V,\partial V} \to \mathbb{D}(e)$. To do this, we need to understand better the morphism $\iota^!_{\leq 0}$.

From [Gro13, Corollary 3.8] we have that the $(\iota_{\leq 0,*}, \iota^!_{\leq 0})$ adjunction factors as

$$\begin{tikzcd}
\mathbb{D}^{V^{\leq 0}} \arrow[d, bend right, "\iota_{\leq 0,*}"'] \\
\mathbb{D}^{V', V' \setminus V^{\leq 0}} \arrow[u, bend right, "\iota^*_{\leq 0}"'] \arrow[d, bend right, "\mathrm{incl}"'] \\
\mathbb{D}^{V'} \arrow[u, bend right, "R"']
\end{tikzcd}$$

where this bottom adjunction is a vanishing subderivator colocalization. Suppose that $X \in \mathbb{D}^{V,\partial V}$. Then $\iota^* X \in \mathbb{D}^{V'}$ already vanishes on $V' \setminus V^{\leq 0}$, so we know that the colocalization morphism R is isomorphic to the identity on $\iota^* X$. Therefore for any $X \in \mathbb{D}^{V,\partial V}$,

$$(0, 0)^* \iota^!_{\leq 0} \iota^* X \cong (0, 0)^* \iota^*_{\leq 0} \iota^* X = (0, 0)^* X.$$

This shows that L is left adjoint to $(0, 0)^*$ on $\mathbb{D}^{V,\partial V} \subset \mathbb{D}^V$.

Therefore when we consider the corresponding morphism of derivators $L\colon \mathbb{D} \to \operatorname{Sp} \mathbb{D}$, we obtain the total adjunction

(3.4.6)
$$\begin{tikzcd}
\mathbb{D} \arrow[d, bend right, "{(0,0)_!}"'] \arrow[ddd, bend right=80, "L"'] \\
\mathbb{D}^{V^{\leq 0}} \arrow[u, bend right, "{(0,0)^*}"'] \arrow[d, bend right, "\iota_{\leq 0,*}"'] \\
\mathbb{D}^{V', V' \setminus V^{\leq 0}} \arrow[u, bend right, "\iota^*_{\leq 0}"'] \arrow[d, bend right, "\iota_!"'] \\
\operatorname{Sp} \mathbb{D} \arrow[u, bend right, "\iota^*"'] \arrow[uuu, bend right=80, "{(0,0)^*}"']
\end{tikzcd}$$

□

Let us now compute the essential image of L by going step by step. Let $x \in \mathbb{D}$. As indicated above, $\iota_{\leq 0,*}(0,0)_! x$ extends x by zero, giving underlying diagram

$$\begin{tikzcd}
& & & {} & \\
& & 0 \arrow[ur, dotted] & & {} \\
& 0 \arrow[r] & x \arrow[r] \arrow[u] & 0 \arrow[ur, dotted] & \\
& 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

with a constant zero diagram towards the bottom left and zeroes along the diagonals going up and right. This shows (as we claimed above) that the image of L vanishes on ∂V. The last functor $\iota_!$ is some sort of colimit. The next lemma will tell us what exactly it does.

Lemma 3.4.7. Let $i_n \colon \square \to V$ be the inclusion of the square $(a, b) \mapsto (a + n, b + n)$. Then for $n \geq 0$, $i_n^* Lx$ is cocartesian for any $x \in \mathbb{D}$.

Proof. We use the following criterion from Groth. For $K \in \mathbf{Dia}$, define a *square in* K to be a functor $i \colon \square \to K$ which is injective on objects.

Proposition 3.4.8. [Gro13, Proposition 3.10] Let $i \colon \square \to K$ be a square in K and let $u \colon J \to K$ be a functor. Assume that the induced functor $\llcorner \to ((K \setminus i(1,1))/i(1,1))$ has a left adjoint and that $i(1,1)$ does not lie in the image of u. Then for all $Y \in \mathbb{D}(K)$ in the image of $u_! \colon \mathbb{D}(J) \to \mathbb{D}(K)$, the square i^*Y is cocartesian.

To be clear, the category $((K\setminus i(1,1))/i(1,1))$ is the comma category of the two inclusions of subcategories $(K \setminus i(1,1)) \to K$ and $i(1,1) \to K$. The inclusion $\square \to K$ restricts to $\llcorner \to K \setminus i(1,1)$ in a clear way, we obtain a functor to the comma category when we remember that $(1,1)$ was the final object in $\square$ so that every object in $\llcorner$ has a unique map to $(1,1)$.

For us, $J = V'$, $K = V$, u is the inclusion ι, and $i = i_n$. Let us investigate the indicated comma category $((V \setminus i_n(1,1))/i_n(1,1))$. Using Lemma 2.2.5, this comma category is identified with the full subcategory of V on all objects admitting a map to $i_n(1,1) = (n+1, n+1)$. Therefore we introduce the notation

$$V^{<n+1} := \{(i,j) \in V : i \leq n+1, j \leq n+1, (i,j) \neq (n+1,n+1)\} \subset V$$

for the comma category under consideration. We now examine the functor $r \colon \llcorner \to V^{<n+1}$ and construct a left adjoint ℓ to it explicitly.

Let (i,j) denote an arbitrary object of $V^{<n+1}$, and let (a,b) denote an arbitrary object of $\llcorner$. Then we would like

$$\mathrm{Hom}_{V^{<n+1}}((i,j), r(a,b)) \cong \mathrm{Hom}_{\llcorner}(\ell(i,j),(a,b)).$$

We know that $r(i,j) = (a+n, b+n)$, so we can characterise this hom-set as follows:

$$\mathrm{Hom}_{V^{<n+1}}((i,j), r(a,b)) = \begin{cases} * & i \leq a+n \text{ and } j \leq b+n \\ \varnothing & i > a+n \text{ or } j > b+n \end{cases}$$

If $i \leq n$ and $j \leq n$, we will be in the top case, so we will set $\ell(i,j) = (0,0)$ in that case. Since $(0,0)$ is initial in $\llcorner$, we know that there is always a unique morphism to (a,b) no matter what.

Suppose that $(i,j) = (n, n+1)$. Then $\mathrm{Hom}_{V^{<n+1}}((n,n+1),(a+n,b+n))$ is nonempty if and only if $b = 1$. That means that $\ell(n, n+1)$ should have a morphism to $(0,1)$ but not to $(1,0)$. This forces $\ell(n,n+1) = (0,1)$. Similarly, we can see that we need $\ell(n+1,n) = (1,0)$. This defines ℓ on every object and establishes a bijection of hom-sets. As this bijection is necessarily natural in both variables, we obtain the adjunction.

For any square $i_n\colon \square \to V$ as described above, we have that $i_n(1,1) = (n+1, n+1)$ is not in the image of $\iota\colon V' \to V$. Therefore the proposition applies, showing that $i_n^* \iota_! X$ is cocartesian for any $n \in \mathbb{N}$ and any $X \in \mathbb{D}^{V'}$. This in particular applies to $i_n^* Lx$, for $x \in \mathbb{D}$, completing the proof of the lemma. $\square$

We can now write the underlying diagram of Lx. It is zero almost everywhere, except on the entires (i,i) where $i \geq 0$. Write X_i for $(i,i)^* Lx$, and note that $X_0 \cong x$. We first restrict to the square $i_0\colon \square \to V$:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \\
x \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Because this square is cocartesian, we have $X_1 \cong \Sigma x$. Iterating this process, we see that $X_i \cong \Sigma^i x$. Therefore as we claimed above, $L\colon \mathbb{D} \to \operatorname{Sp}\mathbb{D}$ gives a (connective) suspension prespectrum on whatever object we start with.

3.5 Stabilization

We now want to give a formal definition of the subderivator of stable spectra in $\operatorname{Sp}\mathbb{D}$. To do so, we need a better definition than 'all the structure maps are isomorphisms'. This is not too hard to do, but it does require some new diagram notation.

Notation 3.5.1. Recall the notation for the category V of Definition 3.4.2. Let $\sigma\colon V \to V$ be the functor defined by $(i,j) \mapsto (j+1, i+1)$. Since $\sigma(\partial V) \subset \partial V$ and σ^* is a pointed morphism, we obtain an endomorphism $\sigma^*\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}$.

Notation 3.5.2. For any diagram K, let $\operatorname{Sp}\mathbb{D}^K$ denote the derivator $\operatorname{Sp}\mathbb{D}$ shifted by K, not the prespectrum derivator associated to $\mathbb{D}^K$. These are, in fact, exactly the same derivator, but we prefer the former interpretation for the following constructions.

Construction 3.5.3. There exists a functor $w\colon \square \times V \to V$ such that w^* restricts to a morphism $\operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^\square$ and, for any $X \in \operatorname{Sp}\mathbb{D}$, we have

$$w^* X = \begin{tikzcd}
0 \arrow[r] & \sigma^* X \\
X \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

We draw inspiration from Heller's construction of w in [Hel97, §8]. Throughout, (i, j) will denote an object of V and (a, b) will denote an object of $\Box$ (recall Notation 3.4.1). Let $\tau\colon \Box \to \Box$ be defined by $(a, b) \mapsto (b, a)$. We would like w to satisfy the following three properties.

(1) $w(a, b, i, i) = (i + a, i + b)$.

(2) $w(a, b, i, j) \in \partial V$ for $i \neq j$.

(3) $w \circ (\tau \times \sigma) = \sigma \circ w$.

We will give the definition of w first, followed by an illustration.

$$w(a,b,i,j) = \begin{cases} (i,j) & (a,b) = (0,0) \\ (j+1, i+1) & (a,b) = (1,1) \\ (j+1, j) & (a,b) = (1,0) \\ (i, i+1) & (a,b) = (0,1) \end{cases} \tag{3.5.4}$$

To see this in action, let us restrict to the subset of V where $i, j \in \{0, 1, 2, 3\}$. Then the codomain of w (given this restriction) we may write as

$$\begin{tikzcd} & & \iota \arrow[r] & \kappa \\ & \zeta \arrow[r] & \eta \arrow[r] \arrow[u] & \theta \arrow[u] \\ \gamma \arrow[r] & \delta \arrow[r] \arrow[u] & \varepsilon \arrow[u] & \\ \alpha \arrow[r] \arrow[u] & \beta \arrow[u] & & \end{tikzcd} \tag{3.5.5}$$

where $\alpha = (0, 0)$ and $\kappa = (3, 3)$. We will restrict the domain to the subset of V where $i, j \in \{0, 1, 2\}$ and write at each place $(a, b, i, j) \in \Box \times V$ the element $w(a, b, i, j) \in V$ in Diagram 3.5.5. This gives us the underlying diagram of w^*X for a coherent object $X \in \mathbb{D}^V$ with incoherent diagram as above. The starting picture, without decoration yet, is this:

$$\begin{tikzcd}
 & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \Rightarrow & \bullet \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & & & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \\
 & \Uparrow & & & & \Uparrow & \\
 & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \Rightarrow & \bullet \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & & & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] &
\end{tikzcd}$$

The bold arrows $\Rightarrow$ on each side represent the $\square$ dimension of the diagram, which connects the coherent subdiagrams in each corner. They do not define a morphism between the objects they might seem to, but orient the diagram as a whole. This means the lower left corner is $(0, 0, 0, 0)$ and the upper right corner is $(1, 1, 2, 2)$.

Criterion (1) fixes what happens along the diagonal of each corner of the square. The upper right and lower left corners of the square should have underlying diagrams X and $\sigma^* X$ (respectively). We place these objects in the diagram and mark some more for further study.

$$\begin{tikzcd}
 & \triangle_1 \arrow[r] & \iota & & & \theta \arrow[r] & \kappa \\
\bullet \arrow[r] & \zeta \arrow[r] \arrow[u] & \star_1 \arrow[u] & \Rightarrow & \varepsilon \arrow[r] & \eta \arrow[r] \arrow[u] & \iota \arrow[u] \\
\gamma \arrow[r] \arrow[u] & \bullet \arrow[u] & & & \delta \arrow[r] \arrow[u] & \zeta \arrow[u] & \\
 & \Uparrow & & & & \Uparrow & \\
 & \zeta \arrow[r] & \eta & & & \star_2 \arrow[r] & \theta \\
\gamma \arrow[r] & \delta \arrow[r] \arrow[u] & \varepsilon \arrow[u] & \Rightarrow & \bullet \arrow[r] & \varepsilon \arrow[r] \arrow[u] & \triangle_2 \arrow[u] \\
\alpha \arrow[r] \arrow[u] & \beta \arrow[u] & & & \beta \arrow[r] \arrow[u] & \bullet \arrow[u] &
\end{tikzcd}$$

Let us now examine the objects marked $\star$. We see that $\star_1$ must satisfy $\zeta \leq w(\star_1)$ (using $\leq$ for the ordering in the poset), $\varepsilon \leq w(\star_1)$, and $w(\star_1) \leq \iota$. By criterion (2), we must have $w(\star_1) \neq \eta$. This forces $w(\star_1) = \iota$. Similarly, $w(\star_2) = \theta$.

For the objects marked $\triangle$, we have $\zeta \leq w(\triangle_1)$, $w(\triangle_1) \leq \theta$, and $w(\triangle_1) \leq \iota$. Because

$w(\triangle_1) \neq \eta$ by criterion (2), this forces $w(\triangle_1) = \zeta$. Similarly, $w(\triangle_2) = \varepsilon$. We can then define w on the unadorned $\bullet$ objects by using criterion (3). Therefore we complete the picture:

$$\begin{tikzcd}
& \zeta \arrow[r] & \iota & & & \theta \arrow[r] & \kappa & & & & & \\
\gamma \arrow[r] & \zeta \arrow[r] \arrow[u] & \iota \arrow[u] & \Rightarrow & \varepsilon \arrow[r] & \eta \arrow[r] \arrow[u] & \iota \arrow[u] & & & & \iota \arrow[r] & \kappa \\
\gamma \arrow[r] \arrow[u] & \zeta \arrow[u] & & & \delta \arrow[r] \arrow[u] & \zeta \arrow[u] & & & & \zeta \arrow[r] & \eta \arrow[r] \arrow[u] & \theta \arrow[u] \\
& \Uparrow & & & & \Uparrow & & \longrightarrow & \gamma \arrow[r] & \delta \arrow[r] \arrow[u] & \varepsilon \arrow[u] & \\
& \zeta \arrow[r] & \eta & & & \theta \arrow[r] & \theta & & \alpha \arrow[r] \arrow[u] & \beta \arrow[u] & & \\
\gamma \arrow[r] & \delta \arrow[r] \arrow[u] & \varepsilon \arrow[u] & \Rightarrow & \varepsilon \arrow[r] & \varepsilon \arrow[r] \arrow[u] & \varepsilon \arrow[u] & & & & & \\
\alpha \arrow[r] \arrow[u] & \beta \arrow[u] & & & \beta \arrow[r] \arrow[u] & \beta \arrow[u] & & & & & &
\end{tikzcd}$$

As claimed, we have $(0,0)^*w^*X = X$ and $(1,1)^*w^*X = \sigma^*X$. Criterion (2) guarantees that $w(\square \times \partial V) \subset \partial V$, so the pointed morphism w^* restricts to $\operatorname{Sp}\mathbb{D}$.

Remark 3.5.6. Heller at [Hel97, p.127] uses a slightly different definition for σ, namely the 'obvious' shift $s(i,j) = (i+1, j+1)$. He also replaces τ by $\mathrm{id}_\square$ in criterion (3). However, there is no way to construct w such that $(0,0)^*w^*X = X$ and $(1,1)^*w^*X = s^*X$ satisfying $w \circ (\mathrm{id}_\square \times s) = s \circ w$ despite Heller's assertion (without proof) to the contrary.

To give a quick proof, restrict to $i,j \in \{0,1,2\}$. Then Heller's criterion (1) forces the following, where we emphasise the entry $(0,1,1,0)$ in particular:

$$\begin{tikzcd}
\bullet \arrow[r] & \zeta & \Rightarrow & \zeta \arrow[r] & \eta & & & & \iota \arrow[r] & \kappa \\
\gamma \arrow[r] \arrow[u] & \star \arrow[u] & & \delta \arrow[r] \arrow[u] & \varepsilon \arrow[u] & & & \zeta \arrow[r] & \eta \arrow[r] \arrow[u] & \theta \arrow[u] \\
\Uparrow & & & \Uparrow & & \longrightarrow & & & & \\
\gamma \arrow[r] & \delta & \Rightarrow & \bullet \arrow[r] & \varepsilon & & \gamma \arrow[r] & \delta \arrow[r] \arrow[uu] & \varepsilon \arrow[uu] & \\
\alpha \arrow[r] \arrow[u] & \beta \arrow[u] & & \beta \arrow[r] \arrow[u] & \bullet \arrow[u] & & \alpha \arrow[r] \arrow[u] & \beta \arrow[u] & &
\end{tikzcd}$$

We know that $w(\star)$ must satisfy $\beta \leq w(\star) \leq \varepsilon$ and $\gamma \leq w(\star) \leq \zeta$. This forces $w(\star) = \delta$, which contradicts criterion (2) that $w(\square \times \partial V) \subset \partial V$.

Note that on objects $X \in \operatorname{Sp}\mathbb{D}$, s^*X and σ^*X appear identical incoherently, though coherently they differ (subtly) because of the twist. Therefore we may continue following Heller's reasoning with this proper construction of w.

What does this construction accomplish? Recall from the discussion following Definition 3.4.2 that from the coherent square

$$\begin{tikzcd} 0 \arrow[r] & \sigma^* X \\ X \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{3.5.7}$$

we obtain a canonical coherent map $(X \to \Omega\sigma^* X)$ that we will call φ_X. This process is natural in X and gives a morphism of derivators $\varphi\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^{[1]}$.

Definition 3.5.8. An object $X \in \operatorname{Sp}\mathbb{D}$ is called an *Ω-spectrum* if the canonical map $\varphi_X\colon X \to \Omega\sigma^* X$ is an isomorphism.

Note that $(n,n)^*\varphi_X$ is exactly the nth structure map of X, and all the structure maps are isomorphisms if and only if φ_X is by Der2.

Let $\mathcal{S}_K \subset \operatorname{Sp}\mathbb{D}(K)$ be the full subcategory comprised of the Ω-spectra. Note that each subcategory $\mathcal{S}_K$ is replete, as an isomorphism $X \to Y$ in $\operatorname{Sp}\mathbb{D}(K)$ will give an isomorphism $\varphi_X \to \varphi_Y$ of coherent maps, so X is an Ω-spectrum if and only if Y is.

Further, note that $\mathcal{S}_K$ assemble to a prederivator $\operatorname{St}\mathbb{D}$. Suppose that $X \in \operatorname{St}\mathbb{D}(K)$, and let $u\colon J \to K$ be a functor. Then because $\varphi\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^{[1]}$ is a morphism of derivators, $\varphi_{u^*X} \cong u^*\varphi_X$. Because φ_X is an isomorphism, so is $u^*\varphi_X$ and hence u^*X is an Ω-spectrum as well.

Before stating the main theorem of this section, we need to define a new property of a derivator and prove a small lemma about it.

Definition 3.5.9. A derivator $\mathbb{D}$ is called *regular* if filtered colimits commute with finite limits. That is, for any filtered category C and finite category D, the canonical morphism

$$\pi_{C,!}(\mathrm{id}_C \times \pi_D)_* Y \to \pi_{D,*}(\pi_C \times \mathrm{id}_D)_! Y \tag{3.5.10}$$

is an isomorphism for any $Y \in \mathbb{D}(C \times D)$, where π_C, π_D are the usual projections to the final category.

This canonical morphism arises from the calculus of mates in the following way: we begin with the square

$$\begin{tikzcd}
C \times D \arrow[r, "\mathrm{id}_C \times \pi_D"] \arrow[d, "\pi_C \times \mathrm{id}_D"'] & C \arrow[d, "\pi_C"] \\
D \arrow[r, "\pi_D"'] & e
\arrow[from=1-2, to=2-1, Rightarrow, shorten <=8mm, shorten >=8mm, "\mathrm{id}"']
\end{tikzcd}$$

and apply any derivator $\mathbb{D}$:

$$\begin{tikzcd}
\mathbb{D}(e) \arrow[r, "\pi_C^*"] \arrow[d, "\pi_D^*"'] & \mathbb{D}(C) \arrow[d, "(\mathrm{id}_C \times \pi_D)^*"] \\
\mathbb{D}(D) \arrow[r, "(\pi_C \times \mathrm{id}_D)^*"'] & \mathbb{D}(C \times D)
\arrow[from=1-2, to=2-1, Rightarrow, shorten <=8mm, shorten >=8mm, "\mathrm{id}^*"']
\end{tikzcd}$$

We now take the right mate to obtain

$$\begin{tikzcd}
\mathbb{D}(D) \arrow[r, "\pi_{D,*}"] \arrow[d, "(\pi_C \times \mathrm{id}_D)^*"'] & \mathbb{D}(e) \arrow[d, "\pi_C^*"] \\
\mathbb{D}(C \times D) \arrow[r, "(\mathrm{id}_C \times \pi_D)_*"'] & \mathbb{D}(C)
\arrow[from=1-2, to=2-1, Rightarrow, shorten <=8mm, shorten >=8mm, "\mathrm{id}_*"']
\end{tikzcd}$$

The transformation id_* is an isomorphism because pullback functors commute with right Kan extensions. We can therefore take the left mate of $(\mathrm{id}_*)^{-1}$ (i.e. we flip the transformation but not the square) to obtain

$$\begin{tikzcd}
\mathbb{D}(C \times D) \arrow[r, "(\pi_C \times \mathrm{id}_D)_!"] \arrow[d, "(\mathrm{id}_C \times \pi_D)_*"'] & \mathbb{D}(D) \arrow[d, "\pi_{D,*}"] \\
\mathbb{D}(C) \arrow[r, "\pi_{C,!}"'] & \mathbb{D}(e)
\arrow[from=2-1, to=1-2, Rightarrow, shorten <=8mm, shorten >=8mm, "(\mathrm{id}_*^{-1})_!"]
\end{tikzcd}$$

This is the transformation we would like to be a natural isomorphism. Note that the above construction generalizes to arbitrary left and right Kan extensions (not just limits and colimits), so we can speak of the ‘canonical comparison map’ whenever we try to commute left and right Kan extensions in unrelated diagram dimensions.

In a stable derivator, all colimits commute with finite limits; see [Gro16a, Theorem 3.15]. Hence we can view regularity as a test of pre-stability. Requiring that our derivators be regular is not too restrictive in practice. This condition is satisfied for an enormous class of examples, broadest of all being locally presentable categories; see [AR94, Proposition 1.59]. The statement of that proposition is for directed colimits, but the corollary to Theorem 1.5 in that reference shows there is no distinction. The same is true generally in any n-topos, for $n \in \mathbb{N}$ or $n = \infty$; see Example 7.3.4.7 in combination with the dual of Proposition 5.3.2.9 in [Lur09].

We will only need that sequential colimits commute with pullbacks for Lemma 3.5.19, and this is (more or less) Heller's original definition at [Hel88, §5]. But there is no distinction in examples between sequential and filtered colimits and between pullbacks and finite limits. We pick this definition not because we must at the moment, but for a future application of stabilization to derivator K-theory (see Chapter 4), which we anticipate will require the stronger axiom.

Proposition 3.5.11. Let $\mathbb{D}$ be a regular pointed derivator. Then $\operatorname{Sp}\mathbb{D}$ is also a regular pointed derivator.

Proof. First, if $\mathbb{D}$ is regular, so is $\mathbb{D}^V$, in the spirit of Proposition 2.4.9. Let us keep the notation of Equation 3.5.10 but suppress id_C and id_D for simplicity. The comparison morphism for $Y \in \mathbb{D}^V(C \times D)$ gives rise to an isomorphism after applying $(i,j)^*$ for any $(i,j) \in V$

$$(i,j)^*\pi_{C,!}\pi_{D,*}Y \xrightarrow{\cong} \pi_{C,!}\pi_{D,*}(i,j)^*Y \xrightarrow{\cong} \pi_{D,*}\pi_{C,!}(i,j)^*Y \xrightarrow{\cong} (i,j)^*\pi_{D,*}\pi_{C,!}Y$$

where the first and last isomorphisms follow from the bicontinuity of $(i,j)^*$ and the middle isomorphism from the regularity of $\mathbb{D}$. This means that the comparison morphism for $\mathbb{D}^V$ is a pointwise isomorphism, hence must be an isomorphism by Der2.

Now consider the inclusion $\iota\colon \operatorname{Sp}\mathbb{D} \to \mathbb{D}^V$. We use critically that a vanishing subderivator localization is also a colocalization, so that the inclusion preserves both limits and colimits. For $X \in \operatorname{Sp}\mathbb{D}(C \times D)$, we need to show that the below square commutes up to isomorphism:

$$\begin{tikzcd}
\operatorname{Sp}\mathbb{D}(C \times D) \arrow[r, "\pi_{D,*}"] \arrow[d, "\pi_{C,!}"'] \arrow[dr, phantom, "\swarrow"] & \operatorname{Sp}\mathbb{D}(C) \arrow[d, "\pi_{!,C}"] \\
\operatorname{Sp}\mathbb{D}(D) \arrow[r, "\pi_{D,*}"'] & \operatorname{Sp}\mathbb{D}(e)
\end{tikzcd}$$

Using ι, we can build a cube involving the derivator $\mathbb{D}^V$ (omitting the natural transformations on each face):

$$\begin{tikzcd}
& \mathbb{D}^V(C \times D) \arrow[rr] \arrow[dd] & & \mathbb{D}^V(C) \arrow[dd] \\
\operatorname{Sp}\mathbb{D}(C \times D) \arrow[ur, "\iota_{C\times D}"] \arrow[rr, "\pi_{D,*}"] \arrow[dd, "\pi_{C,!}"'] & & \operatorname{Sp}\mathbb{D}(C) \arrow[ur, "\iota_C"] \arrow[dd, "\pi_{!,C}"] & \\
& \mathbb{D}^V(D) \arrow[rr] & & \mathbb{D}(e) \\
\operatorname{Sp}\mathbb{D}(D) \arrow[ur, "\iota_D"] \arrow[rr, "\pi_{D,*}"'] & & \operatorname{Sp}\mathbb{D}(e) \arrow[ur, "\iota_e"'] &
\end{tikzcd}$$

where the horizontal and vertical functors on the back face are the same Kan extensions as those on the front face. The back face commutes up to isomorphism because $\mathbb{D}^V$ is regular. Moreover, the top, bottom, left, and right faces commute up to isomorphism because ι is bicontinuous. This implies that the front face commutes up to isomorphism after applying ι_e. But since this functor is fully faithful, the front face must already commute up to isomorphism and so we conclude that $\operatorname{Sp}\mathbb{D}$ is regular. □

Theorem 3.5.12. Let $\mathbb{D}$ be a regular pointed derivator. Then inclusion $i\colon \operatorname{St}\mathbb{D} \to \operatorname{Sp}\mathbb{D}$ admits a left adjoint $\operatorname{loc}\colon \operatorname{Sp}\mathbb{D} \to \operatorname{St}\mathbb{D}$. That is, $\operatorname{St}\mathbb{D}$ is a localization of $\operatorname{Sp}\mathbb{D}$.

We will give the proof of this theorem in a series of constructions, with the ultimate goal of an explicit formula for the localization. Heller again gives us an idea for what the adjoint $\operatorname{loc}\colon \operatorname{Sp}\mathbb{D} \to \operatorname{St}\mathbb{D}$ should be. We are looking for something of the form

$$\operatorname{loc} X = \operatorname{hocolim}(X \to \Omega s^* X \to \Omega^2 (s^*)^2 X \to \cdots), \tag{3.5.13}$$

where $s\colon V \to V$ is Heller's incorrect shift functor of Remark 3.5.6. Heller gives this as a definition, but offers no way to obtain the *coherent* diagram of which we would like to take the homotopy colimit. The content of the following construction is to repair these errors.

We know that φ_X gives us, more or less, the first stage of the homotopy colimit. We are free to iterate this process, and in fact for any $n \in \mathbb{N}$ we can obtain a coherent diagram

$$X \to \Omega\sigma^* X \to (\Omega\sigma^*)^2 X \to \cdots \to (\Omega\sigma^*)^n X.$$

However, we cannot end up with an infinite coherent diagram like the one we need through this induction.

Construction 3.5.14. There is a morphism of derivators $F\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^{\mathbb{N}}$ given incoherently by

$$X \mapsto (X \to \Omega\sigma^* X \to \Omega^2(\sigma^*)^2 X \to \cdots).$$

We will modify our functor $w\colon \square \times V \to V$ from Construction 3.5.3 to give us an infinite diagram of the form we want. To begin with, let us define the full subcategory $V^{\geq 0} \subset V$ on (i,j) such that $i, j \geq 0$. We can see $\square \subset V^{\geq 0}$ by Notation 3.4.1. We want to define a functor $\omega\colon V^{\geq 0} \times V \to V$ with the aim that, for every square $i_n\colon \square \to V^{\geq 0}$ given by $(a,b) \mapsto (a+n, b+n)$,

$$(i_n)^* \omega^* X = (\sigma^*)^n w^* X.$$

That is, ω^* should be an infinite version of w^*.

Note that, by the cartesian closed structure on **Cat**, the functor $w\colon \square \times V \to V$ corresponds canonically to a functor $V \to V^{\square}$. Specifically,

$$(i,j) \mapsto w(-,-,i,j).$$

Rather than define ω as we did in Equation 3.5.4, we will define the functor $V \to V^{V^{\geq 0}}$ that corresponds to it. As we did above, we will write the functor $V^{\geq 0} \to V$ by giving a diagram of shape $V^{\geq 0}$ with entries given by the value of the functor. To demonstrate this for w, our alternative definition $V \to V^{\square}$ is

$$(i,j) \mapsto \begin{tikzcd} (i,i+1) \arrow[r] & (j+1,i+1) \\ (i,j) \arrow[r] \arrow[u] & (j+1,j) \arrow[u] \end{tikzcd}$$

so that $w\sigma^n$ for n odd is defined by

$$(i,j) \mapsto \begin{tikzcd} (j+n,j+n+1) \arrow[r] & (i+n+1,j+n+1) \\ (j+n,i+n) \arrow[r] \arrow[u] & (i+n+1,i+n) \arrow[u] \end{tikzcd}$$

and $w\sigma^n$ for n even is defined by

$$(i,j) \mapsto \begin{tikzcd} (i+n,i+n+1) \arrow[r] & (j+n+1,i+n+1) \\ (i+n,j+n) \arrow[r] \arrow[u] & (j+n+1,j+n) \arrow[u] \end{tikzcd}$$

Without further ado, ω will be defined by

$$\begin{tikzcd}
& & & & & {} \\
& & & (i+2,i+3) \arrow[r] & (j+3,i+3) \arrow[ur, dotted] & \\
(i,j) \mapsto & & (j+1,j+2) \arrow[r] & (i+2,j+2) \arrow[r] \arrow[u] & (j+3,j+2) \arrow[u] & \\
& (i,i+1) \arrow[r] & (j+1,i+1) \arrow[r] \arrow[u] & (i+2,i+1) \arrow[u] & & \\
& (i,j) \arrow[r] \arrow[u] & (j+1,j) \arrow[u] & & &
\end{tikzcd}$$

It may be apparent why we chose this format rather than the other definition. Examination of the above diagram (or direct calculation) shows an equality of compositions $\omega i_n = w\sigma^n$, thus $(i_n)^*\omega^*X = (\sigma^*)^n w^*X$ as we had hoped.

We now need to construct the $\mathbb{N}$-shaped coherent morphism suggested by Equation 3.5.13. First, we need to embed ω^*X into a larger diagram shape. Let Z be the full subcategory of $V^{\geq 0} \times \mathbb{N}$ consisting of all (i,j,k) for $i \neq j$ but restricting to $k \leq i$ if $i = j$.

There is a projection $\zeta\colon Z \to V^{\geq 0}$ that forgets the k component. For $X \in \operatorname{Sp}\mathbb{D}$ and any $k \in \mathbb{N}$, we have that

$$(i,j,k)^*\zeta^*\omega^*X = (i,j)^*\omega^*X$$

so ζ^* gives us a constant diagram functor of variable lengths depending on the value of (i,j). To illustrate this, we draw $\zeta^*\omega^*X$ for $i,j,k \leq 2$:

$$\begin{tikzcd}
& & & & & & & {} \\
& & & & 0 \arrow[rr] \arrow[dd] & & (\sigma^*)^2X \arrow[dd] \arrow[ur, dotted] & \\
& 0 \arrow[rr] \arrow[dd] & & \sigma^*X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & {} \\
X \arrow[ur] \arrow[rr] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr] \arrow[dd] & & (\sigma^*)^2X \arrow[dd] \arrow[ur, dotted] & \\
& 0 \arrow[rr] \arrow[dd] & & \sigma^*X \arrow[ur] \arrow[rr] & & 0 \arrow[ur] \arrow[dd] & & {} \\
& & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr] \arrow[d, dotted] & & (\sigma^*)^2X \arrow[ur, dotted] & \\
& 0 \arrow[urrr] \arrow[d, dotted] & & & {} & 0 \arrow[ur] \arrow[d, dotted] & & \\
& {} & 0 \arrow[urrr] \arrow[d, dotted] & & & {} & & \\
& & {} & & & & &
\end{tikzcd} \tag{3.5.15}$$

From this complicated shape, we will attempt to find our desired coherent diagram. Let $f\colon Z \to V^{\geq 0} \times \mathbb{N}$ be the natural embedding. The last step of our construction will be to take the right Kan extension f_* and restrict to the subcategory $\{(0,0)\} \times \mathbb{N}$, which should give us the desired coherent diagram.

We will again use Groth's detection lemma to make sure that we obtain cartesian squares where we would like them. We state the dual of Proposition 3.4.8 for reference:

Proposition 3.5.16. Let $i\colon \square \to J$ be a square in K and let $u\colon J \to K$ be a functor. Assume that the induced functor $\urcorner \to (i(0,0)/(K \setminus i(0,0)))$ has a right adjoint and that $i(0,0)$ does not lie in the image of u. Then for all $Y \in \mathbb{D}(K)$ in the image of $u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$, the square i^*Y is cartesian.

In our notation, we have $K = V^{\geq 0} \times \mathbb{N}$, $J = Z$, and $u = f$. The squares that we care about being cartesian are constant in k, so let us use the notation $i_{n,l}\colon \square \to V^{\geq 0} \times \mathbb{N}$ for the inclusion $(a,b) \mapsto (a+n, b+n, l)$ where $l > n$. Let us examine the category $(i_{n,l}(0,0)/(V^{\geq 0} \times \mathbb{N} \setminus i_{n,l}(0,0)))$. Because $V^{\geq 0} \times \mathbb{N}$ is a poset, this comma category will be a (full) subposet of Z (Lemma 2.2.5 again). In particular, it consists of $(i,j,k) \in V^{\geq 0} \times \mathbb{N}$ such that $i, j \geq n$ but $(i,j) \neq (n,n)$ and $k \geq l$. Let us call this subcategory $Z^{>n}$.

We will construct the right adjoint r directly, similar to the proof of Lemma 3.4.7. Let $(a,b) \in \urcorner$ and let $(i,j,k) \in Z^{>n}$. Call the induced functor of the proposition $\ell\colon \urcorner \to Z^{>n}$. Then we want

$$\operatorname{Hom}_{Z^{>n}}(\ell(a,b),(i,j,k)) \cong \operatorname{Hom}_{\urcorner}((a,b), r(i,j,k)).$$

We know that

$$\operatorname{Hom}_{Z^{>n}}(\ell(a,b),(i,j,k)) = \begin{cases} * & i \geq a+n, j \geq b+n, \text{ and } k \geq l \\ \varnothing & \text{otherwise} \end{cases}$$

The k coordinate is irrelevant, since $k \geq l$ is always satisfied. Because $(a,b) = (1,1)$ is final in $\urcorner$, if $i \geq n+1$ and $j \geq n+1$, we must have $r(i,j,k) = (1,1)$. Therefore we only need to consider $(n, n+1, k)$ and $(n+1, n, k)$. We see that $\ell(0,1)$ has a map to $(n, n+1, k)$ but not to $(n+1, n, k)$, so we must have $r(n, n+1, k) = (0,1)$, and similarly $r(n+1, n, k) = (1,0)$. This gives us the desired right adjoint, as again naturality of the hom-set bijection is automatic for posets.

Having established that all squares $i_{n,l}^* f_* \zeta^* \omega^* X$ are cartesian, we should see what this gives us in cash. Let $\widetilde{X} = f_* \zeta^* \omega^* X$ for ease of notation. If we look particularly at the cartesian square given by $i_{n-1,n}^* \widetilde{X}$,

$$\begin{tikzcd}
0 \arrow[r] & (\sigma^*)^n X \\
Y_1 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

we have $Y_1 \cong \Omega(\sigma^*)^n X$. We can now examine the square $i_{n-2,n}^* \widetilde{X}$. Because we have established that $(n-1, n-1, n)^* \widetilde{X} \cong \Omega(\sigma^*)^n X$, this square is of the form

$$\begin{tikzcd}
0 \arrow[r] & \Omega(\sigma^*)^n X \\
Y_2 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

so that $Y_2 \cong \Omega^2(\sigma^*)^n X$. Repeating this process shows that

$$(0, 0, n)^* \widetilde{X} \cong \Omega^n(\sigma^*)^n X.$$

The last step is to restrict to the column $\{(0, 0, k)\} \subset V^{\geq 0} \times \mathbb{N}$, which is precisely the diagram we need. Hence we obtain coherently

$$X \to \Omega\sigma^* X \to \Omega^2(\sigma^*)^2 X \to \cdots \to \Omega^n(\sigma^*)^n X \to \cdots \tag{3.5.17}$$

where each of these maps is a shift of the map φ_X we developed for Definition 3.5.8.

Recall that we named the functor sending X to the diagram of Equation 3.5.17 $F\colon \mathrm{Sp}\,\mathbb{D} \to \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}}$. We let $\mathrm{loc}\colon \mathrm{Sp}\,\mathbb{D} \to \mathrm{Sp}\,\mathbb{D}$ be given by the composition

$$\mathrm{Sp}\,\mathbb{D} \xrightarrow{\omega^*} \mathrm{Sp}\,\mathbb{D}^{V^{\geq 0}} \xrightarrow{\zeta^*} \mathrm{Sp}\,\mathbb{D}^{Z} \xrightarrow{f_*} \mathrm{Sp}\,\mathbb{D}^{V^{\geq 0} \times \mathbb{N}} \xrightarrow{(0,0)^*} \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}} \xrightarrow{\pi_!} \mathrm{Sp}\,\mathbb{D} \tag{3.5.18}$$

so that $\mathrm{loc} = \pi_! F$, where $\pi\colon \mathbb{N} \to e$ is the projection. We now need to make sure loc is a good start for our localization.

Lemma 3.5.19. The morphism $\mathrm{loc}\colon \mathrm{Sp}\,\mathbb{D} \to \mathrm{Sp}\,\mathbb{D}$ has essential image $\mathrm{St}\,\mathbb{D}$.

Proof. Let $X \in \mathrm{Sp}\,\mathbb{D}$. We need to show that $\varphi_{\mathrm{loc}\,X}\colon \mathrm{loc}\,X \to \Omega\sigma^* \mathrm{loc}\,X$ is an isomorphism. We do this by showing that it is (incoherently) a composition of isomorphisms:

$$\begin{tikzcd}
\mathrm{loc}\,X \arrow[r, "\mathrm{loc}(\varphi_X)"] \arrow[dr, bend right, "\varphi_{\mathrm{loc}\,X}"'] & \mathrm{loc}\,\Omega\sigma^* X \arrow[d] \\
& \Omega\sigma^* \mathrm{loc}\,X
\end{tikzcd}$$

where the vertical map is the canonical comparison map.

We first prove that $\operatorname{loc}(\varphi_X)\colon \operatorname{loc} X \to \operatorname{loc} \Omega\sigma^* X$ is an isomorphism. To first give the picture of $F\varphi_X$ in $\operatorname{Sp}\mathbb{D}(\mathbb{N} \times [1])$,

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \Omega\sigma^* X \arrow[r] \arrow[d] & \Omega^2(\sigma^*)^2 X \arrow[r] \arrow[d] & \cdots \\
\Omega\sigma^* X \arrow[r] & \Omega^2(\sigma^*)^2 X \arrow[r] & \Omega^3(\sigma^*)^3 X \arrow[r] & \cdots
\end{tikzcd}$$

The horizontal morphisms and the vertical morphisms with the same domain and codomain are precisely the same map. Therefore $\operatorname{loc}(\varphi_X)$ is the shift map between two sequential colimits, which is an isomorphism as the value of the sequential colimit does not depend on what finite stage we start at.

We now need to prove that loc commutes with $\Omega\sigma^*$, and do this we must look at the construction of loc. Because σ^* is a pullback morphism, we know that $\Omega\sigma^* \operatorname{loc} X \cong \Omega \operatorname{loc} \sigma^* X$. On $\operatorname{Sp}\mathbb{D}$, Ω is a continuous morphism, so it commutes with right Kan extensions as well as pullback morphisms. Therefore the only sticking point is whether Ω commutes with the sequential colimit $\pi_!\colon \operatorname{Sp}\mathbb{D}^{\mathbb{N}} \to \operatorname{Sp}\mathbb{D}$.

We combine the new Notation 3.4.1 and Definition 2.8.3 to recall how Ω is constructed. It is the composition

$$\operatorname{Sp}\mathbb{D} \xrightarrow{(1,1)_!} \operatorname{Sp}\mathbb{D}^{\neg} \xrightarrow{i_{\neg,*}} \operatorname{Sp}\mathbb{D}^{\square} \xrightarrow{(0,0)^*} \operatorname{Sp}\mathbb{D}.$$

Investigating this, truly the only sticking point is commuting $\pi_!$ with the middle morphism. But since we have assumed $\mathbb{D}$ is regular, $\operatorname{Sp}\mathbb{D}$ is also regular by Proposition 3.5.11. If the sequential (hence filtered) colimit $\pi_!$ commutes with finite limits, it also commutes with $i_{\neg,*}$ using Der4. Hence $\varphi_{\operatorname{loc} X}$ is an isomorphism and $\operatorname{loc} X \in \operatorname{St}\mathbb{D}$.

For $Y \in \operatorname{St}\mathbb{D}$, $\operatorname{loc} Y \cong Y$ as each of the morphisms $\Omega^n(\sigma^*)^n Y \to \Omega^{n+1}(\sigma^*)^{n+1} Y$ in $F(Y)$ is an isomorphism, and the colimit of a sequence of isomorphisms is again an isomorphism. Therefore the essential image of loc is all of $\operatorname{St}\mathbb{D}$. □

We have one more lemma to conclude the theorem.

Lemma 3.5.20. There exists a modification $\eta\colon \mathrm{id}_{\operatorname{Sp}\mathbb{D}} \Rightarrow \operatorname{loc}$ such that loc is a localization morphism.

Proof. We construct $\eta\colon \mathrm{id}_{\mathrm{Sp}\,\mathbb{D}} \Rightarrow \mathrm{loc}$ such that η_{loc} and $\mathrm{loc}\,\eta$ are isomodifications. Applying Proposition 3.2.6, we will obtain a localization with essential image $\mathrm{St}\,\mathbb{D}$, so the inclusion is right adjoint to loc.

First, consider the category A given by the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, shift left] & 2' \arrow[l, shift left] \arrow[r, shift left] & 1' \arrow[l, shift left] \arrow[r, shift left] & 0' \arrow[l, shift left] \arrow[r] & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots
\end{tikzcd}$$

where the composition of any two stacked morphisms is the identity (in either direction). There is a natural projection $p\colon A \to \mathbb{N}$ onto the unmarked objects, where $p(n') = 0$ by necessity. If we have an object $Y = (Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots) \in \mathbb{E}(\mathbb{N})$ in some (pre)derivator $\mathbb{E}$, then p^*Y has the form

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "="] & Y_0 \arrow[l, shift left, "="] \arrow[r, shift left, "="] & Y_0 \arrow[l, shift left, "="] \arrow[r, shift left, "="] & Y_0 \arrow[l, shift left, "="] \arrow[r, very thick, "="] & Y_0 \arrow[r, "f_0"] & Y_1 \arrow[r, "f_1"] & Y_2 \arrow[r] & \cdots
\end{tikzcd}$$

We will fold this diagram along the morphism $0' \to 0$, bolded above. Note that there is exactly one morphism $n' \to n$ for every n, and so we may rewrite our diagram with this in mind.

$$\begin{tikzcd}
Y_0 \arrow[r, "f_0"] & Y_1 \arrow[r, "f_1"] & Y_2 \arrow[r] & \cdots \\
Y_0 \arrow[u, "="] \arrow[r, shift left, "="] & Y_0 \arrow[u, "f_0"] \arrow[l, shift left, "="] \arrow[r, shift left, "="] & Y_0 \arrow[u, "f_1 f_0"] \arrow[l, shift left, "="] \arrow[r, shift left, "="] & \cdots \arrow[l, shift left, "="]
\end{tikzcd}$$

If we now forget the leftward arrows on the bottom row, we obtain the category $\mathbb{N} \times [1]$. This gives us in total a functor $a\colon \mathbb{N} \times [1] \to A \to \mathbb{N}$. Now, recall that $\mathrm{loc} = \pi_! F$ and let $X \in \mathrm{Sp}\,\mathbb{D}$. Then $F(X) \in \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}}$, and so we may apply a^* to obtain an object in $\mathrm{Sp}\,\mathbb{D}^{\mathbb{N}\times[1]}$, i.e. a coherent morphism of $\mathbb{N}$-shaped diagrams. To examine this more closely, recall that $F(X)$ has underlying diagram

$$\begin{tikzcd}
X \arrow[r, "f_0"] & \Omega\sigma^* X \arrow[r, "f_1"] & \Omega^2(\sigma^*)^2 X \arrow[r] & \cdots
\end{tikzcd}$$

Then $a^*F(X)$ has underlying diagram

$$\begin{tikzcd}
X \arrow[r, "f_0"] & \Omega\sigma^* X \arrow[r, "f_1"] & \Omega^2(\sigma^*)^2 X \arrow[r] & \cdots \\
X \arrow[u, "="] \arrow[r, "="] & X \arrow[u, "f_0"] \arrow[r, "="] & X \arrow[u, "f_1 f_0"] \arrow[r, "="] & \cdots
\end{tikzcd} \tag{3.5.21}$$

Applying $\pi_!$ gives rise to a morphism of derivators $\mu = \pi_! a^* F \colon \mathrm{Sp}\,\mathbb{D} \to \mathrm{Sp}\,\mathbb{D}^{[1]}$, which we can view as a coherent modification. The underlying actual modification is $\operatorname{dia}\mu \colon s^*\mu \Rightarrow t^*\mu$, where $s, t \colon e \to [1]$ are the source and target functors.

The functor $\pi \colon \mathbb{N} \to e$ admits a left adjoint, namely $0 \colon e \to \mathbb{N}$ which classifies the initial object. By Proposition 2.6.2, the canonical modification $\varepsilon \colon \pi_!\pi^* \Rightarrow \mathrm{id}_{\mathbb{E}}$ is an isomodification for any (left) derivator $\mathbb{E}$. In our case, this shows that $s^*\mu = \pi_!\pi^* \cong \mathrm{id}_{\mathrm{Sp}\,\mathbb{D}}$. By construction $t^*\mu = \mathrm{loc}$, so we obtain a a modification $\eta\colon \mathrm{id}_{\mathrm{Sp}\,\mathbb{D}} \Rightarrow \mathrm{loc}$ by precomposing $\operatorname{dia}\mu$ with the isomodification $\varepsilon^{-1}\colon \mathrm{id}_{\mathrm{Sp}\,\mathbb{D}} \Rightarrow \pi_!\pi^*$.

Let $X \in \mathrm{Sp}\,\mathbb{D}$. We now need to show that $\eta_{\mathrm{loc}\,X}$ and $\mathrm{loc}\,\eta_X$ are both isomorphisms. For the first, because $\mathrm{loc}\,X$ is a stable spectrum, each of the maps f_i in Diagram 3.5.21 are isomorphisms. Therefore each of the vertical maps of $a^*F(X)$ are isomorphisms, and the colimit of isomorphisms is again an isomorphism. Therefore $\eta_{\mathrm{loc}\,X}$ is an isomorphism.

Now, we would like to compare the formulae for η_{loc} and $\mathrm{loc}\,\eta$. Specifically, we have a diagram

$$\begin{tikzcd}
 & & \mathrm{Sp}\,\mathbb{D} \arrow[r, "F"] & \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}} \arrow[dr, bend left, "a^*"] & & \\
\mathrm{Sp}\,\mathbb{D} \arrow[r, "F"] & \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}} \arrow[ur, bend left, "\pi_!"] \arrow[dr, bend right, "a^*"'] & {} \arrow[r, phantom, "\Swarrow"] & {} & \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}\times[1]} \arrow[r, "\pi_!^{[1]}"] & \mathrm{Sp}\,\mathbb{D}^{[1]}, \\
 & & \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}\times[1]} \arrow[r, "\pi_!"'] & \mathrm{Sp}\,\mathbb{D}^{[1]} \arrow[ur, bend right, "F^{[1]}"'] & &
\end{tikzcd} \tag{3.5.22}$$

where the top composition is $\mu\,\mathrm{loc}$ and the bottom is $\mathrm{loc}^{[1]}\,\mu$. The middle transformation encodes the canonical isomorphism given by the zigzag

$$a^*F\pi_!Y \xrightarrow{\cong} F^{[1]}a^*\pi_!Y \xleftarrow{\cong} F^{[1]}\pi_!a^*Y$$

for any $Y \in \mathrm{Sp}\,\mathbb{D}^{\mathbb{N}}$ (as a^* is bicontinuous). We obtain η_{loc} by precomposing $\operatorname{dia}(\mu\,\mathrm{loc})$ by $\varepsilon_{\mathrm{loc}}^{-1}$ and $\mathrm{loc}\,\eta$ by precomposing $\operatorname{dia}(\mathrm{loc}^{[1]}\,\mu)$ by $\mathrm{loc}\,\varepsilon^{-1}$. Because $\varepsilon_{\mathrm{loc}}^{-1}$ and η_{loc} are isomorphisms, this means that $\mu\,\mathrm{loc}\,X$ is a coherent isomorphism for any $X \in \mathrm{Sp}\,\mathbb{D}$. But this means that $\mathrm{loc}^{[1]}\,\mu X$ is a coherent isomorphism as well by Diagram 3.5.22, so the composition $\mathrm{loc}\,\varepsilon^{-1}\operatorname{dia}(\mathrm{loc}^{[1]}\,\mu)X = \mathrm{loc}\,\eta_X$ is an isomorphism for any $X \in \mathrm{Sp}\,\mathbb{D}$.

Thus we have established that Proposition 3.2.6 applies, and so complete the proof of Theorem 3.5.12. □

To complete our goal for this section, we need one more proposition.

Proposition 3.5.23. $\operatorname{St}\mathbb{D}$ is a stable derivator.

Proof. Lemma 3.2.1 tells us that $\operatorname{St}\mathbb{D}$ is a derivator. It is also clear that $0 \in \operatorname{Sp}\mathbb{D}(e)$ satisfies $0 \cong \Omega\sigma^*0$, so $0 \in \operatorname{St}\mathbb{D}(e)$ and $\operatorname{St}\mathbb{D}$ is pointed.

To show that $\operatorname{St}\mathbb{D}$ is stable, in accordance with our agenda we will show that (Σ, Ω) is an adjoint equivalence on $\operatorname{St}\mathbb{D}$.

Recall that $\mathrm{id}_{\operatorname{St}\mathbb{D}} \Rightarrow \Omega\sigma^*$ is an invertible modification of endomorphisms of $\operatorname{St}\mathbb{D}$. Moreover, σ^* is an automorphism of $\operatorname{St}\mathbb{D}$. The map of diagrams $\sigma\colon V \to V$ is invertible, namely by $(i, j) \mapsto (j-1, i-1)$. Precomposition with the modification above gives us $(\sigma^{-1})^* = (\sigma^*)^{-1} \Rightarrow \Omega$ is an isomodification, hence Ω is an automorphism as well (and a fortiori an equivalence). This forces Σ to be an equivalence as well. ☐

Corollary 3.5.24. σ^* and Σ are canonically isomorphic in $\operatorname{St}\mathbb{D}$.

In general any two adjoints to Ω must be canonically isomorphic, and the above shows that these are indeed two choices of left adjoint.

All in all we have two adjunctions

$$\begin{tikzcd}
\mathbb{D} \arrow[d, bend right, "L"'] \\
\operatorname{Sp}\mathbb{D} \arrow[u, bend right, "(0{,}0)^*"'] \arrow[d, bend right, "\mathrm{loc}"'] \\
\operatorname{St}\mathbb{D} \arrow[u, bend right, "i"']
\end{tikzcd}$$

Definition 3.5.25. Let $\mathbb{D}$ be a regular pointed derivator. The *stabilization of* $\mathbb{D}$ is the left adjoint morphism $\mathrm{stab} = \mathrm{loc}\, L\colon \mathbb{D} \to \operatorname{St}\mathbb{D}$ constructed above. Its right adjoint is $(0,0)^* i\colon \operatorname{St}\mathbb{D} \to \mathbb{D}$.

Remark 3.5.26. While $\mathbb{D} \to \operatorname{St}\mathbb{D}$ is not a localization, the map $\mathbb{D}^V \to \operatorname{Sp}\mathbb{D} \to \operatorname{St}\mathbb{D}$ is a composition of localizations, hence is itself a localization. Therefore we can think of the stabilization of a derivator in terms of a localization of a certain diagram category on that derivator.

Alternatively, there is another model of $\operatorname{St}\mathbb{D}$ as Σ-spectra and a colocalization $\operatorname{Sp}\mathbb{D} \to \operatorname{St}\mathbb{D}$ onto the Σ-spectra. This would give us a different stabilization morphism $\operatorname{stab}_\Sigma \colon \mathbb{D} \to \operatorname{St}\mathbb{D}$ for *coregular* pointed derivators $\mathbb{D}$ which would be right adjoint to $(0,0)^*$. But because regularity seems a more natural condition (as far as examples go), we have chosen to prove everything through the lens of Ω-spectra.

3.6 The universal property of stabilization

We begin with a proposition to confirm that we have not developed an aberrant stabilization theory.

Proposition 3.6.1. If $\mathbb{S}$ is a stable derivator, then $\operatorname{stab}\colon \mathbb{S} \to \operatorname{St}\mathbb{S}$ is an equivalence of derivators.

Proof. First, a stable derivator is both pointed and regular, so this statement makes sense. Regularity follows by [Gro16a, Theorem 3.15(iii.b)]. As we noted above, in a stable derivator, (homotopy) finite limits (such as pullbacks) commute with arbitrary colimits.

We would like to check that the unit and counit of $(\operatorname{stab}, (0,0)^* i)$ are isomodifications. To that end, let $\eta\colon \operatorname{id}_\mathbb{S} \Rightarrow (0,0)^* \operatorname{stab}$ be the unit and $\varepsilon\colon \operatorname{stab}(0,0)^* i \Rightarrow \operatorname{id}_{\operatorname{St}\mathbb{S}}$ the counit. Let $x \in \mathbb{S}$, so that the unit evaluates to

$$\eta_x \colon x \to (0,0)^* i \operatorname{loc}(Lx),$$

where $L\colon \mathbb{S} \to \operatorname{Sp}\mathbb{S}$ is the morphism taking x to its connective suspension spectrum. Because $(\operatorname{stab}, (0,0)^* i)$ is a composite of adjunctions, its unit is the composition of the units (appropriately whiskered) $\eta_1\colon \operatorname{id}_\mathbb{S} \Rightarrow (0,0)^* L$ and $\eta_2\colon \operatorname{id}_{\operatorname{Sp}\mathbb{S}} \Rightarrow i \operatorname{loc}$:

$$x \xrightarrow{\eta_{1,x}} (0,0)^* Lx \xrightarrow{(0,0)^*\eta_{2,Lx}} (0,0)^* i \operatorname{loc}(Lx).$$

The first unit η_1 is an isomodification because L is fully faithful, as we proved it was a composite of fully faithful functors; see Diagram 3.4.6. Recall from Lemma 3.5.20 that η_2 is a composition of the isomodification $\varepsilon^{-1}\colon \operatorname{id}_{\operatorname{Sp}\mathbb{S}} \Rightarrow \pi_! \pi^*$ with a modification $\pi_! \pi^* \Rightarrow \pi_! F$, so let us examine first the map $\pi^* Lx \to F(Lx)$ before applying $\pi_!$. We know that $F(Lx)$ has the form

$$Lx \longrightarrow \Omega\sigma^* Lx \longrightarrow \Omega^2(\sigma^*)^2 Lx \longrightarrow \cdots$$

Now, rather than compute $(0,0)^*\pi_!\pi^* Lx \to (0,0)^*\pi_! F(Lx)$ (where we suppress the inclusion i), we can check the map $(0,0)^*\pi^* Lx \to \ (0,0)^* F(Lx)$ and take $\pi_!$ of this map. Because Lx is a suspension spectrum, $(0,0)^*(\sigma^*)^n Lx = (n,n)^* Lx \cong \Sigma^n x$. Therefore

$$(0,0)^* F(Lx) = (x \to \Omega\Sigma x \to \Omega^2\Sigma^2 x \to \cdots) \in \mathbb{S}^{\mathbb{N}}$$

and each map is an isomorphism because $\mathbb{S}$ is stable. Hence the map in question

$$\begin{tikzcd}
x \arrow[r, "\cong"] & \Omega\Sigma x \arrow[r, "\cong"] & \Omega^2\Sigma^2 x \arrow[r, "\cong"] & \cdots \\
x \arrow[u, "="] \arrow[r, "="'] & x \arrow[u] \arrow[r, "="'] & x \arrow[u] \arrow[r, "="'] & \cdots
\end{tikzcd}$$

is an isomorphism at every $n \in \mathbb{N}$, so that $\pi_!$ of this map is still an isomorphism. Therefore $(0,0)^*\eta_{2,Lx}$ is an isomorphism and thus η_x is as well.

To examine the counit, let $Y \in \operatorname{St}\mathbb{S}$. Then the counit is

$$\varepsilon_Y \colon \operatorname{loc}(L(0,0)^* iY) \to Y.$$

For simplicity, write $Y_0 = (0,0)^* iY$ as usual. The counit is the composition of $\varepsilon_1 \colon L(0,0)^* \Rightarrow \mathrm{id}_{\operatorname{Sp}\mathbb{S}}$ and $\varepsilon_2 \colon \operatorname{loc} i \Rightarrow \mathrm{id}_{\operatorname{St}\mathbb{S}}$:

$$\operatorname{loc}(LY_0) \xrightarrow{\operatorname{loc}\varepsilon_{1,iY}} \operatorname{loc} iY \xrightarrow{\varepsilon_{2,Y}} Y$$

The second counit ε_2 is an isomodification, so we need only to understand the first counit. In particular, we need to show that it is an isomorphism under loc.

The object LY_0 is the connective suspension spectrum on Y_0, which is isomorphic to the truncation of the original Y at the 0th level. The map $LY_0 \to Y$ encodes the (adjoints of the) canonical comparison isomorphisms $\Sigma^n Y_0 \to Y_n$ for $n \geq 0$ and is the zero map otherwise. We now need to compute loc applied to this map, which is $\pi_!$ applied to $F(LY_0) \to \pi^* Y$ followed by the canonical isomorphism $\pi_!\pi^* Y \to Y$.

We now examine the coherent diagram $F(LY_0)$. The spectrum $\Omega\sigma^* LY_0$ is given by

$$(i,i)^*\Omega\sigma^* LY_0 \cong \Omega\Sigma^{i+1}Y_0 \cong \Sigma^i Y_0$$

for any $i \geq -1$ and zero elsewhere. In general, $\Omega^n(\sigma^*)^n LY_0$ is isomorphic to the truncation of the original spectrum Y at the $-n$th level. We can compute the first few entries of the underlying diagram of $F(LY_0) \in \mathbb{S}(V \times \mathbb{N})$, letting $\rho\colon \mathrm{id}_{\mathbb{S}} \Rightarrow \Omega\Sigma$ be the unit of the (Σ, Ω) equivalence on $\mathbb{S}$.

$$\begin{tikzcd}
i & \vdots & \vdots & \vdots & \\
-2 & 0 \arrow[r] & 0 \arrow[r] & \Omega^2 Y_0 \arrow[r] & \cdots \\
-1 & 0 \arrow[r] & \Omega Y_0 \arrow[r, "\Omega\rho_{Y_0}"] & \Omega^2\Sigma Y_0 \arrow[r] & \cdots \\
0 & Y_0 \arrow[r, "\rho_{Y_0}"] & \Omega\Sigma Y_0 \arrow[r, "\Omega\rho_{\Sigma Y_0}"] & \Omega^2\Sigma^2 Y_0 \arrow[r] & \cdots \\
1 & \Sigma Y_0 \arrow[r, "\rho_{\Sigma Y_0}"] & \Omega\Sigma^2 Y_0 \arrow[r, "\Omega\rho_{\Sigma^2 Y_0}"] & \Omega^2\Sigma^3 Y_0 \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{3.6.2}$$

The vertical chains represent $\Omega^n(\sigma^*)^n LY_0$ (from left to right), where we write only the (i,i) entries and mark the value of i on the lefthand side. The colimit $\pi_! F(LY_0)$ is computed using the horizontal morphisms, which we see are eventually isomorphisms for any $i \in \mathbb{Z}$.

We will now prove that $\pi_! F(LY_0) \to \pi_!\pi^* Y$ is an isomorphism by showing that it is a pointwise isomorphism. We may do this by computing $(i,i)^*$ applied to the above map $F(LY_0) \to \pi^* Y$ and then applying $\pi_!$, as $\pi_!$ commutes with $(i,i)^*$ up to canonical isomorphism. For $i \geq 0$, this map is

$$\begin{tikzcd}
\Sigma^i Y_0 \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Omega\Sigma^{i+1} Y_0 \arrow[r, "\cong"] \arrow[d] & \Omega^2\Sigma^{i+2} Y_0 \arrow[r, "\cong"] \arrow[d] & \cdots \\
Y_i \arrow[r, "="] & Y_i \arrow[r, "="] & Y_i \arrow[r, "="] & \cdots
\end{tikzcd}$$

which we see consists solely of isomorphisms, so $\pi_!$ of this map is an isomorphism. For $i < 0$, we have a leading trail of zeroes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \Omega^{-i} Y_0 \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Omega^{-i}\Sigma Y_0 \arrow[r, "\cong"] \arrow[d] & \Omega^{-i}\Sigma^2 Y_0 \arrow[r, "\cong"] \arrow[d] & \cdots \\
Y_i \arrow[r, "="] & \cdots \arrow[r] & Y_i \arrow[r, "="] & Y_i \arrow[r, "="] & Y_i \arrow[r, "="] & Y_i \arrow[r, "="] & \cdots
\end{tikzcd}$$

These maps are not all isomorphisms, but they are eventually isomorphisms. Therefore $\pi_!$ applied to this map is an isomorphism as well, which shows that $\mathrm{loc}(\varepsilon_{1,Y})$ is an isomorphism by Der2.

Having shown that both the unit and the counit of the adjunction are isomodifications, this proves the equivalence. □

We would now like to prove a universal property of stabilization of the following form: let $\mathbb{S}$ be a stable derivator, and let $\mathbb{D}$ be a regular pointed derivator. Then we have an equivalence of categories

$$\mathrm{stab}^* \colon \mathrm{Hom}_\bullet(\mathrm{St}\,\mathbb{D}, \mathbb{S}) \to \mathrm{Hom}_\bullet(\mathbb{D}, \mathbb{S})$$

given by precomposition with $\mathrm{stab}\colon \mathbb{D} \to \mathrm{St}\,\mathbb{D}$, where $\mathrm{Hom}_\bullet \subset \mathrm{Hom}$ is the full subcategory on a certain class of morphisms. It would be surprising if $\mathrm{Hom}_\bullet$ were everything, as different morphisms $\mathrm{St}\,\mathbb{D} \to \mathbb{S}$ which do not involve the stability of the domain and the codomain should not necessarily compose with stab to different morphisms $\mathbb{D} \to \mathrm{St}\,\mathbb{D} \to \mathbb{S}$. We take as inspiration [Hel97, Theorem 8.1] that $\bullet = !$, the subcategory of cocontinuous morphisms.

We deviate from Heller's proof, however, for two reasons. First, Heller's proof relies on $\mathbb{D}$ being a strong derivator. If $\mathbb{D}$ is strong, $\mathrm{St}\,\mathbb{D}$ is a strong, stable derivator so $\mathrm{St}\,\mathbb{D}(e)$ admits a canonical triangulated structure; see [Mal07, §3] or Proposition 2.8.9. A study of 'stable equivalences' and 'stably trivial objects' gives him a kind of Verdier quotient from $\mathrm{Sp}\,\mathbb{D} \to \mathrm{St}\,\mathbb{D}$. But our derivators are not assumed strong, and stability alone is not sufficient to guarantee a triangulation.

The second and more important reason is that Heller's proof lacks key details. In particular, consider [Hel97, Diagram 9.3], an infinite diagram which he constructs incoherently. Despite following his reference to [Hel88, III §3], Heller does not prove that a diagram of shape U (in his terminology) will lift coherently even if we assume that $\mathrm{dia}_{[n]}$ is full and essentially surjective for any $n \in \mathbb{N}$. Moreover, modern derivator literature has produced no (infinite) lifting results of the sort Heller would require, despite his hope that strong derivators have dia_K full and essentially surjective 'for a much larger class of categories' K than free ones [Hel88, III §3]. Because we have not assumed our derivators strong, we would not have access to such results in any case.

Definition 3.6.3. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a pointed morphism of regular pointed derivators. Then define $\operatorname{St}\Phi\colon \operatorname{St}\mathbb{D} \to \operatorname{St}\mathbb{E}$ by the composition

$$\operatorname{St}\mathbb{D} \xrightarrow{i_{\mathbb{D}}} \operatorname{Sp}\mathbb{D} \xrightarrow{\Phi^V} \operatorname{Sp}\mathbb{E} \xrightarrow{\operatorname{loc}_{\mathbb{E}}} \operatorname{St}\mathbb{E}.$$

We require that Φ be pointed in order that $\Phi^V\colon \operatorname{Sp}\mathbb{D} \to \mathbb{E}^V$ have its image in $\operatorname{Sp}\mathbb{E}$.

Lemma 3.6.4. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a cocontinuous morphism of regular pointed derivators. Then the following square commutes up to invertible modification:

$$\begin{tikzcd}
\operatorname{Sp}\mathbb{D} \arrow[r, "\Phi^V"] \arrow[d, "\operatorname{loc}_{\mathbb{D}}"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow}"] & \operatorname{Sp}\mathbb{E} \arrow[d, "\operatorname{loc}_{\mathbb{E}}"] \\
\operatorname{St}\mathbb{D} \arrow[r, "\operatorname{St}\Phi"'] & \operatorname{St}\mathbb{E}
\end{tikzcd}$$

Proof. This modification is given by the unit of the localization $\eta_{\mathbb{D}}\colon \operatorname{id}_{\operatorname{Sp}\mathbb{D}} \Rightarrow i_{\mathbb{D}} \operatorname{loc}_{\mathbb{D}}$. This gives us a map

$$\operatorname{loc}_{\mathbb{E}} \Phi^V \xrightarrow{\operatorname{loc}_{\mathbb{E}} \Phi^V \eta_{\mathbb{D}}} \operatorname{loc}_{\mathbb{E}} \Phi^V i_{\mathbb{D}} \operatorname{loc}_{\mathbb{D}} = \operatorname{St}\Phi \operatorname{loc}_{\mathbb{D}}. \tag{3.6.5}$$

We just need to prove that this is an isomodification. We will explicitly prove that, for any $X \in \operatorname{Sp}\mathbb{D}$, the map $\operatorname{loc}_{\mathbb{E}} \Phi^V \eta_{\mathbb{D},X}$ is an isomorphism, though this will involve changing its domain and codomain up to isomorphism.

As we will be working with many different diagram categories shortly, we will also write Φ for the appropriate Φ^K. Remembering $\operatorname{loc} = \pi_! F$, we know that $\Phi\pi_! \cong \pi_!\Phi$ because Φ is cocontinuous, so our codomain is isomorphic to $\pi_! F \pi_! \Phi F(X)$. We claim that we may commute the middle F and $\pi_!$ by regularity.

Let $Y \in \operatorname{Sp}\mathbb{E}^{\mathbb{N}}$ and examine the comparison map $\pi_! F(Y) \to F\pi_!(Y)$. The underlying diagram is

$$\begin{tikzcd}
\pi_! Y \arrow[r] \arrow[d] & \pi_!\Omega\sigma^* Y \arrow[r] \arrow[d] & \pi_!\Omega^2(\sigma^*)^2 Y \arrow[r] \arrow[d] & \cdots \\
\pi_! Y \arrow[r] & \Omega\sigma^*\pi_! Y \arrow[r] & \Omega^2(\sigma^*)^2\pi_! Y \arrow[r] & \cdots
\end{tikzcd}$$

Hence $\pi_!$ commutes with F so long as Ω (and σ^*) commutes with $\pi_!$. Since we have assumed $\mathbb{E}$ is regular, this is so.

Therefore we may compute $\operatorname{loc}_{\mathbb{E}} \Phi^V i_{\mathbb{D}} \operatorname{loc}_{\mathbb{D}}(X)$ as $\pi_!\pi_! F\Phi F(X)$, that is, the colimit of an $\mathbb{N}^2$-shaped diagram, where for convenience in what follows we will flip the vertical dimension. Let us adopt the shorthand $X^i = \Omega^i(\sigma^*)^i X$. Then the underlying diagram of $F\Phi F(X)$ is

(3.6.6)
$$\begin{tikzcd}
\Phi X^0 \arrow[r] \arrow[d] & \Phi X^1 \arrow[r] \arrow[d] & \Phi X^2 \arrow[r] \arrow[d] & \cdots \\
\Omega\sigma^*\Phi X^0 \arrow[r] \arrow[d] & \Omega\sigma^*\Phi X^1 \arrow[r] \arrow[d] & \Omega\sigma^*\Phi X^2 \arrow[r] \arrow[d] & \cdots \\
\Omega^2(\sigma^*)^2\Phi X^0 \arrow[r] \arrow[d] & \Omega^2(\sigma^*)^2\Phi X^1 \arrow[r] \arrow[d] & \Omega^2(\sigma^*)^2\Phi X^2 \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

In short, $(p,q)^* F\Phi F(X) = \Omega^q(\sigma^*)^q\Phi(X^p)$. Note that the lefthand column ($p = 0$) is $F\Phi^V(X)$, the domain of Equation 3.6.5 before applying $\pi_!$. Indeed, the map $\operatorname{loc}_{\mathbb{E}} \Phi^V \eta_{\mathbb{D},X}$ is the colimit over $\mathbb{N}^2$ of the map from π^* applied to the first column to the diagram as a whole. For a restricted portion of $\mathbb{N}^2$, this map is

$$\begin{tikzcd}
\Phi X^0 \arrow[r, "="] \arrow[d] & \Phi X^0 \arrow[r, "="] \arrow[d] & \Phi X^0 \arrow[d] & & \Phi X^0 \arrow[r] \arrow[d] & \Phi X^1 \arrow[r] \arrow[d] & \Phi X^2 \arrow[d] \\
\Omega\sigma^*\Phi X^0 \arrow[r, "="] & \Omega\sigma^*\Phi X^0 \arrow[r, "="] & \Omega\sigma^*\Phi X^0 & \longrightarrow & \Omega\sigma^*\Phi X^0 \arrow[r] & \Omega\sigma^*\Phi X^1 \arrow[r] & \Omega\sigma^*\Phi X^2
\end{tikzcd}$$

We will show that $\operatorname{loc}_{\mathbb{E}} \Phi^V \eta_{\mathbb{D},X}$ is an isomorphism directly by decomposing it into the composition of two canonical colimit comparison maps which we prove are isomorphisms, namely Equations 3.6.10 and 3.6.12.

In order to do this we show that Diagram 3.6.6 computes the same colimit as another diagram which is essentially [Hel97, Diagram 9.3]. In the theme of this chapter, Heller's argument is upgradeable to a coherent version. The following argument is an extra-dimensional enhancement of Construction 3.5.14.

Recall from Equation 3.5.18 that the functor $F\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^{\mathbb{N}}$ had as its last steps a right Kan extension and a restriction to $\{(0,0)\} \times \mathbb{N} \subset V^{\geq 0} \times \mathbb{N}$. Let us denote by $G = f_*\zeta^*\omega^*\colon \operatorname{Sp}\mathbb{D} \to \operatorname{Sp}\mathbb{D}^{V^{\geq 0}\times\mathbb{N}}$ all but the ultimate step of that construction. For an object $X \in \operatorname{Sp}\mathbb{D}$, $G(X)$ has the following form:

$$\begin{tikzcd}
 & & & & 0 \arrow[rr] \arrow[dd] & & (\sigma^*)^2X \arrow[dd] \\
 & 0 \arrow[rr] \arrow[dd] & & \sigma^*X \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr, crossing over] \arrow[dd] & & (\sigma^*)^2X \arrow[dd] \\
 & 0 \arrow[rr, crossing over] \arrow[dd] & & \sigma^*X \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
\Omega\sigma^*X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr, crossing over] \arrow[d, dotted] & & (\sigma^*)^2X \arrow[d, dotted] \\
 & 0 \arrow[rr, crossing over] \arrow[d, dotted] & & \Omega(\sigma^*)^2X \arrow[rr] \arrow[ur] \arrow[d, dotted] & {} & 0 \arrow[ur] \arrow[d, dotted] & {} \\
\Omega^2(\sigma^*)^2X \arrow[ur] \arrow[rr] \arrow[d, dotted] & {} & 0 \arrow[ur] \arrow[d, dotted] & {} & & {} & \\
{} & & {} & & & &
\end{tikzcd}$$

We now apply the morphism Φ to the entire diagram. The $(0,0) \times \mathbb{N}$ column is now of the form $\Phi(X^i)$, so it is the first row of Diagram 3.6.6. We now need to construct coherently the rest of Diagram 3.6.6, and to do so we will need to add yet another dimension to this diagram.

Consider the following subcategory of $V^{\geq 0} \times \mathbb{N}_1 \times \mathbb{N}_2$, where we imagine $\Phi G(X)$ embedding into the first two coordinates and aiming for a constant diagram in the $\mathbb{N}_2$-direction. At each $k \in \mathbb{N}_1$, we restrict to $\{0, \ldots, i\} \subset \mathbb{N}_2$ for $(i,i) \in V^{\geq 0}$ with $i < k$ and $\{0, \ldots, k\} \subset \mathbb{N}_2$ otherwise. Call this shape B_1 and let $b_1 \colon B_1 \to V^{\geq 0} \times \mathbb{N}$ be the projection forgetting the last coordinate. Then $b_1^* \Phi G(X)$ is equal to $\Phi G(X)$ when the last coordinate is zero, and constant of variable lengths in the $\mathbb{N}_2$-direction.

For example, at $k = 2$ we have the subcategory of $V^{\geq 0} \times \{0, 1, 2\}$:

(3.6.7)

$$\begin{tikzcd}
 & & & & & & & {} \\
 & & & & 0 \arrow[rr] \arrow[dd] & & \Phi(\sigma^*)^2X \arrow[dd] \arrow[ur, dotted] & \\
 & 0 \arrow[rr] \arrow[dd] & & \Phi\Omega(\sigma^*)^2X \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & {} \\
\Phi\Omega^2(\sigma^*)^2X \arrow[ur] \arrow[rr] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr, crossing over] \arrow[dd] & & \Phi(\sigma^*)^2X \arrow[dd] \arrow[ur, dotted] & \\
 & 0 \arrow[rr, crossing over] \arrow[dd] & & \Phi\Omega(\sigma^*)^2X \arrow[rr] \arrow[ur] & & 0 \arrow[ur] \arrow[dd] & & {} \\
 & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr, crossing over] & & \Phi(\sigma^*)^2X \arrow[ur, dotted] & \\
 & 0 \arrow[urrr, crossing over] & & & & 0 \arrow[ur] & & \\
 & & 0 \arrow[urrr] & & & & &
\end{tikzcd}$$

continuing in a constant vertical diagram of shape [2] in the $V^{\geq 0}$ dimension but not downward in the $\mathbb{N}_2$ dimension. This is an analogous construction to Diagram 3.5.15, but depending on both $(i, j) \in V^{\geq 0}$ and $k \in \mathbb{N}_1$.

Let B_2 be the subdiagram $V^{\geq 0} \times \mathbb{N}_1 \times \mathbb{N}_2$ which has at $k \in \mathbb{N}_1$ the entire $\{0, \ldots, k\} \subset \mathbb{N}_2$ regardless of $(i, j) \in V^{\geq 0}$, and let $b_2 \colon B_1 \to B_2$ be the inclusion. Then we would like to compute $b_{2,*}b_1^*\Phi G(X)$. Specifically, we would like to investigate the squares

$$\begin{tikzcd}
(i, i+1, k, l) \arrow[r] & (i+1, i+1, k, l) \\
(i, i, k, l) \arrow[r] \arrow[u] & (i+1, i, k, l) \arrow[u]
\end{tikzcd}$$

for $i < l \leq k$. The lower left corner does not lie in the image of b_2, so we will again use Proposition 3.5.16 to determine that these squares are cartesian. Though we have another dimension, we are still working in a poset, so the comma categories we construct will be subcategories of B_2.

We once again use Lemma 2.2.5 to describe $((i, i, k, l)/(B_2 \setminus (i, i, k, l)))$: it is the full subposet of B_2 on objects admitting a map from (i, i, k, l). We can now construct the right adjoint r to $\urcorner \to ((i, i, k, l)/(B_2 \setminus (i, i, k, l)))$. The adjoint is nearly the same as the one we constructed in our first application of Proposition 3.5.16 in the Construction 3.5.14. We use the same notation for $\urcorner$ as we did there, and we temporarily abandon entirely our conventions, having run out of plausible variable names:

$$r(w, x, y, z) = \begin{cases} (1,1) & w, x \geq i+1 \\ (1,0) & (w, x) = (i+1, i) \\ (0,1) & (w, x) = (i, i+1) \end{cases}$$

As in the earlier argument, the $\mathbb{N}$-dimensions do not matter, because any $(w, x, y, z) \in B_2$ which is in the subcategory $((i, i, k, l)/(B_2 \setminus (i, i, k, l)))$ will always satisfy $(y, z) \geq (k, l)$. That this is a right adjoint can be verified in a way identical to the proofs earlier in this chapter.

This proves that all such squares are cartesian. Aiming again to calculate the values of our diagram at $(0, 0, k, l)$, we can first look at the square with lower left corner $(l-1, l-1, k, l)$.

The rest of this square is entirely in B_1, with underlying diagram

$$\begin{tikzcd}
0 \arrow[r] & \Phi\Omega^{k-l}(\sigma^*)^k X \\
& 0 \arrow[u]
\end{tikzcd}$$

This makes sense because we necessarily have $l \leq k$. After applying $b_{2,*}$ we obtain a cartesian square, hence

$$(l-1, l-1, k, l)^* b_{2,*} b_1^* \Phi G(X) \cong \Omega\Phi\Omega^{k-l}(\sigma^*)^k X$$

and so by induction

$$(0, 0, k, l)^* b_{2,*} b_1^* \Phi G(X) \cong \Omega^l \Phi\Omega^{k-l}(\sigma^*)^k X.$$

To give the result of Diagram 3.6.7 after $b_{2,*}$:

(3.6.8)

$$\begin{tikzcd}
& & & & & & & {} \\
& & & & 0 \arrow[rr] \arrow[dd] & & \Phi(\sigma^*)^2 X \arrow[ur, dotted] \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dd] & & \Phi\Omega(\sigma^*)^2 X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & {} \\
\Phi\Omega^2(\sigma^*)^2 X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr] \arrow[dd] & & \Phi(\sigma^*)^2 X \arrow[ur, dotted] \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dd] & & \Phi\Omega(\sigma^*)^2 X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & {} \\
\Omega\Phi\Omega(\sigma^*)^2 X \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & & 0 \arrow[rr] & & \Phi(\sigma^*)^2 X \arrow[ur, dotted] & \\
& 0 \arrow[rr] & & \Omega\Phi(\sigma^*)^2 X \arrow[ur] \arrow[rr] & & 0 \arrow[ur] & & \\
\Omega^2\Phi(\sigma^*)^2 X \arrow[ur] \arrow[rr] & & 0 \arrow[ur] & & & & &
\end{tikzcd}$$

Up to the precise location of σ^*, we obtain all elements that appear in Diagram 3.6.6. Our final step of this construction is to restrict to all elements with $(0, 0)$ in the $V^{\geq 0}$ coordinate. This has underlying diagram

(3.6.9)

$$\begin{tikzcd}
\Phi X \arrow[r] \arrow[dr] & \Phi\Omega\sigma^* X \arrow[r] \arrow[d] & \Phi\Omega^2(\sigma^*)^2 X \arrow[r, dotted] \arrow[d] & {} \\
& \Omega\Phi\sigma^* X \arrow[r] \arrow[dr] & \Omega\Phi\Omega(\sigma^*)^2 X \arrow[r, dotted] \arrow[d] & {} \\
& & \Omega^2\Phi(\sigma^*)^2 X \arrow[r, dotted] \arrow[dr, dotted] & {} \\
& & & {}
\end{tikzcd}$$

where now the horizontal direction is $\mathbb{N}_1$ and the vertical direction $\mathbb{N}_2$. For example, the three objects in Diagram 3.6.8 in the $(0, 0)$ column form the third column of the above diagram. Call this triangular diagram shape U, as Heller does, and let $\mathbf{X} = (0, 0)^* b_{2,*} b_1^* \Phi G(X) \in \operatorname{Sp} \mathbb{D}^U$.

Let $d\colon \mathbb{N} \to U$ denote the diagonal in Diagram 3.6.9. Consider any comma category (a/d) for $a \in U$. These all admit an initial object, namely the point on the diagonal immediately 'below' a in U and the unique map from a to it. Thus the projection from that comma category to the final category e is right adjoint to the inclusion of that initial object. By Proposition 2.6.2, this makes (a/d) homotopy contractible, making d homotopy final by Proposition 2.6.18.

Therefore for any $\mathbf{Y} \in \operatorname{Sp}\mathbb{D}^U$, e.g. the diagram $\mathbf{X}$ we have constructed above, the canonical map

$$\pi_{\mathbb{N},!}d^*\mathbf{Y} \xrightarrow{\cong} \pi_{U,!}\mathbf{Y} \tag{3.6.10}$$

is an isomorphism. In our case, $d^*\mathbf{X} \cong F\Phi^V(X)$, so $\pi_{\mathbb{N},!}d^*\mathbf{X} \cong \operatorname{loc}_{\mathbb{E}} \Phi^V(X)$, precisely the domain of the map we are constructing, Equation 3.6.5.

We now claim that the original Diagram 3.6.6 can be found in U as well. To see this, we stretch the shape U:

$$\begin{tikzcd}
\Phi X^0 \arrow[r] \arrow[d] & \Phi X^1 \arrow[r] \arrow[dl] \arrow[d, dashed] & \Phi X^2 \arrow[r] \arrow[dl] \arrow[d, dashed] & \cdots \\
\Omega\sigma^*\Phi X^0 \arrow[r] \arrow[d] & \Omega\sigma^*\Phi X^1 \arrow[r] \arrow[dl] \arrow[d, dashed] & \Omega\sigma^*\Phi X^2 \arrow[r] \arrow[dl] \arrow[d, dashed] & \cdots \\
\Omega^2(\sigma^*)^2\Phi X^0 \arrow[r] \arrow[d] & \Omega^2(\sigma^*)^2\Phi X^1 \arrow[r] \arrow[d, dashed] & \Omega^2(\sigma^*)^2\Phi X^2 \arrow[r] \arrow[d, dashed] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd} \tag{3.6.11}$$

The first row of Diagram 3.6.11 corresponds to the first row of Diagram 3.6.9 and the first column corresponds to the diagonal. The vertical arrows in Diagram 3.6.9 thus have stretched to the diagonal arrows in Diagram 3.6.11, and the dashed vertical arrows are just compositions. In this way, we can see $\mathbb{N}^2$ is a non-full subcategory of U, and we let $i_2\colon \mathbb{N}^2 \to U$ be the inclusion. This gives in total

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "i_1"'] \arrow[rr, bend left, "d"] & \mathbb{N}^2 \arrow[r, "i_2"'] & U,
\end{tikzcd}$$

where $\mathbb{N} \to \mathbb{N}^2$ is the map into the first column and $\mathbb{N}^2 \to U$ is the map indicated by the above diagram. The composition is the diagonal map $d\colon \mathbb{N} \to U$.

We now show that i_2 is a homotopy final functor. To do so, let us label the objects of U as indicated by Diagram 3.6.11 by (p, q). We need to prove that each of the categories $((p, q)/i_2)$ is homotopy contractible for any $(p, q) \in U$. But since $\mathbb{N}^2 \to U$ is surjective (though not full), we have that $(p, q) = i_2(p, q)$ is an element of each $((p, q)/i_2(p, q))$, which is an initial object. This makes the projection to the terminal category again a right adjoint, so by the same reasoning above (Propositions 2.6.2 and 2.6.18) we conclude that i_2 is homotopy final.

Thus we have an isomorphism

$$\pi_{\mathbb{N}^2,!}i_2^*\mathbf{Y} \xrightarrow{\cong} \pi_{U,!}\mathbf{Y} \tag{3.6.12}$$

for any $\mathbf{Y} \in \operatorname{Sp}\mathbb{D}^U$. Using our constructed $\mathbf{X}$, we have $\pi_{\mathbb{N}^2,!}i_2^*\mathbf{X} \cong \pi_!\pi_!F\Phi F(X)$, which is the codomain of Equation 3.6.5. Putting these two isomorphisms together, we have the required isomorphism

$$\begin{tikzcd}
\mathrm{loc}_{\mathbb{E}}\,\Phi^V(X) \arrow[r, phantom, "\cong"] \arrow[rrrr, bend right=15, "\mathrm{loc}\,\Phi^V\eta_{\mathbb{D},X}"'] & \pi_{\mathbb{N},!}d^*\mathbf{X} \arrow[r, "\cong"] & \pi_{!,U}\mathbf{X} & \pi_{\mathbb{N}^2,!}i_2^*\mathbf{X} \arrow[l, "\cong"'] \arrow[r, phantom, "\cong"] & \mathrm{loc}_{\mathbb{E}}\,\Phi^V i_{\mathbb{D}}\,\mathrm{loc}_{\mathbb{D}}(X).
\end{tikzcd}$$

This completes the proof. □

Notation 3.6.13. Let $\mathbf{Der}_!$ be the 2-category with objects regular pointed derivators, cocontinuous morphisms, and all modifications. Let $\mathbf{StDer}_! \subset \mathbf{Der}_!$ be the full 2-subcategory of stable derivators. We write $\operatorname{Hom}_!(\mathbb{D}, \mathbb{E})$ for the category of cocontinuous morphisms between $\mathbb{D}$ and $\mathbb{E}$ in either case.

Proposition 3.6.14. The assignment $\mathbb{D} \mapsto \operatorname{St}\mathbb{D}$ and $\Phi \mapsto \operatorname{St}\Phi$ gives a pseudofunctor

$$\operatorname{St}\colon \mathbf{Der}_! \longrightarrow \mathbf{StDer}_!.$$

Proof. First, we need to show that for any cocontinuous morphism $\Phi \in \operatorname{Hom}_!(\mathbb{D}, \mathbb{E})$, the morphism $\operatorname{St}\Phi$ is still cocontinuous. To see this, let $u\colon J \to K$ be any functor in $\mathbf{Dia}$. Using the natural isomorphism of Lemma 3.6.4, we know that for any $X \in \operatorname{Sp}\mathbb{D}(J)$,

$$\mathrm{loc}_{\mathbb{E}}(\Phi^V(u_!X)) \cong \operatorname{St}\Phi(\mathrm{loc}_{\mathbb{D}}(u_!X)).$$

But $\mathrm{loc}_{\mathbb{E}}\,\Phi^V$ is a composition of cocontinuous morphisms, so

$$\mathrm{loc}_{\mathbb{E}}(\Phi^V(u_!X)) \cong u_!\,\mathrm{loc}_{\mathbb{E}}(\Phi^V(X)).$$

Using again the natural isomorphism,

$$u_! \operatorname{loc}_{\mathbb{E}}(\Phi^V(X)) \cong u_! \operatorname{loc}_{\mathbb{D}}(\operatorname{St}\Phi(X)).$$

Because $\operatorname{loc}_{\mathbb{D}}$ is cocontinuous, it also commutes with $u_!$. Putting this all together we conclude that

$$\operatorname{St}\Phi(u_! \operatorname{loc}_{\mathbb{D}}(X)) \cong u_! \operatorname{St}\Phi(\operatorname{loc}_{\mathbb{D}}(X)).$$

Therefore $\operatorname{St}\Phi$ preserves left Kan extensions along u on the essential image of $\operatorname{loc}_{\mathbb{D}}$, which is all of $\operatorname{St}\mathbb{D}$.

To show that St is a pseudofunctor, we also need to give an isomodification $\operatorname{id}_{\operatorname{St}\mathbb{D}} \Rightarrow \operatorname{St}(\operatorname{id}_{\mathbb{D}})$ for any $\mathbb{D}$. But by definition, $\operatorname{St}(\operatorname{id}_{\mathbb{D}}) = \operatorname{loc}_{\mathbb{D}} i_{\mathbb{D}}$, which is isomorphic to $\operatorname{id}_{\operatorname{St}\mathbb{D}}$ via the counit of the $(\operatorname{loc}_{\mathbb{D}}, i_{\mathbb{D}})$ adjunction that we have already constructed.

The last thing to show for pseudofunctoriality is the assignment on morphisms behaves well with respect to composition. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ and $\Psi\colon \mathbb{E} \to \mathbb{F}$ be cocontinuous morphisms. Then we would like to say that $\operatorname{St}\Psi \operatorname{St}\Phi \cong \operatorname{St}(\Psi\Phi)$ naturally in Φ and Ψ. Unwinding the definitions, we have the following situation:

$$\begin{tikzcd}
\operatorname{St}\mathbb{D} \arrow[r, "i_{\mathbb{D}}"] & \operatorname{Sp}\mathbb{D} \arrow[rr, "\Psi^V\Phi^V = (\Psi\Phi)^V"] \arrow[d, "\Phi^V"'] \arrow[drr, phantom, "\Swarrow"] & & \operatorname{Sp}\mathbb{F} \arrow[r, "\operatorname{loc}_{\mathbb{F}}"] & \operatorname{St}\mathbb{F} \\
& \operatorname{Sp}\mathbb{E} \arrow[r, "\operatorname{loc}_{\mathbb{E}}"'] & \operatorname{St}\mathbb{E} \arrow[r, "i_{\mathbb{E}}"'] & \operatorname{Sp}\mathbb{E} \arrow[u, "\Psi^V"'] &
\end{tikzcd}$$

where the bottom composition is $\operatorname{St}\Psi \operatorname{St}\Phi$ and the top is $\operatorname{St}(\Psi\Phi)$. The natural transformation is induced by the unit $\eta_{\mathbb{E}}\colon \operatorname{id}_{\operatorname{Sp}\mathbb{E}} \Rightarrow i_{\mathbb{E}} \operatorname{loc}_{\mathbb{E}}$. Specifically, it is $\operatorname{loc}_{\mathbb{F}} \Psi^V \eta_{\mathbb{E},\Phi^V i_{\mathbb{D}}}$. Using essentially the same argument of Lemma 3.6.4, this modification is an isomodification. This gives us the requisite invertible modification defining composition.

We should technically check associativity of composition of morphisms, but the above constructions show that the modifications involved in the associativity diagrams come from the unit and the counit of the adjunction. Therefore all the necessary associativity conditions hold because they do for compositions of modifications. □

Theorem 3.6.15. The pseudofunctor $\mathrm{St}\colon \mathbf{Der}_! \to \mathbf{StDer}_!$ is left adjoint to the (fully faithful) inclusion $\mathbf{StDer}_! \to \mathbf{Der}_!$. That is, for any stable derivator $\mathbb{S}$,

$$\mathrm{stab}^*\colon\ \mathrm{Hom}_!(\mathrm{St}\,\mathbb{D}, \mathbb{S}) \to \mathrm{Hom}_!(\mathbb{D}, \mathbb{S})$$

given by precomposition with the morphism $\mathrm{stab}\colon \mathbb{D} \to \mathrm{St}\,\mathbb{D}$ is an equivalence of categories.

Proof. It suffices to give an inverse equivalence $Q\colon\ \mathrm{Hom}_!(\mathbb{D}, \mathbb{S}) \to \mathrm{Hom}_!(\mathrm{St}\,\mathbb{D}, \mathbb{S})$. Suppose that $\Phi\colon \mathbb{D} \to \mathbb{S}$ is a cocontinuous morphism. Then we let $Q(\Phi)$ be given by the composition

$$\begin{tikzcd}
\mathbb{D} \arrow[r] \arrow[d, "\Phi"'] & \mathrm{Sp}\,\mathbb{D} \arrow[r, "\mathrm{loc}_{\mathbb{D}}"] \arrow[d, "\Phi^V"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \mathrm{St}\,\mathbb{D} \arrow[d, "\mathrm{St}\,\Phi"'] \arrow[dr, bend left, "Q(\Phi)"] & \\
\mathbb{S} \arrow[r] & \mathrm{Sp}\,\mathbb{S} \arrow[r, "\mathrm{loc}_{\mathbb{S}}"'] & \mathrm{St}\,\mathbb{S} \arrow[r, "{(0,0)^*}"'] & \mathbb{S}
\end{tikzcd}$$

Otherwise put, we have $Q(\Phi) := (0,0)^*\,\mathrm{St}\,\Phi$. The middle square commutes up to isomorphism by Lemma 3.6.4. Since the upper composition is exactly $\mathrm{stab}_{\mathbb{D}}\colon \mathbb{D} \to\ \mathrm{St}\,\mathbb{D}$, this shows that we have an isomorphism $Q(\Phi)\,\mathrm{stab}_{\mathbb{D}} \cong (0,0)^*\,\mathrm{stab}_{\mathbb{S}}\,\Phi$. But because $\mathbb{S}$ is already a stable derivator, we have that $(0,0)^*\,\mathrm{stab}_{\mathbb{S}} \cong \mathrm{id}_{\mathbb{S}}$, so that $Q(\Phi)\,\mathrm{stab}_{\mathbb{D}} \cong \Phi$.

For the other direction, suppose that we have a cocontinuous morphism $\Psi\colon\ \mathrm{St}\,\mathbb{D} \to\ \mathbb{S}$. Then we would like to show that $Q(\Psi\,\mathrm{stab}_{\mathbb{D}}) \cong \Psi$. These morphisms fit into the following diagram:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\mathrm{stab}_{\mathbb{D}}"] \arrow[d, "\mathrm{stab}_{\mathbb{D}}"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \mathrm{St}\,\mathbb{D} \arrow[d, "\mathrm{St}\,\mathrm{stab}_{\mathbb{D}}"] \arrow[dr, bend left] & \\
\mathrm{St}\,\mathbb{D} \arrow[r, "\mathrm{stab}_{\mathrm{St}\,\mathbb{D}}"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \mathrm{St}\,\mathrm{St}\,\mathbb{D} \arrow[r, "{(0,0)^*}"] \arrow[d, "\mathrm{St}\,\Psi"] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \mathrm{St}\,\mathbb{D} \arrow[d, "\Psi"] \\
\mathbb{S} \arrow[r, "\mathrm{stab}_{\mathbb{S}}"'] & \mathrm{St}\,\mathbb{S} \arrow[r, "{(0,0)^*}"'] & \mathbb{S}
\end{tikzcd} \tag{3.6.16}$$

The rightmost composition is by definition $Q(\Psi\,\mathrm{stab}_{\mathbb{D}})$. Tracing the diagram gives us

$$\Psi\,\mathrm{stab}_{\mathbb{D}} \cong Q(\Psi\,\mathrm{stab}_{\mathbb{D}})\,\mathrm{stab}_{\mathbb{D}},$$

but this is not quite enough.

Instead, we would like to compare $\mathrm{St}\,\mathrm{stab}_{\mathbb{D}}$ and $\mathrm{stab}_{\mathrm{St}\,\mathbb{D}}$. The first map is defined by

$$\mathrm{St}\,\mathbb{D} \xrightarrow{i_{\mathbb{D}}} \mathrm{Sp}\,\mathbb{D} \xrightarrow{L^V} \mathrm{Sp}(\mathrm{Sp}\,\mathbb{D}) \xrightarrow{\mathrm{loc}^V} \mathrm{Sp}(\mathrm{St}\,\mathbb{D}) \xrightarrow{\mathrm{loc}_{\mathrm{St}\,\mathbb{D}}} \mathrm{St}(\mathrm{St}\,\mathbb{D})$$

and the second defined by

$$\operatorname{St}\mathbb{D} \xrightarrow{L_{\operatorname{St}\mathbb{D}}} \operatorname{Sp}(\operatorname{St}\mathbb{D}) \xrightarrow{\operatorname{loc}_{\operatorname{St}\mathbb{D}}} \operatorname{St}(\operatorname{St}\mathbb{D}).$$

We would like to show that $\operatorname{loc}^V L^V i_{\mathbb{D}}$ and $L_{\operatorname{Sp}\mathbb{D}} i_{\mathbb{D}}$ are isomorphic after applying $\operatorname{loc}_{\operatorname{St}\mathbb{D}}$. Therefore let $X \in \operatorname{St}\mathbb{D}$. Then $L^V(X) \in \operatorname{Sp}(\operatorname{Sp}\mathbb{D}) \subset \mathbb{D}^{V\times V}$, which has underlying diagram

$$\begin{tikzcd}
 & & & & {} \\
 & & 0 \arrow[r] & LX_1 \arrow[ur, dotted] & \\
 & 0 \arrow[r] & LX_0 \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & LX_{-1} \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 {} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

where each object is the connective suspension prespectrum on $X_i = (i,i)^*X$. Applying loc^V to each of these gives back the original spectrum X up to shifting because X was originally a stable spectrum, just as in the proof of Proposition 3.6.1:

$$\begin{tikzcd}
 & & & & {} \\
 & & 0 \arrow[r] & \sigma^* X \arrow[ur, dotted] & \\
 & 0 \arrow[r] & X \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & (\sigma^{-1})^* X \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 {} \arrow[ur, dotted] & & & &
\end{tikzcd} \tag{3.6.17}$$

Call this object $\overline{X}$. Already we have that $\overline{X} \to \Omega\sigma^*\overline{X}$ is an isomorphism, i.e. $\overline{X}$ is a stable spectrum in $\operatorname{St}\mathbb{D}$, so $\operatorname{loc}_{\operatorname{St}\mathbb{D}} \overline{X} \cong \overline{X}$.

Now, consider the same $X \in \operatorname{St}\mathbb{D}$ but now apply $L_{\operatorname{St}\mathbb{D}}$. This is a connective suspension prespectrum on X, which has underlying diagram

$$\begin{tikzcd}
 & & & & {} \\
 & & 0 \arrow[r] & \Sigma X \arrow[ur, dotted] & \\
 & 0 \arrow[r] & X \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 {} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

To analyze $\widetilde{X} = \mathrm{loc}_{\mathrm{St}\,\mathbb{D}}\, L_{\mathrm{St}\,\mathbb{D}} X$, we can look at $(i,i)^*\widetilde{X}$, taking this restriction on the objectwise V dimension. Namely, $(i,i)^*\widetilde{X} \in \mathrm{St}\,\mathbb{D}$ is the localization of

$$\begin{tikzcd}
& & & & {} \\
& & 0 \arrow[r] & \Sigma X_i \arrow[ur, dotted] & \\
& 0 \arrow[r] & X_i \arrow[r] \arrow[u] & 0 \arrow[u] & \\
& 0 \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
{} \arrow[ur, dotted] & & & &
\end{tikzcd}$$

Again, because X was originally a stable spectrum, this is just $\mathrm{loc}_{\mathbb{D}}\, LX_i$, which is isomorphic to $(\sigma^*)^i X$. Therefore $\widetilde{X}$ is isomorphic to $\overline{X}$.

But now we use the fact that $(0,0)^*$ is a quasiinverse for $\mathrm{stab}_{\mathrm{St}\,\mathbb{D}}$, i.e. the middle composition of Diagram 3.6.16. This makes $(0,0)^*$ a quasiinverse for $\mathrm{St}\,\mathrm{stab}_{\mathbb{D}}$ as well, so that the top-right triangular horizontal composition is isomorphic to the identity:

$$\begin{tikzcd}
\mathrm{St}\,\mathbb{D} \arrow[d, "\mathrm{St}\,\mathrm{stab}_{\mathbb{D}}"'] \arrow[dr, bend left, "="] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \\
\mathrm{St}\,\mathrm{St}\,\mathbb{D} \arrow[r, "(0,0)^*"] \arrow[d, "\mathrm{St}\,\Psi"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \mathrm{St}\,\mathbb{D} \arrow[d, "\Psi"] \\
\mathrm{St}\,\mathbb{S} \arrow[r, "(0,0)^*"'] & \mathbb{S}
\end{tikzcd}$$

This gives the required isomorphism $Q(\Psi\,\mathrm{stab}_{\mathbb{D}}) = (0,0)^*\,\mathrm{St}\,\Psi\,\mathrm{St}\,\mathrm{stab}_{\mathbb{D}} \to \Psi$ and completes the proof of [Hel97, Theorem 8.1]. $\square$

CHAPTER 4

K-theory

4.1 Agenda

We first present a new (maximal) class of derivators on which derivator K-theory can be defined and spend some time proving interesting features of this 2-category. We then give the construction of derivator K-theory and present previously-known results. Then we get to the main new theorems of derivator K-theory.

4.2 Half pointed derivators

To motivate the following definition, we recall the definition of K_0 of an abelian category $\mathcal{A}$. It is constructed as the free abelian group on (isomorphism classes of) objects $A \in \mathcal{A}$, written $[A] \in K_0(\mathcal{A})$, under the relation that if $0 \to A \to B \to C \to 0$ is a short exact sequence, we have $[B] = [A] + [C]$. A short exact sequence is equivalently a cocartesian square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
0 \arrow[r] & C
\end{tikzcd}$$

under the assumption that $A \to B$ is a monomorphism. Thus if we are to construct even K_0 for a derivator, it needs to admit a notion of (coherent) cocartesian squares as above.

Definition 4.2.1. A prederivator $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$ is a *left pointed derivator* if it is a left derivator, $\mathbb{D}(e)$ is pointed, and for every sieve $u\colon J \to K$, u^* admits a right adjoint u_* satisfying Der4R.

A prederivator $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$ is a *right pointed derivator* if it is a right derivator, $\mathbb{D}(e)$ is pointed, and for every cosieve $u\colon J \to K$, u^* admits a left adjoint $u_!$ satisfying Der4L.

If we are referring to either a left or a right pointed derivator we will use the general term *half pointed derivator*.

Remark 4.2.2. Recalling Proposition 2.6.8, it is actually superfluous to ask that the right Kan extensions in a left pointed derivator satisfy Der4R, as the relevant comma squares are already $\mathbb{D}$-exact for any half derivator $\mathbb{D}$. The dual of that proposition shows that the dual request for right pointed derivators is also superfluous.

Left pointed derivators are the important class for derivator K-theory, so we will focus more on them. The existence of the specified u_* means that a left pointed derivator $\mathbb{D}$ admits right extension by zero morphisms (recall Proposition 2.7.5) that can be computed pointwise.

Recall from the proof of Theorem 3.3.4 the construction of the cone morphism $C\colon \mathbb{D}^{[1]} \to \mathbb{D}$: beginning with a coherent map $(f\colon a \to b) \in \mathbb{D}^{[1]}$, we first extend by zero using $i_{[1],*}\colon \mathbb{D}^{[1]} \to \mathbb{D}^{\ulcorner}$ and then compute the pushout using $i_{\ulcorner,!}\colon \mathbb{D}^{\ulcorner} \to \mathbb{D}^{\square}$ (using the original Notation 2.8.1). These two steps yield the (coherent) cocartesian square

(4.2.3)
$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d] & b \arrow[d] \\
0 \arrow[r] & C(f)
\end{tikzcd}$$

which should be an important part of defining $K_0(\mathbb{D})$. Without access to the extension by zero morphism $i_{[1],*}$ we would be lost from the outset.

But all this discussion begs the question: why not just use an ordinary pointed derivator, which would have all the right Kan extensions we could ever want? There is an important class of examples that are *not* full derivators but are left pointed, which we hinted at in Remark 2.4.13.

Lemma 4.2.4 (Lemme 4.3, [Cis10])**.** Let $\mathcal{W}$ be a saturated Waldhausen category satisfying the cylinder axiom. Then the associated prederivator $\mathbb{D}_{\mathcal{W}}\colon K \mapsto \mathrm{Ho}(\mathrm{Fun}(K, \mathcal{W}))$ defined on $\mathbf{Dir_f}$ is a (strong) left pointed derivator.

Recall that $\mathbf{Dir_f}$ is the diagram 2-category of finite direct categories. These derivators $\mathbb{D}_{\mathcal{W}}$ cannot be full derivators in general because arbitrary Waldhausen categories admit no notion of product or pullback square, although most examples of Waldhausen categories that one thinks of come from exact or abelian categories which do. To challenge our intuition, recall that in [Wal85, §1.7] Waldhausen defines for any Waldhausen category $\mathcal{W}$ and (co)homology theory on $\mathcal{W}$ the subcategory of n-spherical objects, i.e. objects which (co)homology concentrated in degree n. These categories are still Waldhausen but do not admit products, as homology need not be appropriately concentrated. Waldhausen uses these cellular filtrations to prove various theorems in the rest of Section 1 of [Wal85]. As an analogous example, categories of chain complexes valued in an abelian or exact category with bounded (co)homology are Waldhausen categories that do not admit all (fiber) products.

To investigate derivator K-theory as an extension of Waldhausen K-theory, we will compare $K(\mathbb{D}_{\mathcal{W}})$ with Waldhausen K-theory $K(\mathcal{W})$, which is only possible if we are as general as possible with the input to derivator K-theory.

Remark 4.2.5. In the reformulation of derivator K-theory in [MR17], the authors define derivator K-theory on all 'pointed right derivators', which in our terms (as we warned in Remark 2.4.6) is a left derivator $\mathbb{D}$ such that $\mathbb{D}(e)$ is pointed. The cocartesian squares of the form of Diagram 4.2.3 still exist in such a derivator, but they are impossible to construct given just the information of $(f\colon a \to b) \in \mathbb{D}^{[1]}$. They are still identifiable as cocartesian squares $X \in \mathbb{D}(\square)$ satisfying $(0,1)^*X = 0 \in \mathbb{D}(e)$.

We choose the order of our adjectives to emphasize that $\mathbb{D}$ is half of a pointed derivator, not a half derivator that happens to be pointed (as in [MR17]). In particular, a Muro-Raptis 'pointed right derivator' may not be strongly pointed, as the construction of exceptional left adjoints to left extension by zero morphisms requires particular right extension by zero morphisms to exist. However, our left pointed derivators do not have this (potential) defect, as the next proposition shows.

Proposition 4.2.6. A half pointed derivator $\mathbb{D}$ is half strongly pointed. That is, in a left pointed derivator, left Kan extensions along cosieves admit exceptional left adjoints. In a right pointed derivator, right Kan extensions along sieves admit exceptional right adjoints.

Proof. First, although we only required that $\mathbb{D}(e)$ be pointed, each level $\mathbb{D}(K)$ of a half pointed derivator is pointed. The map $i_\varnothing \colon \varnothing \to K$ is both a sieve and a cosieve, so the functors $i_{\varnothing,*}, i_{\varnothing,!} \colon \mathbb{D}(\varnothing) \to \mathbb{D}(K)$ defining the initial and final object in $\mathbb{D}(K)$ always exist for any half pointed derivator. Moreover, they are isomorphic in any prederivator satisfying Der2 with pointed base: there is a canonical transformation $i_{\varnothing,*} \Rightarrow i_{\varnothing,!}$ which is an isomorphism under k^* for every $k \in K$, as $k^* i_{\varnothing,!} \cong \pi_{\varnothing,!}$ where $\pi_\varnothing \colon \varnothing \to e$ is the empty inclusion/projection into the final category, and similarly $k^* i_{\varnothing,*} \cong \pi_{\varnothing,*}$. Because $\mathbb{D}(e)$ is pointed, the canonical natural transformation $\pi_{\varnothing,!} \Rightarrow \pi_{\varnothing,*}$ is an isomorphism, hence the same thing is true for $k^* i_{\varnothing,*} \Rightarrow k^* i_{\varnothing,!}$

By Der2, this means that initial objects in $\mathbb{D}(K)$ are also final, making $\mathbb{D}(K)$ a pointed category. Left and right Kan extensions preserve final and initial objects (respectively), so they also preserve zero objects. Pullback functors preserve either initial or final objects depending on whether $\mathbb{D}$ is a left or right pointed derivator, but in either case they preserve zero objects as well.

In the right case, the construction of the exceptional right adjoint to a sieve requires only right Kan extensions and left Kan extensions along cosieves, as we illustrated in the proof of Theorem 3.3.4. Therefore the same construction still works in a right pointed derivator. For more details in both the left and right cases see [Gro13, §3]. ☐

Remark 4.2.7. We might hope that in a *left* pointed derivator, right Kan extensions along sieves would still admit exceptional right adjoints. There does not seem any reason for this to be true. To give an explicit counterexample, consider the sieve $i_{[1]} \colon [1] \to \ulcorner$ and consider an object $\mathbb{D}(\ulcorner)$ given (incoherently) by

$$\begin{tikzcd}
& a \arrow[r] \arrow[dl] & b \\
c & &
\end{tikzcd}$$

Working through the formula for $i_{[1]}^!$, as we did in Theorem 3.3.4, we have a composition

$$\begin{tikzcd}
& & & & & & & & & P \arrow[r] \arrow[dl] \arrow[ddd] & b \arrow[ddd, "="] & & & \\
& a \arrow[r] \arrow[dl] & b & & 0 \arrow[dd] & & & & 0 \arrow[ddd] & & & & & \\
c & & & \mapsto & & a \arrow[r] \arrow[dl] & b & \mapsto & & & & \mapsto & P \arrow[r] & b \\
& & & & c & & & & & a \arrow[r] \arrow[dl] & b & & & \\
& & & & & & & & c & & & & &
\end{tikzcd}$$

The lefthand (slanted) square is cartesian, i.e. P is the pullback of $a \to c$ along zero. But in an arbitrary left pointed derivator, this pullback has no reason to exist.

Having established the objects we will use as input to derivator K-theory, we now need to discuss the morphisms. Remaining a bit vague, such morphisms will need to preserve cocartesian squares and the zero object, so that they preserve objects like Diagram 4.2.3. The common adjective for such morphisms is *right exact*, which will remain (for the moment) an example of the left/right notational hazard.

Any cocontinuous morphism will be right exact, i.e. pullbacks of functors $u\colon J \to K$ in **Dia** and the associated left Kan extensions. But we no longer have exceptional right adjoints to our right Kan extensions along sieves, so these morphisms are not left adjoints and hence not automatically cocontinuous. Fortunately, we have the following.

Proposition 4.2.8. Let $\mathbb{D}$ be a left pointed derivator. Then for any sieve $i\colon A \to B$, $i_*\colon \mathbb{D}^A \to \mathbb{D}^B$ is a cocontinuous morphism of derivators.

Proof. It is enough to show that for any $K \in \mathbf{Dia}$, i_* preserves colimits of shape K by Proposition 2.5.8. Specifically, if we let $\pi_K\colon K \to e$ be the projection, we need the canonical comparison map

$$(\mathrm{id}_B \times \pi_K)_!(i \times \mathrm{id}_K)_* X \to i_*(\mathrm{id}_A \times \pi_K)_! X$$

to be an isomorphism for any $X \in \mathbb{D}(A \times K)$. It suffices to check that this map is an isomorphism at every $b \in B$ by Der2, so therefore we consider

$$b^*(\mathrm{id}_B \times \pi_K)_!(i \times \mathrm{id}_K)_* X \to b^* i_*(\mathrm{id}_A \times \pi_K)_! X \tag{4.2.9}$$

for any $X \in \mathbb{D}(A \times K)$.

We begin by analyzing the lefthand side. Even though b^* does not necessarily admit a right adjoint, it is still cocontinuous by [Gro13, Proposition 2.5]. That proposition is stated for full derivators, but its proof only depends on Proposition 2.6.8, which we have shown is satisfied for half derivators. Therefore

$$b^*(\mathrm{id}_B \times \pi_K)_!(i \times \mathrm{id}_J)_* X \cong \pi_{K,!}(b \times \mathrm{id}_J)^*(i \times \mathrm{id}_J)_* X.$$

Using the properties of the extension by zero morphism i_* from Proposition 2.7.5, we have that

$$\pi_{K,!}(b \times \mathrm{id}_K)^*(i \times \mathrm{id}_K)_* X \cong \begin{cases} \pi_{K,!}(b \times \mathrm{id}_K)^* X & b \in A \\ 0 & b \notin A \end{cases}$$

For the righthand side of Equation 4.2.9, again using that i_* is extension by zero,

$$b^* i_* (\mathrm{id}_A \times \pi_K)_! X \cong \begin{cases} b^*(\mathrm{id}_A \times \pi_K)_! X & b \in A \\ 0 & b \notin A \end{cases}$$

So if $b \notin A$, the canonical transformation in Equation 4.2.9 is an isomorphism of zero objects. If $b \in A$, then the transformation is isomorphic to

$$\pi_{K,!}(b \times \mathrm{id}_K)^* X \to \ b^*(\mathrm{id}_A \times \pi_K)_! X$$

which is still an isomorphism because the square

$$\begin{tikzcd}
K \arrow[r, "b\times \mathrm{id}_K"] \arrow[d, "\pi_K"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\swarrow}"] & B \times K \arrow[d, "\mathrm{id}_B \times \pi_K"] \\
e \arrow[r, "b"'] & B
\end{tikzcd}$$

is homotopy exact. Specifically, it is the strict pullback along the Grothendieck opfibration $B \times K \to K$, so is homotopy exact by [Gro13, Proposition 1.24]. □

As another ingredient, we would like to know that extension by zero morphisms commute with cocontinuous morphisms. In fact, extension by zero morphisms commute with merely pointed morphisms of derivators, i.e. morphisms of prederivators that additionally send zero to zero. This is proven for (two-sided) pointed derivators at [Gro16b, Corollary 8.2], but its proof should not rely on the existence of arbitrary left and right Kan extensions. Nonetheless, we give an explicit proof because our situation is somewhat novel.

Proposition 4.2.10. Suppose that $\Phi\colon \mathbb{D} \to \mathbb{E}$ is a pointed morphism of half pointed derivators. Then Φ commutes with extension by zero morphisms (along both sieves and cosieves).

Proof. Let $u\colon J \to K$ be a cosieve, so that $u_!$ is extension by zero. Then we have a canonical transformation

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "u_!"] \arrow[d, "\Phi_J"'] \arrow[dr, phantom, "\gamma^{\Phi}_{u,!}\Nearrow"] & \mathbb{D}(K) \arrow[d, "\Phi_K"] \\
\mathbb{E}(J) \arrow[r, "u_!"'] & \mathbb{E}(K)
\end{tikzcd}$$

which we will prove is an isomorphism. To do so, we paste onto that square: for any $k \in K$, we obtain

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "u_!"] \arrow[d, "\Phi_J"'] \arrow[dr, phantom, "\gamma^{\Phi}_{u,!}\Nearrow"] & \mathbb{D}(K) \arrow[r, "k^*"] \arrow[d, "\Phi_K"] \arrow[dr, phantom, "\cong\Nearrow"] & \mathbb{D}(e) \arrow[d, "\Phi_e"] \\
\mathbb{E}(J) \arrow[r, "u_!"'] & \mathbb{E}(K) \arrow[r, "k^*"'] & \mathbb{E}(e)
\end{tikzcd}$$

The added square commutes up to isomorphism because Φ is a morphism of prederivators. The total pasting depends on whether $k \in J$ or $k \in K \setminus J$. In the first case, $k^*u_! \cong k^*$, so the total pasting is isomorphic to

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "k^*"] \arrow[d, "\Phi_J"'] \arrow[dr, phantom, "\cong\Nearrow"] & \mathbb{D}(e) \arrow[d, "\Phi_e"] \\
\mathbb{E}(J) \arrow[r, "k^*"'] & \mathbb{E}(e)
\end{tikzcd}$$

which is an isomorphism (again because Φ is a morphism of prederivators). This gives us a composition

$$\begin{tikzcd}
k^*u_!\Phi_J \arrow[r, "k^*\gamma^{\Phi}_{u,!}"'] \arrow[rr, bend left, "\cong"] & k^*\Phi_K u_! \arrow[r, "\cong"'] & \Phi_e k^* u_!
\end{tikzcd}$$

so that $k^*\gamma^{\Phi}_{u,!}$ is an isomorphism for any $k \in J$.

In the second case, $k^*u_! = 0$. Specifically, using Der4, we have an isomorphism of functors $k^*u_! \cong \pi_{(k/u),!}\mathrm{pr}^*$. This yields an isomorphism of pastings

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "u_!"] \arrow[d, "\Phi_J"'] \arrow[dr, phantom, "\gamma^{\Phi}_{u,!}\Nearrow"] & \mathbb{D}(K) \arrow[r, "k^*"] \arrow[d, "\Phi_K"] \arrow[dr, phantom, "\cong\Nearrow"] & \mathbb{D}(e) \arrow[d, "\Phi_e"] \\
\mathbb{E}(J) \arrow[r, "u_!"'] & \mathbb{E}(K) \arrow[r, "k^*"'] & \mathbb{E}(e)
\end{tikzcd}
\cong
\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "\mathrm{pr}^*"] \arrow[d, "\Phi_J"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(\varnothing) \arrow[r, "\pi_{\varnothing,!}"] \arrow[d, "\Phi_\varnothing"] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(e) \arrow[d, "\Phi_e"] \\
\mathbb{E}(J) \arrow[r, "\mathrm{pr}^*"'] & \mathbb{E}(\varnothing) \arrow[r, "\pi_{\varnothing,!}"'] & \mathbb{E}(e)
\end{tikzcd}$$

where we identify the comma category $(k/u) = \varnothing$ when $k \in K \setminus J$ because the functor u is a cosieve. For this righthand diagram, the left square commutes up to isomorphism because

Φ is a morphism of prederivators, and the right square does as well because Φ is pointed. Thus $k^*\gamma_{u_!}^{\Phi}$ is an isomorphism for $k \in K \setminus J$.

Putting these together, we complete the proof for extensions by zero along cosieves. For extensions by zero along sieves, the proof is identical. $\square$

The last fact that we need is a long-sought result for 'small' derivators, i.e. those defined on $\mathbf{Dir_f}$. In [PS16, Theorem 7.1], Ponto and Shulman prove that a morphism of derivators preserves cocartesian squares and initial objects (i.e. is right exact) if and only if it preserves homotopy finite colimits. The proof uses some heavy machinery developed by Cisinski in [Cis06] that, in particular, depends on the derivators being defined on $\mathbf{Dia} = \mathbf{Cat}$. This is an inappropriate setting for K-theory; the existence of infinite coproducts gives us the usual issue of an Eilenberg swindle which zeroes out K-theory.

Luckily, the proof of Ponto-Shulman can be adapted to our situation. The following statement appears in the doctoral thesis of Ioannis Lagkas, but we will give the full proof here, filling in all the details omitted there.

Theorem 4.2.11 (Theorem 1.4.8(ii), [Lag18])**.** Let $\mathbb{D}, \mathbb{E}$ be left pointed derivators with domain $\mathbf{Dir_f}$. Then a pointed morphism $\Phi\colon \mathbb{D} \to \mathbb{E}$ preserves cocartesian squares if and only if is cocontinuous.

Proof. The backwards direction is trivial, so we immediate proceed with the forward direction. The proof in [Lag18] relies also on [GPS14, Corollary 4.11] and [PS16, Theorem 7.1], but we will spell out all relevant arguments.

Let $K \in \mathbf{Dir_f}$. We need a bit of notation: define the *length of* K to be the length of the longest chain of non-identity morphisms in K, which is finite by assumption. Write $\ell(K)$ for this number. For example, $\ell([n]) = n$ and $\ell(\square) = 2$. We proceed by induction on $n = \ell(K)$.

Recalling $i_\ulcorner\colon \ulcorner \to \square$ is a key ingredient of getting cocartesian squares, consider the homotopy exact square in **Dia**

$$\begin{tikzcd}
(i_{\ulcorner}/(1,1)) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\mathrm{ex}}{\swarrow}"] & \ulcorner \arrow[d, "i_{\ulcorner}"] \\
e \arrow[r, "{(1,1)}"'] & \square
\end{tikzcd}$$

Because $(1,1)$ is the final object in $\square$, using Lemma 2.2.5 we can identify $(i_{\ulcorner}/(1,1))$ exactly as $\ulcorner$ and $\mathrm{pr} = \mathrm{id}_{\ulcorner}$. Thus the left mate $\pi_! \, \mathrm{id}_{\ulcorner}^* \Rightarrow (1,1)^* i_{\ulcorner,!}$ is an isomorphism. That Φ preserves cocartesian squares means by definition that Φ commutes with $i_{\ulcorner,!}$, and Φ commutes with $(1,1)^*$ because all morphisms of (pre)derivators do, so we conclude that Φ commutes with colimits of shape $\ulcorner$. This is a slightly more convenient formulation for the base case of our induction.

A diagram with $\ell(K) = 0$ is just a discrete category, and colimits of discrete categories are just finite coproducts. Starting with $K = e \sqcup e$ consisting of two disjoint objects, consider the inclusion $i \colon K \to \ulcorner$ into the objects $(1,0)$ and $(0,1)$. Then we have that $\pi_{K,!} \cong \pi_{\ulcorner,!} i_!$ because $\pi_K = \pi_{\ulcorner} \circ i$. Because $i_!$ is just an extension by zero morphism, Φ commutes with it by Proposition 4.2.10. Moreover, Φ commutes with $\pi_{\ulcorner,!}$ by the above, so Φ commutes with $\pi_{K,!}$. This proves that Φ preserves binary coproducts. Since any finite coproduct is an iterated binary coproduct (and we only have finite coproducts because $\mathbf{Dia} = \mathbf{Dir_f}$), this settles $\ell(K) = 0$.

We will now show that colimits of (diagram categories of) length n can be built from colimits of length $n-1$, and we will further reduce to one sufficient instance of this phenomenon. To begin, let K be such a length n category. Then we can look at the category of nondegenerate simplices in K, which we name $\widetilde{K}$, whose objects are chains $\{k_0 \to \cdots \to k_m\}$ with no maps identities and $m \leq n$ and whose maps are 'face maps' which decrease the length of chains by composing adjacent maps. Note that since K is finite direct, we can never end up with a degenerate simplex after composing, and moreover this category of nondegenerate simplices is also finite direct.

Let $q \colon \widetilde{K} \to K$ be the map sending a chain to its first vertex k_0. Then q is homotopy final. To see this, consider any (k/q) for $k \in K$. This category has a final object, namely the pair $\{k_0 = k\}$ with the identity map $k = q(\{k_0 = k\})$. Thus it is homotopy contractible, and by Proposition 2.6.18 for any $X \in \mathbb{D}(K)$ we have an isomorphism $\pi_{K,!}X \cong \pi_{\widetilde{K},!}q^*X$. While $\widetilde{K}$ is much bigger-looking than K, it is simpler, so we can get a handle on this second colimit more easily.

More precisely, let $\Delta_n \subset \Delta^{\mathrm{nd}}$ be the full subcategory of the nondegenerate simplex category on the objects $\{[0], \ldots, [n]\}$ and maps the *injective* order-preserving maps. Because we have only taken injections, this category is direct (and obviously finite). Since $\ell(K) = n$, there is a projection $p \colon \widetilde{K} \to \Delta_n^{\mathrm{op}}$ sending $\{k_0 \to \cdots \to k_m\}$ to $[m]$. Thus the colimit $\pi_{\widetilde{K},!}$ we may factor as $\pi_{\Delta_n^{\mathrm{op}},!}p_!$. But let us look more closely at $[\ell]^*p_!q^*X$ for some $X \in \mathbb{D}(K)$ and $[\ell] \in \Delta_n^{\mathrm{op}}$. Using Der4, we use the diagram below

$$\begin{tikzcd}
(p/[\ell]) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi_{(p/[\ell])}"'] \arrow[dr, phantom, "\Swarrow"] & \widetilde{K} \arrow[d, "p"] \\
e \arrow[r, "{[\ell]}"'] & \Delta_n^{\mathrm{op}}
\end{tikzcd}$$

to conclude that $[\ell]^*p_!q^*X \cong \pi_{(p/[\ell]),!}\mathrm{pr}^*q^*X$. But now we need to understand the structure of $(p/[\ell])$. By definition its objects are $\{k_0 \to \cdots \to k_m\}$ along with a map $[m] \leftarrow [\ell]$ in Δ_n^{op}, i.e. an injection $[\ell] \to [m]$. This means, in particular, that $m \geq \ell$. On the other hand, maps in $(p/[\ell])$ must come from maps in $\widetilde{K}$, which strictly decrease the length of chains. Therefore there is a discrete category $\widetilde{K}_\ell \subset (p/[\ell])$ consisting of the length ℓ objects in $\widetilde{K}$ along with the map $[\ell] = [\ell]$ in Δ_n^{op}.

We claim now that the inclusion of this subcategory admits a left adjoint. In particular, $\widetilde{K}_\ell \subset (p/[\ell])$ is a cosieve, which is in particular a discrete Grothendieck opfibration. Then by results in [Qui73, §1] near Theorem A, the inclusion of the strict fiber $p^{-1}(\ell) = \widetilde{K}_\ell$ into the comma category $(p/[\ell])$ admits a left adjoint, which we name $j \colon (p/[\ell]) \to \widetilde{K}_\ell$.

As an illustration: let $K = \ulcorner$ with the labelling $c \leftarrow a \to b$. Then the category of simplices $\widetilde{K}$ has the form

$$\begin{tikzcd}
 & (a \to b) \arrow[dl] \arrow[dr] & & (a \to c) \arrow[dl] \arrow[dr] & \\
b & & a & & c
\end{tikzcd}$$

If we then look at $(p/[0])$, we will have $\widetilde{K}_0 = \{a, b, c\}$ as a discrete subcategory. The other objects are $(a \to b)$ and $(a \to c)$ along with an injection $[0] \to [1]$. There are two options for this, which we name s, t, as they will correspond to the source and target of the 1-simplex in $\widetilde{K}$. To get a handle on the maps in $(p/[0])$: the map $(a \to b) \to a$ in $\widetilde{K}$ could induce a map $((a \to b), s) \to a$ or $((a \to b), t) \to a$. But only this first option is a valid map in $(p/[0])$; we would need $(a \to b) \to a$ to correspond to the map $t \colon [0] \to [1]$, but this is not the case. We can therefore give the structure of $(p/[0])$ with this in mind:

$$\begin{tikzcd}
(a \to b), t \arrow[d] & (a \to b), s \arrow[dr] & & (a \to c), s \arrow[dl] & (a \to c), t \arrow[d] \\
b & & a & & c
\end{tikzcd}$$

Now it is clear that $\widetilde{K}_0$ is a cosieve, and the left adjoint $(p/[0]) \to \widetilde{K}_0$ sends every object to the 0-simplex at the end of its connected component.

Having given some evidence for Quillen's result, by Proposition 2.6.2, we can conclude that $\pi_{(p/[\ell]),!}\mathrm{pr}^*q^*X \cong \pi_{\widetilde{K}_\ell,!}j^*\mathrm{pr}^*q^*X$. The important point here is that we can compute $p_!q^*X$ pointwise as a colimit over a discrete category, i.e. pointwise this object is a finite coproduct, a case which we have already concluded. Thus to complete the proof, we need to show that any pointed morphism of left pointed derivators which preserves cocartesian squares will also preserve colimits of shape Δ_n^{op}.

To do so, we will need more diagram categories. Consider the arrow category $\mathrm{Ar}\,\Delta_n^{\mathrm{op}}$, which is still finite direct. Let $A_n \subset \mathrm{Ar}\,\Delta_n^{\mathrm{op}}$ be the full subcategory on non-identity arrows with domain $[n] \in \Delta_n^{\mathrm{op}}$ as well as the identity $([\ell] = [\ell])$ for all $0 \leq \ell \leq n-1$. To illustrate for $n = 2$, we have

$$\begin{tikzcd}
& {[1] = [1]} \arrow[ddd, bend right=90] \arrow[ddd, bend left=90] & \\
{[2] \leftarrow [1]} \arrow[ur] \arrow[d] \arrow[dr] & {[2] \leftarrow [1]} \arrow[u] \arrow[dl] \arrow[dr] & {[2] \leftarrow [1]} \arrow[ul] \arrow[dl] \arrow[d] \\
{[2] \leftarrow [0]} \arrow[dr] & {[2] \leftarrow [0]} \arrow[d] & {[2] \leftarrow [0]} \arrow[dl] \\
& {[0] = [0]} &
\end{tikzcd}$$

where we recall that there are three injections $[1] \to [2]$ and three injections $[0] \to [2]$ and each injection $[1] \to [2]$ is only compatible with two injections $[0] \to [2]$. The two arrows on the outside come from the two injections $[0] \to [1]$.

We then form a new category B_n by adding a new object ∞ and a unique map from each arrow with domain $[n]$ to ∞. There is a functor $A_n \to \Delta_n^{\mathrm{op}}$ which sends each object-arrow to its source, which we can extend to $s\colon B_n \to \Delta_n^{\mathrm{op}}$ by letting $s(\infty) = [n]$. We claim that this functor is homotopy final, and will show this by again proving that each comma category $([\ell]/s)$ is homotopy contractible for all $[\ell] \in \Delta_n^{\mathrm{op}}$.

Consider first $0 \leq \ell \leq n-1$. In order for there to be a map $[\ell] \leftarrow s(\beta)$ in Δ_n^{op} for some $\beta \in B_n$, we must have $s(\beta) \leq \ell$. Therefore the only β we are allowed are of the form $([m] = [m])$ for $m \leq \ell$. The elements of $([\ell]/s)$ therefore are $([m] = [m])$ with a map $[\ell] \leftarrow [m]$ in Δ_n^{op}. There are no interesting maps in B_n between such objects, so the maps between $([m_1] = [m_1], [\ell] \leftarrow [m_1])$ and $([m_2] = [m_2], [\ell] \leftarrow [m_2])$ are induced by $[m_1] \leftarrow [m_2]$ in Δ_n^{op} compatible with the maps from $[\ell]$. Thus we have an isomorphism $([\ell]/s) \cong ([\ell]/\Delta_n^{\mathrm{op}})$, and this latter category has an initial object $([\ell], \mathrm{id}_{[\ell]})$, so $([\ell]/s)$ is homotopy contractible.

We now need to show that $([n]/s)$ is homotopy contractible. There are three types of objects in this category: first, the object $(\infty, \mathrm{id}_{[n]})$; second, $(a, \mathrm{id}_{[n]})$ for some map $a\colon [n] \leftarrow [\ell]$ in Δ_n^{op}; and third, $([m] = [m], b)$ for some map $b\colon [n] \leftarrow [m]$. Maps in $([n]/s)$ come from maps in B_n, so in particular there are no maps $([m] = [m], b) \to (a, \mathrm{id}_{[n]})$ or $([m] = [m], b) \to (\infty, \mathrm{id}_{[n]})$.

Consider the full subcategory $C_n \subset ([n]/s)$ on objects of the form $(\infty, \mathrm{id}_{[n]})$ and $(a, \mathrm{id}_{[n]})$. The inclusion $C_n \to ([n]/s)$ admits a right adjoint $r\colon ([n]/s) \to C_n$. It is the identity on objects of C_n, so we need only to define $r([m] = [m], b) = (b, \mathrm{id}_{[n]})$. To sketch the bijection on hom-sets, a map $(a, \mathrm{id}_{[n]}) \to ([m] = [m], b)$ is a commutative diagram with all maps injections

$$\begin{tikzcd}
 & {[n]} \arrow[dl, "="'] & {[\ell]} \arrow[l, "a"'] \\
{[n]} & & \\
 & {[m]} \arrow[ul, "b"] \arrow[uu] & {[m]} \arrow[l, "="] \arrow[uu]
\end{tikzcd}$$

which forces the vertical map $[n] \leftarrow [m]$ to be b. A map $(a, \mathrm{id}_{[n]}) \to r([m] = [m], b)$ is a commutative diagram with all maps injections

$$\begin{tikzcd}
& {[n]} \arrow[dl, "="'] & {[\ell]} \arrow[l, "a"'] \\
{[n]} & & \\
& {[n]} \arrow[ul, "="] \arrow[uu, "="'] & {[m]} \arrow[l, "b"] \arrow[uu]
\end{tikzcd}$$

It is clear that the only data in both cases is an injection $[m] \to [\ell]$ such that composing with a gives b, which gives the bijection. That this is natural in both argument is again easy to check but we will not do so here.

We now note that C_n has the final object $(\infty, \mathrm{id}_{[n]})$, so is homotopy contractible. This gives us a pasting

$$\begin{tikzcd}
(n/s) \arrow[r, "r"] \arrow[d, "\pi_{(n/s)}"'] \arrow[dr, phantom, "\overset{\text{ex}}{\swarrow}"] & C_n \arrow[r, "\pi_{C_n}"] \arrow[d, "\pi_{C_n}"'] \arrow[dr, phantom, "\overset{\text{ex}}{\swarrow}"] & e \arrow[d] \\
e \arrow[r] & e \arrow[r] & e
\end{tikzcd}$$

The righthand square is homotopy exact because C_n is homotopy contractible, and the lefthand square is homotopy exact by Proposition 2.6.2. Thus we conclude that the total pasting is exact, i.e. that $([n]/s)$ is homotopy contractible, which finishes the proof that $s \colon B_n \to \Delta_n^{\mathrm{op}}$ is homotopy final.

To finish off the argument, we now consider the functor $c \colon B_n \to \ulcorner$ defined by $c(\infty, \mathrm{id}_{[n]}) = (1,0)$, $c(a, \mathrm{id}_{[n]}) = (0,0)$, and $c([m] = [m], b) = (0,1)$. It is immediate that $\pi_{\ulcorner,!} c_! s^* X \cong \pi_{B_n,!} X$ for any $X \in \mathbb{D}(B_n)$, as $\pi_{\ulcorner} c = \pi_{B_n}$, and we know that our morphism of derivators Φ commutes with $\pi_{\ulcorner,!}$ by assumption. The last step of this proof is to show that Φ commutes with $c_!$. Once again, we will prove this pointwise, i.e. that Φ commutes with colimits of shape $(c/(0,0))$, $(c/(1,0))$, and $(c/(0,1))$. Recall from Lemma 2.2.5 that comma categories with codomain a poset can be viewed as full subcategories of the domain.

To begin with $(c/(0,0))$, because $(0,0)$ is initial we can identify $(c/(0,0)) \cong c^{-1}(0,0)$, which is the full subcategory of B_n on the objects $(a, \mathrm{id}_{[n]})$. A maximal chain of morphisms in this category has the form

$$([n] \leftarrow [0], \mathrm{id}_{[n]}) \longrightarrow ([n] \leftarrow [1], \mathrm{id}_{[n]}) \longrightarrow \cdots \longrightarrow ([n] \leftarrow [n-1], \mathrm{id}_{[n]})$$

for some compatible family of injections $[0] \to [1] \to \cdots \to [n-1]$. In particular, the length of this category is $n-1$. By induction, Φ commutes with colimits of length less than n.

To now examine $(c/(1,0))$, note that it contains both $c^{-1}(0,0)$ and $c^{-1}(1,0) = \{(\infty, \mathrm{id}_{[n]})\}$ (and nothing else), so we can conclude that $(c/(1,0)) \cong C_n$ (in the above notation). Since C_n has a final object, we have an isomorphism $\pi_{C_n,!} \cong (\infty, \mathrm{id}_{[n]})^*$ by Corollary 2.6.3, and Φ thus commutes with colimits of shape $(c/(1,0))$ as well.

Finally, $(c/(0,1)) \cong A_n$, as it is the full subcategory on all objects besides $(\infty, \mathrm{id}_{[n]})$. Note that there is a fully faithful functor $i_{n-1} \colon \Delta^{\mathrm{op}}_{n-1} \to A_n$ given by $[\ell] \mapsto ([\ell] = [\ell])$. We claim that this functor admits a left adjoint, which sends an object of $A_n \subset \mathrm{Ar}\,\Delta^{\mathrm{op}}_n$ to its codomain. To again sketch the bijection on hom-sets, a map in A_n between $a \colon [n] \leftarrow [\ell]$ and $[m] = [m]$ is a commutative diagram

$$\begin{tikzcd}
{[n]} & {[\ell]} \arrow[l, "a"'] \\
{[m]} \arrow[u] & {[m]} \arrow[l, "="'] \arrow[u]
\end{tikzcd}$$

Once we fix the injection $[m] \to [\ell]$, the injection $[m] \leftarrow [n]$ must be the composite. Thus such a map is determined completely by the domain of these injections, which are the codomains in the opposite category.

Since i_{n-1} is a right adjoint, it is homotopy final by Proposition 2.6.2. Thus we have an isomorphism $\pi_{\Delta^{\mathrm{op}}_{n-1},!} i^*_{n-1} X \cong \pi_{A_n,!} X$ for any $X \in \mathbb{D}(A_n) = \mathbb{D}((c/(0,1))$. We now note that $\ell(\Delta^{\mathrm{op}}_{n-1}) = n-1$ (essentially by definition), and Φ commutes with length $n-1$ colimits by induction. This is the last ingredient and completes the proof. $\square$

Hereafter we let $\mathbf{Der}_K$ be the 1-category with objects left pointed derivators and morphisms cocontinuous morphisms of derivators up to isomodification. That is, we consider $\Phi, \Psi \colon \mathbb{D} \to \mathbb{E}$ to be the same if there exists a zig-zag of isomodifications from Φ to Ψ. We do this because such morphisms will induce homotopic maps on K-theory, as we will show in Corollary 4.3.2 shortly.

4.3 Derivator K-theory

It is helpful at this point to recall Waldhausen's K-theory for a category with cofibrations and weak equivalences from [Wal85]. To such a category $\mathcal{W}$ we assign a simplicial object in

Waldhausen categories $\mathbf{S}_\bullet\mathcal{W}$, where $\mathbf{S}_n\mathcal{W}$ is the category of 'exact functors' from the arrow category $\mathrm{Ar}[n]$ to $\mathcal{W}$. Taking the wide subcategory with only maps the weak equivalences $w\mathbf{S}_\bullet\mathcal{W}$, we obtain again a simplicial Waldhausen category. Then we define K-theory as follows:

$$K(\mathcal{C}) := \Omega|N_\bullet w\mathbf{S}_\bullet\mathcal{W}|,$$

the loop space of the (diagonal) geometric realisation of the bisimplicial set given by the nerve.

In [MR17], Muro and Raptis improved upon a construction of Garkusha in [Gar05] which generalises Waldhausen's $\mathbf{S}_\bullet$ construction. First, we can restate the $\mathbf{S}_\bullet$ construction in the language of derivators. To help with notation, for a category $[n] \in \Delta$, let the elements of its arrow category $\mathrm{Ar}[n]$ be written (i, j) for $i \to j$.

Let $\mathbb{D}$ be a left pointed derivator. We let $\mathbf{S}_n\mathbb{D}$ be the full subcategory of $\mathbb{D}(\mathrm{Ar}[n])$ of objects X such that:

(1) For every $0 \leq i \leq n$, $(i, i)^*X \in \mathbb{D}(e)$ is a zero object.

(2) For every fully faithful inclusion $\iota \colon \square \to \mathrm{Ar}[n]$, the object $\iota^*X \in \mathbb{D}(\square)$ is cocartesian.

In (2), it suffices to check only the inclusions such that $\iota(0, 1) = (i, i)$ by [Gro13, Proposition 3.13]. We then define *derivator K-theory* by

$$K(\mathbb{D}) = \Omega|N_\bullet i\mathbf{S}_\bullet\mathbb{D}|$$

where $i\mathbf{S}_n\mathbb{D} \subset \mathbf{S}_n\mathbb{D}$ is the wide subcategory consisting only of isomorphisms, in analogy with $w\mathbf{S}_n\mathcal{C}$, and the geometric realization is taking diagonally.

To give a few examples, first we have that $\mathbf{S}_0\mathbb{D} \subset \mathbb{D}(\mathrm{Ar}[0]) = \mathbb{D}(e)$ is trivial: it has only one object and property (1) above requires it to be a zero object of $\mathbb{D}(e)$. The category $\mathbf{S}_1\mathbb{D} \subset \mathbb{D}(\mathrm{Ar}[1])$ is slightly more interesting: it is a staircase with one nontrivial object:

$$\begin{tikzcd}
0 \arrow[r] & a \arrow[d] \\
 & 0
\end{tikzcd}$$

There are no fully faithful inclusions $\square \to \mathrm{Ar}[1]$, so there is nothing else to require. We can see that $\mathbf{S}_1\mathbb{D} = \mathbb{D}(e)$ as a category, an observation we will use later. The first interesting category is $\mathbf{S}_2\mathbb{D} \subset \mathbb{D}(\mathrm{Ar}[2])$, whose objects have the form

$$\begin{tikzcd}
0 \arrow[r] & a \arrow[r, "f"] \arrow[d] & b \arrow[d, "g"] \\
& 0 \arrow[r] & c \arrow[d] \\
& & 0
\end{tikzcd}$$

with the requirement that the square is cocartesian. This is where we see $K_0(\mathbb{D})$ being encoded: the zero simplices of $K(\mathbb{D}) = \Omega|N_\bullet i\mathbf{S}_\bullet\mathbb{D}|$ come from $\mathbf{S}_1\mathbb{D}$ (because Ω gives a dimension shift) and these zero simplices are identified in $\pi_0 K(\mathbb{D})$ due to the existence of a path, i.e. an element of $\mathbf{S}_2\mathbb{D}$ relating them. Thus three objects appearing a cocartesian square in $\mathbb{D}(\square)$ leads to a relation on the homotopy classes of zero simplices in $\pi_0 K(\mathbb{D})$. The same thing happens at each $\pi_n K(\mathbb{D})$, but the relationship is more difficult to describe for large values of n.

We said above that in order for a morphism $\Phi\colon \mathbb{D} \to \mathbb{E}$ to induce a map on K-theory, it needs to preserve cocartesian squares and the zero object, which we could then conclude was equivalent to asking for Φ to be cocontinuous. However, there is another problem. If we have a cocontinuous morphism $\Phi : \mathbb{D} \to \mathbb{E}$ that is not *strict*, then Φ may not induce a map of simplicial sets $\mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{E}$. If we take (for example) the face map $d_i : [n] \to [n+1]$, then we obtain a diagram

$$\begin{tikzcd}
\mathbf{S}_{n+1}\mathbb{D} \arrow[r, "\Phi_{\mathrm{Ar}[n+1]}"] \arrow[d, "d_i^*"'] \arrow[dr, phantom, "\overset{\gamma^{\Phi}_{d_i}}{\Swarrow}"] & \mathbf{S}_{n+1}\mathbb{E} \arrow[d, "d_i^*"] \\
\mathbf{S}_n\mathbb{D} \arrow[r, "\Phi_{\mathrm{Ar}[n]}"'] & \mathbf{S}_n\mathbb{E}
\end{tikzcd}$$

But this diagram commutes only up to coherent natural isomorphism, so Φ does not give us an honest natural transformation of the bisimplicial sets $N_\bullet i\mathbf{S}_\bullet\mathbb{D} \to N_\bullet i\mathbf{S}_\bullet\mathbb{E}$. However, we have the following proposition to aid us.

Proposition 4.3.1 (Proposition 10.14, [CN08]). Let $\Phi : \mathbb{D} \to \mathbb{E}$ be a morphism of prederivators. Then there exists a prederivator $\widetilde{\mathbb{D}}$, a strict equivalence of derivators $\Pi_\Phi\colon \widetilde{\mathbb{D}} \to \mathbb{D}$, and a strict morphism $\widetilde{\Phi} : \widetilde{\mathbb{D}} \to \mathbb{E}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \widetilde{\mathbb{D}} \arrow[dl, "\Pi_\Phi"', "\sim"] \arrow[dr, "\widetilde{\Phi}"] & \\
\mathbb{D} \arrow[rr, "\Phi"'] & & \mathbb{E}
\end{tikzcd}$$

commutes.

We name the equivalence Π_Φ because it is some sort of projection, though we will not need the precise formula. If Φ is cocontinuous, $\widetilde{\Phi}$ is also cocontinuous because it is the composition of cocontinuous morphisms. The following corollary is found at [CN08, Corollary 10.19] or [MR17, Remark 5.1.4].

Corollary 4.3.2. Any cocontinuous morphism of left pointed derivators $\Phi : \mathbb{D} \to \mathbb{E}$ gives rise to a morphisms on derivator K-theory $K(\Phi)\colon K(\mathbb{D}) \to K(\mathbb{E})$ in $\mathcal{T}$, the homotopy category of spaces. Moreover, this association is functorial, in the sense that we have a 1-functor $\mathbf{Der}_K \to \mathcal{T}$ after inverting isomodifications in $\mathbf{Der}_K$ to obtain a 1-category.

An immediate consequence is that equivalent left pointed derivators have equivalent K-theories.

There are some first results that are worth collecting. Maltsiniotis in [Mal07] proved that, if $\mathbb{D}$ is a triangulated derivator, then $K_0(\mathbb{D}(e))$ of the underlying triangulated category is equivalent to $K_0(\mathbb{D})$. He also established a comparison map from Quillen's K-theory of an exact category to derivator K-theory, which was subsequently extended to Waldhausen categories by Garkusha in [Gar06].

Specifically, let $\mathcal{W}$ be a saturated Waldhausen category satisfying the cylinder axiom as in Lemma 4.2.4 so that it gives rise to a left pointed derivator $\mathbb{D}_{\mathcal{W}}$ (hereafter we will leave these adjectives on $\mathcal{W}$ implicit). Then we have an obvious map

$$\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{W}) \to \mathbb{D}_{\mathcal{W}}(\mathrm{Ar}[n]) = \mathrm{Ho}(\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{W}))$$

sending a diagram to its homotopy class. This map restricts to $\mathbf{S}_n\mathcal{W} \to \mathbf{S}_n\mathbb{D}_{\mathcal{W}}$, sends weak equivalences to isomorphisms, and behaves well with respect to the simplicial structures, so we obtain a map of spaces

$$\mu\colon K(\mathcal{W}) \to K(\mathbb{D}_{\mathcal{W}})$$

Maltsiniotis' proof is easily rewritten to imply that $\mu_0 := \pi_0\mu$ is an isomorphism, and Muro in [Mur08] proved that μ_1 is an isomorphism as well. Muro's techniques are a bit ad hoc, but upcoming (unpublished) work of Raptis implies that the comparison map μ is 2-connected, recovering Muro's result on μ_1 and proving that μ_2 is surjective. Maltsiniotis conjectured that μ should be a weak homotopy equivalence in the case that $\mathbb{D}_{\mathcal{W}}$ is triangulated, and the same question can be asked in general.

Unfortunately, the conjecture fails almost totally. Muro and Raptis together in [MR11] show that μ will generally not be an equivalence for triangulated derivators arising from stable module categories. In that same work, the authors use an example of Schlichting in [Sch02] to prove that derivator K-theory invariant under equivalences of derivators cannot satisfy both agreement and send Verdier localizations of triangulated derivators to long exact sequences in K-theory. Such a localization theorem was also conjectured by Maltsiniotis.

One positive result is a theorem of Cisinski and Neeman [CN08] that derivator K-theory of triangulated derivators satisfies an additivity theorem, to be made more explicit shortly. Muro and Raptis in their second paper on derivator K-theory [MR17] asked whether this additivity proof could be adapted to the more general context of left pointed derivators. The positive answer to this question occupies the next section.

4.4 Additivity

Rather than approach the problem as Cisinski and Neeman did in [CN08] using Neeman's method of regions, we will prove additivity in a novel way. Throughout, let $\mathbb{D}$ be a left pointed derivator defined on $\mathbf{Dia} = \mathbf{Dir_f}$. We adapt this first definition from [CN08, Definition 11.7]. We return to the original Notation 2.8.1 for the category $\square$.

Definition 4.4.1. Let $\mathbb{D}$ be a left pointed derivator. We define the corresponding *cofiber sequence category* for each $K \in \mathbf{Dir_f}$ by $\mathbb{D}_{\mathrm{cof}}(K) \subset \mathbb{D}^K(\square)$ the full subcategory of cocartesian squares X such that $(0,1)^*X = 0 \in \mathbb{D}(K)$.

Lemma 4.4.2. The cofiber sequence categories assemble to a prederivator $\mathbb{D}_{\text{cof}}$. Moreover, there is an equivalence $\mathbb{D}^{[1]} \to \mathbb{D}_{\text{cof}}$ which is pseudonatural with respect to cocontinuous morphisms of derivators, which makes $\mathbb{D}_{\text{cof}}$ a left pointed derivator as well.

Proof. For any $u\colon J \to K$ in $\mathbf{Dir_f}$, $u^*\colon \mathbb{D}^{\square}(K) \to \mathbb{D}^{\square}(J)$ is left adjoint to u_*, so u^* is cocontinuous. Therefore u^* preserves cocartesian squares and the zero object, so restricts to $u^*\colon \mathbb{D}_{\text{cof}}(K) \to \mathbb{D}_{\text{cof}}(J)$. This makes $\mathbb{D}_{\text{cof}}$ a prederivator.

Recall the construction of the cone morphism $C\colon \mathbb{D}^{[1]} \to \mathbb{D}$ from the proof of Theorem 3.3.4. The equivalence between $\mathbb{D}^{[1]}$ and $\mathbb{D}_{\text{cof}}$ is given by

$$\mathbb{D}^{[1]} \xrightarrow{i_{[1],*}} \mathbb{D}^{\ulcorner} \xrightarrow{i_{\ulcorner,!}} \mathbb{D}_{\text{cof}} \subset \mathbb{D}^{\square}$$

which is all but the last step $(1,1)^*$ of the cone morphism.

By definition the image of this composite consists of cocartesian squares with the zero object in the $(0,1)$ position. Since $i_{[1]}$ and $i_{\ulcorner}$ are fully faithful, their left and right Kan extensions are fully faithful by Proposition 2.6.19, hence the above composite induces an equivalence onto its image, which is precisely $\mathbb{D}_{\text{cof}} \subset \mathbb{D}^{\square}$.

For the pseudonaturality, consider a cocontinuous morphism of derivators $\Phi\colon \mathbb{D} \to \mathbb{E}$. Then for $\Phi^{\square}\colon \mathbb{D}^{\square} \to \mathbb{E}^{\square}$ to restrict to $\Phi\colon \mathbb{D}_{\text{cof}} \to \mathbb{E}_{\text{cof}}$, it would have to send cocartesian squares to cocartesian squares, and it would have to send the zero object of $\mathbb{D}$ to the zero object of $\mathbb{E}$. But this is precisely a cocontinuous morphism of derivators with domain $\mathbf{Dir_f}$ by Theorem 4.2.11. $\square$

Remark 4.4.3. The pseudonaturality with respect to cocontinuous morphisms is the most important takeaway of the preceding lemma. In the below constructions, we will construct morphisms $\Phi\colon \mathbb{D}_{\text{cof}} \to (\mathbb{D}_{\text{cof}})^K$ for various categories $K \in \mathbf{Dir_f}$, but often we will have to define these morphisms first as $\Phi\colon \mathbb{D} \to \mathbb{D}^K$. We may then extend Φ to a morphism $\mathbb{D}_{\text{cof}} \to (\mathbb{D}_{\text{cof}})^K$ if Φ is a cocontinuous morphism.

Definition 4.4.4. Let $\mathbb{D}, \mathbb{E}$ be left pointed derivators. We define a *cofibration morphism of derivators* to be a strict cocontinuous morphism $\Xi\colon \mathbb{D} \to \mathbb{E}_{\text{cof}}$. To Ξ we associate three strict cocontinuous morphisms $\mathbb{D} \to \mathbb{E}$

$$S := (0,0)^*\Xi \qquad T := (1,0)^*\Xi \qquad Q := (1,1)^*\Xi$$

and two strict cocontinuous morphisms $\alpha, \beta\colon \mathbb{D} \to \mathbb{E}^{[1]}$ given by restricting to the top and right arrows of the coherent square, respectively. Incoherently, we have

$$\begin{tikzcd} & S(a) \arrow[r, "\alpha_a"] \arrow[d] & T(a) \arrow[d, "\beta_a"] & \\ a \in \mathbb{D} \mapsto & 0 \arrow[r] & Q(a) & \in \mathbb{E}_{\text{cof}} \end{tikzcd}$$

This is a coherent version of a cofibration sequence of exact morphisms of Waldhausen categories in [Wal85, p. 331]. We prove a similar theorem to [Wal85, Proposition 1.3.2].

Theorem 4.4.5. Let $\mathbb{D}$ and $\mathbb{E}$ be left pointed derivators. The following are equivalent:

(1) The map

$$\mathbb{D}_{\text{cof}} \xrightarrow{(0,0)^* \times (1,1)^*} \mathbb{D} \times \mathbb{D}$$

induces a homotopy equivalence on derivator K-theory.

(2) If $\Xi\colon \mathbb{D} \to \mathbb{E}_{\text{cof}}$ is a cofibration morphism of derivators, then there exists a homotopy

$$K(T) \simeq K(S) \vee K(Q) \; (\cong K(S \sqcup Q))$$

where $\wedge$ denotes the coproduct of pointed spaces.

The first statement is the statement of additivity à la Garkusha, Maltsiniotis, and Cisinski-Neeman, first found in [Mal07, Conjecture 3] and similar to [Wal85, Proposition 1.3.2(2)]. The latter is a reinterpretation of [Wal85, Proposition 1.3.2(4)]. Our proof follows the strategy set out by Waldhausen.

To expand a little on (1), it will be helpful to use that $\mathbf{S}_\bullet\mathbb{D} \times \mathbf{S}_\bullet\mathbb{D} \cong \mathbf{S}_\bullet\mathbb{D}^{e\sqcup e}$. First, we know that

$$\mathbf{S}_n\mathbb{D} \times \mathbf{S}_n\mathbb{D} \subset \mathbb{D}(\text{Ar}[n]) \times \mathbb{D}(\text{Ar}[n]) \cong \mathbb{D}(\text{Ar}[n] \sqcup \text{Ar}[n]) \cong \mathbb{D}^{e\sqcup e}(\text{Ar}[n]) \supset \mathbf{S}_n\mathbb{D}^{e\sqcup e}.$$

Second, for a diagram $X \in \mathbb{D}^{e\sqcup e}(\mathrm{Ar}[n])$, the condition of being in $\mathbf{S}_n\mathbb{D}^{e\sqcup e}$ coincides with each projection to $\mathbb{D}(\mathrm{Ar}[n])$ being in $\mathbf{S}_n\mathbb{D}$. This makes it easier for us to define maps into $\mathbf{S}_\bullet\mathbb{D} \times \mathbf{S}_\bullet\mathbb{D}$; they can arise from (strict) cocontinuous morphisms into $\mathbb{D}^{e\sqcup e}$.

In particular, in the spirit of Remark 4.4.3, morphisms arising from left adjoint functors are cocontinuous, so any $u^*, u_!$ induce maps on K-theory. Since we are working in a left pointed derivator, by Proposition 4.2.8, the extension by zero morphisms i_* for any sieve i are also cocontinuous. Put another way, $\mathbf{Der}_K$ contains all left and right Kan extension morphisms available to left pointed derivators.

Proof.

(2) $\implies$ (1)

The map $\rho := (0,0)^* \times (1,1)^*$ admits a section $\sigma\colon \mathbb{D}^{e\sqcup e} \to \mathbb{D}_{\mathrm{cof}}$ on K-theory. Incoherently for $(a,c) \in \mathbb{D}^{e\sqcup e}$, the functor σ is roughly (but not precisely)

$$(a,c) \mapsto \begin{tikzcd} a \arrow[r] \arrow[d] & a \sqcup c \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd}$$

We will write σ as a composite of morphisms of derivators coming from diagram functors, but we will need some diagram notation first.

Recall from Notation 2.8.1 the functor $i_\ulcorner\colon \ulcorner \to \square$. Further, let $i\colon e \sqcup e \to \ulcorner$ be the inclusion into $(1,0)$ and $(0,1)$. We also consider the category $[1] \times [2]$, with labelling

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r] \arrow[d] & (1,1) \arrow[d] \\ (0,2) \arrow[r] & (1,2) \end{tikzcd}$$

Let J be the full subcategory of $[1] \times [2]$ without the element $(1,2)$. Finally, let $i_\square\colon \square \to J$ and $j\colon J \to [1] \times [2]$ be the obvious inclusions, and let $r\colon \square \to [1] \times [2]$ be the inclusion of the bottom square. Note that i is a cosieve, and $i_\ulcorner$, $i_\square$, and j are sieves.

At the level of the derivators, the section σ is given by

$$\mathbb{D}^{e\sqcup e} \xrightarrow{i_!} \mathbb{D}^{\ulcorner} \xrightarrow{i_{\ulcorner,!}} \mathbb{D}^{\square} \xrightarrow{i_{\square,*}} \mathbb{D}^{J} \xrightarrow{j_!} \mathbb{D}^{[1]\times[2]} \xrightarrow{r^*} \mathbb{D}^{\square} \supset \mathbb{D}_{\mathrm{cof}} \tag{4.4.6}$$

All of these maps are cocontinuous, though not all are strict. As we mentioned in Corollary 4.3.2, this means that σ will give rise to a well-defined map in $\mathcal{T}$.

For $(a,c) \in \mathbb{D}^{e \sqcup e}$, σ is explicitly

$$\begin{tikzcd}
 & & & & & & & & 0 \arrow[r] \arrow[d] & c \arrow[d] & & 0 \arrow[r] \arrow[d] & c \arrow[d] & & & \\
(a,c) & \mapsto & 0 \arrow[r] \arrow[d] & c & \mapsto & 0 \arrow[r] \arrow[d] & c \arrow[d] & \mapsto & a \arrow[r] \arrow[d] & a \sqcup c & \mapsto & a \arrow[r] \arrow[d] & a \sqcup c \arrow[d] & \mapsto & a \arrow[r] \arrow[d] & a \sqcup c \arrow[d] \\
 & & a & & & a \arrow[r] & a \sqcup c & & 0 & & & 0 \arrow[r] & c' & & 0 \arrow[r] & c'
\end{tikzcd}$$

By construction, the image of this composite lands in $\mathbb{D}_{\mathrm{cof}} \subset \mathbb{D}^{\square}$, as the square $r^* j_! X$ is easily shown to be cocartesian using Proposition 3.4.8 for any $X \in \mathbb{D}^J$, and $(1,0)^* r^* j_! X = 0$ as long as X is in the image of $i_{\square,*}$ (as is our case). Note further that the composite map $c \to a \sqcup c \to c'$ is an isomorphism, as it is the pushout of the isomorphism $0 \to 0$.

All this data together gives us the section $\sigma\colon \mathbb{D}^{e \sqcup e} \to \mathbb{D}_{\mathrm{cof}}$. Hence we obtain an isomodification $\mathrm{id}_{\mathbb{D}^{e \sqcup e}} \Rrightarrow \rho\sigma$, as the canonical isomorphism $c \to c'$ gives rise to an isomorphism $(a,c) \to \rho\sigma(a,c) = (a,c')$ natural in $(a,c) \in \mathbb{D}^{e \sqcup e}$. On K-theory (after strictifying the non-strict morphisms and passing to $\mathcal{T}$), this gives a homotopy $K(\rho\sigma) \simeq K(\mathrm{id}_{\mathbb{D}^{e \sqcup e}})$. Therefore it suffices to construct a homotopy in the reverse direction, i.e. $K(\sigma\rho) \simeq K(\mathrm{id}_{\mathbb{D}_{\mathrm{cof}}})$.

To that end, we use our assumption. We will construct a cofibration morphism of derivators such that $S \sqcup Q \cong \sigma\rho$ and $T \cong \mathrm{id}_{\mathbb{D}_{\mathrm{cof}}}$. Our cofibration morphism $\Xi\colon \mathbb{D}_{\mathrm{cof}} \to (\mathbb{D}_{\mathrm{cof}})_{\mathrm{cof}}$ should look like, for $X \in \mathbb{D}_{\mathrm{cof}}$,

$$\begin{tikzcd}
 & & & a \arrow[rr, "="] \arrow[d] & & a \arrow[d] \arrow[rr, "{\mathrm{id}_a, f}"] & & a \arrow[rr, "f"] \arrow[d] & & b \arrow[d, "g"] \\
 & & & 0 \arrow[rr] & {} \arrow[ddd] & 0 & & 0 \arrow[rr] & {} \arrow[ddd, "{g, \mathrm{id}_c}"] & c \\
a \arrow[r, "f"] \arrow[d] & b \arrow[d, "g"] & \mapsto & & & & & & & \\
0 \arrow[r] & c & & & & & & & & \\
 & & & 0 \arrow[rr] \arrow[d] & & 0 \arrow[d] \arrow[rr] & & 0 \arrow[rr] \arrow[d] & & c \arrow[d, "="] \\
 & & & 0 \arrow[rr] & & 0 & & 0 \arrow[rr] & & c
\end{tikzcd} \tag{4.4.7}$$

where all squares commute.

We can accomplish this as the pullback along a single functor in $\mathbf{Dir_f}$. In the diagram above of shape $\square \times \square$, let the first $\square$ denote the inner square dimension and the second the outer dimension. We now define $\xi\colon \square \times \square \to \square$ by

$$\xi(a_1, b_1, a_2, b_2) = \begin{cases} (0,0) & (a_1, b_1, a_2, b_2) = (0,0,0,0), (0,0,1,0), (0,1,0,0) \\ (1,0) & (a_1, b_1, a_2, b_2) = (0,1,0,1) \\ (1,1) & (a_1, b_2, a_2, b_2) = (1,0,1,1), (1,1,1,0), (1,1,1,1) \\ (0,1) & \text{otherwise} \end{cases}$$

In plain language, we make this definition so that ξ behaves on objects as sketched in Diagram 4.4.7, and it is also the appropriate functor for the maps. For example, consider the map $(1,0,0,0) \to (1,0,1,0)$ in $\square \times \square$. The diagram says that we want

$$(1,0,0,0)^*\xi^*X \to (1,0,1,0)^*\xi^*X = f\colon a \to b$$

We see that $\xi(1,0,0,0) = (0,0)$ and $\xi(1,0,1,0) = (1,0)$, and thus the natural transformation $(\xi(1,0,0,0))^* \Rightarrow (\xi(1,0,1,0))^*$ induces the map $f\colon a \to b$ when applied to $X \in \mathbb{D}_{\mathrm{cof}}$. Checking that we also have g and identities where required can be done similarly.

We define a strict cocontinuous morphism of derivators $\Xi := \xi^*\colon \mathbb{D}^{\square} \to \mathbb{D}^{\square\times\square}$. By construction, assuming we restrict our domain to $\mathbb{D}_{\mathrm{cof}}$ (as illustrated), the image will be a global cocartesian square in $\mathbb{D}_{\mathrm{cof}}(\square)$ with zero in the bottom-left corner, so we obtain the required (strict cocontinuous) morphism $\Xi\colon \mathbb{D}_{\mathrm{cof}} \to (\mathbb{D}_{\mathrm{cof}})_{\mathrm{cof}}$.

Our assumption gives us that $K(T) \simeq K(S \sqcup Q)$, and clearly $T = \mathrm{id}_{\mathbb{D}_{\mathrm{cof}}}$. It is also evident that we have an isomodification $\sigma\rho \Rightarrow S \sqcup Q$ using again the naturality of the comparison isomorphism $c' \to c$. This shows that $K(\mathrm{id}_{\mathbb{D}_{\mathrm{cof}}}) \simeq K(\sigma\rho)$ as required, which proves additivity in the historical sense for derivator K-theory.

$(1) \implies (2)$

First, consider the two maps $(1,0)^*\colon \mathbb{E}_{\mathrm{cof}} \to \mathbb{E}$ and $\overline{\rho}\colon \mathbb{E}_{\mathrm{cof}} \to \mathbb{E}^{e\sqcup e} \to \mathbb{E}$, where $\overline{\rho}$ is defined to be the composite cocontinuous morphism (see Equation 4.4.6) $(1,1)^* i_{\ulcorner,!} i_! \rho$ which computes the coproduct of $(0,0)^*X$ and $(1,1)^*X$ for any $X \in \mathbb{E}_{\mathrm{cof}}$.

We claim that these maps are homotopic. If we precompose with the map $\sigma\colon \mathbb{E}^{e\sqcup e} \to \mathbb{E}_{\mathrm{cof}}$, it is immediate that $(1,0)^*\sigma$ and $\overline{\rho}\sigma$ are isomorphic, as they both compute the coproduct of $(a, c) \in \mathbb{E}^{e \sqcup e}$. Thus if σ is a homotopy equivalence (on K-theory), $(1,0)^*$ and $\overline{\rho}$ are still homotopic. But by our assumption (1), σ the section of the homotopy equivalence ρ, so it σ too is a homotopy equivalence.

Statement (2) then follows immediately by precomposing these two homotopic maps by any cofibration morphism $\Xi\colon \mathbb{D} \to \mathbb{E}_{\mathrm{cof}}$, which yields

$$K(\overline{\rho}\,\Xi) \cong K(S \sqcup Q) \simeq K(T) = K((1,0)^*\Xi)$$

□

This new reformulation of the additivity theorem produces a proof that differs greatly from [CN08] and [Gar05], the latter of which includes a gap which seems irreparable, see [MR17, §6.3]. We will now prove Theorem 4.4.5(2) in the remainder of this section, following a strategy given by Grayson in [Gra11]. Grayson's paper proves the (classical) additivity theorem for Waldhausen K-theory using an explicit combinatorial homotopy. We are able to take this diagram-flavored argument and make it coherent enough for application to derivator K-theory.

Let Y be a simplicial set. Without any loss of generality, we may extend $Y\colon \Delta^{\mathrm{op}} \to \mathbf{Sets}$ to $Y\colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Sets}$, where **Ord** is the category of (nonempty) finite totally ordered sets. For $A \in \mathbf{Ord}$, we let $Y(A) := Y([n])$, where $[n]$ is the unique element of $\Delta \subset \mathbf{Ord}$ isomorphic to A. We do this in order to introduce a binary operation $*$ on **Ord**, which otherwise would cause us problems. We let $A * B$ be concatenation, that is,

$$A * B := (\{0\} \times A) \cup (\{1\} \times B) \subset [1] \times (A \cup B)$$

with lexicographical ordering. This results in each element of A being smaller than each element of B, but within A and B the ordering does not change.

Definition 4.4.8. Let Y be a simplicial set (on **Ord**). The *two-fold edge-wise subdivision* $\mathrm{sub}_2 Y$ of Y is the simplicial set defined by $\mathrm{sub}_2 Y(A) := Y(A * A)$.

There is a natural homeomorphism $|\,\mathrm{sub}_2 Y| \to |Y|$ defined in [Gra89, §4] whose construction we do not recall here. The important thing to note is that we do not change the homotopy type (or even homeomorphism type) of our simplicial set by subdividing.

Now we can begin to bring derivators back into the conversation. Let $\Phi, \Psi\colon \mathbb{D} \to \mathbb{E}$ be two strict cocontinuous morphisms of derivators. These induce morphisms of simplicial categories $\mathbf{S}_\bullet\Phi, \mathbf{S}_\bullet\Psi\colon \mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{E}$. We define a new map of simplicial categories $\nabla_{\Phi,\Psi}\colon \operatorname{sub}_2\mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{E}$ in the following way.

The totally ordered set $A * A$ has two full subcategories $i_0, i_1\colon A \to A * A$, given by $a \mapsto (0, a), (1, a)$ respectively. These extend to functors on the arrow categories $\operatorname{Ar}(A) \to \operatorname{Ar}(A * A)$, and give $i_0^*, i_1^*\colon \mathbb{D}(\operatorname{Ar}(A * A)) \to \mathbb{D}(\operatorname{Ar}(A))$. Since restriction morphisms are strict and cocontinuous, i_0^* and i_1^* define morphisms of simplicial categories $\mathbf{S}_\bullet\mathbb{D}(A * A) \to \mathbf{S}_\bullet\mathbb{D}(A)$.

Consider an object $X \in \operatorname{sub}_2\mathbf{S}_\bullet\mathbb{D}(A) = \mathbf{S}_\bullet\mathbb{D}(A * A)$. Then let

$$\nabla_{\Phi,\Psi}(X) := \Phi(i_0^*(X)) \sqcup \Psi(i_1^*(X))$$

This indeed lands in $\mathbf{S}_\bullet\mathbb{E}(A)$ as the coproduct of any two cocartesian squares is again cocartesian. We have, in essence, doubled $\mathbf{S}_\bullet\mathbb{D}$ and applied Φ to the first half and Ψ to the second, then took the coproduct of the results.

Unfortunately, the described map does not exist on the level of simplicial categories. While the functors Φi_0^* and Ψi_1^* are still strict, taking the coproduct is not a strict operation. Therefore we will need to strictify this last map, as in Proposition 4.3.1, so we do not honestly get a map of simplicial sets with codomain $\mathbf{S}_\bullet\mathbb{E}$. But on K-theory the map we want does exist, given by the zig-zag

(4.4.9)

$$\Omega|\operatorname{sub}_2\mathbf{S}_\bullet\mathbb{D}| \xrightarrow{K(\Phi i_0^*)\times K(\Psi i_1^*)} K(\mathbb{E}) \times K(\mathbb{E}) \cong K(\mathbb{E}^{e\sqcup e}) \xleftarrow{\simeq} K\left(\widetilde{\mathbb{E}^{e\sqcup e}}\right) \xrightarrow{\widetilde{\sqcup}} K(\mathbb{E})$$

where $\widetilde{\mathbb{E}^{e\sqcup e}}$ is the prederivator constructed in Proposition 4.3.1 to strictify the coproduct map $\sqcup = (1,1)^* i_{\ulcorner,!} i_! : \mathbb{E}^{e\sqcup e} \to \mathbb{E}$.

Now, we need to construct a cylinder object for $\mathbf{S}_\bullet\mathbb{D}$ that will allow us to use $\nabla_{\Phi,\Psi}$ as a replacement for $\Phi \sqcup \Psi$. We need three definitions to get us there.

Definition 4.4.10 (Definition 1.2, [Gra11])**.** For $A, B \in \mathbf{Ord}$, let $A \ltimes B$ be the set $A \times B$ with lexicographic ordering. That is, $(a, b) \leq (a', b')$ if and only if $a < a'$ or $a = a'$ and $b \leq b'$.

Note that the map $A \ltimes B \to A$ is order-preserving, hence a morphism in **Ord**, but the other 'projection' $A \ltimes B \to B$ is generally not.

Definition 4.4.11 (Definition 1.3, [Gra11])**.** Given two maps $\varphi\colon A \to C$ and $s\colon B \to C$ in **Ord**, define $\varphi^{-1}(s) \in \mathbf{Ord}$ to be the subset of $A \ltimes B$ given by $\{(a, b) : \varphi(a) = s(b)\}$.

Definition 4.4.12 (Definition 1.4, [Gra11])**.** Let $s\colon [2] \to [1]$ be the morphism defined by $s(0) = 0$ and $s(1) = s(2) = 1$. For any simplicial set Y, define a new simplicial set IY by $IY(A) := \{(\varphi, y) : \varphi\colon A \to [1], y \in Y(\varphi^{-1}(s))\}$. The definition of IY on morphisms in **Ord** extends by naturality.

Remark 4.4.13. To see how IY is a useful object, notice that $\varphi^{-1}(s) = \varphi^{-1}(0) * \varphi^{-1}(1) * \varphi^{-1}(1)$. Therefore the choice $\varphi = 0$ gives $\varphi^{-1}(s) = A$, and the choice $\varphi = 1$ gives $\varphi^{-1}(s) = A * A$. The simplicial subset of IY at $\varphi = 0$ is isomorphic to X, and the simplicial subset of IY at $\varphi = 1$ is isomorphic to $\mathrm{sub}_2 Y$. Any other morphism $\varphi\colon A \to [1]$ gives a totally ordered set $\varphi^{-1}(s)$ interpolating between these two endpoints.

Lemma 4.4.14 (Lemma 1.6, [Gra11])**.** There is a homeomorphism $|IY| \to |\Delta^1| \times |Y|$.

We do not include the proof because nothing is changed in the context of derivators. This shows that IY is indeed a cylinder object for Y, so we may prove the main proposition.

Proposition 4.4.15. Let $\Xi\colon \mathbb{D} \to \mathbb{E}_{\mathrm{cof}}$ be a cofibration morphism of derivators, and let $S, T, Q\colon \mathbb{D} \to \mathbb{E}$ be the corresponding morphisms of derivators. Then there is a map of simplicial categories $\Theta\colon I\mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{E}$ such that Θ agrees with T on the simplicial subcategory where $\varphi = 0$ and Θ agrees with $\nabla_{Q,S}$ on the simplicial subcategory where $\varphi = 1$.

Remark 4.4.16. Just as $\nabla_{Q,S}$ is not an honest morphism of simplicial categories, Θ will not be well-defined per se but will induce a map on K-theory via a zigzag coming from strictification. We will continue to abuse notation in this fashion.

Proof. We will construct Θ in two steps.

First, we define a morphism $P\colon \mathbb{D}^{[1]} \to \mathbb{E}$. For a coherent morphism $(f\colon a \to b) \in \mathbb{D}^{[1]}$, we may apply $\Xi^{[1]}$ to obtain an object in $\mathbb{E}^{[1]}_{\mathrm{cof}} \subset \mathbb{E}^{[1]\times\square}$. Specifically, its underlying diagram takes the form (where we do not label every arrow)

$$\begin{tikzcd}
S(a) \arrow[rr, "\alpha_a"] \arrow[dd, "S(f)"'] \arrow[dr] & & T(a) \arrow[dd] \arrow[dr, "\beta_a"] & \\
& 0 \arrow[rr, crossing over] & & Q(a) \arrow[dd, "Q(f)"] \\
S(b) \arrow[rr] \arrow[dr] & & T(b) \arrow[dr] & \\
& 0 \arrow[from=uu, crossing over] \arrow[rr] & & Q(b)
\end{tikzcd}$$

We may consider the functor $\ulcorner \to [1] \times \square$ by the inclusion into the upper-left corner of the cube given above. Restriction along this functor gives the coherent diagram in $\mathbb{E}^{\ulcorner}$

$$\begin{tikzcd}
S(a) \arrow[r, "\alpha_a"] \arrow[d, "S(f)"'] & T(a) \\
S(b) &
\end{tikzcd} \tag{4.4.17}$$

We may then apply $(1,1)^* i_{\ulcorner,!}$ to first take the pushout of Diagram 4.4.17 and then restrict to the new object. The composition is a cocontinuous morphism which we denote $P\colon \mathbb{D}^{[1]} \to \mathbb{E}$.

We point out two special cases. If $f \in \mathbb{D}^{[1]}$ is a coherent isomorphism, then $P(f) \cong T(a)$, as pushing out along an isomorphism is still an isomorphism. Second, if f is a zero morphism, then there is a natural isomorphism $P(f) \cong Q(a) \sqcup S(b)$. This arises from a factorization of f as follows

$$\begin{tikzcd}
S(a) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "S(f)"'] & T(a) \\
0 \arrow[d] & \\
S(b) &
\end{tikzcd}$$

If we take pushouts one square at a time, we first obtain an object isomorphic to $Q(a)$ and second compute the pushout of $Q(a)$ with $S(b)$:

$$\begin{tikzcd}
S(a) \arrow[r] \arrow[d] & T(a) \\
0 \arrow[d] & \\
S(b) &
\end{tikzcd}
\mapsto
\begin{tikzcd}
S(a) \arrow[r] \arrow[d] & T(a) \arrow[d] \\
0 \arrow[r] \arrow[d] & Q(a) \\
S(b) &
\end{tikzcd}
\mapsto
\begin{tikzcd}
S(a) \arrow[r] \arrow[d] & T(a) \arrow[d] \\
0 \arrow[r] \arrow[d] & Q(a) \arrow[d] \\
S(b) \arrow[r] & S(b) \sqcup Q(a)
\end{tikzcd}$$

But the composition of two pushouts is again a pushout, which implies that $S(b) \sqcup Q(a)$ is the pushout of the total upper-right corner, which is exactly Diagram 4.4.17. Thus we conclude $P(f) \cong S(b) \sqcup Q(a)$.

Now, we construct the second step of Θ. Recall the definition of $s\colon [2] \to [1]$. Now we define two sections of s: first, $d\colon [1] \to [2]$ with $d(0) = 0$, $d(1) = 1$; second, $e\colon [1] \to [2]$ with $e(0) = 0$, $e(1) = 2$. For any $a \in A$, $(d\varphi(a), a)$ and $(e\varphi(a), a)$ are both elements of $\varphi^{-1}(s)$ as $se = sd = \mathrm{id}_{[1]}$. This gives two inclusions from A into $\varphi^{-1}(s)$, to which we also name d and e (as the other functors will not be returning). There is also a unique natural transformation $\zeta\colon d \Rightarrow e$ as $d\varphi(a) \leq e\varphi(a)$ for any $a \in A$.

We may first extend $d, e : \mathrm{Ar}(A) \to \mathrm{Ar}(\varphi^{-1}(s))$ by naturality, and we still have the natural transformation $\zeta\colon d \Rightarrow e$. We can better view ζ as a functor $\zeta\colon \mathrm{Ar}(A) \times [1] \to \mathrm{Ar}(\varphi^{-1}(s))$, where (in particular) the original data of the natural transformation is retained by

$$\zeta\big[((a \to b), 0) \to ((a \to b), 1)\big] = d(a \to b) \xrightarrow{\zeta_{(a\to b)}} e(a \to b).$$

Therefore we obtain a strict cocontinuous morphism $\zeta^*\colon \mathbb{D}(\mathrm{Ar}(\varphi^{-1}(s))) \to \mathbb{D}(\mathrm{Ar}(A)\times[1])$. As we vary $\varphi\colon A \to [1]$, we obtain a map out of $I\mathbf{S}_\bullet\mathbb{D}(A)$, and by cocontinuity we can restrict the codomain to $\mathbf{S}_\bullet\mathbb{D}^{[1]}(A)$. This tells us how to extend the various ζ^* to a map $Z\colon I\mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{D}^{[1]}$ (read: capital ζ).

We now need to check that Z is a map of simplicial categories. Suppose we have a map $\psi\colon B \to A$ in **Ord**. Then we need to check that the following square commutes:

(4.4.18)
$$\begin{tikzcd}
I\mathbf{S}_\bullet\mathbb{D}(A) \arrow[r, "\psi_1^*"] \arrow[d, "Z_A"'] & I\mathbf{S}_\bullet\mathbb{D}(B) \arrow[d, "Z_B"] \\
\mathbf{S}_\bullet\mathbb{D}^{[1]}(A) \arrow[r, "\psi_2^*"'] & \mathbf{S}_\bullet\mathbb{D}^{[1]}(B)
\end{tikzcd}$$

where we use ψ_1^*, ψ_2^* to denote the maps induced by ψ on the two different simplicial categories $I\mathbf{S}_\bullet\mathbb{D}$ and $\mathbf{S}_\bullet\mathbb{D}^{[1]}$.

Let $(\varphi\colon A \to [1], X \in \mathbf{S}_\bullet\mathbb{D}(\varphi^{-1}(s))) \in I\mathbf{S}_\bullet\mathbb{D}(A)$. Then taking the upper composition of Diagram 4.4.18 we first get the object $\psi_1^*(\varphi, X) = (\varphi\psi, (\psi')^*X \in \mathbf{S}_\bullet\mathbb{D}((\varphi\psi)^{-1}(s))$, where ψ' is the natural map $(\varphi\psi)^{-1}(s) \to \varphi^{-1}(s)$ induced by ψ, $(n, b) \mapsto (n, \psi(b))$. If we apply Z_B to $(\varphi\psi, (\psi')^*X)$, we obtain the coherent object in $\mathbf{S}_\bullet\mathbb{D}^{[1]}(B)$

$$d_B^*(\psi')^*X \xrightarrow{\zeta_B^*} e_B^*(\psi')^*X$$

where d_B and e_B are the specific instances of d, e in this case.

If we traverse Diagram 4.4.18 using the lower composition, we first take $Z_A(\varphi, X)$ to obtain $\zeta_A^* \colon d_A^* X \to e_A^* X$. Then applying ψ_2^* we obtain

$$\psi^* d_A^* X \xrightarrow{\psi^* \zeta_A^*} \psi^* e_A^* X.$$

But by strict 2-functorality, these two coherent maps are equal on the nose, as they are just pullbacks induced by maps in $\mathbf{Dir_f}$. Therefore Z is indeed a morphism of simplicial categories.

This finishes the definition of $\Theta = (\mathbf{S}_\bullet P) Z \colon I\mathbf{S}_\bullet \mathbb{D} \to \mathbf{S}_\bullet \mathbb{D}^{[1]} \to \mathbf{S}_\bullet \mathbb{E}$. We now need to check that Θ in fact interpolates between T and $\nabla_{Q,S}$. Suppose we take some $A \in \mathbf{Ord}$, and let $(\varphi, X) \in I\mathbf{S}_\bullet \mathbb{D}(A)$.

If $\varphi = 0$ is the zero map, then $\varphi^{-1}(s) = A$ by Remark 4.4.13. In this case, $\zeta \colon d \Rightarrow e$ is the identity natural transformation, as $d\varphi(a) = e\varphi(a)$ always. So for $(0, X \in \mathbf{S}_\bullet \mathbb{D}(A))$, we see that $Z_A(0, X)$ is nothing else but the constant map in the [1]-direction. Thus $Z_A(0, X) = \mathrm{id}_X$. This is an isomorphism, one of the special cases we noted following Diagram 4.4.17, and we conclude that $\mathbf{S}_\bullet P(Z_A(0, X))$ is naturally isomorphic to $\mathbf{S}_\bullet T(X)$.

If $\varphi = 1$, then $\varphi^{-1}(s) = A * A$ by the same remark. Let $(1, X \in \mathbf{S}_\bullet \mathbb{D}(A * A)) \in I\mathbf{S}_\bullet \mathbb{D}(A)$. The resulting $Z_A(1, X)$ is an element of $\mathbf{S}_\bullet \mathbb{D}^{[1]}(A)$, which is a subcategory of $\mathbb{D}(\mathrm{Ar}(A) \times [1])$. Incoherently, if we write $X = (x \to y)$, as $X \in \mathbb{D}(\mathrm{Ar}(A * A))$, then $Z_A(1, X)$ looks like

$$\begin{tikzcd}
d^* x \arrow[r, "\zeta_x^*"] \arrow[d] & e^* x \arrow[d] \\
d^* y \arrow[r, "\zeta_y^*"'] & e^* y
\end{tikzcd}$$

where the vertical maps are the elements of $\mathrm{Ar}(A * A)$. But since $\varphi = 1$, we know that $e^* x \geq d^* y$. Therefore the map above factors as

$$\begin{tikzcd}
d^* x \arrow[r] \arrow[d] & d^* y \arrow[r] \arrow[d] & e^* x \arrow[d] \\
d^* y \arrow[r] & d^* y \arrow[r] & e^* y
\end{tikzcd}$$

Therefore our overall (coherent) map $Z_A(1, X)$ factors through the object $(d^* y \to d^* y) \in \mathbf{S}_\bullet \mathbb{D}^{[1]}(A)$. But this is a zero object by assumption, so $Z_A(1, X)$ is a zero map.

This was our second special case, and we conclude that $\mathbf{S}_\bullet P(Z_A(1,X))$ naturally isomorphic to $\mathbf{S}_\bullet \nabla_{Q,S}(X)$. This completes the proof. □

Proving the additivity of derivator K-theory is now immediate.

Theorem 4.4.19. Let $\Xi\colon \mathbb{D} \to \mathbb{E}_{\mathrm{cof}}$ be a cofibration morphism of derivators, and let S, T, Q be the corresponding morphisms of derivators. Then $S \sqcup Q$ and T induce homotopic maps $K(\mathbb{D}) \to K(\mathbb{E})$.

Proof. From Lemma 4.4.14 and Proposition 4.4.15, the morphism of simplicial categories Θ above gives a homotopy between $\mathbf{S}_\bullet T$ and $\nabla_{\mathbf{S}_\bullet Q, \mathbf{S}_\bullet S}$ on $|\mathbf{S}_\bullet \mathbb{D}| \to |\mathbf{S}_\bullet \mathbb{E}|$. We now need to prove that there exists a cofibration morphism of derivators whose associated functors are $S, S \sqcup Q, Q$.

From the proof of Proposition 4.4.5, recall that we constructed a functor $\sigma\colon \mathbb{E}^{e \sqcup e} \to \mathbb{E}_{\mathrm{cof}}$ whose image is precisely what we desire. Let us precompose σ by the morphism $(S \sqcup Q)\colon \mathbb{D} \to \mathbb{E}^{e \sqcup e}$. The target map of $\sigma(S \sqcup Q)\colon \mathbb{D} \to \mathbb{E}_{\mathrm{cof}}$ is isomorphic to $S \sqcup Q$, and so $\nabla_{\mathbf{S}_\bullet Q, \mathbf{S}_\bullet S}$ and $\mathbf{S}_\bullet S \sqcup \mathbf{S}_\bullet Q$ induce homotopic maps on K-theory as well. Combining these homotopies, we see that T and $S \sqcup Q$ induce homotopic maps $K(\mathbb{D}) \to K(\mathbb{E})$. □

In particular, by Proposition 4.4.5, this proves additivity as it was conjectured by Maltsiniotis in [Mal07].

4.5 Further properties

As additivity sits as the central property of any flavor of algebraic K-theory, we can now see what else we have obtained. We proceed in the spirit of Waldhausen in [Wal85], beginning first with the delooping of the K-theory space $K(\mathbb{D})$. In order to do so, we need a construction of relative K-theory.

To begin, we note that the $\mathbf{S}_\bullet$-construction on derivators not only gives us a simplicial category, but in fact a simplicial left pointed derivator.

Proposition 4.5.1. Let $j\colon [n-1] \to \mathrm{Ar}[n]$ denote the inclusion $i \mapsto (0, i+1)$. Then $j^*\colon \mathbf{S}_n\mathbb{D} \to \mathbb{D}([n-1])$ is an equivalence of categories. Thus we can give $\mathbf{S}_n\mathbb{D} \subset \mathbb{D}^{\mathrm{Ar}[n]}$ the structure of a left pointed derivator via the equivalence with $\mathbb{D}^{[n-1]}$.

Proof. We will construct the quasiinverse directly.

First, consider the functor $i_0\colon [n-1] \to [n]$ defined by $i \mapsto i+1$. This map is a cosieve, so the morphism $i_{0,!}$ is extension by zero.

Second, consider the subcategory $D \subset \mathrm{Ar}[n]$ which contains the top row $(0, i)$ as well as all diagonal entries (i, i). The inclusion $i_1\colon [n] \to J$ is a sieve, so $i_{1,*}$ is extension by zero.

The last step is to take the inclusion $i_2\colon D \to \mathrm{Ar}[n]$ and compute $i_{2,!}$. We claim that the image of this composition lies in $\mathbf{S}_n\mathbb{D}(e) \subset \mathbb{D}(\mathrm{Ar}[n])$. To see this, we use Proposition 3.4.8 to detect cocartesian squares. We note that since we have flipped $\square$ back to the original notation, the statement of Proposition 3.4.8 must be flipped accordingly.

Let $\iota_{i,j}\colon \square \to \mathrm{Ar}[n]$ be the square given by $(a, b) \mapsto (i+b, j+a)$ for $0 \leq i < j \leq n-1$ (where the 'flip' of a and b is introduced due to slightly incompatible labelling). It suffices to prove each of these squares is cocartesian, as any other square will be a composite of such squares. If a larger square can be subdivided into cocartesian squares, then it too is cocartesian.

Because $\mathrm{Ar}[n]$ is a poset, Lemma 2.2.5 applies and we can identify the comma category

$$(\mathrm{Ar}[n] \setminus (i+1, j+1)/(i+1, j+1))$$

as the full subcategory of $\mathrm{Ar}[n]$ on (p, q) admitting a map to $(i+1, j+1)$, i.e. $p \leq i+1$ and $q \leq j+1$, excluding $(p, q) = (i+1, j+1)$ as it has been removed. Call the resulting category $B^{i,j}$. We now construct a left adjoint for $\iota_{i,j}$ directly. We define $\ell\colon B^{i,j} \to \ulcorner$ by

$$\ell(p, q) = \begin{cases} (0,0) & p \leq i \text{ and } q \leq j \\ (0,1) & q = j+1 \\ (1,0) & p = i+1 \end{cases}$$

Direct computation shows that $\mathrm{Hom}(\ell(p,q),(a,b)) = \mathrm{Hom}((p,q),\iota_{i,j}(a,b))$ for any elements $(p,q) \in B^{i,j}$ and $(a,b) \in \ulcorner$. We could also construct the unit and the counit directly; the counit is just the identity on $\ulcorner$, and the unit can only be the unique map $(p,q) \to \iota(\ell(p,q))$ at each $(p,q) \in B^{i,j}$.

This proves that $\iota_{i,j}^* i_{2,!}X$ is a cocartesian square for any $X \in \mathbb{D}(D)$. Pasting these squares together shows that any square in $i_{2,!}X$ is cocartesian. Moreover, if we have $X = i_{1,*}i_{0,!}Y$ for some $Y \in \mathbb{D}([n-1])$, then $(i,i)^*X = 0$ by construction. Therefore $X \in \mathbf{S}_n\mathbb{D}(e)$. Call this total morphism $f\colon \mathbb{D}([n-1]) \to \mathbf{S}_n\mathbb{D}$.

Because f is constructed as left and right Kan extensions of fully faithful functors, it too is fully faithful. Moreover, it is left adjoint to j^*, with the counit $\mathrm{id}_{\mathbb{D}([n-1])} = j^*f$ the identity modification. Because the left adjoint morphism is fully faithful, the unit is an isomorphism. Because both the unit and counit are invertible modifications, this gives an equivalence of categories. □

Remark 4.5.2. We can give an alternative proof that $\mathbf{S}_n\mathbb{D}$ has the structure of a left pointed derivator. First, for $K \in \mathbf{Dir_f}$, we define $\mathbf{S}_n\mathbb{D}(K) \subset \mathbb{D}^{\mathrm{Ar}[n]}(K)$ to be the full subcategory on objects X such that $k^*X \in \mathbf{S}_n\mathbb{D}$ for any $k \in K$. This makes $\mathbf{S}_n\mathbb{D}$ a prederivator on $\mathbf{Dir_f}$. Der1 and Der2 follow immediately from its definition as a certain levelwise subcategories of a derivator, and it is also (weakly) pointed because the 0 object of $\mathbb{D}(\mathrm{Ar}[n])$ is in $\mathbf{S}_n\mathbb{D}(e)$.

For the remainder of the axioms, it suffices to show that the left and right Kan extensions in $\mathbb{D}^{\mathrm{Ar}[n]}$ land in the appropriate subcategory. Let $X \in \mathbf{S}_n\mathbb{D}(J)$ and $u\colon J \to K$ a functor. We only know for sure that $u_!X \in \mathbb{D}^{\mathrm{Ar}[n]}(K)$, so we need to check that for all $k \in K$, $(i,i)^*k^*u_!X = 0$ for all $i \in [n]$ and for all squares $\iota\colon \square \to \mathrm{Ar}[n]$, $\iota^*k^*u_!X \in \mathbb{D}^{\square}$ is cocartesian.

For the first point, let us just examine $(i,i)^*u_!X \in \mathbb{D}(K)$. Because $(i,i)^*$ is cocontinuous, we have $(i,i)^*u_!X \cong u_!(i,i)^*X$. We know that $(i,i)^*X = 0 \in \mathbb{D}(J)$ because $X \in \mathbf{S}_n\mathbb{D}(J)$, so because $u_!$ is pointed we have $u_!(i,i)^*X = 0 \in \mathbb{D}(K)$. Therefore $k^*(i,i)^*u_!X = 0 \in \mathbb{D}(e)$, and these first two morphisms commute because they are pullback morphisms in unrelated diagrammatic directions, giving us $(i,i)^*k^*u_!X = 0$ for any $i \in [n]$ and $k \in K$.

For the second point, we have

$$\iota^* k^* u_! X \cong k^* \iota^* u_! X \cong k^* u_! \iota^* X$$

for reasons identical to the above. We know that $\iota^* X$ is a cocartesian square in $\mathbb{D}(\square \times J)$, and $u_!$ preserves cocartesian squares. This implies that each $k^* u_! \iota^* X$ is cocartesian in $\mathbb{D}(\square)$, and following the chain of isomorphisms backwards finishes the proof.

There is no difference if we replace $u_!$ by u_* along a cosieve $u\colon J \to K$, as the corresponding morphism $u_*\colon \mathbb{D}^{\mathrm{Ar}[n]}(J) \to \mathbb{D}^{\mathrm{Ar}[n]}(K)$ is still cocontinuous by Proposition 4.2.8. That was the only fact we used about $u_!$, and so we complete the proof.

Because $\mathbf{S}_n\mathbb{D}$ has the structure of a left pointed derivator, it means that $\mathbf{S}_\bullet\mathbb{D}$ is actually a simplicial object in left pointed derivators. This means we can iterate the $\mathbf{S}_\bullet$ construction, and will do so shortly. But before that, we will define our relative K-theory construction. To do so, we need the following general simplicial constructions.

For any simplicial set Y, we may define a new simplicial set PX (of paths in Y) by precomposing Y by the functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ which sends $[n]$ to $[n+1]$ via $i \mapsto i+1$.

Lemma 4.5.3 (Lemma 1.5.1, [Wal85]). *PY is simplicially homotopy equivalent to the constant simplicial set on Y_0.*

There is a projection map $PY \to Y$ induced by the 0-face map. Moreover, there is a functor $Y_1 \to PY$ which is the inclusion of 0-simplices, as $(PY)_0 = Y_1$. This gives a sequence $Y_1 \to PY \to Y$ for any simplicial set Y.

Now suppose that $\Phi\colon \mathbb{D} \to \mathbb{E}$ is a strict cocontinuous morphism of left pointed derivators. We then define the simplicial category $\mathbf{S}_\bullet\Phi$ by the following 2-pullback in **CAT**, sometimes called the iso-comma, which lies between the lax pullback and the strict pullback:

$$\begin{tikzcd}
\mathbf{S}_\bullet\Phi \arrow[r] \arrow[d] & P\mathbf{S}_\bullet\mathbb{E} \arrow[d, "d_0"] \\
\mathbf{S}_\bullet\mathbb{D} \arrow[r, "\Phi"'] \arrow[ur, phantom, "\overset{\cong}{\nearrow}"] & \mathbf{S}_\bullet\mathbb{E}
\end{tikzcd}$$

Specifically, at each $[n] \in \Delta^{\mathrm{op}}$, we have a square

$$\begin{tikzcd}
\mathbf{S}_n\Phi \arrow[r] \arrow[d] & (P\mathbf{S}_\bullet\mathbb{E})_n = \mathbf{S}_{n+1}\mathbb{E} \arrow[d, "d_0"] \\
\mathbf{S}_n\mathbb{D} \arrow[r, "\Phi"'] \arrow[ur, phantom, "\overset{\cong}{\nearrow}"] & \mathbf{S}_n\mathbb{E}
\end{tikzcd}$$

in which the top-left corner is explicitly the following: an object in $\mathbf{S}_n\Phi$ is a pair $(A \in \mathbf{S}_n\mathbb{D}, B \in \mathbf{S}_{n+1}\mathbb{E})$ along with an isomorphism $f_{A,B}\colon \Phi(A) \to d_0(B) \in \mathbf{S}_n\mathbb{E}$. Note that if Φ is not a strict morphism, then there are naturality problems with the face and degeneracy maps of $\mathbf{S}_\bullet\Phi$.

However, if Φ is not a strict morphism, then by Proposition 4.3.1 there are two strict (cocontinuous) morphisms $\Pi_\Phi\colon \widetilde{\mathbb{D}} \to \mathbb{D}$ and $\widetilde{\Phi}\colon \widetilde{\mathbb{D}} \to \mathbb{E}$ such that Π_Φ is a weak equivalence and $\Phi\Pi_\Phi = \widetilde{\Phi}$. While $\mathbf{S}_\bullet\Phi$ will not be defined directly, it will have the same homotopy type as $\mathbf{S}_\bullet\widetilde{\Phi}$. Therefore it is not an issue for us to assume Φ strict for the rest of this argument.

In Waldhausen K-theory, $\mathbf{S}_\bullet F$ for $F\colon \mathcal{C} \to \mathcal{D}$ an exact morphism of Waldhausen categories is again a simplicial Waldhausen category. For us, it is not immediately clear that $\mathbf{S}_\bullet\Phi$ should be a simplicial left pointed derivator, so we prove that now.

Proposition 4.5.4. The simplicial category $\mathbf{S}_\bullet\Phi$ underlies a simplicial object in left pointed derivators (which we give the same name).

Proof. For $K \in \mathbf{Dir_f}$, the category $\mathbf{S}_n\Phi(K)$ will have objects a triple

$$A \in \mathbf{S}_n\mathbb{D}(K),\ B \in \mathbf{S}_{n+1}\mathbb{E}(K),\ f_{A,B}\colon \Phi(A) \xrightarrow{\cong} d_0(B),$$

which we will shorten to $(A, B, f_{A,B})$. For $u\colon J \to K$, we define $u^*(A, B, f_{A,B})$ to be the triple $(u^*A, u^*B, g_{A,B})$, where $g_{A,B}$ is the map filling in the commutative diagram of isomorphisms below:

$$\begin{tikzcd}
u^*\Phi(A) \arrow[r, "u^*f_{A,B}"] \arrow[d, "\gamma_u^\Phi"'] & u^*d_0(B) \arrow[d, "\gamma_u^{d_0}"] \\
\Phi(u^*A) \arrow[r, dashed, "g_{A,B}"'] & d_0(u^*B)
\end{tikzcd}$$

The vertical isomorphisms are actually equalities because Φ is assumed to be strict (and d_0 is in any case), so $g_{A,B} = u^*f_{A,B}$. We include the full picture for analogy with what follows. This proves that $\mathbf{S}_n\Phi$ has the structure of a prederivator.

A fair question at this point is why we need the flexibility of an isomorphism $f_{A,B}$ if Φ is assumed to be strict. If we required $f_{A,B}$ to be the identity, then we know that $u^*(f_{A,B})$ would also be the identity by strict 2-functoriality. The issue arises for the left and right Kan extensions, which we address now. Suppose now that $(A, B, f_{A,B}) \in \mathbf{S}_n\Phi(J)$. Then we define $u_!(A, B, f_{A,B})$ to be $(u_!A, u_!B, h_{A,B})$, where $h_{A,B}$ is the map filling in a similar commutative diagram of isomorphisms:

$$\begin{tikzcd}
u_!\Phi(A) \arrow[r, "u_!f_{A,B}"] \arrow[d, "(\gamma_u^\Phi)_!"'] & u_!d_0(B) \arrow[d, "(\gamma_u^{d_0})_!"] \\
\Phi(u_!A) \arrow[r, dashed, "h_{A,B}"'] & d_0(u_!B)
\end{tikzcd}$$

Because Φ and d_0 are both cocontinuous, Φ by assumption and d_0 by [Gro13, Proposition 2.5] because it is a pullback functor, the left mates of the structure isomorphisms γ_u are isomorphisms (by definition). However, even though $\gamma_u^{d_0}$ may be the identity, there is no reason to believe that its mates are also identities; they are only guaranteed to be isomorphisms. Therefore we have

$$h_{A,B} = (\gamma_u^{d_0})_! \circ u_!f_{A,B} \circ (\gamma_u^\Phi)_!^{-1}$$

This explains the definition of $\mathbf{S}_\bullet\Phi$ as an iso-comma (simplicial) category instead of a strict pullback.

These left Kan extensions for $\mathbf{S}_n\Phi$ are natural and (moreover) are the only ones that makes sense. The construction of left Kan extensions also generalises to right Kan extensions along sieves $u\colon J \to K$, as the right mates $(\gamma_u^\Phi)_*$ and $(\gamma_u^{d_0})_*$ will also be isomorphisms by Der4R. Finally, $\mathbf{S}_n\Phi(e)$ is pointed by $(0, 0, \cong)$, where the isomorphism is unique, which gives $\mathbf{S}_\bullet\Phi$ the structure of a simplicial left pointed derivator. □

We can now formulate the statement of relative derivator K-theory. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a strict cocontinuous morphism of left pointed derivators. First, there is an inclusion $\mathbf{S}_1\mathbb{E} \to P\mathbf{S}_\bullet\mathbb{E}$ as zero simplices, where we view $\mathbf{S}_1\mathbb{E}$ as a constant simplicial left pointed derivator. Composing this with the map $d_0\colon P\mathbf{S}_\bullet\mathbb{E} \to \mathbf{S}_\bullet\mathbb{E}$ we obtain a sequence

$$\mathbf{S}_1\mathbb{E} \to P\mathbf{S}_\bullet\mathbb{E} \to \mathbf{S}_\bullet\mathbb{E}$$

the composition of which is constant. We can lift the map $\mathbf{S}_1\mathbb{E} \to P\mathbf{S}_\bullet\mathbb{E}$ to $\mathbf{S}_1\mathbb{E} \to \mathbf{S}_\bullet\Phi$ using the pullback defining $\mathbf{S}_\bullet\Phi$ (using the constant map $\mathbf{S}_1\mathbb{E} \to \mathbf{S}_\bullet\mathbb{D} \to \mathbf{S}_\bullet\mathbb{E}$). Composing with the projection from $\mathbf{S}_\bullet\Phi$ to $\mathbf{S}_\bullet\mathbb{D}$ we obtain a sequence

$$\mathbf{S}_1\mathbb{E} \to \mathbf{S}_\bullet\Phi \to \mathbf{S}_\bullet\mathbb{D}$$

the composition of which is again constant. Iterating the $\mathbf{S}_\bullet$ construction, we have the following proposition.

Theorem 4.5.5. The sequence

$$i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \to i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi \to i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D} \tag{4.5.6}$$

is a homotopy fibration, that is, after (diagonally) geometrically realizing nerves and passing to the homotopy category of spaces $\mathcal{T}$.

Proof. We proceed as in [Wal85, Proposition 1.5.5]. We use the following 'realization lemma' from [Wal78]:

Lemma 4.5.7. Let $X_{\bullet\bullet} \to Y_{\bullet\bullet} \to Z_{\bullet\bullet}$ be a sequence of bisimplicial sets so that $X_{\bullet\bullet} \to Z_{\bullet\bullet}$ is constant. Suppose that $X_{\bullet n} \to Y_{\bullet n} \to Z_{\bullet n}$ is a homotopy fibration for every n. Suppose further that $Z_{\bullet n}$ is connected for every n. Then the original sequence is also a homotopy fibration.

We are indeed in this situation, up to a little unpacking. We have a sequence of bisimplicial categories, which we will turn into a sequence of trisimplicial sets by taking the nerve:

$$N_\bullet i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \to N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi \to N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}.$$

However, let us treat this as a bisimplicial set by considering the first two simplicial directions as one diagonal direction, i.e. if we let $\bullet$ and $\star$ denote the two different directions, we have

$$N_\bullet i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \to N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\star\Phi \to N_\bullet i\mathbf{S}_\bullet\mathbf{S}_\star\mathbb{D}$$

which is now a sequence of bisimplicial sets. The first term appears the same because $\mathbf{S}_1\mathbb{E}$ was constant in the $\star$ direction. As geometric realization may be taken variable-by-variable or diagonally (in fact, these give homeomorphic spaces), it suffices to show that this second sequence of bisimplicial sets is a homotopy fibration.

The last thing to check is that $N_\bullet i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D}$ is connected for all n. But $N_0 i\mathbf{S}_0\mathbf{S}_n\mathbb{D}$ consists of the zero objects in $\mathbf{S}_n\mathbb{D}(e)$, all of which are isomorphic, hence there is a 1-simplex in $N_1 i\mathbf{S}_0\mathbf{S}_n\mathbb{D}$ which is this isomorphism. Applying a degeneracy map in the $\mathbf{S}_0$-direction will give us a 1-simplex in the diagonal simplicial set $N_1 i\mathbf{S}_1\mathbf{S}_n\mathbb{D}$ which connects these 0-simplices, showing that this simplicial set is indeed connected.

Therefore let us fix an n and consider the sequence

$$i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \to i\mathbf{S}_\bullet\mathbf{S}_n\Phi \to i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D}$$

of simplicial left pointed derivators. We will make our argument here and pass to the nerve and the corresponding diagonal simplicial sets as we outlined above.

We will show, as Waldhausen does, that this relative K-theory sequence is homotopic to the trivial homotopy fibration. We will do so using the additivity theorem. Recall from Notation 2.8.1 that the category $\square$ is defined to be

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

which, as we have noted, has notation/orientation unfortunately inconsistent with that of $\mathrm{Ar}[n]$.

We need to define a cofibration morphism of derivators

$$\Xi_n \colon \mathbf{S}_n\Phi \to (\mathbf{S}_n\Phi)_{\mathrm{cof}}$$

such that $(0,0)^*\Xi_n$ takes values in a copy of $\mathbf{S}_1\mathbb{E}$ inside $\mathbf{S}_n\Phi$, $(1,0)^*\Xi_n = \mathrm{id}_{\mathbf{S}_n\Phi}$, and $(1,1)^*\Xi_n$ takes values in a copy of $\mathbf{S}_n\mathbb{D}$ inside $\mathbf{S}_n\Phi$.

The sketch for the construction of Ξ_n is the following: for $(A, B, f_{A,B}) \in \mathbf{S}_n\Phi$,

$$\Xi_n(A, B, f_{A,B}) = \begin{tikzcd}
(0, s_n \cdots s_1(0 \to (0,1)^*B \to 0), \cong) \arrow[r] \arrow[d] & (A, B, f_{A,B}) \arrow[d] \\
(0, 0, \cong) \arrow[r] & (A, s_0 d_0(B), f_{A,B})
\end{tikzcd}$$

where s_i are the degeneracy maps of the simplicial set $P\mathbf{S}_n\mathbb{E}$. The entry in the top left is a degenerate n-simplex in $P\mathbf{S}_n\mathbb{E}$ which comes from $(0 \to (0,1)^*B \to 0) \in P\mathbf{S}_0\mathbb{E}$. The isomorphisms between zero objects on the lefthand side are more subtle than they appear, and we will address this below.

To begin with the $\mathbf{S}_n\mathbb{D}$ component of $\mathbf{S}_n\Phi$, we define a cofibration morphism $\mathbf{S}_n\mathbb{D} \to (\mathbf{S}_n\mathbb{D})_{\mathrm{cof}}$. Consider the map $p_n \colon \mathrm{Ar}[n] \times \square \to \mathrm{Ar}[n]$ defined as follows: for $(a,b) = (1,0)$ or $(1,1)$, we let $p_n(i,j,a,b) = (i,j)$. Further, we let $p_n(i,j,0,0) = (0,0)$ and $p_n(i,j,0,1) = (0,0)$ be constant. We illustrate the functor $p_2 \colon \mathrm{Ar}[2] \times \square \to \mathrm{Ar}[2]$, using bold arrows for the $\square$ dimension of the diagram. Similar to Construction 3.5.3 above, we label the objects of the domain according to where they map in the codomain, which shows better how the pullback p_2^* behaves:

$$\begin{tikzcd}
0_1 \arrow[r] & 0_1 \arrow[r] \arrow[d] & 0_1 \arrow[d] & & 0_1 \arrow[r] & a \arrow[r] \arrow[d] & b \arrow[d] & & & & \\
& 0_1 \arrow[r] & 0_1 \arrow[d] & \Longrightarrow & & 0_2 \arrow[r] & c \arrow[d] & & & & \\
& & 0_1 & & & & 0_3 & & 0_1 \arrow[r] & a \arrow[r] \arrow[d] & b \arrow[d] \\
& \Downarrow & & & & \Downarrow & & \to & & 0_2 \arrow[r] & c \arrow[d] \\
0_1 \arrow[r] & 0_1 \arrow[r] \arrow[d] & 0_1 \arrow[d] & & 0_1 \arrow[r] & a \arrow[r] \arrow[d] & b \arrow[d] & & & & 0_3 \\
& 0_1 \arrow[r] & 0_1 \arrow[d] & \Longrightarrow & & 0_2 \arrow[r] & c \arrow[d] & & & & \\
& & 0_1 & & & & 0_3 & & & &
\end{tikzcd}$$

The horizontal arrows in the $\square$ dimension are necessarily zero maps, and the vertical arrows are identity maps, as $p_n(-,-,1,-) \colon \mathrm{Ar}[n] \times [1] \to \mathrm{Ar}[n]$ defining the righthand vertical map is just the projection $\mathrm{id}_{\mathrm{Ar}[n]} \times \pi_{[1]}$ and $p_n(-,-,0,-)$ defining the lefthand vertical map is the constant map $\mathrm{Ar}[n] \times [1] \to e$. This square is cocartesian, and establishes the construction for $\mathbf{S}_n\mathbb{D}$. Note that the definition of p_n implicitly uses that $n \geq 1$, but for the case $n = 0$, $p_0 \colon \square \to \mathrm{Ar}[0] = e$ is the constant map by necessity.

For the $P\mathbf{S}_n\mathbb{E} = \mathbf{S}_{n+1}\mathbb{E}$ component of the derivator $\mathbf{S}_n\Phi$, we define a map $q_n \colon \mathrm{Ar}[n{+}1] \times \square \to \mathrm{Ar}[n{+}1]$ computing what we want. First, we will have $q_n(i,j,1,0) = (i,j)$, just as p_n was defined. To deal with the other $(a,b) \in \square$, we start with $(a,b) = (0,0)$:

$$q_n(i,j,0,0) = \begin{cases} (0,0) & (i,j) = (0,0) \\ (0,1) & i = 0,1 \text{ and } j \geq 1 \\ (1,1) & \text{otherwise} \end{cases}$$

We want $q_n(-,-,0,1)$ to be a constant functor (as $p_n(-,-,0,1)$ was) but in this case we let $q_n(i,j,0,1) = (1,1)$. Finally, for the case $(a,b) = (1,1)$,

$$q_n(i,j,1,1) = \begin{cases} (1,1) & (i,j) = (0,0) \\ (1,j) & i = 0 \text{ and } j \geq 1 \\ (i,j) & \text{otherwise} \end{cases}$$

To illustrate q_2, we have the following picture, where we label the zeroes:

$$\begin{tikzcd}
0'_0 \arrow[r] & \alpha \arrow[r, "="] \arrow[d] & \alpha \arrow[r, "="] \arrow[d] & \alpha \arrow[d] & & 0'_0 \arrow[r] & \alpha \arrow[r] \arrow[d] & \beta \arrow[r] \arrow[d] & \gamma \arrow[d] \\
& 0'_1 \arrow[r] & 0'_1 \arrow[r] \arrow[d] & 0'_1 \arrow[d] & & & 0'_1 \arrow[r] & \delta \arrow[r] \arrow[d] & \varepsilon \arrow[d] \\
& & 0'_1 \arrow[r] & 0'_1 \arrow[d] & \Longrightarrow & & & 0'_2 \arrow[r] & \zeta \arrow[d] \\
& & & 0'_1 & & & & & 0'_3 & & 0'_0 \arrow[r] & \alpha \arrow[r] \arrow[d] & \beta \arrow[r] \arrow[d] & \gamma \arrow[d] \\
& \Downarrow & & & & & \Downarrow & & & \to & & 0'_1 \arrow[r] & \delta \arrow[r] \arrow[d] & \varepsilon \arrow[d] \\
0'_1 \arrow[r] & 0'_1 \arrow[r] \arrow[d] & 0'_1 \arrow[r] \arrow[d] & 0'_1 \arrow[d] & & 0'_1 \arrow[r] & 0'_1 \arrow[r] \arrow[d] & \delta \arrow[r] \arrow[d, "="] & \varepsilon \arrow[d, "="] & & & & 0'_2 \arrow[r] & \zeta \arrow[d] \\
& 0'_1 \arrow[r] & 0'_1 \arrow[r] \arrow[d] & 0'_1 \arrow[d] & & & 0'_1 \arrow[r] & \delta \arrow[r] \arrow[d] & \varepsilon \arrow[d] & & & & & 0'_3 \\
& & 0'_1 \arrow[r] & 0'_1 \arrow[d] & \Longrightarrow & & & 0'_2 \arrow[r] & \zeta \arrow[d] \\
& & & 0'_1 & & & & & 0'_3
\end{tikzcd}$$

To check that this defines the functor $\Xi_n \colon \mathbf{S}_n\Phi \to (\mathbf{S}_n\Phi)_{\text{cof}}$, we need to construct the isomorphisms $\Phi(p_n^*A) \to d_0(q_n^*B)$ corresponding to $\Xi_n(A, B, f_{A,B})$, and we will demonstrate how do so at each corner of $\square$. The isomorphisms at $(1,0)$ and $(1,1)$ are just $f_{A,B}$, but the morphisms at $(0,0)$ and $(0,1)$ are between specific zero objects. In the above diagrams, $f_{A,B}$ defines an isomorphism $\Phi(0_i) \to 0'_i$ for each i. By construction, $(0,0)^*p_n^*A$ and $(0,1)^*p_n^*A$

are the constant diagram on the zero object 0_1, and $d_0((0,0)^*q_n^*B)$ and $d_0((0,1)^*q_n^*B)$ are the constant diagram on the zero object $0'_1$. Therefore $(0,0)^*f_{A,B}$ is still the appropriate isomorphism of zero objects and no additional data is required. Indeed, if we were forced to compose isomorphisms of the form $\Phi(0_1) \to 0'_1 \to 0'_2$, we could not complete this construction coherently.

Thus we define

$$\Xi_n(A, B, f_{A,B}) = (p_n^*A, q_n^*B, p_n^*f_{A,B}).$$

The structure isomorphisms $\gamma_u^{\Xi_n}$ come directly from the structure isomorphisms $\gamma_u^{p_n}$ and $\gamma_u^{q_n}$.

The source morphism $(0,0)^*\Xi_n$ has image a subcategory of $\mathbf{S}_n\Phi$ equivalent to $\mathbf{S}_1\mathbb{E}$, and the quotient morphism $(1,1)^*\Xi_n$ has image a subcategory equivalent to $\mathbf{S}_n\mathbb{D}$. In particular, every object $(A, s_0d_0(B), f_{A,B})$ is isomorphic to the object $(A, s_0\Phi(A), \mathrm{id})$. The target morphism $(1,0)^*\Xi$ has image precisely $\mathbf{S}_n\Phi$. These morphisms are moreover essentially surjective.

Now, there are two projections $\pi_1 \colon \mathbf{S}_n\Phi \to \mathbf{S}_1\mathbb{E}$ and $\pi_2 \colon \mathbf{S}_n\Phi \to \mathbf{S}_n\mathbb{D}$. The first is defined by $\pi_1(A, B, f_{A,B}) = (0 \to (0,1)^*B \to 0)$ and the second defined by $\pi_2(A, B, f_{A,B}) = A$. This gives a total projection

$$\rho \colon \mathbf{S}_n\Phi \to \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D}.$$

This map has a section $\sigma \colon \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \to \mathbf{S}_n\Phi$ given by

$$\sigma((0 \to b \to 0), A) = (A, s_n \cdots s_1(0 \to b \to 0) \sqcup s_0\Phi(A), \mathrm{id}).$$

For example, for $A = (a_1 \to a_2 \to a_3) \in \mathbf{S}_2\mathbb{D}$, the component of $\sigma(b, A)$ in $P\mathbf{S}_2\mathbb{E}$ is

$$\begin{tikzcd}
0 \arrow[r] & b \arrow[r] \arrow[d] & b \sqcup \Phi(a_1) \arrow[r] \arrow[d] & b \sqcup \Phi(a_2) \arrow[d] \\
 & 0 \arrow[r] & \Phi(a_1) \arrow[r] \arrow[d] & \Phi(a_2) \arrow[d] \\
 & & 0 \arrow[r] & \Phi(a_3) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

The actual construction of σ relies on the additive structure of $\mathbf{S}_n\Phi$. We first define $\mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \to \mathbf{S}_n\Phi \times \mathbf{S}_n\Phi$ by the two morphisms

$$\mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \longrightarrow \mathbf{S}_1\mathbb{E} \xrightarrow{S'} \mathbf{S}_n\Phi \quad \text{and} \quad \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \longrightarrow \mathbf{S}_n\mathbb{D} \xrightarrow{Q'} \mathbf{S}_n\Phi$$

where the first map in each case is the projection. The morphism S' is defined by

$$S'(0 \to b \to 0) = (0, s_n \cdots s_1(0 \to b \to 0), \cong)$$

and the morphism Q' is defined by

$$Q'(A) = (A, s_0\Phi(A), \mathrm{id}).$$

We then compose $(S', Q')\colon \mathbf{S}_1\mathbb{E} \times \mathbf{S}_n\mathbb{D} \to \mathbf{S}_n\Phi \times \mathbf{S}_n\Phi$ with the coproduct map $\mathbf{S}_n\Phi \times \mathbf{S}_n\Phi \cong \mathbf{S}_n\Phi^{e \sqcup e} \to \mathbf{S}_n\Phi$. The definition of this map and how to strictify it is contained in Equation 4.4.9 above.

By construction, $\rho\sigma \simeq \mathrm{id}_{\mathbf{S}_1\mathbb{E}\times\mathbf{S}_n\mathbb{D}}$. We now to show that the composition $\sigma\rho$ is homotopic to the identity.

By applying the additivity theorem to Ξ_n, the identity on $\mathbf{S}_n\Phi$ is homotopic to the sum of the inclusion of $\mathbf{S}_1\mathbb{E}$ and $\mathbf{S}_n\mathbb{D}$, and this a morphism isomorphic to $\sigma\rho$. Therefore map of sequences

$$\begin{tikzcd}
i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \arrow[r] \arrow[d, "="] & i\mathbf{S}_\bullet\mathbf{S}_n\Phi \arrow[r] \arrow[d, "\sim"] & i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D} \arrow[d, "="] \\
i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \arrow[r] & i\mathbf{S}_\bullet\mathbf{S}_1\mathbb{E} \times i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D} \arrow[r] & i\mathbf{S}_\bullet\mathbf{S}_n\mathbb{D}
\end{tikzcd}$$

has all vertical maps equivalences. The bottom sequence is a trivial homotopy fibration, so the top sequence is also a homotopy fibration, completing the proof. ☐

Definition 4.5.8. For $\Phi\colon \mathbb{D} \to \mathbb{E}$ a strict cocontinuous morphism of left pointed derivators, define

$$K(\Phi) := \Omega^2|i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi|.$$

There is one more corollary of relative K-theory that bears mentioning before using this definition.

Corollary 4.5.9. The topological space $K(\mathbb{D})$ is an infinite loop space.

Proof. If we take the case $\Phi = \mathrm{id}_\mathbb{D}\colon \mathbb{D} \to \mathbb{D}$, we can identify $\mathbf{S}_\bullet\Phi$ with $P\mathbf{S}_\bullet\mathbb{D}$. There is certainly an equivalence of simplicial categories $\mathbf{S}_\bullet\mathrm{id}_\mathbb{D}(K) \to P\mathbf{S}_\bullet\mathbb{D}(K)$ for each $K \in$ **Dia** as the pullback of the equivalence $\mathrm{id}_{\mathbb{D}}\colon \mathbf{S}_{\bullet}\mathbb{D}(K) \to \mathbf{S}_{\bullet}\mathbb{D}(K)$ is still an equivalence. But because the morphism $\mathbf{S}_{\bullet}\mathrm{id}_{\mathbb{D}} \to P\mathbf{S}_{\bullet}\mathbb{D}$ is defined globally and is levelwise an equivalence of categories, we get an equivalence of simplicial left pointed derivators.

Using this replacement, we have a fibration

$$i\mathbf{S}_{\bullet}\mathbf{S}_{1}\mathbb{D} \to P(i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D}) \to i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D},$$

where P modifies the first simplicial direction. But now the middle term is contractible, giving a homotopy equivalence

$$|i\mathbf{S}_{\bullet}\mathbb{D}| \cong |i\mathbf{S}_{\bullet}\mathbf{S}_{1}\mathbb{D}| \xrightarrow{\sim} \Omega|i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D}|$$

We use here that the bisimplicial category $i\mathbf{S}_{\bullet}\mathbf{S}_{1}\mathbb{D}$ is homotopy equivalent to the simplicial category $i\mathbf{S}_{\bullet}\mathbb{D}$. The equivalence is given (using morphisms of derivators and passing to maps up to homotopy in the geometric realization) by the forgetful functor $\mathbf{S}_{1}\mathbb{D} \to \mathbb{D}$ on the one hand and an iterated extension by zero morphism for the converse.

Specifically, consider $t\colon e \to [1]$ the cosieve given by the inclusion of the target and $s\colon [1] \to \mathrm{Ar}([1])$ the sieve given by the inclusion into $(0,0) \to (0,1)$. Then the map $s_{*}t_{!}\colon \mathbb{D} \to \mathbb{D}(\mathrm{Ar}[1])$ is, for $a \in \mathbb{D}$,

$$\begin{tikzcd} & & 0 \arrow[r] & a \arrow[d] \\ a \arrow[r, mapsto] & 0 \arrow[r] \;a \arrow[r, mapsto] & & 0 \end{tikzcd}$$

This justifies the isomorphism above.

We now replace $\mathbb{D}$ by the simplicial left pointed derivator $\mathbf{S}_{\bullet}\mathbb{D}$ and repeat the process, obtaining

$$|i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D}| \cong |i\mathbf{S}_{\bullet}\mathbf{S}_{1}\mathbf{S}_{\bullet}\mathbb{D}| \xrightarrow{\sim} \Omega|i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D}|$$

which implies that $|i\mathbf{S}_{\bullet}\mathbb{D}|$ is equivalent to $\Omega^{2}|i\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbf{S}_{\bullet}\mathbb{D}|$.

By induction, we then conclude

$$K(\mathbb{D}) = \Omega|i\mathbf{S}_{\bullet}\mathbb{D}| \xrightarrow{\sim} \Omega^{(n)}|i\mathbf{S}_{\bullet}^{(n)}\mathbb{D}|$$

and so $K(\mathbb{D})$ is an infinite loop space. In particular, we can view $K(\mathbb{D})$ as a connective Ω-spectrum. □

Corollary 4.5.10. Let $\Phi\colon \mathbb{D} \to \mathbb{E}$ be a strict cocontinuous morphism of left pointed derivators. Then there is a homotopy fibration

$$K(\Phi) \to K(\mathbb{D}) \to K(\mathbb{E}).$$

Proof. If we rotate the fibration sequence of Equation 4.5.6 to the left twice (and replace $\mathbf{S}_1\mathbb{E}$ by $\mathbb{E}$), we have a fibration sequence

$$\Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\Phi| \to \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}| \to |i\mathbf{S}_\bullet\mathbb{E}|.$$

By the above corollary, $|i\mathbf{S}_\bullet\mathbb{D}| \to \Omega|i\mathbf{S}_\bullet\mathbf{S}_\bullet\mathbb{D}|$ is a homotopy equivalence. Replacing the middle term and applying Ω everywhere, the corollary follows. $\square$

As one last remark, the next logical step is to use Theorem 4.5.5 to prove a localization theorem in K-theory and answer (positive or negatively) Maltsiniotis' conjecture that Verdier quotients of triangulated derivators get sent to long exact sequences in K-theory. Unfortunately, the necessity of coherent diagrams in derivator K-theory obstructs [Wal85, Theorem 1.6.4] from proceeding verbatim. In particular, that technique being directly translatable would mean that Waldhausen K-theory agrees with derivator K-theory in general, which has been proven false in [TV04] and [MR11]. Nonetheless, future work will address the issue of localization in a novel way which should avoid this obstruction.

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