A SIMPLE PROOF OF A THEOREM OF CALABI

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Dedicated to John D’Angelo on the occasion of his 60th birthday

ABSTRACT. We give a simple and more or less elementary proof of a classical result of E. Calabi on the global extension of a local holomorphic isometry into a complex space form.

1. Introduction

Let \((M, \omega)\) be a (connected) complex manifold equipped with a real analytic Kähler metric \(\omega\). Write \((S, \omega_{\text{st}})\) for one of the three complex space forms equipped with the standard canonical metrics \(\omega_{\text{st}}\). More precisely, when \(S = \mathbb{C}^n\), write \(\omega_{\text{st}}\) for the Euclidean metric; when \(S\) is the unit ball in \(\mathbb{C}^n\), \(\omega_{\text{st}}\) is the Poincaré metric; and when \(S\) is the complex projective space \(\mathbb{CP}^n\), \(\omega_{\text{st}}\) stands for the Fubini–Study metric. In this short paper, we give a simple, self-contained and, more or less, elementary proof of the following celebrated theorem of Calabi [Ca] (see Theorems 2, 6, 9, 12 in [Ca]).

**Theorem 1.1 (Calabi [Ca]).** Assume the above notation. Let \(U \subset M\) be a connected open subset and let \(F : U \to S\) be a holomorphic isometric embedding in the sense that \(F^*(\omega_{\text{st}}) = \omega\) over \(U\). Then \(F\) extends holomorphically along any continuous curve \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in U\). In particular, when \(M\) is simply connected, \(F\) extends to a globally defined holomorphic map from \(M\) into \(S\). Moreover, let \(G : U \to S\) be another holomorphic isometric embedding. Then \(F\) and \(G\) are different by a rigid motion in the sense that there is a holomorphic isometry \(T\) of \((S, \omega_{\text{st}})\) such that \(G = T \circ F\).
2. Proof of the theorem

Step I. We first prove the part for uniqueness (up to a rigid motion). We need only to consider the case when $S = \mathbb{CP}^N$ equipped with the standard Fubini–Study metric and the other case can be done in the same way. Let $p \in U$ and consider a holomorphic chart near $p$ with holomorphic coordinates $z = (z_1, \ldots, z_n)$ and $0 \leftrightarrow p$. Let $F, G : U \to S$ be holomorphic isometric embeddings. Composing $F$ and $G$ with isometries of $S = \mathbb{CP}^N$ if necessary, we can assume that $F(0) = G(0) = [1, 0, \ldots, 0]$. For a small neighborhood $U'$ of $0$ in $U$, we have $F(U'), G(U') \subset \{[1, w_1, \ldots, w_N] \in \mathbb{CP}^N\}$, which is identified in a nature way with $\mathbb{C}^N$ with holomorphic coordinates $(w_1, \ldots, w_N)$. In what follows, we also identify a Kähler metric tensor with its associated positive $(1, 1)$-Kähler form.

Then in the $(w_1, \ldots, w_N)$-coordinates, the Fubini–Study metric can be written as $\omega_{st} = i\partial\bar{\partial}\log(1 + \sum_{j=1}^{N} |w_j|^2)$. Now, near $0$, we can write $F = [1, f_1, \ldots, f_N], G = [1, g_1, \ldots, g_N]$ with $F(0), G(0) = [1, 0, \ldots, 0]$ and $f_j, g_j$ holomorphic near $0$.

The assumption that $F^*(\omega_{st}) = G^*(\omega_{st}) = \omega$ gives immediately that

\[ (2.1) \quad \sum_{j=1}^{N} |f_j(z)|^2 \leq \sum_{j=1}^{N} |g_j(z)|^2, \quad \text{or} \quad \sum_{j=1}^{N} f_j(z) \cdot f_j(\xi) = \sum_{j=1}^{N} g_j(z) \cdot g_j(\xi) \]

for $z, \xi$ in a neighborhood of $0 \in U$. By a lemma of D’Angelo [DA] (see also Calabi [Ca] Bochner–Martin [Section 5 of Chapter 2, BM] for some related results), from (2.1) we conclude $F$ and $G$ differ by a unitary matrix. More precisely, there is an $N \times N$ unitary matrix $V$ such that $(g_1, \ldots, g_N) = (f_1, \ldots, f_N) \cdot V$. Hence, we see that $F = G \cdot \text{diag}(1, V)$, which proves that $F$ and $G$ differ by a rigid motion. Next, for completeness, we include in the following paragraph a detailed proof of the above mentioned lemma:

First, we define $\mathcal{H}_{F,p} := \text{span}_{\mathbb{C}}\{D^\alpha(f_1, \ldots, f_N)|_0\}$, which is regarded as a Hermitian subspace of the stand complex Euclidean space $\mathbb{C}^N$. Here, for $\alpha = (\alpha_1, \ldots, \alpha_N)$ with each $\alpha_j$ a nonnegative integer, as usual, we define $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$. Notice that $F(z) = \sum_{|\alpha| > 0} \frac{D^\alpha F(0)}{\alpha!} z^\alpha$ for $z \approx 0$. Hence, $\mathcal{H}_{F,p}$ is the smallest linear subspace of $\mathbb{C}^N$ containing $F(z)$ for $z \approx 0$ and thus containing $F(z)$ for all $z \in U$ by the uniqueness of holomorphic functions. We can now similarly define $\mathcal{H}_{G,p}$. Write $u_\alpha = D^\alpha(f_1, \ldots, f_N)(0)$ and $v_\alpha = D^\alpha(g_1, \ldots, g_N)(0)$. Define $\mathcal{I} : \mathcal{H}_{F} \to \mathcal{H}_{G}$ by linearly extending the map sending $u_\alpha$ to $v_\alpha$. To see $\mathcal{I}$ is indeed a well-defined linear operator, it suffices to show that $\sum_j a_j u_{\alpha_j} = 0$ if and only if $\sum_j a_j v_{\alpha_j} = 0$ for $a_j^j s \in \mathbb{C}$. But this follows trivially from (2.1), for (2.1) gives that $u_\alpha \cdot \overline{u}_\beta = v_\alpha \cdot \overline{v}_\beta$ and thus $\|\sum_j a_j u_{\alpha_j}\|^2 = \|\sum_j a_j v_{\alpha_j}\|^2$. Moreover, this also demonstrates that $\mathcal{I}$ is a linear isometry from $\mathcal{H}_{F}$ to $\mathcal{H}_{G}$, which of course can be extended to a unitary self-transformation of $\mathbb{C}^N$. By the Taylor expansion and by the linearity of
We have \( \mathcal{I}(F(z)) = G(z) \). This thus shows that there is a \( N \times N \) unitary matrix \( V \) such that \( G(z) = F(z) \cdot V \) for \( z \approx 0 \).

**Step II.** We next present the proof for the extension part. We use all the notation set up above.

For any \( q \in U \), we can find a holomorphic isometry \( T_q \) of \( S \) such that \( T_q(F(q)) = [1, 0, \ldots, 0] \). Also choose a holomorphic coordinates near \( q \) with \( q \leftrightarrow 0 \). We can then similarly define \( \mathcal{H}_{F,q} \). Though \( \mathcal{H}_{F,q} \) depends only on the choice of \( T_q \), it is easy to see, from the uniqueness of holomorphic functions, that the complex dimension of \( \mathcal{H}_{F,q} \) is independent of the choice of \( T_q \), the holomorphic coordinates near \( q \) and the point \( q \in U \) itself. We write this dimension as \( d_F \). Again, from the uniqueness property for holomorphic functions, it is clear that if \( F^* \) is a holomorphic map from a domain \( U^* \subset M \) into \( S \), that is obtained by holomorphically continuing \( F \) along a certain curve, then \( d_{F^*} = d_F \). Also, if \( d_F < N \), then we can compose \( F \) with a certain holomorphic isometry \( T \) of \( S \) such that \( T \circ F = [1, f_1, \ldots, f_{d_F}, 0, \ldots, 0] \) for \( z \approx 0 \). Now, to prove the theorem, we need only consider the map \([1, f_1, \ldots, f_{d_F}] \) from a small neighborhood of \( p \in U \) into \( \mathbb{C}P^{d_F} \) equipped with the Fubini–Study metric. Hence, without loss of generality, we can assume at the beginning that \( N = d_F \).

Let \( \gamma : [0, 1] \to M \) be a continuous curve with \( \gamma(0) \in U \). Seeking a contradiction, suppose \( F \) does not extend holomorphically along \( \gamma \). Then there is a point \( c \in [0, 1] \) such that \( F \) extends holomorphically along \( \gamma([0, t]) \) for \( t < c \) but not along \( \gamma([0, c]) \). Choose \( t_j(\in [0, c]) \to c^- \) and let \( F_{t_j} \) be a holomorphic map from a small neighborhood of \( \gamma(t_j) \) in \( M \) into \( S \), obtained by holomorphically continuing \( F \) along \( [0, t_j] \). Let \( T_j \) be a holomorphic isometry of \( S \) such that \( G_j := T_j \circ F_{t_j} \) maps \( \gamma(t_j) \) to \([1, 0, \ldots, 0] \). Choose a holomorphic coordinates \((z_1, \ldots, z_n)\) in a neighborhood of \( \gamma(c) \) with \( \gamma(c) \leftrightarrow 0 \) and write \( G_j = [1, g_{j,1}, \ldots, g_{j,N}] \) in a small neighborhood of \( \gamma(t_j) \). Also, let \( \phi(z, \overline{z}) \) be a positive-valued real analytic function in a certain fixed neighborhood \( U_0 \) of \( 0 \) such that \( \omega = i\partial \overline{\partial} \log \phi(z, \overline{z}) \) near \( \gamma(c) \). Since \( G_j^*(\omega_{nt}) = \omega \) over a small neighborhood of \( \gamma(t_j) \) for \( j \gg 1 \), we get, near \( \gamma(t_j) \), the following:

\[ (2.2) \quad \partial \overline{\partial} \log \phi(z, \overline{z}) = \partial \overline{\partial} \log \left( 1 + \sum_{k=1}^N |g_{j,k}|^2 \right). \]

Now, we write \( \phi(z, \overline{z}) = \Re(h_j(z)) + \phi_j(z, \overline{z}) \), where \( h_j(z) \) is holomorphic in a certain fixed neighborhood \( U_0^* \) of \( \gamma(c) \) (independent of \( j \) for \( j \gg 1 \)) with \( \Re(h_j(z)) > 0 \) over \( U_0^* \). Also, over \( U_0^* \), we have the following convergent power series expansion:

\[ \phi_j(z, \overline{z}) = \sum_{|\alpha|, |\beta| > 0} a_{\alpha\beta}(z - z(\gamma(t_j)))^\alpha \overline{(z - z(\gamma(t_j)))}^\beta. \]
Then, one easily derives the following equation

\[
\sum_{k=1}^{N} g_{j,k}(z) \overline{g_{j,k}(\xi)} = \psi_j(z, \overline{\xi}) = \frac{2\phi_j(z, \overline{\xi})}{h_j(z) + \overline{h_j(\xi)}}.
\]

Since \(d_{F_j} = N\), there are \(\{\alpha_1, \ldots, \alpha_N\}\) such that

\[
u_{\alpha_k,j} = \left. D^{\alpha_k} (g_{j,1}, \ldots, g_{j,N}) \right|_{z = \gamma(t_j)}
\]
for \(j = 1, \ldots, N\) form a linearly independent family. Write \(A_j\) for the constant \(N \times N\) matrix with \(u_{\alpha_k,j}\) as its \(k\)th-row. Applying \(D^{\alpha_j}\) to (2.3) and then substituting \(z\) by \(\gamma(t_j)\), we get the following equation:

\[
A_j \cdot \begin{pmatrix} g_{j,1}(\xi) \\ \vdots \\ g_{j,N}(\xi) \end{pmatrix} = \begin{pmatrix} \left. D^{\alpha_1} \psi_j(z, \overline{\xi}) \right|_{z = \gamma(t_j)} \\ \vdots \\ \left. D^{\alpha_N} \psi_j(z, \overline{\xi}) \right|_{z = \gamma(t_j)} \end{pmatrix}.
\]

Aprio, (2.4) only holds for \(\xi \approx \gamma(t_j)\). However, since \(\psi_j(z, \overline{\xi})\) is real-analytic for \(z, \xi \in U^*\), the left-hand side of (2.4) is well defined and is holomorphic for \(\xi \in U^*\). The crucial point for our simple proof is that the left-hand side is linear in \(g_{j,k}\) and thus no implicit function theorem is needed for solving \(g_{j,k}\). Hence, the solution is well defined over the same defining domain of the right-hand side, whenever the coefficient matrix \(A_j\) is invertible. \(A_j\) is indeed invertible by our arrangement, though its determinant may approach to 0 as \(j \to \infty\). Hence, from (2.4), we conclude that \(g_{j,k}\) extends to a holomorphic function, for each \(k\) and \(j \gg 1\), to the fixed neighborhood \(U^*\) of \(\gamma(c)\). That shows that \(F\) also admits a holomorphic continuation along \(\gamma\) all the way across \(\gamma(c)\). This is a contradiction. The extension part is also proved.

**Remark.** (1) The same argument above (with a little more care on the convergence) also applies to the case when \(S\) is a Hilbert space form as obtained in the original paper of Calabi. (2) For late work along the lines of Calabi in [Ca], we refer the reader to [MN] and many references therein. (3) The simplicity for the argument here makes it applicable in other more complicated settings. We refer the reader to a recent preprint [HY].

**References**


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