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Regular multi-types and the Bloom conjecture

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ABSTRACT

We prove that the commutator type, the regular contact type and the Levi form type of order $s = (n-2)$ are the same for a smooth pseudoconvex real hypersurface in $\mathbb{C}^n$ with $n \geq 3$. In particular, this provides, in the case of complex dimension three, a complete solution of a long standing conjecture of Bloom formulated in his famous and important 1981 paper [12]. When $n \geq 4$, our theorem provides the first result along the lines of the Bloom conjecture in any dimensions in a case where the pseudoconvexity assumption of the hypersurface starts to be crucial.

RÉSUMÉ

Pour toute hypersurface rèlle lisse et pseudoconvexe de $\mathbb{C}^3$, nous considérons trois notions de type : le type obtenu en considérant les crochets de Lie de champs de vecteurs CR, le type obtenu à partir de l'ordre de contact avec les courbes holomorphes régulières et le type associé à forme de Levi. Notre résultat principal établit l'égalité entre ces trois types, apportant ainsi une solution complète à une ancienne conjecture de Bloom en dimension trois. En dimension n supérieure, nous vérifions aussi la conjecture de Bloom pour $s = n - 2$, obtenant ainsi la première solution à la conjecture de Bloom pour laquelle l'hypothése de pseudoconvexité est nécessaire.

1. Introduction

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$. Many analytic and geometric properties of D are determined by its boundary holomorphic invariants. To generalize his subelliptic estimate for the $\overline{\partial}$-Neumann problem from bounded strongly pseudoconvex domains [24] to bounded weakly pseudocon-

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vex domains in $\mathbb{C}^2$, Kohn in a fundamental paper [34] investigated three different boundary invariants for $D \subset \mathbb{C}^2$. These invariants describe, respectively, the maximum order of contact with smooth holomorphic curves at a boundary point, degeneracy of the Levi-form along the CR directions and the length of the iterated Lie brackets of boundary CR vector fields as well as their conjugates needed to recover the boundary contact direction. Kohn proved that all these invariants are in fact the same, called the type value of a point on $\partial D \subset \mathbb{C}^2$. When this type value is finite at each point, Kohn's work in [34] together with that of Greiner [30] (see also Rothschild-Stein [41]) gives the precise information of how much subelliptic gain one obtains for the $\overline{\partial}$-Neumann problem for a smoothly bounded weakly pseudoconvex domain in $\mathbb{C}^2$. For decades, the finite type condition initiated by the work of Kohn has been playing fundamental roles in many problems in Several Complex Variables, CR Geometry and Analysis as well as the theory of Subelliptic Partial Differential Equations. For instance, Bedford-Fornaess [7], Fornaess-Sibony [23] studied peak functions on weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ and discovered close connections between the type value of the boundary and the Hölder-continuity of the peak functions up to the boundary.

Generalizations of Kohn's notion of the boundary finite type condition to higher dimensions have been a subject under extensive investigations in the past 40 years in Several Complex Variables. Kohn later introduced a finite type condition in higher dimensions through the subelliptic multiplier ideals [35]. The understanding of this type has later revived to be a very active field of studies through the work of many people including Diederich-Fornaess [20], Siu [43], Kim-Zaistev [32][33], Zaistev [45], as well as the references therein. Bloom [11] and Bloom-Graham [9] established Kohn's original notion of types in $\mathbb{C}^2$ to any dimensions. Namely, for each integer $s \in [1, n-1]$ and for a smooth real hypersurface $M \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in M$, Bloom-Graham and Bloom defined the vector field commutator type $t^{(s)}(M,p)$, the Levi-form type $c^{(s)}(M,p)$ and the regular contact type $a^{(s)}(M,p)$ of M at p, which are called the regular multi-types of Kohn [34], Bloom-Graham [9] and Bloom [12]. Bloom-Graham [9][10] showed that when $s = n-1$, all these types are also the same as in the case of $n = 2$ by Kohn. However, without pseudoconvexity for M, Bloom [12] showed that when $s \neq n-1$, while the contact type $a^{(s)}$ may be finite, the commutator type $t^{(s)}$ and the Levi-form type $c^{(s)}$ can be infinite in many examples. The commutator type is intrinsically defined only through the Lie bracket of CR or conjugate CR vector fields of M valued in some smooth subbundle of $T^{(1,0)}M \oplus T^{(0,1)}M$. It is an important object in the fields such as sub-elliptic analysis and PDEs. In Adwan-Berhanu [1], the commutator type was crucially used to obtain analytic hyper-ellipticity of solutions of non-linear PDEs. An excellent description on this matter can be found in the work of Derridj [19] and the book of Berhanu-Cordaro-Hounie [5]. The other two types are more on the emphasis of complex analysis, defined through the complex structure of the ambient space. D'Angelo [17] introduced his famous notion of finite type condition by considering the order of contact with not just smooth complex manifolds but possibly singular complex analytic varieties, which is a singular contact type condition and turns out to be equivalent to the existence of the subelliptic estimate by the work of Kohn [35], Diederich-Fornaess [20] and Catlin [15]. Catlin in [14] studied his version of multi-types as well as its connection with the boundary stratification in terms of the degeneracy of Levi forms. Catlin's types are more along the lines of differentiation of Levi forms and thus more along the lines of Levi-form types. There was also a very useful type condition called holomorphically finite non-degeneracy condition in [4] which has late played a fundamental role in understanding various problems in CR geometry in the work of Berhanu-Xiao [6] Lamel-Mir [37], etc. Other studies involving various type conditions as well as their applications at least include the work in D'Angelo [18], Sibony [42], McNeal [39], Boas-Straube [8], Fu-Isaev-Krantz [26], Baouendi-Ebenfelt-Rothschild [3], Bove-Derridj-Kohn-Tartakoff [13], Berhanu-Xiao [6], Lamel-Mir [38], Gong-Stolotvich [28,29], Gong-Lanzani [27], etc., and many references therein.

All these type conditions mentioned above were introduced through different aspects of studies. Revealing the connections among them always resulted in a deeper understanding of the subject. For instance, proving that the Kohn multiplier ideal type is equivalent to the finite D'Angelo type would provide a new and much more direct solution of the $\overline{\partial}$-Neumann problem.

In this paper, we are interested in the three multi-regular types of Kohn-Bloom-Graham. We will be concerned with the question when all these types are equivalent, known as the Bloom conjecture formulated in Bloom's famous 1981 paper [12]. We will show that the $(n-2)$-commutator type $t^{(s)}(M,p)$, also called $(n-2)$-Hörmander type, coincides with the $(n-2)-$ Levi-form type $c^{(n-2)}(M,p)$ and the regular $(n-2)-$contact type $a^{(n-2)}(M,p)$ for any pseudo-convex hypersurface M in $\mathbb{C}^n$ with $n \geq 3$ and with $p \in M$. When $s = n-1$, these three regular types were proved to be the same by Bloom-Graham [9][10] more than 40 years ago, and in the Bloom-Graham case, the pseudo-convexity for M is not needed. Hence, our main theorem provides the first equivalence result of these three types in any dimensions in the case where the pseudoconvexity starts to play a fundamental role after the work of Bloom-Graham [9][10] more than 40 years ago. In the $\mathbb{C}^3$ case, Bloom obtained in 1981 the equality of the Levi-form type and the regular contact type. However, Bloom left the important open question when the commutator type is also the same as the regular contact type. As an immediate consequence of our main theorem, in the case of complex dimension three, our result finally provides a complete solution of the famous Bloom conjecture posed in 1981 [12].

Our focus in this paper will be on the understanding of commutator types. The other types will be reduced immediately to the study of commutator types.

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2. Statement of the main theorem

Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface with $p \in M$. Then $\dim_{\mathbb{C}} T_p^{1,0}M = n-1$ for $p \in M$. For any $1 \leq s \leq n-1$, we have the following three sets of important local holomorphic invariants ([12]), used to describe the holomorphic non-degeneracy of M at p.

(i): The s-contact type $a^{(s)}(M,p)$:

$$a^{(s)}(M,p) = \sup_X \big\{r \mid \exists \text{ an } s\text{-dimensional complex submanifold } X \\ \text{whose order of contact with } M \text{ at } p \text{ is } r\big\}. \tag{2.1}$$

Let ρ be a defining function of M near p, namely, $\rho \in C^\infty(U)$ with U an open neighborhood of $p \in \mathbb{C}^n$ and $U \cap M = \{\rho = 0\} \cap U$, $d\rho|_{U \cap M} \neq 0$. Remark that the order of contact of X with M at p is defined as the order of vanishing of $\rho|_X$ at p.

(ii) The s-vector field commutator type $t^{(s)}(M,p)$:

Let B be an s-dimensional subbundle of $T^{1,0}M$. We let $\mathcal{M}_1(B)$ be the $C^\infty(M)$-module spanned by the smooth tangential $(1,0)$ vector fields L with $L|_q \in B|_q$ for each $q \in M$, together with the conjugate of these vector fields.

For $\mu \geq 1$, we let $\mathcal{M}_\mu(B)$ denote the $C^\infty(M)$-module spanned by commutators of length less than or equal to μ of vector fields from $\mathcal{M}_1(B)$. A commutator of length μ of vector fields in $\mathcal{M}_1(B)$ is a vector field of the following form: $[Y_\mu, [Y_{\mu-1}, \cdots, [Y_2, Y_1] \cdots]$. Here $Y_j \in \mathcal{M}_1(B)$. Define $t^{(s)}(B,p) = m$ if $\langle F, \partial\rho\rangle(p) = 0$ for any $F \in \mathcal{M}_{m-1}(B)$ but $\langle G, \partial\rho\rangle(p) \neq 0$ for a certain $G \in \mathcal{M}_m(B)$. Then

$$t^{(s)}(M,p) = \sup_B \{t(B,p) \mid B \text{ is an } s\text{-dimensional subbundle of } T^{1,0}M\}. \tag{2.2}$$

$t^{(s)}(B,p)$ is the smallest length of the commutators by vector fields in $\mathcal{M}_1(B)$ to recover the complex contact direction in $\mathbb{C}T_pM$. $t^{(s)}(M,p)$ is the largest possible value among all $t^{(s)}(B,p)'s$. Namely, $t^{(s)}(M,p)$ describes the degeneracy of the most degenerate s-subbundles of $T^{1,0}M$ in terms of the commutators of its smooth sections. Notice that it is intrinsically defined, independent of the ambient embedded space.

(iii) The s-Levi form type $c^{(s)}(M,p)$:

Let B be as in (ii). Let $\mathcal{L}_M$ be a Levi form associated with a defining function ρ near p of M. For $V_B=\{L_1,\cdots,L_s\}$, a basis of smooth sections of B near p, we define the trace of $\mathcal{L}_M$ along V_B by

$$\mathrm{tr}_{V_B}\mathcal{L}_M(q)=\sum_{j=1}^{s}\langle[L_j,\overline{L_j}],\partial\rho\rangle(q),\quad q\approx p. \tag{2.3}$$

We define $c(B,p)=m$ if for any $m-3$ vector fields $F_1,\cdots,F_{m-3}$ of $\mathcal{M}_1(B)$ and any basis V_B, it holds that

$$F_{m-3}\cdots F_1\big(\mathrm{tr}_{V_B}\mathcal{L}_M\big)(p)=0;$$

and for a certain choice of $m-2$ vector fields $G_1,\cdots,G_{m-2}$ of $\mathcal{M}_1(B)$ and a certain basis V_B, we have

$$G_{m-2}\cdots G_1\big(\mathrm{tr}_{V_B}\mathcal{L}_M\big)(p)\neq 0.$$

Then

$$c^{(s)}(M,p)=\sup_B\{c(B,p):\ B\ \text{ is an } s\text{-dimensional subbundle of } T^{1,0}M\}. \tag{2.4}$$

In his fundamental paper [34], when $n=2$, Kohn showed that $t^{(1)}(M,p)=c^{(1)}(M,p)=a^{(1)}(M,p)$. Bloom-Graham [10] and Bloom [11] proved that

$$t^{(n-1)}(M,p)=c^{(n-1)}(M,p)=a^{(n-1)}(M,p)\ \text{ for } M\subset\mathbb{C}^n.$$

And for any $1\le s\le n-2$, Bloom in [12] observed that $a^{(s)}(M,p)\le c^{(s)}(M,p)$ and $a^{(s)}(M,p)\le t^{(s)}(M,p)$. For these results to hold there is no need to assume the pseudoconvexity of M. However, the following example of Bloom shows that for $n\ge 3$, when M is not pseudoconvex, it may happen that $a^{(s)}(M,p)<c^{(s)}(M,p)$ and $a^{(s)}(M,p)<t^{(s)}(M,p)$ for $1\le s\le n-2$.

Example 2.1 *(Bloom, [12]).* Let $\rho=2\mathrm{Re}(w)+(z_2+\overline{z_2}+|z_1|^2)^2$ and let $M=\{(z_1,z_2,w)\in\mathbb{C}^3|\ \rho=0\}$. Let $p=0$. Then $a^{(1)}(M,p)=4$ but $c^{(1)}(M,p)=t^{(1)}(M,p)=\infty$.

With the pseudoconvexity assumption of M, Bloom in [12] showed that when $M\subset\mathbb{C}^3$, $a^{(1)}(M,p)=c^{(1)}(M,p)$. Motivated by this result, Bloom in 1981 [12] formulated the following famous conjecture:

Conjecture 2.2. *Let $M\subset\mathbb{C}^n$ be a pseudoconvex real hypersurface with $n\ge 3$. Then for any $1\le s\le n-2$ and $p\in M$,*

$$t^{(s)}(M,p)=c^{(s)}(M,p)=a^{(s)}(M,p).$$

The goal of the present paper is to prove the following theorem:

Theorem 2.3. *Let $M \subset \mathbb{C}^n$ be a smooth pseudoconvex real hypersurface with $n \geq 3$. Then for $s = n-2$ and any $p \in M$, it holds that*

$$t^{(n-2)}(M,p) = a^{(n-2)}(M,p) = c^{(n-2)}(M,p).$$

In particular, we answer affirmatively the Bloom conjecture in the case of complex dimension three (namely, $n = 3$):

Theorem 2.4. *The Bloom conjecture holds in the case of complex dimension three. Namely, for a smooth pseudoconvex real hypersurface $M \subset \mathbb{C}^3$ and $p \in M$, it holds that*

$$t^{(1)}(M,p) = a^{(1)}(M,p) = c^{(1)}(M,p).$$

Our proof of Theorem 2.3 is a combination of analytic and geometric arguments along the lines of singular foliation theory and CR geometry. Our arguments are quite different from what has appeared in the literature in many aspects. Our paper focuses on understanding the commutator of vector fields evaluated in a certain subbundle, for the Levi form type can be easily reduced to the study of the commutator type case. Notice that Kohn's multiple ideal sheaf type and Catlin's type are more about differentiations of the Levi form in a certain way and thus are more relate to the Levi form type here. Commutators of vector fields are not just important in complex analysis but also play a fundamental role in many problems bordering complex analysis and sub-elliptic analysis. In the paper of Adwan-Berhanu [1], the commutator type condition of vector fields is crucially applied to get various real analytic hypo-ellipticity results. See also the book of Berhanu-Cordaro-Hounie [5] and a paper of Derridj [19] for many references and historical discussions on this matter. In §3, we give a general set-up and provide a normalization of the related vector fields. In §4, we give a proof of Theorem 2.3 assuming Theorem 6.1. §5 and §6 are dedicated to the long and very much involved proof of Theorem 6.1 which is about a weak version of the uniqueness property of a complex linear PDE associated with a CR singular submanifold contained in a pseudoconvex hypersurface [22][28][29].

Already from the work of Chern-Moser [16], it is clear that a good weight system is always important to single out the boundary holomorphic invariants for real hypersurfaces in a complex Euclidean space. In this regard, we mention at least the works in [12,10,14,28,29,40,31,36,37] and many references therein concerning different weight systems used in different settings. In this work, for a smooth subbundle B of $T^{(1,0)}M$ of complex dimension $s \leq n-1$, the CR directions along the subbundle are assigned weight one and the missing CR direction is then assigned to have the weight equal to the first Hörmander number [2] of $B \oplus \overline{B}$. Once the weights of the CR directions are determined, the weight of the complex normal direction is determined by the order of the degree of the weighted lowest order of the defining function of the hypersurface. These weights can be used to apply the singular Frobenius-Nagano theorem to the truncated manifold if the theorem fails. Then we are led to two very different scenarios: the CR setting [2] and the CR singular setting [31,22,28,29]. To attack the Bloom conjecture, it is crucial to find a good use of the pseudo-convexity. Our fundamental new ideas for applying pseudoconvexity are to deduce the problem to the setting where the classical Hopf lemma (Proposition 4.6) can be applied in the first scenario; and to deduce the problem to a weak version of the uniqueness theorem for solutions of a certain geometrically oriented complex linear equation with real part plurisubharmonic (Theorem 6.1) in the second scenario. The other new ingredients in this work include the crucial use of the Euler vector field, which does not seem to have appeared before in the study of the finite type conditions.

Before proceeding to the proof of our main theorem, we mention the work of D'Angelo [18] and Fassina [21], where results have been obtained related to the following question: Let $M \subset \mathbb{C}^n$ $(n \geq 3)$ and for $p \in M$, when does it hold that $t^{(1)}(B_1,p) \geq c^{(1)}(B_1,p)$? Here B_1 is a one dimensional complex smooth subbundle of $T^{(1,0)}M$.

3. Normalization of CR vector fields

In this section, we present a normalization for a basis of cross sections of a complex subbundle B of $T^{(1,0)}M$ of CR codimension one. This will lead us to define the right weight system needed for the purpose of applying the Nagano theorem. The basic idea behind the complicated normalization procedure in this section is to apply holomorphic changes of coordinates to normalize as much as possible the lowest order holomorphic terms in the coefficients of the vector fields with respect to a standard CR frame of M.

Denote by $(z_1, \cdots, z_{n-1}, w) = (z, w)$ the coordinates in $\mathbb{C}^n$. Let $M \subset U$ be a smooth real hypersurface in $\mathbb{C}^n$ with $p \in M$ and let ρ be a defining function of M near p. After a holomorphic change of coordinates, we may assume that $p = 0$ and ρ takes the following form:

$$\rho(z, w, \overline{z}, \overline{w}) = -2\mathrm{Re}(w) + \chi(z, w, \overline{z}, \overline{w}),\ \ \chi(z, w, \overline{z}, \overline{w}) = O(|z^2| + |zw|). \tag{3.1}$$

In what follows, when there is no risk of causing confusion, we use 0 to denote the number 0 or the origin of $\mathbb{C}^n$. We will assume that $a^{(n-2)}(M, 0) < \infty$ in all that follows, for otherwise

$$t^{(n-2)}(M, 0), c^{(n-2)}(M, 0) \geq a^{(n-2)}(M, 0) = \infty$$

and thus all these invariants coincide. After a holomorphic change of coordinates of the form $(z', w') = (z, w + O(2))$, we assume that

$$\chi(z, 0, \cdots, 0) = O(a^{(n-2)}(M, 0) + 1) \tag{3.2}$$

in the sense that the partial derivatives of χ up to order $a^{(n-2)}(M, 0)$ along z-directions vanish at 0. Shrinking U if necessary, we assume $\frac{\partial \rho}{\partial w} \neq 0$ for $(z, w) \in U$. For a defining function ρ defined over U as in (3.1), write

$$L_i = \frac{\partial}{\partial z_i} - \frac{\partial \rho}{\partial z_i}\Big(\frac{\partial \rho}{\partial w}\Big)^{-1}\frac{\partial}{\partial w} \quad \text{for } i = 1, \cdots, n-1. \tag{3.3}$$

Then $\{L_i\}_{i=1}^{n-1}$ forms a basis for the space of CR vector fields along M. Let B be an $(n-2)$ dimensional subbundle of $T^{1,0}M$. Assume that the sections of B are generated by a certain linearly independent smooth CR vector fields $S_1, \cdots, S_{n-2}$ along M near 0. After a linear holomorphic change of coordinates, we assume that $S_j(0) = L_j(0) = \frac{\partial}{\partial z_j}|_0$ for $1 \leq j \leq n-2$. Write

$$S_j = \sum_{h=1}^{n-1} a_{jh} L_h \text{ with } a_{jh}(0, \cdots, 0) = \delta_{jh} \text{ for } 1 \leq j, h \leq n-2. \tag{3.4}$$

We start with the following simple transformation law for $\{L'_j, \rho'\}$ and $\{L_j, \rho\}$ under a holomorphic change of coordinates $(z', w') = F(z, w)$ with $\rho = \rho' \circ F$, $F(0) = 0$.

Lemma 3.1. *Let $(z', w') = F(z, w) = (z'_1, \cdots, z'_{n-1}, w')$ be a new holomorphic coordinate system where $z'_j = z'_j(z_1, \cdots, z_{n-1})$ for $j = 1, \cdots, n-1$, $w' = w$ with $z'(0) = 0$. Then we have*

$$F_*(L_i) = \sum_{j=1}^{n-1} \frac{\partial z'_j}{\partial z_i} L'_j \quad \textit{for } i = 1, \cdots, n-1. \tag{3.5}$$

With S_j and the frame $\{L_j\}$ being given as above, we define

$$\ell_0^* =: \min\{k_j : k_j = \text{vanishing order of } a_{j(n-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}. \cdots, \overline{z_{n-2}}, 0, 0) \text{ at } 0\}. \tag{3.6}$$

Here $a_{j(n-1)}(z, w, \overline{z}, \overline{w})$ for $j = 1, \cdots, n-2$ are as in (3.4). In this section, for a smooth function A, we write $A^{(\tau)}(z, \overline{z})$ for the sum of monomials of (ordinary) degree τ in its Taylor expansion at 0; also when we mention a holomorphic change of coordinates, we refer to a special type of holomorphic maps of the form $(z', w') = F(z, w)$ as in Lemma 3.1.

Lemma 3.2. *Suppose $\ell_0^* \neq \infty$. After a holomorphic change of coordinates, we have*

$$a_{j(n-1)}^{(\ell_0^*)}(0, \cdots, 0, z_j, \cdots, z_{n-2}, 0, \cdots, 0) = 0 \textit{ for all } \ 1 \leq j \leq n-2.$$

Proof. Let

$$z_j' = z_j \text{ for } 1 \leq j \leq n-2,\ z_{n-1}' = z_{n-1} - \int_0^{z_1} a_{1(n-1)}^{(\ell_0^*)}(\xi, z_2, \cdots, z_{n-2}, 0, \cdots, 0) d\xi,\ w = w'.$$

Then in the new coordinates (z', w'), we have

$$\begin{aligned}
\frac{\partial}{\partial z_1} &= \frac{\partial}{\partial z_1'} - a_{1(n-1)}^{(\ell_0^*)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) \frac{\partial}{\partial z_{n-1}'}, \\
\frac{\partial}{\partial z_j} &= \frac{\partial}{\partial z_j'} + O(\ell_0^*) \frac{\partial}{\partial z_{n-1}'} \text{ for } 2 \leq j \leq n-2, \\
\frac{\partial}{\partial z_{n-1}} &= \frac{\partial}{\partial z_{n-1}'}.
\end{aligned} \tag{3.7}$$

In the new coordinates, by Lemma 3.1, we have

$$\begin{aligned}
S_1 = \sum_{h=1}^{n-1} a_{1h} L_h &= a_{11}(L_1' - a_{1(n-1)}^{(\ell_0^*)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) L_{n-1}') \\
&+ \sum_{h=2}^{n-2} a_{1h}(L_h' + O(\ell_0^*) L_{n-1}') + a_{1(n-1)} L_{n-1}'.
\end{aligned} \tag{3.8}$$

Hence in the new coordinates, the coefficient $a_{1(n-1)}$ is changed to

$$-a_{11} a_{1(n-1)}^{(\ell_0^*)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) + \sum_{h=2}^{n-2} a_{1h} \cdot O(\ell_0^*) + a_{1(n-1)}. \tag{3.9}$$

Recall that $a_{1j} = \delta_{1j} + O(1)$ for $1 \leq j \leq n-2$. Hence in these new coordinates, which are still denoted by (z, w), we have $a_{1(n-1)}^{(\ell_0^*)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) = 0$. We remark that with such a change of holomorphic coordinates, the non-holomorphic terms remain the same for $a_{1(n-1)}^{(\ell_0^*)}$.

Suppose that we have achieved $a_{h(n-1)}^{(\ell_0^*)}(0, \cdots, 0, z_h, \cdots, z_{n-2}, 0, \cdots, 0) = 0$ for $1 \leq h \leq j-1$. We next show that we can make $a_{j(n-1)}^{(\ell_0^*)}(0, \cdots, 0, z_j, \cdots, z_{n-2}, 0, \cdots, 0) = 0$ after a holomorphic change of coordinates. Set $w = w'$ and

$$z_j' = z_j, 1 \le j \le n-2,\ z_{n-1}' = z_{n-1} - \int_0^{z_j} a_{j(n-1)}^{(\ell_0^*)}(0,\cdots,0,\xi,z_{j+1},\cdots,z_{n-2},0,\cdots,0)d\xi.$$

By a similar argument as in the proof for $a_{1(n-1)}^{(\ell_0^*)}(z_1,\cdots,z_{n-2},0,\cdots,0)=0$, we have in the new coordinates that

$$a_{j(n-1)}^{(\ell_0^*)}(0,\cdots,0,z_j,\cdots,z_{n-2},0,\cdots,0)=0.$$

Notice that this transformation of coordinates preserves the property:

$$a_{h(n-1)}^{(\ell_0^*)}(0,\cdots,0,z_h,\cdots,z_{n-2},0,\cdots,0)=0 \text{ for } 1\le h\le j-1.$$

By induction, this completes the proof of Lemma 3.2. □

Next, when $\ell_0^*=\infty$, we set $\ell_0 = a^{(n-2)}(M,0)$. Otherwise, we define

$$\begin{aligned} \ell_0' &:= \min_{1\le j\le n-2}\{k_j :\ k_j = \mathrm{ord}_{z=0}\ a_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)\}, \\ \ell_0 &= \min\{\ell_0', a^{(n-2)}(M,0)\}, \end{aligned} \tag{3.10}$$

where $a_{j(n-1)}$'s are normalized as in Lemma 3.2.

Proposition 3.3. *Assume that $\ell_0 \le a^{(n-2)}(M,0)-1$. After a holomorphic change of coordinates we can normalize the coefficients of $\{S_j\}$ to further satisfy one of the following two normalization properties with ℓ_0 being unchanged.*

(I) $a_{j(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ *is holomorphic in* $z_1,\cdots,z_{n-2}$ *for each* j*, and there exists* $j_0\in[2,n-2]$ *such that* $a_{j(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)=0$ *for* $1\le j\le j_0-1$, $a_{j_0(n-1)}^{(\ell_0)}(0,\cdots,0,z_{j_0},\cdots,z_{n-2},0,\cdots,0)=0$, *but* $a_{j_0(n-1)}^{(l_0)}(z_1,\cdots,z_{n-2},0,\cdots,0)\not\equiv 0$.

(II) $a_{1(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ *is not a holomorphic polynomial and* $a_{1(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,\cdots,0)=0$.

Proof. (I): First, we assume that each $a_{j(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ is holomorphic and each $a_{j(n-1)}^{(\ell_0)}$ satisfies the properties as in Lemma 3.2. Then

$$a_{1(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,\cdots,0)=0.$$

By the definition of ℓ_0, we can find the smallest $j_0\in[2,n-2]$ such that

$$a_{j(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,\cdots,0)\equiv 0$$

for all $1\le j\le j_0-1$, but $a_{j_0(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,\cdots,0)\not\equiv 0$. By Lemma 3.2, this j_0 satisfies the property in part (I) of the proposition.

(II): Next, assume that $a_{j(n-1)}^{(\ell_0)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ is not holomorphic for a certain $j\in[1,n-2]$. Switching j with the index 1 and repeating the proof in Lemma 3.2, we can make

$a^{(\ell_0)}_{1(n-1)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) = 0$ and achieve the other normalization properties as in Lemma 3.2. Notice that ℓ_0 is not changed after this normalization procedure. This completes the proof of the proposition. $\square$

Define the weight of z_j and $\overline{z_j}$ for $1 \le j \le n-2$ to be 1. The weight of z_{n-1} and $\overline{z_{n-1}}$ is defined to be ℓ_0+1 and the weight of w is defined to be m that is the lowest weighted vanishing order of ρ in the expansion of $\rho(z, 0, \overline{z}, 0)$ at 0 with respect to the weights of $\{z_1, \cdots, z_{n-2}, z_{n-1}\}$ just defined. Later, we will see the only non-trivial weight $\ell_0 + 1$ for the missing CR direction along z_{n-1} is precisely the first Hörmander number associated with the lowest part of the system $\{S_j\}$. In what follows, for a smooth function A, we write $A^{[\sigma]}(z_1, \cdots, z_{n-1}, \overline{z_1}, \cdots, \overline{z_{n-1}})$ for the weighted homogeneous part of weighted degree σ with the weight system just defined in its Taylor expansion at 0. Notice that when A does not contain z_{n-1} then $A^{[\sigma]} = A^{(\sigma)}$. Then we have the following

Proposition 3.4. *In the case of Proposition 3.3 (II), we can further apply a holomorphic transformation of coordinates and change the basis $\{S_j\}$ if needed to make the coefficients of $\{S_j\}$ in the expansion with respect to $\{L_j\}$ satisfy one of the following two normalizations with ℓ_0 being unchanged:*

(1) $a^{(\ell_0)}_{1(n-1)}(z_1, 0, \cdots, 0, \overline{z_1}, 0, \cdots, 0) \not\equiv 0$, $a^{(\ell_0)}_{1(n-1)}(z_1, 0, \cdots, 0) = 0$
$(a^{(\ell_0)}_{1(n-1)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) = 0$, *in fact),*

$$\rho^{[m]}(z_1, 0, \cdots, 0, z_{n-1}, 0, \overline{z_1}, 0, \cdots, 0, \overline{z_{n-1}}, 0)$$

is not identically zero (and contains no non-trivial holomorphic terms).

(2) *For a certain $j \in [1, n-2]$, $a^{(\ell_0)}_{j(n-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0, 0)$ is not holomorphic,*
$\sum_{k=1}^{n-2} z_k a^{(\ell_0)}_{k(n-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0, 0) = 0,$
$\rho^{[m]}(z_1, \cdots, z_{n-1}, 0, \overline{z_1}, \cdots, \overline{z_{n-1}}, 0)$ *is not identically zero (and contains no non-trivial holomorphic terms).*

Proof. Consider the following change of coordinates:

$$z'_1 = z_1,\ z'_j = z_j - \alpha_j z_1,\ \text{for } 2 \le j \le n-2,\ z'_{n-1} = z_{n-1},\ w' = w. \tag{3.11}$$

We first give a sufficient condition under which, for a generic choice of α_j with $2 \le j \le n-2$, we have

$$\begin{aligned}&\rho^{[m]}(z_1, 0, \cdots, 0, z_{n-1}, 0, \overline{z_1}, 0, \cdots, 0, \overline{z_{n-1}}, 0) \not\equiv 0,\\ &a^{(\ell_0)}_{1(n-1)}(z_1, 0, \cdots, 0, \overline{z_1}, 0, \cdots, 0) \text{ contains a non holomorphic term.}\end{aligned} \tag{3.12}$$

Notice that

$$\begin{aligned}&\rho^{[m]}(z_1, \cdots, z_{n-1}, 0, \overline{z_1}, \cdots, \overline{z_{n-1}}, 0)\\ =&\rho^{[m]}(z'_1, z'_2 + \alpha_2 z'_1, \cdots, z'_{n-2} + \alpha_{n-2} z'_1, z'_{n-1}, 0, \overline{z'_1}, \overline{z'_2} + \overline{\alpha_2}\overline{z'_1}, \cdots, \overline{z'_{n-2}} + \overline{\alpha_{n-2}}\overline{z'_1}, \overline{z'_{n-1}}, 0).\end{aligned}$$

The coefficient of ${z'}_1^t {z'}_{n-1}^{\mu} \overline{z'_1}^s \overline{z'_{n-1}}^{\nu}$ with $t + s + (\ell_0 + 1)(\mu + \nu) = m$ in its Taylor expansion is

$$\sum_{\substack{\sum h_\lambda = t, \sum j_\lambda = s\\ H=(h_1,\cdots,h_{n-2}), J=(j_1,\cdots,j_{n-2})}} \rho^{[m]}_{(H\mu 0)(J\nu 0)} \alpha^H \overline{\alpha}^J.$$

Here $\alpha_1 = 1$, $\alpha = (\alpha_1, \cdots, \alpha_{n-2})$, and $\rho^{[m]}_{(H\mu 0)(J\nu 0)}$ is the coefficient of

$$z_1^{h_1}\cdots z_{n-2}^{h_{n-2}}z_{n-1}^{\mu}\overline{z_1^{j_1}}\cdots\overline{z_{n-2}^{j_{n-2}}}\,\overline{z_{n-1}^{\nu}}$$

in the Taylor expansion of ρ at 0.

Notice that this term is 0 for a generic choice of α if and only if $\rho^{[m]}_{(H\mu 0)(J\nu 0)}=0$ for any pair (H,J) with $\sum h_\lambda=t,\ \sum j_\lambda=s$. By our choice of the weight m, there exists a pair $(H\mu 0)(J\nu 0)$ with $|J|+(\ell_0+1)\nu>0$ such that $\rho^{[m]}_{(H\mu 0)(J\nu 0)}\neq 0$. Thus for a generic choice of α_j's, we have

$$\rho'^{[m]}(z_1',0,\cdots,0,z_{n-1}',0,\overline{z_1'},0,\cdots,0,\overline{z_{n-1}'},0)\not\equiv 0.$$

Since $\rho^{[m]}$ contains no holomorphic terms, so is

$$\rho'^{[m]}(z_1',0,\cdots,0,z_{n-1}',0,\overline{z_1'},0,\cdots,0,\overline{z_{n-1}'},0).$$

Hence for a generic choice of α, the statement in the first line of (3.12) holds.

Next notice that

$$\frac{\partial}{\partial z_1}=\frac{\partial}{\partial z_1'}-\sum_{\lambda=2}^{n-2}\alpha_\lambda\frac{\partial}{\partial z_\lambda'},\ \ \frac{\partial}{\partial z_j}=\frac{\partial}{\partial z_j'}\ \text{for}\ 2\le j\le n-1. \tag{3.13}$$

And by Lemma 3.1, we have

$$L_1=L_1'-\sum_{\lambda=2}^{n-2}\alpha_\lambda L_\lambda',\ \ L_j=L_j'\ \text{for}\ 2\le j\le n-1. \tag{3.14}$$

Set $S_1'=S_1+\sum_{\lambda=2}^{n-2}\alpha_\lambda S_\lambda$, $S_j'=S_j$. Then

$$\begin{aligned}S_1'=&\sum_{h=1}^{n-1}a_{1h}L_h+\sum_{\lambda=2}^{n-2}\alpha_\lambda\sum_{h=1}^{n-1}a_{\lambda h}L_h\\ =&\big(a_{11}+\sum_{\lambda=2}^{n-2}\alpha_\lambda a_{\lambda 1}\big)\big(L_1'-\sum_{\lambda=2}^{n-2}\alpha_\lambda L_\lambda'\big)+\sum_{h=2}^{n-2}\big(a_{1h}+\sum_{\lambda=2}^{n-2}\alpha_\lambda a_{\lambda h}\big)L_h'\\ &+\big(a_{1(n-1)}+\sum_{\lambda=2}^{n-2}\alpha_\lambda a_{\lambda(n-1)}\big)L_{n-1}':=\sum_{\lambda=1}^{n-1}a_{1\lambda}'L_\lambda'.\end{aligned} \tag{3.15}$$

Hence

$$\begin{aligned}&a_{1(n-1)}'(z_1',0,\cdots,0,\overline{z_1}',0,\cdots,0)\\ =&a_{1(n-1)}(z_1',\alpha_2z_1',\cdots,\alpha_{n-2}z_1',0,0,\overline{z_1'},\overline{\alpha_2z_1'},\cdots,\overline{\alpha_{n-2}z_1'},0,0)\\ &+\sum_{\lambda=2}^{n-2}\alpha_\lambda a_{\lambda(n-1)}(z_1',\alpha_2z_1',\cdots,\alpha_{n-2}z_1',0,0,\overline{z_1'},\overline{\alpha_2z_1'},\cdots,\overline{\alpha_{n-2}z_1'},0,0).\end{aligned} \tag{3.16}$$

Then the coefficient of ${z_1}'^t\overline{{z_1}'^s}$ with $t+s=\ell_0$ in $a_{1(n-1)}'(z_1',0,\cdots,0,\overline{z_1}',0,\cdots,0)$ is the following

$$\sum_{\lambda=1}^{n-2}\sum_{|H|=t,|J|=s}(a^{(\ell_0)}_{\lambda(n-1)})_{HJ}\alpha^{H+e_\lambda}\overline{\alpha}^J=\sum_{\lambda=1}^{n-2}\sum_{|H|=t+1,|J|=s}(a^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)J}\alpha^H\overline{\alpha}^J, \tag{3.17}$$

where $e_\lambda = (0, \cdots, 0, 1, 0 \cdots, 0)$ with 1 at the λ-th position. This term is 0 for a generic choice of α if and only if $\sum_{\lambda=1}^{n-2} (a^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)J} = 0$. We next proceed in two steps:

(1). First, we suppose that there exists a pair (H, J) with $|J| \neq 0$ such that

$$\sum_{\lambda=1}^{n-2} (a^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)J} \neq 0.$$

Then for a generic choice of α, $a'^{(\ell_0)}_{1(n-1)}(z'_1, 0, \cdots, 0, \overline{z_1}', 0, \cdots, 0)$ contains a non-holomorphic term. Through the normalization procedure as in Lemma 3.2, we can make

$$a'^{(\ell_0)}_{1(n-1)}(z'_1, \cdots, z'_{n-2}, 0, \cdots, 0) = 0$$

and thus, in particular, $a'^{(\ell_0)}_{1(n-1)}(z'_1, 0, \cdots, 0) = 0$. We point out that this transformation preserves the statement in the first line of (3.12). Then by (3.9), the new $a^{(\ell_0)}_{1(n-1)}$ and $\rho^{[m]}$ satisfy the desired properties in (1) of Proposition 3.4 and thus ℓ_0 is not changed. Next, we can repeat the same argument in Lemma 3.2 to normalize $a^{(\ell_0)}_{j(n-1)}$ for $j \geq 2$ and thus obtain the normalization for $a^{(\ell_0)}_{j(n-1)}$ with $j = 2, \cdots, n-2$.

(2). We now suppose

$$\sum_{\lambda=1}^{n-2} (a^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)J} = 0 \text{ for any } |H| + |J| = \ell_0 + 1, \ |J| \neq 0. \tag{3.18}$$

We will show that by a suitable change of coordinates of the form $z'_j = z_j$, $z'_{n-1} = z_{n-1} + g(z_1, \cdots, z_{n-2})$, $w' = w$, we can make

$$\sum_{\lambda=1}^{n-2} (a^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)0} = 0 \text{ for any } |H| = \ell_0 + 1. \tag{3.19}$$

Here $g(z_1, \cdots, z_{n-2})$ is a homogeneous holomorphic polynomial of degree $\ell_0 + 1$.

In fact, under this transformation, we have

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial z'_j} + g_{z_j} \frac{\partial}{\partial z'_{n-1}}, \quad \frac{\partial}{\partial z_{n-1}} = \frac{\partial}{\partial z'_{n-1}}.$$

And by Lemma 3.2, we have

$$\begin{aligned} S_j &= \sum_{j=1}^{n-1} a_{jh} L_h = \sum_{j=1}^{n-2} a_{jh}(L'_h + g_{z_h} L'_{n-1}) + a_{j(n-1)} L'_{n-1} \\ &= \sum_{j=1}^{n-2} a_{jh} L'_h + (a_{j(n-1)} + \sum_{j=1}^{n-2} a_{jh} g_{z_h}) L'_{n-1}. \end{aligned}$$

Hence

$$a'^{(\ell_0)}_{\lambda(n-1)} = a^{(\ell_0)}_{\lambda(n-1)} + g_{z_\lambda}. \tag{3.20}$$

Thus $\sum_{\lambda=1}^{n-2} (a'^{(\ell_0)}_{\lambda(n-1)})_{(H-e_\lambda)0} = 0$ for any H with $|H| = \ell_0 + 1$, which is equivalent to $\sum_{\lambda=1}^{n-2} z_\lambda a'^{(l_0)}_{\lambda(n-1)}(z_1, \cdots, z_{n-2}, 0, \cdots, 0) = 0$, if and only if

$$\sum_{\lambda=1}^{n-2} z_\lambda g_{z_\lambda} + \sum_{\lambda=1}^{n-2} z_\lambda a^{(\ell_0)}_{\lambda(n-1)}(z_1,\cdots,z_{n-2},0,\cdots,0) = 0.$$

This is the well-known Euler equation and can be solved as follows:

Notice that if we write $g = \sum_{|J|=\ell_0+1} \Gamma_J z^J$, then

$$\sum_{\lambda=1}^{n-2} z_\lambda g_{z_\lambda} = \sum_{\lambda=1}^{n-2} \sum_{|J|=\ell_0+1} j_\lambda \Gamma_J z^J = (\ell_0+1)g.$$

Hence g can be uniquely solved as

$$g = -\frac{1}{\ell_0+1} \sum_{\lambda=1}^{n-2} z_\lambda a^{(\ell_0)}_{\lambda(n-1)}(z_1,\cdots,z_{n-2},0,\cdots,0).$$

Thus we get the desired normalization property in (3.19). Notice that by (3.20), we conclude that ℓ_0 is not changed because any non-holomorphic term in $a^{\ell_0}_{\lambda(n-1)}$ can not removed by this transform. Moreover, $(a^{(l_0)}_{\lambda(n-1)})_{(H-e_\lambda)J}$ with $|H| + |J| = \ell_0 + 1$, $|J| \neq 0$ is not changed under this transformation. Hence (3.18) still holds to be true.

Notice that (3.18) and (3.19) are equivalent to the normalization property in (2) of Proposition 3.4. In fact,

$$\begin{aligned}&\sum_{j=1}^{n-2} z_j a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)\\ =&\sum_{|H|+|J|=\ell_0+1} \sum_{j=1}^{n-2} (a^{(\ell_0)}_{j(n-1)})_{(H-e_j)J} z^H \overline{z}^J = 0.\end{aligned}$$

This completes the proof of Proposition 3.4. □

We summarize what we did in this section in the following theorem to facilitate our future quotation:

Theorem 3.5. *Let $M \subset \mathbb{C}^n$ be as defined in (3.1) and let B be a smooth subbundle of $T^{(1,0)}M$ of complex dimension $s = n-2$. Let $\{L_h\}_{h=1}^{n-1}$, $\{S_j\}_{j=1}^{n-2}$ and $\{a_{jh}\}$ be as in (3.3) and (3.4). Suppose that ℓ_0 is defined as in (3.10) and m is the weight of w. Assumption that $\ell_0 \le a^{(n-2)}(M,0) - 1$. Then, after a holomorphic change of coordinates and after re-choosing a suitable basis $\{S_j\}_{j=1}^{n-2}$ of the cross sections of B, if needed, we have one of the following three normalizations for the system*

$$\{a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0), \rho^{[m]}(z,0,\overline{z},0)\}_{j=1}^{n-2}:$$

(I) $a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ *is holomorphic in $z_1,\cdots,z_{n-2}$ for each j, and there exists $j_0 \in [2, n-2]$ such that*

$$a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0) = 0$$

for $1 \le j \le j_0 - 1$, $a^{(\ell_0)}_{j_0(n-1)}(0,\cdots,0,z_{j_0},\cdots,z_{n-2},0,\cdots,0) = 0$, but $a^{(\ell_0)}_{j_0(n-1)}(z_1,\cdots,z_{n-2},0,\cdots,0) \not\equiv 0$.

(II) $a^{(\ell_0)}_{1(n-1)}(z_1,0,\cdots,0,\overline{z_1},0,\cdots,0)\not\equiv 0$, $a^{(\ell_0)}_{1(n-1)}(z_1,0,\cdots,0)=0$, $\rho^{[m]}(z_1,0,\cdots,0,z_{n-1},0,\overline{z_1},0,\cdots,0,\overline{z_{n-1}},0)$ *is not identically zero (and contains no non-trivial holomorphic terms).*

(III) *For a certain* $j\in[1,n-2]$, $a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)$ *is not holomorphic,* $\sum_{j=1}^{n-2} z_j a^{(\ell_0)}_{j(n-1)}(z_1,\cdots,z_{n-2},0,0,\overline{z_1},\cdots,\overline{z_{n-2}},0,0)=0$. *Moreover*

$$\rho^{[m]}(z_1,\cdots,z_{n-1},0,\overline{z_1},\cdots,\overline{z_{n-1}},0)$$

is not identically zero and contains no non-trivial holomorphic terms.

4. Proof of Theorem 2.3

We now present a proof of Theorem 2.3, assuming Theorem 6.1 whose proof is very much involved and will be given in §5 and §6. As we mentioned before, our focus is on the equality of the commutator type with the contact type. The Levi-form type can be easily reduced to the commutator type case.

Proof of the equality: $t^{(n-2)}(M,p)=a^{(n-2)}(M,p)$**.** We keep the notations set up in §2 and §3 with $p=0$. Assume that M is defined as in (3.1) and (3.2). As we mentioned there, we assume that $a^{(n-2)}(M,p=0)<\infty$. Supposing that $t^{(n-2)}(M,0)>a^{(n-2)}(M,0)$, we will then seek a contradiction.

Let B be an $(n-2)$-dimensional smooth vector subbundle of $T^{1,0}M$ such that $t^{(n-2)}(M,0)=t^{(n-2)}(B,0)$. By the assumption that $t^{(n-2)}(M,0)>a^{(n-2)}(M,0)$, for any $l\le a^{(n-2)}(M,0)$ we have

$$\langle F,\partial\rho\rangle(0)=0 \text{ for any } \ F=[F_l,F_{l-1},\cdots[F_2,F_1]\cdots] \text{ with } F_1,\cdots,F_l\in\mathcal{M}_1(B). \tag{4.1}$$

We assume the normalization of §2 up to (3.10) such that we can well define ℓ_0.

Recall that the weight of z_j for $1\le j\le n-2$ and their conjugates is 1. Define the weight of z_{n-1} and its conjugate to be $k=\ell_0+1$. Denote the weight of w to be m, which is the lowest weighted vanishing order of $\rho(z,0,\overline{z},0)$ with respect to the weights just given. We also define

$$\begin{aligned}&\mathrm{wt}(\frac{\partial}{\partial z_j})=\mathrm{wt}(\frac{\partial}{\partial\overline{z_j}})=-1 \text{ for } 1\le j\le n-2,\\ &\mathrm{wt}(\frac{\partial}{\partial z_{n-1}})=\mathrm{wt}(\frac{\partial}{\partial\overline{z_{n-1}}})=-k,\ \mathrm{wt}(\frac{\partial}{\partial w})=\mathrm{wt}(\frac{\partial}{\partial\overline{w}})=-m.\end{aligned}\tag{4.2}$$

By the definition of $a^{(n-2)}(M,0)$, when restricted to the $(n-2)$-manifold $\{(z,w):\ z_{n-1}=w=0\}$, the vanishing order of ρ is bounded by $a^{(n-2)}(M,0)$. Thus $m\le a^{(n-2)}(M,0)$. When $k\le a^{(n-2)}(M,0)$, we further assume that S_j and ρ are normalized as in Theorem 3.5.

Write

$$S_j^0=\frac{\partial}{\partial z_j}+a^{[k-1]}_{j(n-1)}\frac{\partial}{\partial z_{n-1}}+a^{[m-1]}_{jn}\frac{\partial}{\partial w}.$$

Then S_j^0 is the sum of terms in S_j of weighted degree -1.

Now, let $\mathcal{M}^0$ be the $C^\infty(M^0)$-module spanned by S_j^0 and $\overline{S_j^0}$ for all $1\le j\le n-2$, where $M^0=\{(z,w):\rho^{[m]}=-2\mathrm{Re}w+\chi^{[m]}(z,0,\overline{z},0)=0\}$ and $\mathcal{M}^0_l$ be the $C^\infty(M^0)$ module formed by taking the Lie bracket of length $\le l$ of sections from $\mathcal{M}^0$ for $l=2,\cdots$. $\mathcal{M}^0_\infty=\cup_{l\in\mathbb{N}}\mathcal{M}^0_l$. Notice that S_j^0 is a CR vector field along M^0 for each j. We start with the following lemma:

Lemma 4.1. *It holds that* $k<m$*, namely,* $\ell_0<m-1$.

Proof. Suppose that $k \geq m$. Then the weight of z_{n-1} is no less than m. Hence $\chi^{[m]}(z, 0, \overline{z}, 0)$ is independent of z_{n-1}. Write

$$\widetilde{S_j^0} = S_j^0 - a_{j(n-1)}^{[k-1]} \frac{\partial}{\partial z_{n-1}} = \frac{\partial}{\partial z_j} + a_{jn}^{[m-1]} \frac{\partial}{\partial w}.$$

Since S_j^0 is tangent to M^0, whose defining function is independent of z_{n-1}, we see that $\widetilde{S_j^0}$ is also tangent to M^0 and $a_{jn}^{[m-1]} = \frac{\partial \chi^{[m]}}{\partial z_j}$. Hence $a_{jn}^{[m-1]}$ is independent of z_{n-1}.

Regarding M^0 as a real hypersurface in $\mathbb{C}^{n-1}$. Let $\widetilde{\mathcal{M}}^0$ be the $C^\infty(M^0)$ module spanned by $\widetilde{S_j^0}$ and $\overline{\widetilde{S_j^0}}$ for all $1 \leq j \leq n-2$. Define $Q : \mathcal{M}^0 \to \widetilde{\mathcal{M}}^0$ by sending $\sum d_j(z, \overline{z}) \frac{\partial}{\partial z_j} \in \mathcal{M}^0$ to $\sum_{j \neq n-1} d_j(z_1, \cdots, z_{n-2}, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0) \frac{\partial}{\partial z_j} \in \widetilde{\mathcal{M}}^0$. Then by (4.1), for any $Z_j^0 \in \widetilde{\mathcal{M}}^0$, there exists $Y_j^0 \in \mathcal{M}^0$ with $Q(Y_j^0) = Z_j^0$ such that

$$\langle [Z_j^0, [Z_{j_1}^0, \cdots, [Z_2^0, Z_1^0] \cdots], \partial\rho \rangle(0) = \langle [Y_j^0, [Y_{j-1}^0, \cdots, [Y_2^0, Y_1^0] \cdots], \partial\rho \rangle(0) = 0 \text{ for } j \leq m.$$

(Indeed, we can simply take Y_j^0 to be Z_j^0, but regard it as a CR vector field of M^0 as a real hypersurface in $\mathbb{C}^n$.) Hence we have $t^{((n-1)-1)}(M^0, 0) > m$. However, by our construction, $a^{((n-1)-1)}(M^0, 0) = m$. This contradicts a result of Bloom-Graham for the equalities of regular $(n-1)$-types in [9], which says that $t^{((n-1)-1)}(M^0, 0) = a^{((n-1)-1)}(M^0, 0)$ for $M^0 \subset \mathbb{C}^{n-1}$. $\square$

Lemma 4.2. *For any $Y^0 \in \mathcal{M}_l^0$, we have $\langle Y^0, \partial\rho^{[m]} \rangle(0) = 0$.*

Proof. We can assume, without loss of generality, that $Y^0 = [X_l^0, \cdots, [X_2^0, X_1^0] \cdots]$ with $X_j^0 \in \mathcal{M}^0$ being weighted homogeneous of degree -1. Write

$$X_j^0 = Z_j^0 + B_j \frac{\partial}{\partial w} + C_j \frac{\partial}{\partial \overline{w}} \text{ with } Z_j^0 = \sum_{k=1}^{n-1} (b_{jk} \frac{\partial}{\partial z_k} + c_{jk} \frac{\partial}{\partial \overline{z_k}}).$$

Here Z_j^0 is weighted homogeneous of degree -1 and $\mathrm{wt}(B_j) = \mathrm{wt}(C_j) = m-1$. A direct computation shows

$$[X_2^0, X_1^0] = (Z_2^0(B_1) - Z_1^0(B_2)) \frac{\partial}{\partial w} \mod (\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{w}})$$

and by an induction,

$$Y^0 = C_l^0 \frac{\partial}{\partial w} \mod (\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{w}})$$

with C_l^0 a weighted homogeneous polynomial of weighted degree equal to $-l+m$. Hence $Y^0 \equiv 0$ when $l > m$ and $Y^0|_0 = 0$ when $l < m \mod (\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{w}})$.

When $l = m$, suppose that $Z_j \in \mathcal{M}_1$ such that $(Z_j)^0 = X_j^0$. Then $[Z_j, Z_k]^0 = [X_j^0, X_k^0]$. Hence if $Z \in \mathcal{M}_l$ with $l = m$ such that $(Z)^0 = Y^0$, then $Z = C_m^0 Y^0 + D_m \frac{\partial}{\partial w} \mod (\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{w}})$ with $\mathrm{wt}(Y^0) = -m$ and thus $\mathrm{wt}(D_m) > \mathrm{wt}(C_m^0) = 0$. From (4.1), $Z|_0 = 0 \mod (\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{w}})$. Thus we obtain $C_m^0 \equiv 0$. Hence $\langle Y^0, \partial\rho^{[m]} \rangle(0) = 0$ for all $l \in \mathbb{N}$. $\square$

Then we have $\frac{\partial}{\partial v}|_0 \notin \mathcal{M}_\infty^0$. Now applying the Nagano theorem (see [2]) to $\mathcal{M}_{\infty,\omega}^0$ we obtain a unique real analytic integral submanifold N^0 with $0 \in N^0 \subset M^0 = \{-2\mathrm{Re} w + \chi^{[m]}(z, 0, \overline{z}, 0) = 0\}$. Moreover, $\dim_{\mathbb{R}} N^0 = \dim_{\mathbb{R}} \mathcal{M}_{\infty,\omega}^0 = \dim_{\mathbb{R}} \mathcal{M}_\infty^0$. Here $\mathcal{M}_{\infty,\omega}^0 \subset \mathcal{M}_\infty^0$ is the submodule generated by the aforementioned homogenous frames over $C^\omega(M^0)$.

Since $\frac{\partial}{\partial v}|_0 \notin T_0 N^0$, N^0 is contained in the graph of $v = f_1(z, \overline{z}, u)$ for a certain real analytic function f_1 near 0. Since $u = \frac{1}{2}\chi^{[m]}(z, 0, \overline{z}, 0)$, we conclude that N^0 is contained in the graph of

$$w = f(z, \overline{z}) = \frac{1}{2}\chi^{[m]}(z, 0, \overline{z}, 0) + i f_1\big(z, \overline{z}, \frac{1}{2}\chi^{[m]}(z, 0, \overline{z}, 0)\big).$$

We mention that from the pseudoconvexity of M, we immediately conclude the pseudoconvexity of M^0, which is equivalent to the plurisubharmonicity of $\mathrm{Re}(f) = \chi^{[m]}(z, 0, \overline{z}, 0)$.

Lemma 4.3. *The real dimension of N^0 is either $2n-3$ or $2n-2$.*

Proof. The proof is carried out in two steps according to the properties of

$$a^{(\ell_0)}_{j(n-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0, 0)$$

in Proposition 3.3.

(1): Suppose we have the normalization in (I) of Proposition 3.3. We suppose that $(a^{(k-1)}_{j_0(n-1)})_{H+e_\mu} \neq 0$ with $H = (h_1, \cdots, h_{n-2})$ and $1 \leq \mu \leq j_0 - 1$. Then

$$[S^0_\mu, S^0_{j_0}] = \frac{\partial}{\partial z_\mu}(a^{(k-1)}_{j_0(n-1)})\frac{\partial}{\partial z_{n-1}} \mod (\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}).$$

Write

$$(\overbrace{S^0_1, \cdots, S^0_1}^{h_1 \text{ times}}, \overbrace{S^0_2, \cdots, S^0_2}^{h_2 \text{ times}}, \cdots, \overbrace{S^0_{n-2}, \cdots, S^0_{n-2}}^{h_{n-2} \text{ times}}) \text{ as } (X_1, \cdots, X_{|H|}). \tag{4.3}$$

Then

$$\begin{aligned}&[X_1, [\cdots [X_{|H|}, [S^0_\mu, S^0_{j_0}]] \cdots]] \\ =&(h_\mu + 1) \cdot h_1! \cdots h_{n-2}!(a^{(k-1)}_{j_0(n-1)})_{H+e_\mu} \frac{\partial}{\partial z_{n-1}} \mod (\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}).\end{aligned}$$

Since its conjugate is also in $\mathcal{M}^0_\infty$, we conclude that the dimension of N^0 is $2n-2$.

(2): Suppose we have the normalization in (II) of Proposition 3.3. Then there is a $(H, J) = (h_1, \cdots, h_{n-2}, j_1, \cdots, j_{n-2})$ such that $(a^{(k-1)}_{1(n-1)})_{H(J+e_\mu)} \neq 0$. Then

$$[\overline{S^0_\mu}, S^0_1] = \frac{\partial}{\partial \overline{z_\mu}}(a^{(k-1)}_{1(n-1)})\frac{\partial}{\partial z_{n-1}} \mod (\frac{\partial}{\partial \overline{z_{n-1}}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}).$$

Write $(X_1, \cdots, X_{|H|})$ as in (4.3) and write

$$(\overbrace{S^0_1, \cdots, S^0_1}^{j_1 \text{ times}}, \overbrace{S^0_2, \cdots, S^0_2}^{j_2 \text{ times}}, \cdots, \overbrace{S^0_{n-2}, \cdots, S^0_{n-2}}^{j_{n-2} \text{ times}}) \text{ as } (Y_1, \cdots, Y_{|J|}).$$

Then

$$\begin{aligned}Y_{HJ} :=&[X_1, [\cdots, [X_{|H|}, [\overline{Y_1}, [\cdots, [\overline{Y_{|J|}}, [\overline{S^0_\mu}, S^0_1]] \cdots] \\ =&(j_\mu + 1) \cdot h_1! \cdots h_{n-2}! j_1! \cdots j_{n-2}!(a^{(k-1)}_{j(n-1)})_{H(J+e_\mu)} \frac{\partial}{\partial z_{n-1}} \mod (\frac{\partial}{\partial \overline{z_{n-1}}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}).\end{aligned}$$

Hence $Y_{HJ} \notin \mathrm{span}_{\mathbb{C}}\{S_j^0, \overline{S_j^0}, 1 \le j \le n-2\}$. Thus either $\mathrm{Re}Y_{HJ}|_0 \ne 0$ or $\mathrm{Im}Y_{HJ}|_0 \ne 0$.

Since $\frac{\partial}{\partial w}|_0$ and $\frac{\partial}{\partial \overline{w}}|_0$ are not tangent to N^0 at 0, the dimension of N^0 is either $2n-3$ or $2n-2$. □

Lemma 4.4. *When N^0 has real dimension $2n-2$, f is a weighted homogeneous polynomial of weighted degree m.*

Proof. Let X^0 be a (weighted) homogeneous vector field from $\mathcal{M}_\infty^0$. Then from the equality that $X^0(-w+f) = \overline{X^0}(-w+f) \equiv 0$, it follows that $X^0(-w+f^{[m]}) = X^0\overline{(-w+f^{[m]})} \equiv 0$. Hence the manifold defined by $w = f^{[m]}$ is also an integral manifold of the module $\mathcal{M}_\infty^0$ through 0. By the uniqueness of the integrable manifold, we conclude that $f^{[m]} = f$. □

Before proceeding further, we need the following lemma:

Lemma 4.5. *Let $h(z,\overline{z})$ be a real analytic function in $z \in \mathbb{C}^n$ near the origin. Assume that h is holomorphic in its variable $(z_1, \cdots, z_k)$ with $k \le n$ for each fixed $(z_{k+1}, \cdots, z_n)$ near the origin. Assume that $Ref(z,\overline{z})$ is a plurisubharmonic function without non-trivial holomorphic terms in its Taylor expansion at 0. Then $h(z,\overline{z})$ is independent of $z_1, \cdots, z_k$ and $\overline{z_1}, \cdots \overline{z_k}$.*

Proof. We need only to prove the lemma with $k=1$ and the other case follows from an induction argument. Since $\mathrm{Re}h(z,\overline{z})$ is plurisubharmonic, for each j with $2 \le j \le n$, we have

$$(2\mathrm{Re}h)_{z_1\overline{z_1}}(2\mathrm{Re}h)_{z_j\overline{z_j}} - (2\mathrm{Re}h)_{z_1\overline{z_j}}(2\mathrm{Re}h)_{z_j\overline{z_1}} \ge 0. \tag{4.4}$$

Since $h(z,\overline{z})$ is holomorphic in z_1, we have

$$(2\mathrm{Re}h)_{z_1\overline{z_1}} = 0,\ (2\mathrm{Re}h)_{z_1\overline{z_j}} = h_{z_1\overline{z_j}},\ (2\mathrm{Re}h)_{z_j\overline{z_1}} = \overline{h}_{z_j\overline{z_1}}.$$

Substituting these relations back to (4.4), we obtain $-|h_{z_1\overline{z_j}}|^2 \ge 0$. Thus $h_{z_1\overline{z_j}} \equiv 0$. Since $h(z,\overline{z})$ is holomorphic in z_1, we see that

$$g(z,\overline{z}) = h(z,\overline{z}) - h(0, z_2, \cdots, z_n, 0, \overline{z_2}, \cdots, \overline{z_n}) := \sum_{k\ge1} g_k(z_2, \cdots, z_n, \overline{z_2}, \cdots, \overline{z_n}) z_1^k$$

with $(g_k)_{\overline{z_j}} \equiv 0$ for $j = 2, \cdots, n$. Hence g is a holomorphic function. By our assumption,

$$Reh = Reg(z,\overline{z}) + Reh(0, z_2, \cdots, z_n, 0, \overline{z_2}, \cdots, \overline{z_n})$$

contains no non-trivial holomorphic terms. Hence $g(z,\overline{z})$ is independent of z_1. This shows that $h(z,\overline{z})$ is independent of z_1 and $\overline{z_1}$. □

The rest of the argument is carried out according to the dimension of N^0. We remark that when the real dimension of N^0 is $2n-3$, it is a CR submanifold of hypersurface type, for it has a constant CR dimension $n-2$ everywhere. When its dimension is $2n-2$, it has CR dimension $n-1$ at the origin. Since it cannot be Levi-flat due to the fact that $\mathrm{Re}(f) \not\equiv 0$, it is thus a codimension two CR singular submanifold [22].

Step I. In this step, we suppose N^0 is of real dimension $2n-2$. Since $\overline{S_j^0}$ is tangent to N^0, and since N^0 is defined by $w = f(z,\overline{z})$ for $z \approx 0$ in $\mathbb{C}^{n-1}$, we have

$$\frac{\partial}{\partial \overline{z_j}} f(z,\overline{z}) + \overline{a_{j(n-1)}^{(k-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0, 0)} \frac{\partial}{\partial \overline{z_{n-1}}} f(z,\overline{z}) = 0,\ z \in \mathbb{C}^{n-1}. \tag{4.5}$$

By Lemma 4.1, we have $k < m \leq a^{(n-2)}(M,0)$. Our further discussions are divided into the following cases according to the normalizations in Theorem 3.5.

Case (1): In this case, suppose that we have the normalization in (I) of Theorem 3.5. For $1 \leq j \leq j_0 - 1$, $a^{(k-1)}_{j(n-1)} \equiv 0$. Thus (4.5) takes the form $\frac{\partial f}{\partial \overline{z_j}} = 0$. Hence f is holomorphic in $z_1, \cdots, z_{j_0-1}$ for each fixed $z_{j_0}, \cdots, z_{n-1}$. By the following Lemma 4.5, since $\mathrm{Re}(f)$ is plurisubharmonic and contains no non-trivial holomorphic terms, f is in fact independent of $z_1, \cdots, z_{j_0-1}$. Setting $j = j_0$ in (4.5), we obtain

$$\frac{\partial f}{\partial \overline{z_{j_0}}} = -\overline{a^{(k-1)}_{j_0(n-1)}}\frac{\partial f}{\partial \overline{z_{n-1}}}.$$

Notice that the left hand side is independent of $z_1, \cdots, z_{j_0-1}$. On the other hand, the right hand side is divisible by $\overline{a^{(k-1)}_{j_0(n-1)}}(\not\equiv 0)$, in which each term depends on $z_1, \cdots, z_{j_0-1}$. Thus $\frac{\partial f}{\partial \overline{z_{j_0}}} = \frac{\partial f}{\partial \overline{z_{n-1}}} = 0$. Substituting this back to (4.5), we obtain $\frac{\partial f}{\partial \overline{z_j}} = 0$ for each $1 \leq j \leq n-2$. Thus f is holomorphic in $z_1, \cdots, z_{n-1}$. However, $\chi^{[m]} = \mathrm{Re}(f) \neq 0$ does not contain any non-trivial holomorphic term. We thus reach a contraction.

Case (2): In this case, suppose we have the normalization in (II) of Theorem 3.5. Letting $j = 1$ in (4.5) and restricting the equation to z_1 and z_{n-1} spaces, we obtain:

$$\Big(\frac{\partial f}{\partial \overline{z_1}} + \overline{a^{(k-1)}_{1(n-1)}}\frac{\partial}{\partial \overline{z_{n-1}}} f\Big)(z_1, 0, \cdots, 0, z_{n-1}, \overline{z_1}, 0, \cdots, 0, \overline{z_{n-1}}) = 0. \tag{4.6}$$

By our assumption, $a^{(k-1)}_{1(n-1)}(z_1, 0, \cdots, 0, z_{n-2}, 0, 0, \overline{z_1}, 0, \cdots, 0, \overline{z_{n-2}}, 0, 0)$ is not identically zero and contains no non-trivial holomorphic terms. By Theorem 6.1, we know $\chi^{[m]} = \mathrm{Re}(f) = 0$ when restricted to z_1 and z_{n-1} spaces. This contradicts the last normalization in (II) of Theorem 3.5.

Case (3): In this case, suppose we have the normalization in (III) of Theorem 3.5. Then we have $\sum_{j=1}^{n-2} z_j a^{(k-1)}_{j(n-1)}(z_1, \cdots, z_{n-2}, 0, 0, \overline{z_1}, \cdots, \overline{z_{n-2}}, 0, 0) = 0$. Since $f_{\overline{z_j}} + \overline{a^{(k-1)}_{j(n-1)}} f_{\overline{z_{n-1}}} = 0$ and $a^{(k-1)}_{j(n-1)}$ is independent of z_{n-1} and w, we get

$$\sum_{j=1}^{n-2} \overline{z_j} f_{\overline{z_j}}(z_1, \cdots, z_{n-1}, \overline{z_1}, \cdots, \overline{z_{n-1}}) = 0.$$

This is again the well-known Euler equation on f. Write $f(z, \overline{z}) = \sum_{|\alpha|\geq 0} g_\alpha(z)\overline{z}^\alpha$, where $g(z)$ is holomorphic in z. Then

$$\sum_{j=1}^{n-2} \overline{z_j} f_{\overline{z_j}} = \sum_{j=1}^{n-2}\sum_{|\alpha|\geq 0} g_\alpha(z)\alpha_j \overline{z}^\alpha = \sum_{|\alpha|\geq 0}(\sum_{j=1}^{n-2}\alpha_j) g_\alpha(z)\overline{z}^\alpha = 0.$$

Hence $g_\alpha(z) = 0$ for $\sum_{j=1}^{n-2}|\alpha_j| > 0$. Thus $f(z_1, \cdots, z_{n-1}, \overline{z_1}, \cdots, \overline{z_{n-1}})$ is holomorphic in $z_1, \cdots, z_{n-2}$. Hence $f_{\overline{z_j}} = 0$ for each $1 \leq j \leq n-2$. Substituting this back to $f_{\overline{z_j}} + \overline{a^{(k-1)}_{j(n-1)}} f_{\overline{z_{n-1}}} = 0$, we know $\overline{a^{(k-1)}_{j(n-1)}} f_{\overline{z_{n-1}}} = 0$. Recall that at least one $a^{(k-1)}_{j(n-1)}$ is not holomorphic and thus is nonzero. Thus $f_{\overline{z_{n-1}}} = 0$. Hence $f(z_1, \cdots, z_{n-1}, \overline{z_1}, \cdots, \overline{z_{n-1}})$ is holomorphic in $z_1, \cdots, z_{n-1}$. Since $\mathrm{Re}f$ contains no non-trivial holomorphic terms, we reach a contradiction.

Step II. In this step, we suppose N is of real dimension $2n - 3$.

Without loss of generality, we assume $\mathrm{Re}Y_{HJ}|_0 \neq 0$. Then

$$\mathbb{C}TN^0 = \mathrm{Span}_{\mathbb{C}}\{S^0_1, \cdots, S^0_{n-2}, \overline{S^0_1}, \cdots, \overline{S^0_{n-2}}, \mathrm{Re}Y_{HJ}\} \text{ near } 0.$$

Thus N^0 is a CR manifold of hypersurface type of finite type in the sense of Hömander-Bloom-Graham. With a rotation in z_{n-1}-variable, we can assume, without loss of generality, that $\mathrm{Re}Y_{HJ}|_0 = \frac{\partial}{\partial x_{n-1}}|_0$. Now, we define $\pi : N^0 \to \mathbb{C}^{n-1}$ by sending $(z_1, \cdots, z_{n-1}, w)$ to $(z_1, \cdots, z_{n-1})$. π is a CR immersion near 0. Write $\pi(N^0) = \widetilde{N^0} \subset \mathbb{C}^{n-1}$. Then $\widetilde{N^0}$ is a real hypersurface in $\mathbb{C}^{n-1}$ and $\pi^{-1} : \widetilde{N^0} \to N^0$ is a local real analytic CR diffeomorphism with $\pi^{-1}(0) = 0$. Write

$$\pi^{-1}(z_1, \cdots, z_{n-1}) = (z_1, \cdots, z_{n-1}, h(z_1, \cdots, z_{n-1})).$$

Since real analytic CR functions are restrictions of holomorphic functions, we can assume that $h(z_1, \cdots, z_{n-1})$ is a holomorphic function. Notice that $h = O(|z|^2)$ and define $(\xi_1, \cdots, \xi_{n-1}, \eta) = F(z_1, \cdots, z_{n-1}, w) = (z_1, \cdots, z_{n-1}, w - h(z_1, \cdots, z_{n-1}))$. Then

$$F(N^0) \subset \mathbb{C}^{n-1} \times \{0\} = \{(\xi_1, \cdots, \xi_{n-1}, 0) : \ \xi_1, \cdots, \xi_{n-1} \in \mathbb{C}\}.$$

Also, $F(M^0)$ is defined by $-2\mathrm{Re}\eta + 2\mathrm{Re}h(\xi) + \chi^{[m]}(\xi, 0, \overline{\xi}, 0) = 0$ or $2\mathrm{Re}\eta = 2\mathrm{Re}h(\xi) + \chi^{[m]}(\xi, 0, \overline{\xi}, 0) := \widetilde{\rho}(\xi, \overline{\xi})$. Notice that $F(M^0)$ is holomorphically equivalent to M^0. Hence $F(M^0)$ is also pseudo-convex and of finite type in the sense of Hömander-Bloom-Graham. Notice that $\widetilde{N^0} = F(N^0) \subset \widetilde{M_0} = F(M^0)$. Hence, $\forall \xi \in \widetilde{N^0}$, $\widetilde{\rho}(\xi, \overline{\xi}) = 0$. Notice that $\widetilde{\rho} = O(|\xi|^2)$ and is plurisubharmonic. By the following proposition, we reach a contradiction to the assumption that $2\mathrm{Re}h(\xi) + \chi^{[m]} \not\equiv 0$.

Proposition 4.6. *Let N be a real analytic hypersurface in $\mathbb{C}^{n-1}$ with $0 \in N$ with $n \geq 3$. Let $\rho(z, \overline{z})$ be a real analytic plurisubharmonic function with $\rho = O(|z|^2)$ as $z \to 0$ defined over a neighborhood of $\mathbb{C}^{n-1}$. Assume that N is of finite type in the sense of Hömander–Bloom-Graham and $N \subset \{\rho = 0\}$. Then $\rho \equiv 0$.*

Proof. Let $\phi : \Delta \to \mathbb{C}^{n-1}$ be a smooth small holomorphic disk attached to N with $\phi(1) = 0$. Namely, we assume that $\phi \in C^\infty(\overline{\Delta}) \cap \mathrm{Hol}(\Delta)$, $\phi(\partial\Delta) \subset N$, $\phi(1) = 0$, $\phi(\overline{\Delta})$ is close to 0. Since $\rho(\phi(\xi), \overline{\phi(\xi)}) = 0$ on $\partial\Delta$ and $\frac{\partial}{\partial\xi\partial\overline{\xi}}\rho(\phi(\xi), \overline{\phi(\xi)}) \geq 0$ for $\xi \in \Delta$, $\rho(\phi(\xi), \overline{\phi(\xi)})$ is a subharmonic function in Δ smooth up to $\partial\Delta$. By the maximum principle, we have $\rho(\phi(\xi), \overline{\phi(\xi)}) < 0$ for $\xi \in \Delta$ unless $\rho(\phi(\xi), \overline{\phi(\xi)}) \equiv 0$ for $\xi \in \Delta$. Now, we apply the Hopf Lemma to get

$$\frac{d}{d\xi}\rho(\phi(\xi), \overline{\phi(\xi)})|_{\xi=1} \geq 0$$

and the equality holds if and only if $\rho(\phi(\xi), \overline{\phi(\xi)}) \equiv 0$. On the other hand,

$$\rho(\phi(\xi), \overline{\phi(\xi)}) = O(|\phi(\xi)|^2) = O(|\phi(\xi) - \phi(1)|^2) = O(|\xi - 1|^2)$$

as $\xi(\in (0, 1)) \to 1$. We conclude that $\rho(\phi(\xi), \overline{\phi(\xi)}) \equiv 0$.

Next, by a result of Trépreau [44], since the union $\phi(\Delta)$ of all attached discs fill in at least one side of N near 0, we see that $\rho \equiv 0$ in one side of N. Since we assumed that ρ is real analytic, we conclude that $\rho \equiv 0$. This completes the proof of Proposition 4.6. □

We thus complete the proof of the equality that $t^{(n-2)}(M, p) = a^{(n-2)}(M, p)$. □

Proof of the equality: $c^{(n-2)}(M, p) = a^{(n-2)}(M, p)$**.** We will reduce this case to the commutator case that we just achieved.

We continue to use the notations and initial setups as in §2 and §3. By [12], we have $c^{(n-2)}(M, p = 0) \geq a^{(n-2)}(M, p = 0)$. We will seek a contradiction supposing that $c^{(n-2)}(M, 0) > a^{(n-2)}(M, 0)$.

Let B be an $(n-2)$-dimensional smooth subbundle of $T^{1,0}M$ such that $c^{(n-2)}(M, 0) = c^{(n-2)}(B, 0)$. With a biholomorphic change of coordinates, we can find a basis $\{S_j\}$ of B and a defining function ρ that satisfy the normalization conditions up to (3.10) so that ℓ_0 is well defined. Since $c^{(n-2)}(M,0) > a^{(n-2)}(M,0)$, for any $2 \le l \le a^{(n-2)}(M,0)$, we have

$$F_1 \cdots F_{l-2} \sum_{j=1}^{n-2} \partial\overline{\partial}\rho(S_j, \overline{S_j})(0) = 0 \text{ for any } F_1, \cdots, F_{l-2} \in \mathcal{M}_1(B). \tag{4.7}$$

As in the proof of $t^{(n-2)}(M,p) = a^{(n-2)}(M,p)$, we can similarly define the weights of $z_1, \cdots, z_{n-1}, w$, and define S_j^0, $\mathcal{M}^0$, M^0, $\mathcal{M}_l^0$, $\mathcal{M}_\infty^0$. By the same argument as that in Lemma 4.1, we have $k < m$. Then we can further assume the normalization in Theorem 3.5. Similar to Lemma 4.2, we have the following:

Lemma 4.7. *For any l and $Y_1^0, \cdots, Y_{l-2}^0 \in \mathcal{M}_1^0$, we have*

$$Y_1^0 \cdots Y_{l-2}^0 \sum_{j=1}^{n-2} \partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0})(0) = 0.$$

Proof. Similar to the previous case, we can assume that each Y_j^0 is weighted homogeneous of degree -1. First notice that $Y^0 := Y_1^0 \cdots Y_{l-2}^0 \sum_{j=1}^{n-2} \partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0})$ is a weighted homogeneous polynomial of weighted degree $-l+m$. Hence $Y^0 = 0$ when $l > m$ and $Y^0|_0 = 0$ when $l < m$.

Next we suppose $l = m$. For any $1 \le j \le l-2$, suppose $Z_j \in \mathcal{M}_1$ such that $(Z_j)^0 = Y_j^0$. By (4.7), we have

$$Z_1 \cdots Z_{m-2} \sum_{j=1}^{n-2} \partial\overline{\partial}\rho(S_j, \overline{S_j})(0) = 0.$$

Notice that

$$Z_1 \cdots Z_{m-2} \sum_{j=1}^{n-2} \partial\overline{\partial}\rho(S_j, \overline{S_j}) = Y_1^0 \cdots Y_{m-2}^0 \sum_{j=1}^{n-2} \partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0}) + o(1).$$

We thus have $Y^0(0) = 0$ for $l = m$. This completes the proof of Lemma 4.7. □

Now we similarly apply the Nagano theorem to conclude that $\mathcal{M}_{\infty,\omega}^0$ gives a unique real analytic integral submanifold N^0 with $N^0 \subset M^0 = \{-2\mathrm{Re}w + \chi^{[m]}(z,0,\overline{z},0) = 0\}$. Since the tangent space at each point of N^0 is generated by $\mathrm{Re}\mathcal{M}_\infty^0$, by Lemma 4.7, we have

$$\sum_{j=1}^{n-2} \partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0}) \equiv 0 \text{ on } N^0,$$

for $\rho^{[m]}(S_j^0, \overline{S_j^0})$ is real-analytic and it vanishes to infinite order at 0 along N^0. Since $\rho^{[m]}$ is plurisubharmonic, we have $\partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0}) \ge 0$ on M^0. Notice that $N^0 \subset M^0$, we have $\partial\overline{\partial}\rho^{[m]}(S_j^0, \overline{S_j^0}) \equiv 0$ on N^0. Hence $\mathrm{Re}(S_j^0), \mathrm{Im}(S_j^0) \in T^N(M^0)$. By [20, Proposition 2] (see also Freedman [25]), which says the Lie-bracket operation is closed for sections in the null space of Levi-form $T^N M^0$, for any vector field Y^0 in $\mathcal{M}_j^0$, $\mathrm{Re}(Y^0)$, $\mathrm{Im}(Y^0) \in T^N(N^0)$ for each j. Hence for any $Y^0 \in \mathcal{M}_j^0$, we have $\langle Y^0, \partial\rho^{[m]}\rangle(0) = 0$, for both the real part and the imaginary part of $Y^0|_0$ are in $\mathrm{Re}(T_0^{(1,0)}N^0)$. This then reduces the rest of the proof to that in the proof of the equality of $t^{(n-2)}(M,0) = a^{(n-2)}(M,0)$. The proof of the equality $c^{(n-2)}(M,p) = a^{(n-2)}(M,p)$ is now complete. □

Finally, we make a remark for the Hörmander number of a subbundle B of $T^{(1,0)}$ at $p \in M$. Let $\mathcal{M}_1$ be as defined in §2 for B. We define the first Hörmander number $\ell_0(B)$ to be the minimum length of Lie-Bracket of sections of $\mathcal{M}_1$ that produces a vector at p no longer in $\mathcal{M}_1$. From our discussion in this and last sections, it is clear that our weight ℓ_0 is $\ell_0(B)$ at $p = 0$.

5. Further application of positivity: proofs of three lemmas

In this section, we prove three lemmas concerning a homogeneous polynomial whose real part is plurisubharmonic. Plurisubharmonicity is an inequality. The basic idea behind the complicated computations of this section is that when the plurisubharmonicity is combined with weighted homogeneous polynomials, we often obtain identities with the help of Hölder inequality. This idea was already appeared in the proof of Lemma 4.5.

We begin with the following two simple folklore lemmas.

Lemma 5.1. *Let $h(\xi, \overline{\xi})$ be a homogeneous polynomial of $(\xi, \overline{\xi}) \in \mathbb{C} \times \mathbb{C}$. Suppose that*

$$hh_{\xi\overline{\xi}} - h_\xi h_{\overline{\xi}} = 0. \tag{5.1}$$

Then h must be a monomial. Namely, $h = c\xi^j\overline{\xi}^k$ for a certain complex number c.

Proof. Suppose that h is not a monomial and takes the following form:

$$h = \alpha\xi^j\overline{\xi}^h + \beta\xi^t\overline{\xi}^s + O(\xi^{t+1}) \text{ with } j < t,\ \alpha, \beta \neq 0.$$

Here and in what follows, we write $O(\xi^k)$ for a homogeneous polynomial with degree in ξ at least k. Then

$$\begin{pmatrix} h & h_\xi \\ h_{\overline{\xi}} & h_{\xi\overline{\xi}} \end{pmatrix} = \begin{pmatrix} \alpha\xi^j\overline{\xi}^h + \beta\xi^t\overline{\xi}^s + O(\xi^{t+1}) & j\alpha\xi^{j-1}\overline{\xi}^h + t\beta\xi^{t-1}\overline{\xi}^s + O(\xi^t) \\ h\alpha\xi^j\overline{\xi}^{h-1} + s\beta\xi^t\overline{\xi}^{s-1} + O(\xi^{t+1}) & jh\alpha\xi^{j-1}\overline{\xi}^{h-1} + ts\beta\xi^{t-1}\overline{\xi}^{s-1} + O(\xi^t) \end{pmatrix}.$$

Thus

$$hh_{\xi\overline{\xi}} - h_\xi h_{\overline{\xi}} = \alpha\beta(ts + jh - th - js)\xi^{j+t-1}\overline{\xi}^{h+s-1} + O(\xi^{j+t}).$$

On the other hand, $j + h = t + s$, $j < t$. Thus $j \neq t$ and $h \neq s$. Hence $ts + jh - th - js = (j - t)(h - s) \neq 0$. Thus $hh_{\xi\overline{\xi}} - h_\xi h_{\overline{\xi}}$ is not identically 0, which contradicts our hypothesis in (5.1). □

Lemma 5.2. *Let $h(z, \overline{z}) = \sum_{I\overline{J}} a_{I\overline{J}} z^I \overline{z^J}$ be a real nonzero plurisubharmonic polynomial in $(z, \overline{z}) \in \mathbb{C}^n \times \mathbb{C}^n$, where $I = (i_1, \cdots, i_n)$, $J = (j_1, \cdots, j_n)$ with $i_l + j_l$ being a fixed positive integer (independent of I, J) denoted by k_l for each $l \in [1, n]$. Assume that $h_{z_1\overline{z_1}} \not\equiv 0$. Then each k_l is even and the coefficient of $\Pi_{l=1}^n |z_l|^{k_l}$ is positive.*

Proof. By the plurisubharmonicity of $h(z, \overline{z})$, we know $h_{z_1\overline{z_1}} \geq 0$. Since $h_{z_1\overline{z_1}} \not\equiv 0$, each k_l is even. Write $z_i = r_i e^{i\theta_i}$. Then for any $R_i \in (0, \infty)$, we have

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_0^{R_1} \cdots \int_0^{R_n} \cdot \int_0^{2\pi} \cdots \int_0^{2\pi} h_{z_1\overline{z_1}} dr_1 \cdots dr_n d\theta_1 \cdots d\theta_n \\ &= \text{the coefficient of } \Pi|z_j|^{k_i} \cdot \text{some positive constant} \geq 0. \end{aligned} \tag{5.2}$$

If the coefficient of $\Pi_{j=1}^n |z_j|^{k_j}$ is 0, then the above integral is 0. Combining with $h_{z_1\overline{z_1}} \geq 0$, we obtain $h_{z_1\overline{z_1}} \equiv 0$. This contradicts our assumption that $h_{z_1\overline{z_1}} \not\equiv 0$. This proves Lemma 5.2. □

Lemma 5.3. *Let $B(z_1, \overline{z_1})$, $f(z_2, \overline{z_2})$ and $g(z_2, \overline{z_2})$ be three homogeneous polynomials of degree $k \geq 2$, $m \geq 1$ and $m \geq 1$, respectively, in the ordinary sense with $B(z_1, \overline{z_1}) \not\equiv 0$, $f(z_2, \overline{z_2}) \not\equiv 0$. Suppose that $B(z_1, 0) = B(0, \overline{z_1}) = 0$. Suppose that $F = Bf + z_1^k g$ with ReF being a non zero plurisubharmonic polynomial without any non-trivial holomorphic term. Then k and m are even. Moreover $g \equiv 0$ and $ReF = \alpha|z_1|^k|z_2|^m$ for some $\alpha > 0$.*

Proof. By the assumption that ReF is non-zero and plurisubharmonic, $(Re(F))_{z_1\overline{z_1}} \geq 0$. Since $B(z_1, 0) = B(0, \overline{z_1}) = 0$ and ReF contains no non-trivial holomorphic terms, one further concludes that $(Re(F))_{z_1\overline{z_1}}$ is not identically 0. By Lemma 5.2, m and k are even. Set $k = 2k_3$ and $m = 2m_3$. Write

$$B = \sum_{j+h=k} B_{jh} z_1^j \overline{z_1}^h, \; f = \sum_{t+s=m} f_{ts} z_2^t \overline{z_2}^s, \; g = \sum_{t+s=m} g_{ts} z_2^t \overline{z_2}^s.$$

First we claim that $B_{k_3k_3} \neq 0$ and $f_{m_3m_3} \neq 0$. Otherwise the coefficient of the $|z_1|^{2k_3-2}|z_2|^{2m_3}$ in $(Re(F))_{z_1\overline{z_1}}$ is zero, and thus by Lemma 5.2, we reach a contradiction. After writing $F = cB \cdot \frac{1}{c} f + z_1^k g$, we can assume that $B_{k_3k_3} = 1$.

By the plurisubharmonicity of Re(F), we have

$$(\text{Re}F)_{z_1\overline{z_1}}(\text{Re}F)_{z_2\overline{z_2}} - (\text{Re}F)_{z_1\overline{z_2}}(\text{Re}F)_{z_2\overline{z_1}} \geq 0. \tag{5.3}$$

The idea behind the next complicated computation is to write the left hand side of (5.3) into a negative sum of squares modified some terms under control so that the Hölder inequality can be applied. This is made possible due to the homogeneity of the functions under study.

Notice that

$$2(\text{Re}F)_{z_1\overline{z_1}} = B_{z_1\overline{z_1}} f + \overline{B}_{z_1\overline{z_1}} \overline{f}, \; 2(\text{Re}F)_{z_2\overline{z_2}} = B f_{z_2\overline{z_2}} + \overline{Bf}_{z_2\overline{z_2}} + 2\text{Re}(z_1^k g_{z_2\overline{z_2}}). \tag{5.4}$$

Thus

$$4(\text{Re}F)_{z_1\overline{z_1}}(\text{Re}F)_{z_2\overline{z_2}} = 2\text{Re}\Big(BB_{z_1\overline{z_1}} f f_{z_2\overline{z_2}} + \overline{B} B_{z_1\overline{z_1}} f \overline{f}_{z_2\overline{z_2}} + B_{z_1\overline{z_1}} f \cdot 2\text{Re}(z_1^k g_{z_2\overline{z_2}})\Big). \tag{5.5}$$

The coefficients of $|z_1|^{2k-2}$ in $BB_{z_1\overline{z_1}}$ and $\overline{B}B_{z_1\overline{z_1}}$ are, respectively,

$$\sum_{j+h=k} jh B_{jh} B_{hj}, \quad \sum_{j+h=k} jh|B_{hj}|^2.$$

The coefficients of $|z_2|^{2m-2}$ in $f f_{z_2\overline{z_2}}$ and $f\overline{f}_{z_2\overline{z_2}}$ are, respectively,

$$\sum_{t+s=m} ts f_{ts} f_{st}, \quad \sum_{t+s=m} ts|f_{ts}|^2$$

Notice that $B_{z_1\overline{z_1}} f \cdot \text{Re}(z_1^k g_{z_2\overline{z_2}})$ is not divisible by $|z_1|^{2k-2}$ (unless it is identically zero). Hence the coefficient of $|z_1|^{2k-2}|z_2|^{2m-2}$ in $4(\text{Re}f)_{z_1\overline{z_1}}(\text{Re}f)_{z_2\overline{z_2}}$ is

$$\sum_{j+h=k, t+s=m} 2\text{Re}\Big(jh B_{jh} B_{hj} ts f_{ts} f_{st} + jh|B_{hj}|^2 ts|f_{ts}|^2\Big). \tag{5.6}$$

We similarly compute the coefficient of $|z_1|^{2k-2}|z_2|^{2m-2}$ in $4(\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}}$ as follows:

$$\begin{aligned}2(\mathrm{Re}F)_{z_1\overline{z_2}} =& B_{z_1}f_{\overline{z_2}} + \overline{B}_{z_1}\overline{f}_{\overline{z_2}} + kz_1^{k-1}g_{\overline{z_2}},\\ 2(\mathrm{Re}F)_{z_2\overline{z_1}} =& B_{\overline{z_1}}f_{z_2} + \overline{B}_{\overline{z_1}}\overline{f}_{z_2} + k\overline{z_1}^{k-1}\overline{g}_{z_2}.\end{aligned} \tag{5.7}$$

Thus

$$\begin{aligned}4(\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}} =& B_{z_1}B_{\overline{z_1}}f_{z_2}f_{\overline{z_2}} + B_{z_1}\overline{B}_{\overline{z_1}}\overline{f}_{z_2}f_{\overline{z_2}}\\ &+ B_{\overline{z_1}}\overline{B}_{z_1}f_{z_2}\overline{f}_{\overline{z_2}} + \overline{B}_{z_1}\overline{B}_{\overline{z_1}}\overline{f}_{\overline{z_2}}\overline{f}_{z_2}\\ &+ 2\mathrm{Re}\Big(kz_1^{k-1}g_{\overline{z_2}}(B_{\overline{z_1}}f_{z_2} + \overline{B}_{\overline{z_1}}\overline{f}_{z_2})\Big) + k^2|z_1|^{2k-2}|g_{\overline{z_2}}|^2.\end{aligned} \tag{5.8}$$

The coefficients of $|z_1|^{2k-2}$ in $B_{z_1}B_{\overline{z_1}}$, $B_{z_1}\overline{B}_{\overline{z_1}}$, $B_{\overline{z_1}}\overline{B}_{z_1}$ and $\overline{B}_{z_1}\overline{B}_{\overline{z_1}}$ are, respectively

$$\sum_{j+h=k} h^2 B_{hj}B_{jh}, \quad \sum_{j+h=k} h^2|B_{hj}|^2, \quad \sum_{j+h=k} j^2|B_{hj}|^2, \quad \sum_{j+h=k} j^2\overline{B_{hj}B_{jh}}.$$

The coefficients of $|z_2|^{2m-2}$ in $f_{\overline{z_2}}f_{z_2}$, $f_{\overline{z_2}}\overline{f}_{z_2}$, $f_{z_2}\overline{f}_{\overline{z_2}}$ and $\overline{f}_{\overline{z_2}}\overline{f}_{z_2}$ are, respectively,

$$\sum_{t+s=m} s^2 f_{ts}f_{st}, \quad \sum_{t+s=m} s^2|f_{ts}|^2, \quad \sum_{t+s=m} t^2|f_{ts}|^2, \quad \sum_{t+s=m} t^2\overline{f_{ts}f_{st}}.$$

Notice that $kz_1^{k-1}g_{\overline{z_2}}(B_{\overline{z_1}}f_{z_2} + \overline{B}_{\overline{z_1}}\overline{f}_{z_2})$ is not divisible by $|z_1|^{2k-2}$ (when not identically zero). Hence the coefficient of $|z_1|^{2k-2}|z_2|^{2m-2}$ in $4(\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}}$ is

$$\begin{aligned}\sum_{j+h=k,t+s=m} &\Big(h^2B_{jh}B_{hj}s^2f_{ts}f_{st} + h^2|B_{hj}|^2s^2|f_{ts}|^2 + j^2|B_{hj}|^2t^2|f_{ts}|^2\\ &+ j^2\overline{B_{jh}B_{hj}}t^2\overline{f}_{ts}\overline{f}_{st}\Big) + \sum_{t+s=m} k^2s^2|g_{ts}|^2.\end{aligned}$$

Hence the coefficient of $|z_1|^{2k-2}|z_2|^{2m-2}$ in $4(\mathrm{Re}F)_{z_1\overline{z_1}}(\mathrm{Re}F)_{z_2\overline{z_2}} - 4(\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}}$ is

$$\begin{aligned}&\sum_{j+h=k,t+s=m} \Big\{2\mathrm{Re}\big(jhB_{jh}B_{hj}tsf_{ts}f_{st} + jh|B_{hj}|^2ts|f_{ts}|^2\big) - \big(h^2B_{hj}B_{jh}s^2f_{ts}f_{st}\\ &+ h^2|B_{hj}|^2s^2|f_{ts}|^2 + j^2|B_{hj}|^2t^2|f_{ts}|^2 + j^2\overline{B_{jh}B_{hj}}t^2\overline{f}_{ts}\overline{f}_{st}\big)\Big\} - \sum_{t+s=m} k^2s^2|g_{ts}|^2\\ =& -\sum_{j+h=k,t+s=m} \Big\{(hs-jt)^2|B_{hj}|^2|f_{ts}|^2 + hs(hs-jt)B_{jh}B_{hj}f_{ts}f_{st}\\ &+ jt(jt-hs)\overline{B}_{jh}\overline{B}_{hj}\overline{f}_{ts}\overline{f}_{st}\Big\} - \sum_{t+s=m} k^2s^2|g_{ts}|^2\\ =& -\sum_{h\le j,t\le s} \Gamma_{hj}^{ts}\Big\{(hs-jt)^2|B_{hj}|^2|f_{ts}|^2 + (js-ht)^2|B_{jh}|^2|f_{ts}|^2 + (ht-js)^2|B_{hj}|^2|f_{st}|^2\\ &+ (jt-hs)^2|B_{jh}|^2|f_{st}|^2 + \Big(hs(hs-jt) + js(js-ht) + ht(ht-js) + jt(jt-hs)\Big)B_{jh}B_{hj}f_{ts}f_{st}\\ &+ \Big(jt(jt-hs) + ht(ht-js) + js(js-ht) + hs(hs-jt)\Big)\overline{B}_{jh}\overline{B}_{hj}\overline{f}_{ts}\overline{f}_{st}\Big\} - \sum_{t+s=m} k^2s^2|g_{ts}|^2\\ =& -\sum_{h\le j,t\le s} \Gamma_{hj}^{ts}\Big\{(hs-jt)^2|B_{hj}|^2|f_{ts}|^2 + (js-ht)^2|B_{jh}|^2|f_{ts}|^2 + (ht-js)^2|B_{hj}|^2|f_{st}|^2\end{aligned}$$

$$+ (jt-hs)^2|B_{jh}|^2|f_{st}|^2 + \Big((hs-jt)^2+(js-ht)^2\Big)B_{jh}B_{hj}f_{ts}f_{st}$$
$$+ \Big((ht-js)^2+(jt-hs)^2\Big)\overline{B}_{jh}\overline{B}_{hj}\overline{f}_{ts}\overline{f}_{st}\Big\} - \sum_{t+s=m} k^2s^2|g_{ts}|^2.$$

Here we have set

$$\Gamma^{ts}_{hj} = \begin{cases} 1 & h<j,\ t<s, \\ \frac{1}{2} & h=j,\ t<s \text{ or } h<j,\ t=s, \\ 0 & h=j,\ t=s. \end{cases}$$

Notice, by the Hölder inequality, that

$$\begin{aligned} &|\Big((js-ht)^2+(hs-jt)^2\Big)B_{jh}B_{hj}f_{ts}f_{st} + \Big((ht-js)^2+(jt-hs)^2\Big)\overline{B}_{jh}\overline{B}_{hj}\overline{f}_{ts}\overline{f}_{st}| \\ &\le (js-ht)^2(|B_{hj}f_{st}|^2+|B_{jh}f_{ts}|^2) + (jt-hs)^2(|B_{hj}f_{ts}|^2+|B_{jh}f_{st}|^2). \end{aligned} \tag{5.9}$$

Thus we see that the coefficient of $|z_1|^{2k-2}|z_2|^{2m-2}$ in

$$4(\mathrm{Re}F)_{z_1\overline{z_1}}(\mathrm{Re}F)_{z_2\overline{z_2}} - 4(\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}}$$

is non positive. Furthermore, this coefficient is 0 if and only if for $h\le j$, $t\le s$ and for any $j^*+l^*=m-1$ with $l^*\ne 0$:

$$B_{hj}f_{st} = -\overline{B_{jh}f_{ts}} \text{ for } js\ne ht,\ B_{jh}f_{ts} = -\overline{B_{hj}f_{st}} \text{ for } jt\ne hs,\ g_{j^*l^*}=0. \tag{5.10}$$

Hence, we conclude from (5.3) that (5.10) holds and moreover

$$(\mathrm{Re}F)_{z_1\overline{z_1}}(\mathrm{Re}F)_{z_2\overline{z_2}} - (\mathrm{Re}F)_{z_1\overline{z_2}}(\mathrm{Re}F)_{z_2\overline{z_1}} = 0. \tag{5.11}$$

Since ReF and Bf contain no non-trivial holomorphic terms, we see $g\equiv 0$.

We next prove that $\mathrm{Re}F = \alpha|z_1|^k|z_2|^m$ for some $\alpha>0$ to complete the proof of the lemma. To this aim, setting $j=h=k_3$ in (5.10) and using the normalization that $B_{k_3k_3}=1$, we obtain $f_{ts} = -\overline{f_{st}}$ for $t\ne s$.

Now, if f is of the form $f = f_{m_3m_3}|z_2|^m$ and $\mathrm{Re}F = |z_2|^m p(z_1,\overline{z_1})$, then (5.11) is equivalent to

$$pp_{z_1\overline{z_1}} - p_{z_1}p_{\overline{z_1}} = 0.$$

By Lemma 5.1, p is a monomial. On the other hand, since p is real valued, $p=\alpha|z_1|^k$ for some $\alpha>0$. Namely, $\mathrm{Re}F = \alpha|z_1|^k|z_2|^m$. This proves the lemma.

For the rest of the proof, we suppose that f is not of the form $f = f_{m_3m_3}|z_2|^m$. Since $f_{m_3m_3}\ne 0$, f is not a monomial.

We can now write

$$\mathrm{Re}F = z_1^h\overline{z_1}^j q(z_2,\overline{z_2}) + O(z_1^{h+1}),\ q\ne 0. \tag{5.12}$$

Since $B(z_1,0)=B(0,z_1)=0$, we have $h,j\ge 1$. From (5.11), we get

$$hjz_1^{2h-1}\overline{z_1}^{2j-1}(qq_{z_2\overline{z_2}} - q_{z_2}q_{\overline{z_2}}) + O(z_1^{2h}) = 0.$$

This gives

$$qq_{z_2\overline{z_2}} - q_{z_2}q_{\overline{z_2}} = 0,$$

which further forces q to be a monomial. In the following, let h, j be as in (5.12).

(1) If $B_{hj} = 0$ or $B_{jh} = 0$, then $q = \frac{1}{2}\overline{B_{jh}f}$ or $q = \frac{1}{2}B_{hj}f$, respectively. In either case, since f is not a monomial, q is not a monomial and thus we reach a contradiction.

(2) Assume that $B_{hj} \neq 0$, $B_{jh} \neq 0$ and $h < j$. In this case, $B_{hj}f_{m_3m_3} = -\overline{B_{jh}f_{m_3m_3}}$. Hence $q \neq c|z_2|^m$ for some constant c.

Setting $h = j$ in (5.10), we see $f_{ts} = -\overline{f_{st}}$ for $t \neq s$. Thus $B_{hj}f_{m_3m_3}|z_2|^m + \overline{B_{jh}f_{m_3m_3}|z_2|^m} = 0$ and $f - f_{m_3m_3}|z_2|^m = -\overline{(f - f_{m_3m_3}|z_2|^m)}$. Hence Re$F$ can be computed as follows:

$$\begin{aligned} Re(F) =& \frac{1}{2}(B_{hj}f + \overline{B_{jh}f})z_1^h\overline{z_1}^j + O(z_1^{h+1}) \\ =& \frac{1}{2}(B_{hj} - \overline{B_{jh}})z_1^h\overline{z_1}^j \cdot (f - f_{m_3m_3}|z_2|^m) + O(z_1^{h+1}). \end{aligned} \tag{5.13}$$

Thus we conclude that $q = \frac{1}{2}(B_{hj} - \overline{B_{jh}})(f - f_{m_3m_3}|z_2|^m)$, which can not be a monomial for f is not a monomial. This thus gives a contradiction.

Hence we must have $h \geq j$. But from the reality of ReF and our choice of h, we must have $h = j$ and $B = |z_1|^k$. Hence ReF takes the form $\frac{1}{2}|z_1|^k(f(z_2, \overline{z_2}) + \overline{f(z_2, \overline{z_2})})$. Since $f_{ts} = -\overline{f_{st}}$ for $s \neq t$, we conclude that ReF takes the form $\alpha|z_1|^k|z_2|^m$ with $\alpha > 0$.

This finally completes the proof of the lemma. $\square$

6. Proof of Theorem 6.1

In this section, we provide a detailed proof of Theorem 6.1, which played a key role in the proof of our main theorem. We write $z = (z_1, z_2)$ for the coordinates in $\mathbb{C}^2$ in this section.

Theorem 6.1. *Define the weight of z_1 and $\overline{z_1}$ to be 1, the weight of z_2 and $\overline{z_2}$ to be $k \in \mathbb{N}$ with $k > 1$. Let $A = A(z_1, \overline{z_1})$ be a homogenous polynomial of degree $k-1$ in $(z_1, \overline{z_1})$ without non-trivial holomorphic terms. Suppose that f is a weighted homogeneous polynomial in $(z, \overline{z})$ of weighted degree $m > k$. Further assume that $Re(f)$ is plurisubharmonic, contains no non-trivial holomorphic terms and assume that f satisfies the following equation:*

$$f_{\overline{z_1}}(z, \overline{z}) + \overline{A(z_1, \overline{z_1})}f_{\overline{z_2}}(z, \overline{z}) = 0. \tag{6.1}$$

Then $Re(f) \equiv 0$.

Without the plurisubharmonicity on Re(f), the above theorem can not be true as the following simple example demonstrates:

Example 6.2. Let $L = \frac{\partial}{\partial z_1} - |z_1|^2\frac{\partial}{\partial z_2}$, $k = 3$ and let $f = z_1\overline{z_2} + \frac{1}{2}|z_1|^4$. Then $\overline{L}(f) \equiv 0$. Notice that Re(f) is not plurisubharmonic neither is 0. Notice that $A = \overline{A} = -|z_1|^2$, Re$(f)(\not\equiv 0)$ has no non-trivial holomorphic terms.

We also mention that in Theorem 6.1, we can not conclude $f \equiv 0$ as demonstrated by the following example:

Example 6.3. Let $L = \frac{\partial}{\partial z_1} + kz_1^k\overline{z_1^{k-1}}\frac{\partial}{\partial z_2}$ and $f = i(z_2 + \overline{z_2} - |z_1|^{2k})^2$. The weight of z_2 and $\overline{z_2}$ are $2k$. Then $\overline{L}f = 0$ and Re$(f) \equiv 0$. However $f \not\equiv 0$.

Remarks 6.4. Theorem 6.1 also holds if we simply assume that f is real analytic near the origin. Then we just need to do a weighted Taylor expansion of f at the origin and apply Theorem 6.1 inductively on each weighted truncation.

Proof of Theorem 6.1. The proof of Theorem 6.1 is long. The idea is to find a good use of the plurisubharmonicity of $\mathrm{Re}(f)$. We will proceed according to the four different scenarios, two of which are reduced to CR equations along finite type hypersurfaces where Proposition 4.6 can be applied. (Hence, plurisubharmonicity is used to apply the Hopf lemma.) The other two easier scenarios are treated by a formal theory method with the help of Lemma 5.3.

Recall that the degree of A is $k-1$ and the weight of z_2 and $\overline{z_2}$ is k.

For $0 \le j \le [\frac{m}{k}] := m_0$, denoted by $f^{[j]}$ the sum of terms (monomial terms) in f which has ordinary degree j in z_2 and $\overline{z_2}$. Then

$$f = f^{[m_0]} + f^{[m_0-1]} + \cdots + f^{[0]}.$$

In the course of the proof, for $j = 1, 2$, we write $O(|z_j|^k)$ for a homogeneous polynomial with (the ordinary or un-weighted) degree in z_j and $\overline{z_j}$ at least k. We also denote by $L(|z_j|^k)$ a homogeneous polynomial with the un-weighted degree in z_j and $\overline{z_j}$ at most k. For a homogeneous polynomial $P = \sum_{h+j=l} C_{hj} z_1^h \overline{z_1}^j$, we denote the integral of P along $\overline{z_1}$ as

$$F(P) = \sum_{h+j=l} \frac{1}{j+1} C_{hj} z_1^h \overline{z_1}^{j+1}. \tag{6.2}$$

We remark that after a transformation of the form: $(z_1, z_2) \to (z_1, \delta^{-1} z_2)$, A and f, in the new coordinates still denoted by (z_1, z_2), takes the form

$$\delta^{-1} A \text{ and } f(z_1, \delta z_2, \overline{z_1}, \overline{\delta}\overline{z_2}). \tag{6.3}$$

We will need this transformation to normalize certain coefficients in our proof.

Case I: In this case, we suppose $km_0 < m$ or $km_0 = m$, $f^{[m_0]} = 0$.

Suppose h is the largest integer such that $f^{[h]} \not\equiv 0$. From (6.1), $f^{[h]}$ is holomorphic in z_1. We suppose that

$$f^{[h]} = z_1^j \sum_{t+s=h} f_{ts} z_2^t \overline{z_2}^s, \quad \text{here } j + kh = m. \tag{6.4}$$

We then have $j \ge 1$. Since $\mathrm{Re} f$ contains no non-trivial holomorphic terms, $f_{h0} = 0$. In particular, we see that we must have $h \ge 1$. In what follows, we regard any term with a negative power in some variable to be zero to simplify the notations.

First, we claim $f_{ts} = 0$ for any $t \ge 1$. Since $\mathrm{Re} f$ is plurisubharmonic, we obtain

$$\big(\mathrm{Re}(f)\big)_{z_2\overline{z_2}} = \big(\mathrm{Re}(f^{[h]})\big)_{z_2\overline{z_2}} + L(|z_2|^{h-3}) \ge 0.$$

For $\frac{1}{2} \le |z_1| \le 1$, we then have $C_0 >> 1$ such that whenever $|z_2| \ge C_0$ we have

$$z_1^j \sum_{t+s=h,t,s\ge1} ts f_{ts} z_2^{t-1} \overline{z_2}^{s-1} + \overline{z_1}^j \sum_{t+s=h,t,s\ge1} ts \overline{f_{ts}} z_2^{s-1} \overline{z_2}^{t-1} \ge 0.$$

Since $j \geq 1$, this is possible only when the left hand side is identically 0. This implies that $f_{ts} = 0$ for all $t, s \geq 1$ and thus for $t \geq 1$. Thus

$$f^{[h]} = f_{0h} z_1^j \overline{z_2}^h \text{ with } j, h \geq 1,\ j + kh = m.$$

Then

$$\mathrm{Re}(f^{[h]}) = \frac{1}{2} f_{0h} z_1^j \overline{z_2}^h + \frac{1}{2} \overline{f_{0h}} z_2{}^h \overline{z_1}^j.$$

Since $\mathrm{Re}(f)$ is plurisubharmonic, we have

$$\big(\mathrm{Re}(f)\big)_{z_1\overline{z_1}} \big(\mathrm{Re}(f)\big)_{z_2\overline{z_2}} - \big(\mathrm{Re}(f)\big)_{z_1\overline{z_2}} \big(\mathrm{Re}(f)\big)_{z_2\overline{z_1}} \geq 0.$$

Notice that

$$\begin{aligned}
&\big(\mathrm{Re}(f)\big)_{z_1\overline{z_1}} = O(|z_1|^{j-1}),\ \big(\mathrm{Re}(f)\big)_{z_2\overline{z_2}} = O(|z_1|^{j+1}) \\
&\big(\mathrm{Re}(f)\big)_{z_1\overline{z_2}} = \frac{1}{2} f_{0h} j h z_1^{j-1} \overline{z_2}^{h-1} + O(|z_1|^j), \\
&\big(\mathrm{Re}(f)\big)_{z_2\overline{z_1}} = \frac{1}{2} \overline{f_{0h}} j h z_2^{h-1} \overline{z_1}^{j-1} + O(|z_1|^j).
\end{aligned}$$

Hence

$$\begin{aligned}
&\big(\mathrm{Re}(f)\big)_{z_1\overline{z_1}} \big(\mathrm{Re}(f)\big)_{z_2\overline{z_2}} - \big(\mathrm{Re}(f)\big)_{z_1\overline{z_2}} \big(\mathrm{Re}(f)\big)_{z_2\overline{z_1}} \\
&= -\frac{1}{4} j^2 h^2 |f_{0h}|^2 |z_1|^{2j-2} |z_2|^{2h-2} + O(|z_1|^{2j-1}) \geq 0.
\end{aligned} \tag{6.5}$$

Now, for each fixed z_2 and letting $|z_1|$ sufficiently small, we get $-j^2 h^2 |f_{0h}|^2 |z_1|^{2j-2} |z_2|^{2h-2} \geq 0$. Hence $f_{0h} = 0$, which means that $f^{[h]} \equiv 0$. This contradicts our assumption that $f^{[h]} \not\equiv 0$. This completes the proof in Case I, for we must then have $f^{[h]} \equiv 0$ for any h.

Case II: We now assume that $km_0 = m$, $\mathrm{Re}(f^{[m_0]}) \not\equiv 0$.

Suppose

$$f^{[m_0]} = \sum_{t+s=m_0} f_{ts} z_2^t \overline{z_2}^s.$$

Since $\mathrm{Re}(f^{[m_0]})$ contains no non-trivial holomorphic terms, we have $f_{0m_0} = -\overline{f_{m_0 0}}$. By the plurisubharmonicity of $\mathrm{Re}(f)$, we get $(\mathrm{Re}(f^{[m_0]}))_{z_2\overline{z_2}} \geq 0$ and can not be identically zero. By Lemma 5.2, m_0 is even and $\mathrm{Re} f_{m_1 m_1} > 0$. Here $m_0 = 2m_1$.

After a rotational transformation of the form $(z_1, z_2) \to (z_1, \delta^{-1} z_2)$ for some constant $\delta \neq 0$, by (6.3), we can make

$$f_{(m_1-1)(m_1+1)} = c\overline{f_{m_1 m_1}} \text{ for a certain } c \geq 0. \tag{6.6}$$

We remark that this transformation does not change our original hypotheses in this case. Now (6.1) can be solved as

$$f = -F(\overline{A}) f_{\overline{z_2}} + \sum_{j=0}^{m_0} z_1^{jk} h^{[m_0-j]}(z_2, \overline{z_2}),\ h^{[m_0-j]}(z_2, 0) = 0,\ \text{for each } j. \tag{6.7}$$

In particular, we get

$$f^{[m_0-1]} = -F(\overline{A}) \cdot f^{[m_0]}_{\overline{z_2}} + z_1^k g^{[m_0-1]}(z_2, \overline{z_2}),\ g(z_2, 0) = 0. \tag{6.8}$$

By the plurisubharmonicity of Ref, we have $(\mathrm{Re}f)_{z_1\overline{z_1}} \geq 0$. Notice that $F(\overline{A})$ is divisible by $|z_1|^2$. Hence

$$(\mathrm{Re}f)_{z_1\overline{z_1}} = (\mathrm{Re}f^{[m_0-1]})_{z_1\overline{z_1}} + L(|z_2|^{m_0-2}) \geq 0.$$

Hence

$$(\mathrm{Re}f^{[m_0-1]})_{z_1\overline{z_1}} \geq 0.$$

Notice that the (ordinary) degree of $(\mathrm{Re}f^{[m_0-1]})_{z_1\overline{z_1}}$ in z_2 and $\overline{z_2}$ is $m_0 - 1$ which is an odd number, we have $(\mathrm{Re}f^{[m_0-1]})_{z_1\overline{z_1}} \equiv 0$. Again since $F(\overline{A})$ is divisible by $|z_1|^2$, it follows from (6.8) that $\mathrm{Re}(F(\overline{A}) \cdot f^{[m_0]}_{\overline{z_2}}) = 0$.

Next, write $\overline{A} = \sum_{j+h=k-1,h\geq 1} \overline{A_{jh}} z_1^h \overline{z_1}^j$. Then

$$F(\overline{A}) = \sum_{j+h=k-1,h\geq 1} \frac{1}{j+1} \overline{A_{jh}} z_1^h \overline{z_1}^{j+1}.$$

Hence

$$\mathrm{Re}(F(\overline{A}) \cdot f^{[m_0]}_{\overline{z_2}}) = \mathrm{Re}\Big(\sum_{j+h=k-1,h\geq 1} \frac{1}{j+1} \overline{A_{jh}} z_1^h \overline{z_1}^{j+1} \cdot \sum_{t+s=m_0,s\geq 1} s f_{ts} z_2^t \overline{z_2}^{s-1}\Big) = 0. \tag{6.9}$$

Hence for $h + j = k$, $t + s = m_0 - 1$, we have

$$\frac{1}{j}\overline{A_{(j-1)h}} \cdot (s+1) f_{t(s+1)} = -\frac{1}{h}\overline{\overline{A_{(h-1)j}} \cdot (t+1) f_{s(t+1)}}. \tag{6.10}$$

Setting $t = m_1 - 1$, $s = m_1$ in the above equation and making use of (6.6), we get

$$\frac{1}{j}\overline{A_{(j-1)h}} \cdot (m_1 + 1)c = -\frac{1}{h}\overline{\overline{A_{(h-1)j}} \cdot m_1}. \tag{6.11}$$

If $c = 0$, then $A_{(h-1)j} = 0$ for all $h + j = k,\ h \geq 1,\ j \geq 1$. This implies that $A \equiv 0$, which is impossible. Thus $c \neq 0$. From (6.11), we get

$$A_{(j-1)h} A_{(h-1)j} \leq 0 \text{ and the equality holds only when } A_{(j-1)h} = A_{(h-1)j} = 0. \tag{6.12}$$

Next, by (6.1), (6.7), we compute the following:

$$f^{[m_0-2]} = F(\overline{A}F(\overline{A})) \cdot f^{[m_0]}_{\overline{z_2}^2} - F(\overline{A}z_1^k)\phi^{[m_0-1]}_{\overline{z_2}}(z_2, \overline{z_2}) + z_1^{2k}\phi^{[m_0-2]}(z_2, \overline{z_2}).$$

We will compute the coefficient of $|z_1|^{2k}|z_2|^{m_0-2}$ in $f^{[m_0-2]}$. First, the coefficient of $|z_2|^{m_0-2}$ in $f^{[m_0]}_{\overline{z_2}^2}$ is $(m_1 + 1)m_1 f_{(m_1-1)(m_1+1)}$. Notice that

$$\overline{A}F(\overline{A}) = \sum_{j+h=k-1} \overline{A_{jh}} z_1^h \overline{z_1}^j \cdot \sum_{t+s=k-1} \frac{1}{t+1} \overline{A_{ts}} z_1^s \overline{z_1}^{t+1} \tag{6.13}$$

Hence

$$F(\overline{A}F(\overline{A})) = \sum_{j+h=k-1, t+s=k-1} \frac{1}{(t+1)(j+t+2)} \overline{A_{jh}A_{ts}} z_1^{h+s}\overline{z_1}^{j+t+2} \tag{6.14}$$

When $k = h+s = j+t+2,\ j+h = k-1,\ t+s = k-1$, we have $j = k-1-h,\ t = h-1,\ s = k-h$. Hence the coefficient in $F(\overline{A}F(\overline{A}))$ with the factor $|z_1|^{2k}$ is

$$\sum_{1 \le h \le k-1} \frac{1}{hk} \overline{A_{(k-1-h)h}A_{(h-1)(k-h)}} := H.$$

By (6.12), $H \le 0$. Moreover $H = 0$ if and if $A_{(k-1-h)h} = 0$ for all $h \ge 1$, which is equivalent to $A = 0$. This is impossible and thus $H < 0$.

Notice that $\overline{A}$ is divisible by z_1, thus $F(\overline{A}z_1^k)$ does not contain $|z_1|^{2k}$ term.

Thus the coefficient of $|z_1|^{2k}|z_2|^{m_0-2}$ in $f^{[m_0-2]}$ is $(m_1+1)m_1 f_{(m_1-1)(m_1+1)}H$. Recall that $\mathrm{Re} f_{m_1m_1} > 0$ and $c > 0$. Together with (6.6), we get $\mathrm{Re} f_{(m_1-1)(m_1+1)} > 0$. Hence the real part of the coefficient of $|z_1|^{2k}|z_2|^{m_0-2}$ in $f^{[m_0-2]}$ must be negative. This contradicts the following

$$\Big(\mathrm{Re}(f^{[m_0-2]})\Big)_{z_1\overline{z_1}} \ge 0,$$

which is true due to the fact that $\Big(\mathrm{Re}(f^{[m_0]})\Big)_{z_1\overline{z_1}}$ and $\Big(\mathrm{Re}(f^{[m_0-1]})\Big)_{z_1\overline{z_1}} = 0$.

The following two cases are more subtle. Fortunately, we have more geometry in these two settings to enable us to Proposition 4.6.

Case III: $m = km_0$, $f^{[m_0]} \ne 0$, $\mathrm{Re}(f^{[m_0]}) = 0$ and $\mathrm{Re}(f^{[m_0-1]}) \ne 0$.

Here, we reduce f to the solution of a CR vector field of a real hypersurface of finite type in $\mathbb{C}^2$ and then apply Proposition 4.6 to reach a contradiction. Write $B := -F(\overline{A}) = \sum_{j+h=k} B_{jh} z_1^j \overline{z_1}^h$. By Lemma 5.2, both k and $m_0 - 1$ are even. Define $k = 2k_2$, $m_0 = 2m_2 + 1$. Then $B_{k_2k_2} \ne 0$ by Lemma 5.2, which implies that $A_{(k_2-1)k_2} \ne 0$. After a dilation transform of the form as in (6.3), we assume that $A_{(k_2-1)k_2} = -k_2$. Then $B_{k_2k_2} = 1$. A direct computation shows

$$f^{[m_0-1]} = Bf^{[m_0]}_{\overline{z_2}} + z_1^k g(z_2, \overline{z_2}).$$

From our assumption, $\mathrm{Re} f^{[m_0-1]}$ is plurisubharmonic. By Lemma 5.3,

$$g(z_2, \overline{z_2}) = 0,\ \mathrm{Re}(f^{[m_0-1]}) = \mathrm{Re}(Bf^{[m_0]}_{\overline{z_2}}) = \lambda |z_1|^k |z_2|^{m_0-1},\ \lambda > 0. \tag{6.15}$$

Notice that $B_{k_2k_2} = 1$ and $(m_2+1)\mathrm{Re}(f_{m_2(m_2+1)}) = \lambda \ne 0$. Since $\mathrm{Re}(f^{[m_0]}) = 0$, we have $f_{(m_2+1)m_2} + \overline{f_{m_2(m_2+1)}} = 0$. Notice that $f^{[m_0]}_{\overline{z_2}} - (m_2+1)f_{m_2(m_2+1)}|z_2|^{2m_2}$ has no term divisible by $|z_2|^{2m_2}$. Hence we conclude from (6.15)

$$\mathrm{Re}(Bf_{m_2(m_2+1)}(m_2+1))|z_2|^{m_0-1} = \lambda |z_1|^k |z_2|^{m_0-1}.$$

Collecting terms divisible by $z_2^{m_2+1}\overline{z_2}^{m_2-1}$ in (6.15), we get

$$m_2 Bf_{(m_2+1)m_2} + \overline{B(m_2+2)f_{(m_2-1)(m_2+2)}} = 0.$$

Hence B is different from $\overline{B}$ by a constant. Since we normalized $B_{k_2k_2} = 1$, we see that B is real-valued. But $f^{[m_0]}_{\overline{z_2}}$ contains a term of the form $\mu|z_2|^{m_0-1}$ with $\mathrm{Re}\mu \ne 0$. Thus $-F(\overline{A}) = |z_1|^k$, namely, $A = -k_2 z_1^{k_2-1}\overline{z_1}^{k_2}$.

Now, $L = \frac{\partial}{\partial z_1} + A(z_1, \overline{z_1})\frac{\partial}{\partial z_2}$ forms a basis for the sections of CR vector fields along the real algebraic finite type hypersurface M_0 in $\mathbb{C}^2$ defined by $-z_2 - \overline{z_2} = |z_1|^{2k_2}$ and $\overline{L}(f) \equiv 0$. Thus f is a CR polynomial on M_0 and $g = f(z_1, \overline{z_1}, z_2, -z_2 - |z_1|^{2k_2})$ is a weighted homogeneous holomorphic polynomial of degree $m > k$. Since $f - g \equiv 0$ over M_0, M_0 is contained in the zero set of the plurisubharmonic $\rho = \mathrm{Re}(f - g)$ with $0 \in M_0$. Notice that $\rho = O(|z|^2)$, we conclude by Proposition 4.6 that $\rho \equiv 0$ or $\mathrm{Re}(f)$ is pluriharmonic. This is a contradiction. Hence Case III cannot occur.

Case IV: $m = km_0$, $f^{[m_0]} \neq 0$ but $\mathrm{Re}(f^{[m_0]}) = \mathrm{Re}(f^{[m_0-1]}) = 0$.

Write

$$A = \sum_{h+j=k-1} A_{hj} z^h \overline{z}^j, \; B = -F(\overline{A}) = \sum_{h+j=k} B_{hj} z^h \overline{z}^j, \; f^{[m_0]} = \sum_{t+s=m_0} f_{ts} z_2^t \overline{z_2}^s.$$

Then by our assumption that $\mathrm{Re}(f^{[m_0]}) = \mathrm{Re}(f^{[m_0-1]}) = 0$ and $F(\overline{A})$ is divisible by $|z_1|^2$. By Lemma 5.3, as in Case (III), we have the following

$$f^{[m_0-1]} = B \cdot f^{[m_0]}_{\overline{z_2}}. \tag{6.16}$$

Still write $f^{[m_0]} = \sum_{t+s=m_0} f_{ts} z_2^t \overline{z_2^s}$. Then we similarly have from the hypotheses that $\mathrm{Re}(f^{[m_0]}) = \mathrm{Re}(f^{[m_0-1]}) = 0$ the following

$$f_{ts} = -\overline{f_{st}}, \; B_{hj}(s+1) f_{t(s+1)} = -(t+1)\overline{B_{jh} f_{s(t+1)}}. \tag{6.17}$$

Hence for each pair (h, j), if $B_{hj} \neq 0$, then $B_{jh} \neq 0$; for otherwise we get $f_{ts} = 0$ for any $t + s = m_0$ and reach a contradiction. Since B is nonzero, we can suppose there is a pair (h_0, j_0) such that $B_{h_0 j_0} \neq 0$ and thus $B_{j_0 h_0} \neq 0$. Since $f^{[m_0]} \neq 0$, there is a certain $f_{t_0(s_0+1)} \neq 0$ and thus $f_{s_0(t_0+1)} \neq 0$. By (6.17), we have

$$\begin{aligned} B_{h_0 j_0}(s_0+1) f_{t_0(s_0+1)} &= -(t_0+1)\overline{B_{j_0 h_0} f_{s_0(t_0+1)}}, \\ B_{j_0 h_0}(s_0+1) f_{t_0(s_0+1)} &= -(t_0+1)\overline{B_{h_0 j_0} f_{s_0(t_0+1)}}. \end{aligned}$$

Since $f_{t_0(s_0+1)} \neq 0$ and $f_{s_0(t_0+1)} \neq 0$, we have $|B_{h_0 j_0}| = |B_{j_0 h_0}|$. After a rotational transformation as in (6.3) with a suitable choice of δ, we can assume that $B_{h_0 j_0} = \overline{B_{j_0 h_0}}$. Then by (6.17), we have

$$f_{ts} = -\overline{f_{st}}, \; (s+1) f_{t(s+1)} = -(t+1)\overline{f_{s(t+1)}}. \tag{6.18}$$

By (6.18), $f^{[m_0]}_{\overline{z_2}}$ is pure imaginary. Also, it is not identically zero for the absolute value of each coefficient is a non-zero multiple of the others and at least one of them is non-zero. Now, by (6.16), we easily conclude that $B = F(\overline{A})$ is a real-valued homogeneous polynomial divisible by $|z_1|^2$. Hence, $L = \frac{\partial}{\partial z_1} + A(z_1, \overline{z_1})\frac{\partial}{\partial z_2}$ forms a basis for the sections of CR vector fields along the real algebraic finite type hypersurface M_0 in $\mathbb{C}^2$ defined by $z_2 + \overline{z_2} = F(\overline{A})$ and $\overline{L}(f) \equiv 0$. Now, following the same argument as in Case (III), we achieve a contradiction by Proposition 4.6 unless $\mathrm{Re}(f) \equiv 0$.

Combining our arguments in Cases I-IV, we conclude the proof of Theorem 6.1. □

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