ON A CR FAMILY OF COMPACT STRONGLY PSEUDOCONVEX CR MANIFOLDS

Xiaojun Huang, Hing-Sun Luk and Stephen S. T. Yau

Abstract

We study the simultaneous filling and embedding problem for a CR family of compact strongly pseudoconvex CR manifolds of dimension at least 5. We also derive, as a consequence, the normality of the Stein fibers of the filled-in Stein space under the constant dimensionality assumption of the first Kohn-Rossi cohomology group of the fiber CR manifolds. Two main ingredients for our approach are the work of Catlin on the solution of the $\overline{\partial}$-equation with mixed boundary conditions and the work of Siu and Ling on the study of the Grauert direct image theory for a (1,1)-convex-concave family of complex spaces.

1. Introduction

In this paper, we are concerned with a Cauchy-Riemann type of deformation for a compact strongly pseudoconvex manifold of real dimension at least 5. We will address the simultaneous embedding and filling problem of the family, as well as their applications in the deformation theory of isolated complex singularities. To start with, we introduce the following notion: (For more definitions, see §2).

Definition 1.1: Write $\Delta := \{t \in \mathbf{C},\ |t| < 1\}$ and $\Delta_r := \{t \in \mathbf{C},\ |t| < r\}$ for $r > 0$. Assume that $\{M_t\}_{t\in\Delta_r}$ is a parameterized family of connected compact C^∞-smooth strongly pseudoconvex CR manifolds of (real) dimension $2n-1$. The family $\{M_t\}_{t\in\Delta_r}$ is said to be a CR family, or M_{t_1} is said to be a CR deformation of M_{t_2} for any $t_1, t_2 \in \Delta_r$, if there is a C^∞-smooth strongly pseudoconvex CR manifold X_r of real dimension $2n+1$ and a C^∞ CR map $\pi : X_r \to \Delta_r$ such that

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(I) π is a proper submersion; (II) for any $t \in \Delta_r$, $M_t = \pi^{-1}(t)$ and M_t is a C^∞-smooth CR submanifold of X_r. In what follows, we simply say (X_r, π, Δ_r) or $\pi : X_r \to \Delta_r$ is a CR family of compact strongly pseudoconvex CR manifolds.

As compact CR manifolds often come as the smooth boundaries of complex spaces with isolated singularities, the above definition is modeled by the following typical example of the holomorphic deformation of the complex structure of isolated singularities: Let $(\mathcal{V}_r, \pi, \Delta_r)$ be a small deformation of the complex space $V_0 = \pi^{-1}(t_0)$ with an isolated singularity at $p_0 \in V_0$. Assume that $\mathcal{V}_r$ is embedded in $\mathbf{C}^N$. For a positive ϵ, write $S_\epsilon(p_0)$ for the sphere centered at p_0 with radius ϵ. When it cuts $\mathcal{V}_r$ only at smooth points and CR-transversally, then $(\mathcal{V}_r \cap S_\epsilon(p_0), \pi, \Delta_r)$ gives a CR deformation of the strongly pseudoconvex CR manifold $V_0 \cap S_\epsilon(p_0)$, as defined above. It will be seen later in this paper that a CR deformation defined above can be generically realized in such a concrete manner when $2n - 1 \geq 5$. Hence, the study of the deformation of isolated singularities is closely related to the study of the above notion of CR deformation of compact strongly pseudoconvex CR manifolds.

A compact strongly pseudoconvex CR manifold M of dimension at least 5 can be CR diffeomorphically mapped to the smooth boundary of a certain Stein space with at most isolated singularities embedded in some complex Euclidean space $\mathbf{C}^N$, by the work of Boutet de Monvel [8] and Harvey-Lawson [19]. (See §2 for the basic definitions and notations). In general, N well depends on the intrinsic CR structure of M. For a smooth family of strongly pseudoconvex CR manifolds, Tanaka addressed the simultaneous embedding problem under the assumption that the first Kohn-Rossi cohomology group of each fiber has a fixed dimension [35]. (Namely, $\dim H^{(0,1)}_{KR}(M_t)$ is independent of t. For the precise definition of $H^{(0,1)}_{KR}(M_t)$, the reader is referred to [35] [13] or the first paragraph of §4 of the present paper).

However, the methods in [35] cannot be used to deal with the CR dependence on the parameter for the CR families, which turns out to be crucial for many studies in the deformation theory of the complex structure of isolated singularities.

In this paper, we will study the simultaneous embedding and filling problems for a CR family of CR manifolds. We will also give applications to problems concerning the deformation of complex isolated singularities.

Before we give our main result, we briefly recall some basic definitions and results.

Suppose M is a finitely generated module over a local ring (R, m) where m is the maximal ideal in R. Suppose that $\{f_1, \ldots, f_k\}$ in m is a sequence such that f_1 is not a zero divisor for M and f_j is not a zero-divisor for $M \Big/ \sum_{i=1}^{j-1} f_i M$ for $2 \leq j \leq k$ if $k \geq 2$. We then call $\{f_1, \cdots, f_k\}$ an M-sequence. Any permutation of an M-sequence is still an M-sequence. An M-sequence is called maximal if it is not contained in a longer M-sequence. All maximal M-sequences have the same length. This common length is called the homological codimension of M over R, denoted by $\mathrm{codh}_R M$ or simply by $\mathrm{codh}\, M$. (We say that the homological dimension of M over R is zero if there is no M-sequence.)

Definition 1.2 [33]: Let $(X, \mathcal{O})$ be a complex space, $\mathcal{F}$ an analytic sheaf of X, and p a non-negative integer. The p-th absolute gap-sheaf of $\mathcal{F}$, denoted by $\mathcal{F}^{[p]}$, is the analytic sheaf over X defined by the following pre-sheaf: Suppose $U \subset V$ are open subsets of X. Then

$$\mathcal{F}^{[p]}(U) = \underset{A \in \mathcal{U}(U)}{\text{direct limit}}\, \Gamma(U - A, \mathcal{F}),$$

where $\mathcal{U}$ is the directed set of all subvarieties in U of dimension $\leq p$ directed by inclusion. The map $\mathcal{F}^{[p]}(V) \to \mathcal{F}^{[p]}(U)$ is induced by the restriction map.

Let $\mathcal{F}$ be a coherent analytic sheaf over X. Set

$$S_k(\mathcal{F}) = \{x \in X : \mathrm{codh}_{\mathcal{O}_x} \mathcal{F}_x \leq k\}.$$

Then $S_k(\mathcal{F})$ is a subvariety of dimension $\leq k$ in X ([29], Satz 5, p. 81). The following proposition gives a relation for the above introduced objects:

Proposition 1.3 [3.13, 34]: Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then $\mathcal{F}^{[p]} = \mathcal{F}$ if and only if $\dim S_{k+2}(\mathcal{F}) \leq k$ for $-1 \leq k < p$.

The following definition was given in Andreotti and Siu [3].

Definition 1.4 [3]: Let $(X, \mathcal{O})$ be a complex space. We say that X is p-normal at $x \in X$ if $\mathcal{O}_x^{[p]} = \mathcal{O}_x$. We say that X is p-normal if $\mathcal{O}^{[p]} = \mathcal{O}$.

Main Theorem: Let $\pi : X \to \Delta$ be a CR family of compact strongly pseudoconvex CR manifolds of (real) dimension $2n - 1$ ($n \geq 3$). Let $M_t = \pi^{-1}(t)$ for $t \in \Delta$. Then there exists a unique (up to isomorphism) 2-normal Stein complex space $\widehat{X}$, which has X as part of its smooth boundary. The CR structure of X coincides with the inherited CR structure from $\widehat{X}$ and is strongly pseudoconvex with respect to the complex Stein space $\widehat{X}$. Moreover, there is a holomorphic map $\widehat{\pi} : \widehat{X} \to \Delta$ such that the following hold
(I): For any $t \in \Delta$, $\widehat{\pi}^{-1}(t) = \widehat{M_t}$ is a Stein space with $M_t = \pi^{-1}(t)$ as its smooth strongly pseudoconvex boundary.

(II): For $\epsilon < 1$, write $X_\epsilon := \bigcup_{|t|<\epsilon} M_t$ and $\overline{X_\epsilon} = \bigcup_{|t|\le\epsilon} M_t$. Also write $\widehat{X_\epsilon} = \widehat{\pi}^{-1}(\Delta_\epsilon)$ and $\overline{\widehat{X_\epsilon}} = \widehat{\pi}^{-1}(\overline{\Delta_\epsilon}) \cup \overline{X_\epsilon}$. Then there exists a smooth function ρ^ϵ defined in $\overline{\widehat{X_\epsilon}}$, such that (a) ρ^ϵ is strictly plurisubharmonic near $\overline{X_\epsilon}$; (b) $c^* < \rho \le 0$ for some $c^* < 0$; (c) $\rho^\epsilon = 0$ exactly on $\overline{X_\epsilon}$ and $d\rho^\epsilon|_{\overline{X_\epsilon}} \neq 0$.

(III): $\widehat{\pi}$ extends smoothly up to X. Denote its smooth extension over X by $\widehat{\pi}|_X$. Then $\widehat{\pi}|_X \equiv \pi$.

(IV): Assume that for a certain $\epsilon_0 \in (0,1)$, there is a complex manifold Z_{ϵ_0} such that X_{ϵ_0} can be CR embedded into Z_{ϵ_0}. Suppose that f is a smooth CR equivalence map from M_0 to a certain CR submanifold $M_0' \subset\subset \mathbf{C}^m$, that extends holomorphically to $\widehat{M_0}$. Assume that M_0' is the smooth boundary of a certain Stein space V_0' embedded in $\mathbf{C}^m$. (In particular, V_0' is assumed to have only smooth points in a small neighborhood of M_0' in $\mathbf{C}^m$.) Then when $\varepsilon << 1$, there is a CR embedding $T : X_\epsilon \to \mathbf{C}^m \times \mathbf{C}$ such that the following holds:

(IV1): There is a Stein space $\widehat{X_\epsilon'} \subset\subset \mathbf{C}^m \times \mathbf{C}$, which has $X_\epsilon' := T(X_\epsilon)$ as part of its smooth boundary.

(IV2): Let $\widehat{\pi}'$ be the natural projection from $\mathbf{C}^m \times \Delta_\epsilon$ into Δ_ϵ. Then $\widehat{\pi}'^{-1}(t) \cap \widehat{X_\epsilon'} := \widehat{M_t'}$ is a Stein subvariety of $\widehat{X_\epsilon'}$ with M_t' as its strongly pseudoconvex boundary. Moreover, $M_t' = T(M_t)$.

(IV3): T extends to a proper holomorphic map, still denoted by T, from $\widehat{X_\epsilon}$ into $\widehat{X_\epsilon'}$ such that $T = (F, \hat{\pi})$ with $F|_{\widehat{M_0}} = f$.

We call the triplet $(\widehat{X}, \widehat{\pi}, \Delta)$ the Siu-Ling completion of the CR family (X, π, Δ). By the theorems proved in [8], [19], [30-31] and Ling [22], for many interesting families, we can always find the map f as in the Main Theorem (IV), provided that $m >> 1$. This makes our Main Theorem usable in many applications. We will address this issue in §4. Here, we will be content to state the following corollaries:

Corollary 1.5: Let (X, π, Δ) be a CR family of compact strongly pseudoconvex CR manifolds $\{M_t\}$. Suppose that $X_{\epsilon_0}(= \pi^{-1}(\Delta_{\epsilon_0}))$ for a certain $\epsilon_0 \in (0,1)$ can be CR embedded into a complex manifold. Assume that the real dimension of M_t is at least 5 and $\dim H^{(0,1)}_{KR}(M_t)$ is constant. Then any $\widehat{M_t} = \widehat{\pi}^{-1}(t)$ with $t \in \Delta_{\epsilon_0}$, in the Siu-Ling completion of (X, π, Δ), is a normal Stein space.

Corollary 1.6: Let (X, π, Δ) be a CR deformation of a compact strongly pseudoconvex CR manifold M_0. Suppose that $X_{\epsilon_0}(= \pi^{-1}(\Delta_{\epsilon_0}))$ for a certain $\epsilon_0 \in (0,1)$ can be CR embedded into a complex manifold. Assume that the real dimension of M_0 is at least 5 and $\dim H^{(0,1)}_{KR}(M_t)$ is constant. If M_0 can be CR embedded into $\mathbf{C}^m$ by the smooth CR diffeomorphism f_0, then when $0 < \epsilon << \epsilon_0$, there is a CR embedding

$\Psi : X_\epsilon \to \mathbf{C}^m \times \mathbf{C}$ such that $\Psi|_{M_t}$ CR embeds M_t into $\mathbf{C}^m \times \{t\}$. Moreover, we can make $\Psi|_{M_0} = (f_0, 0)$.

Remark 1.7: Let X and X_ϵ be defined as above. By a result to be proved in §2 of this paper, X_ϵ (for any $\epsilon \in (0,1)$) can always be CR embedded into a complex manifold, when the CR structure over X is real analytic. Here, we recall that the CR structure over X is said to be real analytic if X is a real analytic manifold and the bundle $T^{(1,0)}X$ is a real analytic bundle over X. Hence, when the total space X has a real analytic CR structure, the assumption that X_{ϵ_0} can be CR embedded into a complex manifold for a certain ϵ_0 is redundant in Main Theorem (IV), Corollary 1.5 and Corollary 1.6. By a very deep result of Catlin [Theorem 1.1, 10], one also notices that for any $0 < \epsilon_0 < 1$, X_{ϵ_0} can be CR embedded into a complex manifold, even when X is merely assumed to be (C^∞) smooth. (See a detailed discussion on this matter in Remark 2.3 of §2). We should mention that in many applications of the theory on CR manifolds to the study of complex singularities, the total space X comes as the smooth link of complex singularities and thus is naturally embedded in a complex manifold.

A special case of Corollary 1.5 was obtained by a different method by Fujiki [14] when $\dim H^{(0,1)}_{KR}(M_t) \equiv 0$. Corollary 1.6 can be viewed as a Cauchy-Riemann strengthening property for the CR family (or, for a holomorphic family of Stein spaces, respectively) along the parameter space. Corollary 1.6 has an immediate application to the study of the simultaneous blowing-down problem for strongly pseudoconvex complex manifolds, which will be addressed in Corollary 4.4 in §4. Also, the constant dimensionality of $H^{(0,1)}_{KR}(M_t)$ in Corollary 4.6 (and thus Corollaries 1.5-1.6) seems to be important for the results to hold by the work of Knorr-Schneider and Riemenschneider [20] [28] on the conditions for the simultaneous blowing-down problem of a holomorphic family of the exceptional sets .

The key step for the proof of these results is to obtain a CR extension theorem for CR functions from the submanifold M_0 to X. In our argument, it is important to have the dimension of M_t at least 5. However, it is not clear to us whether the Main Theorem still holds when the real dimension of M_t is 3, assuming that each fiber is fillable by complex spaces with isolated singularities, which we state as an open question. We do not know if we can also have some version of Corollary 1.6 when each M_t has real dimension 3, and each M_t is assumed to be globally embeddable. Apparently, by the work of Rossi and Jacobowitz-Treves [18], it cannot be true in general if one just considers the real analytic family.

The basic ingredients for the proof of the Main Theorem and Corollary 1.6 include the work on the embedding of the CR structures and holomorphic completion of the so-called $(1,1)$-convex and concave space. (See the papers by Androitti-Siu [3] Siu [30] and Ling [22], Kuranishi [24], Akaholi [2], Webster [37], Catlin [10], etc). Especially, the work of Catlin [10] for solving the $\bar{\partial}$-equation with mixed boundary conditions and the work of Ling [22], Siu [30-31] on the generalization of the Grauert direct image theorem will be crucial to us. The interaction of the deformation of CR manifolds and the deformation of isolated normal singularities, which in some work is also referred as the Kuranishi program, has attracted some attention in recent years. Related to this work, we would like to mention the long papers by Buchweitz-Millson [9] and Miyajima [25] and the references therein, to name a few. There has also been much work done on the smooth family of CR manifolds, in conjunction with the embedding and related problems of three dimensional compact CR manifolds . Here, we refer to the reader the papers of Lempert [23] and Bland-Epstein [7], and the references therein.

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2. Simultaneous filling of a CR family

In this section, we first recall some definitions and notation. Then we turn to the Hartogs-Rossi type of holomorphic filling of complex manifolds by applying the work of Kuranishi-Akahori-Webster ([24], [2], [37]) and Ling [22].

Let M be a (C^∞) smooth manifold of real dimension $(2n-1)$. A smooth real 1-form θ over M is called a contact form if the $(2n-1)$-form $\theta \wedge \mathrm{d}\theta \wedge \cdots \wedge \mathrm{d}\theta$ vanishes nowhere on M. The complexified contact bundle $C\mathcal{S}$ is then a subbundle of $C\mathrm{T}M$ annihilated by θ. A complex structure J is a base point preserving smooth bundle isomorphism of $C\mathcal{S}$ with $J^2 = -\mathrm{id}$. $T^{(1,0)}M$ is defined to be the eigenspace of i, which is apparently a subbundle of $C\mathcal{S}$ and $\mathrm{T}^{(0,1)}M$ is defined to be the complex conjugate of $T^{(1,0)}M$. J is also required to be integrable in the sense that the space of cross sections of $\mathrm{T}^{(1,0)}M$ is closed under the Lie bracket operation. When M is a real analytic manifold, we say the CR structure J is real analytic if $\mathrm{T}^{(1,0)}M$ is locally generated by real analytic complex vector fields. In this case, we also say M is a real analytic CR manifold. Unless mentioned explicitly, all CR manifolds in this paper are assumed to C^∞-smooth. The Levi form $\mathcal{L}$ associated to θ is a Hermitian form over $T^{(1,0)}M$ such that for any two cross sections, L_1, L_2, $\mathcal{L}(L_1, L_2) = i < \theta, [L_1, \overline{L_2}] >$. For an integral J, M equipped with J is called a pseudoconvex CR manifold if the Levi form defined above is semi-definite and is called strongly pseudoconvex CR manifold if the Levi-form is definite. When M is part of the smooth boundary of a complex manifold V, then we adapt the standard meaning for the notion that V lies on the pseudoconvex side when the CR structure inherted from V is pseudonvex.

Let $N \subset M$ be a smooth submanifold. If for any $p \in N$, $C\mathcal{S}_p \cap CT_pN = J|_{CT_pN \cap \mathcal{S}_p}(C\mathcal{S}_p \cap CT_pN)$ and has complex codimension 1 in CT_pN, then J naturally induces a CR structure (of hypersurface type) on N. N equipped with such a CR structure is called a CR submanifold of M. An important class of functions over M is the class of CR functions which is annihilated by any $(0,1)$-vector field along M. A map from M into $\mathbf{C}^k$ is called a CR map if each of its components is a CR function.

A real hypersurface in $\mathbf{C}^{n+1}$ is strongly pseudoconvex with the inherited complex structure from its ambient space if it can be defined by a strongly plurisubharmonic function. A famous theorem of Kuranishi-Akaholi-Webster states that any strongly pseudoconvex smooth CR manifold of real dimension $2n+1 \geq 7$ can be locally embedded as a real hypersurface in $\mathbf{C}^{n+1}$ through a CR diffeomorphism. More recently, in a very deep paper of Catlin [Theorem 1.1, 10], one sees that any pseudoconvex manifold X with at least three positive Levi eigenvalues can be realized as the smooth pseudoconvex boundary of some complex manifold Z. In the case that we are considering, the construction of Z directly follows from the embedding theorem of Kuranishi-Akaholi-Webster, which we will explain in details as follows. In the rest of this paper, all strongly pseudoconvex CR manifolds are assumed to be connected.

Notice that when a strongly pseudoconvex manifold M is part of the smooth boundary of a certain complex manifold V, then there is a Lewy-type extension phenomenon for CR functions. Here we state the following one which can be easily proved by using the Baouendi-Treves approximation theorem and the so-called analytic disk argument (see [BER]).

Lemma 2.1: Let V be a domain in $\mathbf{C}^n$ with M as part of its smooth strongly pseudoconvex boundary. For any subdomain $M' \subset\subset M$ of M, there is a subdomain $V' \subset V$ such that any CR function over M' can be holomorphically extended to V'. Here V' is assumed to have M' as part of its smooth boundary.

Let (X, π, Δ) be a CR deformation of the strongly pseudoconvex manifold $M_0 = \pi^{-1}(0)$ of dimension $2n-1 \geq 5$, as defined in Definition 1.1.

Since the total space X has real dimension at least 7. By the above mentioned Kuranishi-Akaholi-Webster's embedding theorem, for each $p \in M_0$, there is a neighborhood U_p of p in X and a CR diffeomorphism $F_p : U_p \to \mathbf{C}^{n+1}$ such that $U_p^* = F_p(U_p)$ is a strongly pseudoconvex real hypersurface in $\mathbf{C}^{n+1}$.

Let $0 < \epsilon < 1$ be fixed. There is a finite set of such open pieces U_j of X which covers $\overline{X_\epsilon} := \pi^{-1}(\overline{\Delta}_\epsilon)$. Each of them can be assumed to be connected. Next, we choose a finer finite cover $\{V_j\}$ of $\overline{X_\epsilon}$, for which we can find another finite set of connected open subsets $\{B_j\}$ of X with the following properties: (a): $V_j \subset\subset B_j$; (b) for each j, there is a certain index $L(j)$ such that $B_k \subset\subset U_{L(j)}$ for any k for which there is an l with $B_l \cap B_j \neq \emptyset$ and $B_k \cap B_l \neq \emptyset$.

Now, for each V_j, let F_j be a CR diffeomorphism from $U_{L(j)} (\supset\supset V_j)$ to a certain strongly pseudoconvex hypersurface $U_{L(j)}^* \subset\subset \mathbf{C}^{n+1}$. Let D_j^* be a domain in the pseudoconvex side of $U_{L(j)}^*$ with $V_{L(j)}^*$ as part of its smooth boundary, and define $\delta_j = \sup\{\mathrm{dist}(z, V_{L(j)}^*) : z \in D_j^*\}$, where we write $B_{L(j)}^* = F_j(B_j)$ and $V_{L(j)}^* = F_j(V_j)$. We assume that $\delta_j << 1$ so that for each j, k, $F_{jk} := F_j \circ F_k^{-1}$ extends holomorphically to D_k^* by Lemma 2.1, whenever there is an l with $B_j \cap B_l \neq \emptyset$ and $B_k \cap B_l \neq \emptyset$

Now we let Z^* be the disjoint union of the finite set $\{V_{L(j)}^* \cup D_j^*\}_j$. We say that $p \in V_{L(j)}^* \cup D_j^*$ and $q \in V_{L(k)}^* \cup D_k^*$ are equivalent if (a): $B_j \cap B_k \neq \emptyset$ and (b): $p = F_{jk}(q)$. By the following Lemma 2.2, one sees that this equivalence relation is well-defined when $\delta = \max\{\delta_j\}$ is sufficiently small. Hence we obtain the quotient space, which is denoted by Ω_ϵ.

Lemma 2.2: Suppose $\delta << 1$. Then the above mentioned equivalence relation is well-defined. Moreover, the quotient space Ω_ϵ carries an integrable complex structure which has a smooth piece of its boundary CR-diffeomorphic to a neighborhood of $\overline{X_\epsilon}$ in X.

Proof of Lemma 2.2: Let l_1, l_2, l_3 be such that $B_{l_1} \cap B_{l_2} \neq \emptyset$, $B_{l_2} \cap B_{l_3} \neq \emptyset$, but $B_{l_1} \cap B_{l_3} = \emptyset$. We first claim that when δ is sufficiently small, there are no points $p \in D_{l_1}^*$ and $q \in D_{l_3}^*$ such that $p = F_{l_1 l_3}(q)$. Indeed, suppose not. There would be a sequence $p_j \to p \in \overline{V_{l_1}^*}$ and $q_j \to q \in \overline{V_{l_3}^*}$ such that $p_j = F_{l_1 l_3}(q_j)$. Passing to the limit, it thus follows that $F_{l_1}^{-1}(p) = F_{l_3}^{-1}(q) \in B_{l_1} \cap B_{l_3}$. This is a contradiction. From this claim, Lemma 2.1, as well as the simple fact that $F_{l_1 l_3} = F_{l_1 l_2} \circ F_{l_2 l_3}$, it is easy to see that the equivalence relation is well-defined when $\delta << 1$.

Now, we assume δ is sufficiently small so that the above claim holds. Write the equivalence class of D_l^* by $\widetilde{D_l^*}$. Then what we just obtained shows that $\widetilde{D_{l_1}^*} \cap \widetilde{D_{l_2}^*} = \emptyset$ when $B_{l_1} \cap B_{l_2} = \emptyset$. When $B_{l_1} \cap B_{l_2} \neq \emptyset$, a similar argument shows that for

$$\delta << 1, \qquad \widetilde{D^*_{l_1}} \cap \widetilde{D^*_{l_2}} = \qquad \widetilde{D^*_{l_1} \cap F_{l_1 l_2}}(D^*_{l_2}).$$

Now, we assign the topology to Ω_ϵ so that $\widetilde{D^*_{l_1}}$ is homeomorphic to $D^*_{l_1}$ with the inherited topology from $\mathbf{C}^{n+1}$. Then Ω_ϵ can be easily seen to be a Hausdorff space. Indeed, for $p \in D^*_{l_1}$ and $q \in D^*_{l_2}$ with $\widetilde{p} = [p] \neq \widetilde{q} = [q]$, $\widetilde{p}$ and $\widetilde{q}$ are apparently separated by open subsets when $\widetilde{D^*_{l_1}} \cap \widetilde{D^*_{l_2}} = \emptyset$ or when $p \in F_{l_1 l_2}(D^*_{l_2})$. In case $p \notin F_{l_1 l_2}(D^*_{l_2})$, let U_q be a small open neighborhood of q in $D^*_{l_2}$ with $U_q \subset\subset F_{l_1 l_2}(D^*_{l_2})$, then $\widetilde{p}$ and $\widetilde{q}$ are separated by $\widetilde{U_q}$ and $\widetilde{D^*_{l_2} \setminus \overline{U_q}}$. Moreover, we can see that Ω_ϵ with the local charts $\{\widetilde{D^*_j}\}$ is a complex manifold with holomorphic transition functions F_{jk}. This completes the proof of Lemma 2.2. ∎

Apparently, the complex manifold Ω_ϵ discussed above has a piece of smooth boundary which is diffeomorphic to a neighborhood of X_ϵ in X by the way it was constructed. Next, by shrinking δ thus Ω_ϵ if necessary, we can assume that Ω_ϵ has a topological boundary, which can be decomposed into three pieces Y_0, Y_1 and Y_2, where Y_0 is CR diffeomorphic to X_ϵ. Moreover, Lemma 2.1 can be used to see that π can be extended to a holomorphic submersion $\hat{\pi}$ from Ω_ϵ to Δ_ϵ. Also there is a strongly plurisubharmonic function $\rho^\epsilon \in C^\infty(\overline{\Omega_\epsilon})$ over $\overline{\Omega_\epsilon}$ such that (a): $Y_1 = \hat{\pi}^{-1}(|t| = \epsilon)$, (b): $\rho^\epsilon|_{Y_0} = 0$, $\mathrm{d}\rho^\epsilon|_{Y_0} \neq 0$, $\rho^\epsilon < 0$ over $\overline{\Omega^\epsilon} \setminus Y_0$, and (c) $\rho^\epsilon|_{Y_2} = -\epsilon_2$, with ϵ_2 a sufficiently small positive constant. Also write the naturally defined CR embedding from Y_0 to X_ϵ as Ψ. Then $\pi \circ \Psi = \hat{\pi}$. Hence, $(\Omega_\epsilon, \hat{\pi}, \Delta_\epsilon)$ is a $(1,1)$ convex-concave complex space as defined in [22].

Notice that Ω_ϵ is 2-normal when $n \geq 3$. Namely, for any complex analytic variety of dimension at most two $E \subset \Omega_\epsilon$, any holomorphic function in $\Omega_\epsilon \setminus E$ extends holomorphically to Ω_ϵ.

By the work of Siu [30] and Theorem $(\mathrm{I})_n$ in [22], Ω_ϵ can be completed to a 2-normal Stein space $(\widehat{X_\epsilon}, \widehat{\pi}, \Delta_\epsilon)$. By the uniqueness part of the Ling theorem mentioned above, we can patch all those $(\widehat{X_\epsilon}, \widehat{\pi}, \Delta_\epsilon)$ into the required completion $(\widehat{X}, \widehat{\pi}, \Delta)$, which has Properties (I)-(III) as described in the Main Theorem if we identify in an obvious way X_ϵ with Y_0 defined above. Here, we only mention that the Steinness of $\widehat{X}$ follows from the fact that $\widehat{X}_{\epsilon_1}$ is a holomorphically convex subset of $\widehat{X}_{\epsilon_2}$ for any $\epsilon_2 > \epsilon_1$. (See [§3, 32].)

In what follows, we call such a 2-normal completion the Siu-Ling completion of the family (X, π, Δ).

Further assume that X is a real analytic CR manifold. Then we can similarly define $D^{*0}_j = D^*_j \cup D^{*-}_j \cup V^*_{L(j)}$ such that F_{jk} extends holomorphically to D^{*0}_j, by the reflection principle for strongly pseudoconvex hypersurfaces. (See [4], for instances). Here D_j^{*-} is a certain domain in the pseudoconcave side of $U^*_{L(j)}$, which has $V^*_{L(j)}$ as part of its smooth boundary. We can then define $\delta_j^0 = \sup\{\text{dist}(z, V^*_{L(j)}) : z \in D_j^{*0}\}$ and define the equivalence relation over the space $\cup D_j^{*0}$ by identifying points through F_{jk}. Then when $\delta_j^0 << 1$, the quotient space Ω^0 we got is also a complex manifold whose induced CR structure over X_ϵ coincides with the original one. (We also say X_ϵ is CR embedded into Ω^0 as a CR submanifold). With the same discussions presented above, we conclude that there exists a complex space $\hat{\mathcal{Z}}_\epsilon$, containing $\widehat{X}_\epsilon$, such that (i): $\hat{X}_{\epsilon'} \subset\subset \hat{\mathcal{Z}}_\epsilon$ for any $\epsilon' < \epsilon$; and (ii) the singular set of $\hat{\mathcal{Z}}_\epsilon$ is contained in the singular set of $\hat{X}_\epsilon$.

Remark 2.3: We mention that if one uses [Theorem 1.1, 10], one can conclude the existence of the aforementioned $\hat{\mathcal{Z}}_\epsilon$ even when X is purely smooth. (See already [Theorem 6, 11]). Following the argument in the proof of [Theorem 1.3, 10], we here indicate how the main result of Catlin [Theorem 1.1, 10] can be used to construct $\widehat{\mathcal{Z}}_\epsilon$: One first extends $\widehat{X}$ into Z with the same dimension as that of Ω_ϵ, which is smooth away from the singular set $Sing(\widehat{X})$ of $\widehat{X}$ and has X_ϵ as its interior for $\epsilon < 1$. Let θ be the contact form which makes X strongly pseudoconvex and let T be a real vector field along X such that $<\theta, T> \equiv 1$. Notice that X then has n-negative Levi eigenvalues with $n \geq 3$ with respect to $-\theta$. Now, by [Theorem 1.1, 10], one can find an integrable complex structure over Ω_ϵ (after making $\delta << 1$), whose (complex conjugate) reflection to the other side of X_ϵ in Z extends smoothly to X_ϵ for $\epsilon < 1$ and induces the same CR structure over X_ϵ. Moreover, it is related with the original complex structure over Ω_ϵ so that the formal uniqueness result in [Theorem 4.2, 10] can be applied (with the map G there side-preserving). Now, following the same argument as in the proof of [Theorem 1.3, 10], one sees the existence of the aforementioned $\widehat{\mathcal{Z}}_\epsilon$ by modifying the complex structure in the pseudoconcave side of X.

For any CR family of strongly pseudoconvex family (X, π, Δ) which will appear in the rest of the paper, we always assume that X is smooth with its Siu-Ling completion $(\widehat{X_\epsilon}, \widehat{\pi}, \Delta_\epsilon)$, for a certain $0 < \epsilon < 1$, being contained in a larger complex space $\widehat{\mathcal{Z}}_\epsilon$ as described above.

3. $\overline{\partial}$-equation on a lunar domain and extension of CR functions

We now let (X, π, Δ) be a smooth family of compact strongly pseudoconvex CR manifolds with the dimension of each fiber at least 5.

We now proceed to the study of the simultaneous embedding problem of the CR family. For this, we need to study the solutions of a certain $\overline{\partial}$-equation with good boundary behavior on $\hat{X}_\epsilon$ constructed in §2. However, the non-smooth feature of $\hat{X}_\epsilon$ makes a direct approach difficult. What we will do here is to remove a neighborhood of the singular set of $\hat{X}_\epsilon$ so that we need only to work on a smooth manifold. But, this also brings the problem arising from its boundary. To deal with this, we use the work of Catlin [10] for solving the $\overline{\partial}$-equation with mixed boundary conditions.

As in §2, we first construct a smooth domain $\Omega_\epsilon \subset \hat{X}_\epsilon$, which has three pieces of smooth boundaries, Y_0, Y_1, Y_2, that intersect CR-transversally at their intersections. Namely, they satisfy the following properties: (a): $Y_0 = X_\epsilon$, and thus Y_0 is strongly pseudoconvex with respect to Ω_ϵ, (b): $Y_1 = \hat{\pi}^{-1}(\{|t| = \epsilon\})$, and (c) $Y_2 = \{p \in \widehat{X_\epsilon} \cap \widehat{\pi}^{-1}(\overline{\Delta_{\epsilon_1}}) : \rho^\epsilon = -\epsilon_2\}$ with $\epsilon_2 << \epsilon$.

We use the notation set up before. For instance, we will write $\widehat{M_t} = \widehat{\pi}^{-1}(t)$. Without loss of generality, we will assume, in this section, that $\widehat{X_{\epsilon'}} \subset\subset \widehat{\mathcal{Z}}_{\epsilon_0}$ for any $\epsilon' < \epsilon_0$ with $\epsilon_0 = 3/4$. Here, as above, $(\widehat{X}_{\epsilon_0} \subset)\widehat{\mathcal{Z}}_{\epsilon_0}$ is a complex space with the same singular set as that for $\widehat{X}_{\epsilon_0}$. We also fix $\epsilon = 1/2$ in this section. For simplicity, we write ρ for the ρ^{ϵ_0} constructed at the end of §2, which is defined over $\overline{\widehat{X_{\epsilon_0}}}$.

The main step is to prove the following extension theorem:

Theorem 3.1: Let ϕ be a holomorphic function over $\widehat{M_0}$, which is smooth up to its boundary M_0. Then ϕ admits an extension that is holomorphic over $\widehat{X_\epsilon}$ and is smooth up to the strongly pseudoconvex manifold X_ϵ.

Proof of Theorem 3.1: Let $\{U_j\}_{j=0}^m$ be a finite (open) covering of the compact space $\overline{\widehat{X_\epsilon}}$ and let $\{\chi_j\}$ be a partition of unity with $\mathrm{Supp}\chi_j \subset\subset U_j$ for each j. Here we let each U_j be a connected open subset of $\overline{\widehat{X_{\epsilon_0}}}$ such that $U_j \cap \widehat{X_\epsilon}$ is Stein for each j. Make $U_0 = \widehat{X_{\epsilon_0}} \setminus \{p(\approx X_\epsilon) \in \overline{\widehat{X_\epsilon}} : \rho(p) \geq -\delta_0\}$ with $0 < \delta_0 << 1$ and $\chi_0 \equiv 1$ in a Stein neighborhood of the singular set of $\widehat{X_{\epsilon_0}}$. Assume that U_j for $j \neq 0$ does not cut the singularities of $\widehat{X_\epsilon}$. Without loss of generality, we can also assume that ϕ admits a holomorphic extension ϕ_j to $U_j \cap \widehat{X_\epsilon}$ when $j \neq 0$ and $U_j \cap \widehat{M_0} \neq \emptyset$. Moreover, we can further assume that $\phi_j \in C^\infty(\overline{U_j})$, when $j \neq 0$ and $U_j \cap \widehat{M_0} \neq \emptyset$. (For instance, see [6]).

We let $\phi_j \equiv 0$ when $j \neq 0$ and $U_j \cap \widehat{M_0} = \emptyset$.

Notice that by Cartan's Theorem A and B, ϕ extends to a holomorphic function ϕ_0 in U_0. For $p \in \overline{\widehat{X_\epsilon}}$, write $t(p) := \widehat{\pi}(p)$. Choose

$\chi^*(t) = \chi^*(|t|)$ such that it is identically one for $|t| << 1$ and zero for $|t| > \frac{1}{2}$. Consider the following closed $(0,1)$ form

$$\omega(p) = \bar{\partial}\left(\chi^*(t)\sum \frac{\chi_j(p)\phi_j(p)}{\widehat{\pi}(p)}\right).$$

Then, it can be easily verified that ω is smooth over $\overline{\widehat{X_\epsilon}}\setminus Sing(\widehat{X}_\epsilon)$ and at smooth points in a small Stein neighborhood of $Sing(\widehat{X}_\epsilon)$, $\omega = \overline{\partial}(\chi^*\frac{\phi_0}{t})$.

Notice that ω is compactly supported along the t-direction. For convenience of the reader, we say a few words on the smoothness of ω along M_0. The other cases can be done similarly.

Let $p_0 \in M_0 \cap U_j$ for a certain j. Let $\{z_j\}_{j=1}^{n+1}$ be a set of holomorphic functions over $U_j \cap \widehat{X_\epsilon}$, that are smooth up to a certain small neighborhood of p_0 in X and satisfy the condition: $dz_1 \wedge \cdots \wedge dz_{n+1} \neq 0$ at p_0. Assume that $z_j(p_0) = 0$ for each j. Since $dt|_{p_0} \neq 0$, we can assume, without loss of generality, that $z_1 = t$. Then the map $\Psi = (z_1, \cdots, z_{n+1})$ diffeomorphically maps a small neighborhood of p_0 in $\overline{\widehat{X}_\epsilon}$ to a certain D_j in $\mathbf{C}^{n+1}$. Certainly Ψ is holomorphic in the interior of $\overline{\widehat{X}_\epsilon}$ and CR up to the boundary. For each l with $p_0 \in U_l$, write the formal power series expansion of $\phi_l \circ \Psi^{-1}$ at 0 as $\sum_{k_1\cdots k_{n+1}\geq 0} a^l_{k_1\cdots k_{n+1}} z_1^{k_1}\cdots z_{n+1}^{k_{n+1}}$. Since ϕ_l is the extension of ϕ, we see that $a^l_{0k_2\cdots k_{n+1}}$ are independent of l. Now, still write ω for its push-forward form through Ψ. Then *at* 0, we have the following formal expansion for ω:

$$\omega = \sum_{l;p_0\in U_l}\left(\overline{\partial}\chi_l \sum_{k_1\cdots k_{n+1}\geq 0, k_1\geq 1} a^l_{k_1\cdots k_{n+1}} z_1^{k_1-1}\cdots z_{n+1}^{k_{n+1}}\right).$$

Similarly, we have a formal Taylor series expansion for ω at any nearby point of p_0 in $\overline{\widehat{X_\epsilon}}$. From this, the smoothness of ω at p_0 follows.

We can always find a smooth function u over $\widehat{X}_\epsilon$ which solves the equation $\overline{\partial}(u) = \omega$ over $\widehat{X}\setminus Sing(\widehat{X}_\epsilon)$ as follows: Take a Stein refinement $\{V_l\}$ of $\{U_j \cap \widehat{X_\epsilon}\}$ such that V_0 contains the singular set of $\widehat{X_\epsilon}$ with $\omega = \overline{\partial}\chi^* \cdot \frac{\phi_0}{t}$ over V_0 and any other does not cut the singularity of $\widehat{X_\epsilon}$. Notice that on V_j for $j \neq 0$, there is a smooth solution u_j to the equation $\bar{\partial}u_j = \omega$. Let $u_0 = (\chi^* - 1)\frac{\phi_0}{t}$, which is smooth over V_0. Then it is clear that $u_j - u_l$ is holomorphic over $V_j \cap V_l$. Since $\widehat{X_\epsilon}$ is Stein, we have $h_k \in \mathrm{Hol}(V_k)$ such that $u_j - u_l = h_j - h_l$ over $V_j \cap V_l$. Hence $u := u_j - h_j$, which is smooth over $\widehat{X}_\epsilon$, solves the equation $\bar{\partial}u = \omega$.

The solution produced from above may not have good behavior near X_ϵ. If we can find a solution u^* which is also smooth up to X_ϵ, then

$$\phi^* := \chi^*(t)\sum \chi_j(p)\phi_j(p) - tu^* \tag{3.1}$$

is holomorphic over $\widehat{X}_\epsilon$ and smooth up X_ϵ. Moreover, $\phi^* \equiv \phi$ over $\widehat{M}_0$.

Indeed, if we can find u^* that is continuous over $\widehat{X_\epsilon} \cup X_\epsilon$ and whose restriction to X_ϵ is smooth, then the restriction of ϕ^* to X_ϵ is a smooth CR function. Hence, by the strong pseudoconvexity of X, it follows easily that the ϕ^* in (3.1) must be smooth over $\widehat{X_\epsilon} \cup X_\epsilon$. Hence, the proof of Theorem 3.1 will be complete, if we can prove the following:

Proposition 3.2: Let ω be as above. Then $\overline{\partial} u = \omega$ has a solution u that is continuous up to X_ϵ and whose restriction to X_ϵ is smooth.

Proof of Proposition 3.2: As above, we let $\epsilon_0 = 3/4$ and assume that the Stein space $\hat{\mathcal{Z}}_{\epsilon_0}$ has precisely the same singular set as that of $\widehat{X}_{\epsilon_0}$. Smoothly extend the ρ-function in the end of §2 to $\hat{\mathcal{Z}}_{\epsilon_0}$. Also smoothly extend $t(p)(= \widehat{\pi}(p))$ to $\hat{\mathcal{Z}}_{\epsilon_0}$. Notice that for $p(\notin \overline{\hat{X}_\epsilon}) \approx X_\epsilon$, $\rho(p) > 0$. Define

$$\Omega_\sigma^- = \{p(\approx X_\epsilon) \in \widehat{\mathcal{Z}}_{\epsilon_0} \setminus \widehat{X_{\epsilon_0}} :$$

$$\widehat{\pi}(p) \in \Delta_\epsilon,\ \sigma^4 > r(p) > 0,\ r(p) = \frac{\rho}{(\epsilon^2 - |t|^2)^4}\},$$

where $0 < \sigma << 1$. Then Ω_σ^- is a lunar domain which has the boundary component $Y_\sigma^- := \{r(p) = \sigma^4\}$ strongly pseudoconvex and the boundary component X_ϵ strongly pseudoconcave with at least 3-negative Levi eigenvalues.

Next we let ω be as in Proposition 3.2 and we extend it smoothly to $\Omega_\sigma^0 := \Omega_\sigma^- \cup \Omega_\epsilon \cup X_\epsilon$, where Ω_{ϵ_0} is as defined in §2 and $\Omega_\epsilon = \widehat{\pi}^{-1}(\{|t| < \epsilon\}) \cap \Omega_{\epsilon_0}$. Still write ω for its smooth extension to Ω_σ^0. Then $\overline{\partial}(\omega)$ vanishes to infinite order along X_ϵ. As mentioned before, we can assume that $\omega \equiv 0$ when $|t|$ is sufficiently close to ϵ. Consider the $\overline{\partial}$-equation

$$\overline{\partial}\alpha = \overline{\partial}\omega$$

over Ω_σ^- with the $\overline{\partial}$-Neumann boundary condition along Y_σ^- and the Dirichlet boundary condition along X_ϵ.

More precisely, let M be a real hypersurface defined by $r_0 = 0$ in a complex manifold (or complex space) of dimension $n \geq 2$ with $p \in M$. Let $\{L_j\}_{j=1}^n$ be a smooth basis of the cross sections of $T^{(1,0)}U_p$, where U_p is a small neighborhood of p in the ambient space. Let $\{\omega_j\}$ be its dual frame. Assume that $L_j(r_0) \equiv 0$ when restricted to M for $j \neq n$. For a $(0, q)$-form

$$A = \sum_{i_1 < i_2 < \cdots < i_q} a_{i_1 \cdots i_q} \overline{\omega_{i_1}} \wedge \cdots \wedge \overline{\omega_{i_q}}$$

defined in a certain side of $U_p \cap M$, that is continuous up to M. We say A satisfies the $\overline{\partial}$-Neumann condition along M if $a_I|_M \equiv 0$ whenever $I = (i_1, \cdots, i_q)$ with $i_q = n$. We say that A satisfies the Dirichlet condition along M if $a_I|_M \equiv 0$ when $i_q \neq n$.

Return to the domain Ω_σ^-. After making $\sigma << 1$ we can always find a globally defined $(1,0)$-type smooth vector field L_{n+1} over $\overline{\Omega_\sigma^-}$ such that $L_{n+1}(\rho) \equiv 1$. Fix a smooth Hermitian metric $< \cdot,\cdot >_0$ over $\overline{\Omega_\sigma^-}$ and define a weighted metric $< \cdot,\cdot >$ over Ω_σ^- such that the following holds:
(i). For any $(1,0)$-type vectors $L_1, L_2 \in T_p^{(1,0)}\Omega_\sigma^-$ with $L_1(r)(p) = L_2(r)(p) = 0$,

$$< L_1, L_2 >= \sigma^{-1}(\epsilon^2 - |t|^2)^{-4} < L_1, L_2 >_0;$$

(ii). $< L_{n+1}, L_{n+1} >= \sigma^{-2}(\epsilon^2 - |t|^2)^{-8} < L_{n+1}, L_{n+1} >_0$; and $< L_{n+1}, L >= 0$ for any $L \in T_p^{(1,0)}\Omega_\sigma^-$ with $L(r)(p) = 0$,

Following Catlin in [10], we write $\mathcal{E}_c^k(\Omega_\sigma^-)$ for the collection of smooth $(0,k)$-forms over $\overline{\Omega_\sigma^-}$ that vanish when $|t|$ is sufficiently close to ϵ. Write $\mathcal{B}_k^+(\Omega_\sigma^-)$ for the subset of $\mathcal{E}_c^k(\Omega_\sigma^-)$, whose elements satisfy the Dirichlet boundary condition along X_ϵ. Write $\mathcal{B}_k^-(\Omega_\sigma^-)$ for the subset of $\mathcal{E}_c^k(\Omega_\sigma^-)$, whose elements satisfy the $\overline{\partial}$-Neumann boundary condition along Y_σ^-. Define $\mathcal{B}^k(\Omega_\sigma^-) := \mathcal{B}_k^-(\Omega_\sigma^-) \cap \mathcal{B}_k^+(\Omega_\sigma^-)$.

We define the formal adjoint $\overline{\partial}_k^{f*}$ of the $\overline{\partial}$-operator acting on the (0,k)-form as in the standard way. We say that $U \in L^2_{q-1}(\Omega_\sigma^-)$ is in the domain of the operator $\overline{\partial}_{q-1}^{mix}$, or $U \in \mathrm{Dom}(\overline{\partial}_{q-1}^{mix})$, with $\overline{\partial}_{q-1}^{mix}(U) = F$ if for any $V \in \mathcal{B}_q^-(\Omega_\sigma^-)$, we have $< U, \overline{\partial}_q^{f*} V >=< F, V >$. We write $\overline{\partial}_q^{mix*}$ for the Hilbert space adjoint of $\overline{\partial}_q^{mix}$ by using the norm induced from the inner product defined above.

Then

$$\mathrm{Dom}(\overline{\partial}_k^{mix}) \cap \mathcal{E}_c^k(\Omega_\sigma^-) = \mathcal{B}_k^+(\Omega_\sigma^-),$$

$$\mathrm{Dom}(\overline{\partial}_k^{mix*}) \cap \mathcal{E}_c^k(\Omega_\sigma^-) = \mathcal{B}_k^-(\Omega_\sigma^-), \text{ and}$$

$$\mathrm{Dom}(\overline{\partial}_k^{mix}) \cap \mathrm{Dom}(\overline{\partial}_k^{mix*}) \cap \mathcal{E}_c^k(\Omega_\sigma^-) = \mathcal{B}_k(\Omega_\sigma^-).$$

(See [10]). For $U, V \in \mathrm{Dom}(\overline{\partial}_k^{mix}) \cap \mathrm{Dom}(\overline{\partial}_k^{mix*})$, define

$$Q_k(U,V) = (\overline{\partial}_k^{mix}(U), \overline{\partial}_k^{mix}(V)) + (\overline{\partial}_k^{mix*}(U), \overline{\partial}_k^{mix*}(V)),$$

where the inner product on forms is induced from the above defined Hermitian metric on vectors. Then the following basic estimate is contained in the work of Catlin ([10]) (See [Theorem 7.1, 10]):

Theorem 3.3 (Catlin [10]): When σ is sufficiently small, one has, for a certain constant C, that $Q_2(U,U) \geq C\|U\|^2$ for any $(0,2)$-form $U \in \mathrm{Dom}(\overline{\partial}_2^{mix}) \cap \mathrm{Dom}(\overline{\partial}_2^{mix*})$.

Hence, by the standard Hilbert space theory argument as in [13], Theorem 3.3 shows that for ω introduced above, there is a unique α_2 in the domain of Q_2 such that

$$Q_2(\alpha_2, U) \equiv < \overline{\partial}(\omega), U >$$

for any U in the domain of Q_2.

By the sub-elliptic estimate established in [Theorem 9.2, Lemma 10.1, 10], one concludes that α_2 is smooth over $\Omega_\sigma^- \cup X_\epsilon \cup Y_\sigma^-$ with $\overline{\partial}_2^{mix}\alpha_2 \in \mathrm{Dom}(\overline{\partial}_3^{mix*})$ and $\overline{\partial}_2^{mix*}(\alpha_2) \in \mathrm{Dom}(\overline{\partial}_1^{mix})$. Also, from [Theorems 10.3, 10.5; Cat1], it follows that $\overline{\partial}_3^{mix*}\overline{\partial}_2^{mix}\alpha_2 \equiv 0$. Hence, write $\beta = \overline{\partial}_2^{mix*}\alpha_2$. We thus obtain $\overline{\partial}_1^{mix}(\beta) = \overline{\partial}_1(\beta) = \overline{\partial}(\omega)$ over Ω_σ^-, where $\overline{\partial}_1$ is the regular $\overline{\partial}$-operator acting on $(0,1)$-forms. Notice that β satisfies the Dirichlet boundary condition along X_ϵ.

Next, define $\widetilde{\beta}_0(p)$ to be $-\beta(p)$, for $p \in \Omega_\sigma^-$; and to be 0 for $p \in \widehat{X_\epsilon}$. Define $\widetilde{\beta} = \widetilde{\beta}_0 + \omega$. $\widetilde{\beta}$ is a (locally) L^2-integrable $(0,1)$-form over $\widehat{\Omega} \setminus Sing(\widehat{X}_\epsilon)$ with $\widehat{\Omega} := \Omega_\sigma^- \cup \widehat{X_\epsilon} \cup X_\epsilon$. We also claim that $\overline{\partial}(\widetilde{\beta}) \equiv 0$ in the sense of distribution (over $\widehat{\Omega} \setminus Sing(\widehat{X}_\epsilon)$). Indeed, we need only to verify that for each $p \in X_\epsilon$ and a small neighborhood U_p of $p \in \Omega_\sigma^0$, $< \widetilde{\beta}, \overline{\partial}_2^{f*}\chi >= 0$ for any smooth $(0,2)$-form χ compactly supported in U_p. For this, we can assume without loss of generality that U_p is an open subset in $\mathbf{C}^{n+1}$. Also assume that $\{L_j\}$ is a smooth orthonormal basis of $(0,1)$-vector fields over U_p with L_j tangent to X_ϵ for $j \neq n+1$. Also, we write $\{\omega_j\}$ for its dual basis. Write $\widetilde{\beta}_0 = \sum_{j=1}^{n+1} b_j\overline{\omega_j}$. Notice that $b_j \in C^\infty(U_p \setminus \widehat{X_\epsilon})$ and $b_j(p) = 0$ for $p \in \widehat{X_\epsilon}$. By the Dirichlet condition of β along X_ϵ, $b_j = 0$ along X_ϵ for $j \neq n+1$. Apparently, to prove the above statement, it suffices for us to verify that the distribution $\overline{\partial}\widetilde{\beta}_0$ in U_p coincides with the (locally) L^2-integrable function $\widetilde{\beta}_0^0$, which is $\overline{\partial}(-\beta)$ for $p \in U_p \setminus \widehat{X_\epsilon}$ and is 0 otherwise. Write $\chi = \sum_{j<l} \chi_{jl}\overline{\omega_j} \wedge \overline{\omega_l}$ with $\chi_{jl} \in C_0^\infty(U_p)$. Then a direct verification shows that

$$\overline{\partial}^{f*}(\chi) = \sum_{j<l} L_l(\chi_{jl})\overline{\omega_j} - \sum_{j<l} L_j(\chi_{jl})\overline{\omega}_l + \sum_j K_j(\chi)\overline{\omega_j},$$

where K only linearly involves the zeroth order terms in χ_{jl}.

Hence,

$$(\widetilde{\beta}_0, \overline{\partial}^{f*}(\chi)) := -\sum_{j<l}\int b_l\overline{L_j(\chi_{jl})} + \sum_{j<l}\int b_j\overline{L_l(\chi_{jl})} - \sum_j\int b_j\overline{K_j(\chi)}.$$

When $l \neq n+1$, we have $b_j, b_l = 0$ along X_ϵ and thus

$$\int b_l\overline{L_j(\chi_{jl})} = \int_{U_p\setminus\widehat{X_\epsilon}} L_j^*(b_l)\overline{\chi_{jl}}, \quad \int b_j\overline{L_l(\chi_{jl})} = \int_{U_p\setminus\widehat{X_\epsilon}} L_l^*(b_j)\overline{\chi_{jl}}.$$

where L_j^*, L_l^* is the formal adjoint of L_j and L_l, respectively. When $l = n+1$, since $j < n+1$ and $L_j(\rho) = 0$ along X_ϵ, we see also that in the integrals $\int b_j\overline{L_l(\chi_{jl})}$ and $\int b_l\overline{L_j(\chi_{jl})}$, there are no boundary integral terms after integrating by parts. Therefore, the distribution $\overline{\partial}\beta$ coincides with the (locally) L^2-integrable function defined above.

Finally, we consider the following $\overline{\partial}$-equation

$$\overline{\partial}u = \widetilde{\beta} \quad \text{over } \widehat{\Omega} \setminus Sing(\widehat{X}_\epsilon),$$

with u smooth in a neighborhood of $Sing(\widehat{X}_\epsilon)$.

Since $\widehat{\Omega}$ is also Stein, the same argument at the beginning of this section together with Hörmander's L^2-estimates for the $\overline{\partial}$-equation ([17]) shows that it has a solution u^0, that is in the $L^2_{loc}(\widehat{\Omega} \setminus Sing(\widehat{X}_\epsilon))$ space and is smoth in a neighborhood of the singular set of $\widehat{X_\epsilon}$ in $\widehat{\Omega}$. (The solution must also be smooth away from X_ϵ). Notice that $\overline{\partial}u^0 = \omega$ over $\widehat{X_\epsilon} \setminus Sing(\widehat{X}_\epsilon)$ and

$$\phi^* := \chi^*(t) \sum \chi_j(p)\phi_j(p) - tu^0$$

gives a holomorphic extension of ϕ to $\widehat{X_\epsilon}$. As remarked right before Proposition 3.2, the proof of Proposition 3.2 will be complete if we can prove the following:

Lemma 3.4: Let u^0 be as above. Then $u^0 \in C^0(\widehat{X_\epsilon} \cup X_\epsilon)$ and $u^0|_{X_\epsilon} \in C^\infty(X_\epsilon)$.

Proof of Lemma 3.4: By the construction of $\widetilde{\beta}$, it suffices to prove that for any $p \in X_\epsilon$, there is a small neighborhood U_p of p in $\widehat{\Omega}$ such that $u^0 \in C^0(U_p \cap \overline{\widehat{X_\epsilon}}) \cap C^\infty(X_\epsilon)$. Since the problem under study is purely local, without loss of generality, we can assume that U_p is the Euclidean ball $B_{n+1}(2) := \{z \in \mathbf{C}^{n+1} : |z| < 2\}$ and $p = 0$. Notice that u^0 is in the Sobolev $H^1(B_{n+1}(2))$-space. By the Bochner-Martinelli formula, we have the following:

$$u^0(z) = \frac{1}{(n+1)W(n+1)} \int_{|\xi|=1} \frac{u^0(\xi)\eta(\overline{\xi} - \overline{z}) \wedge \hat{\omega}(\xi)}{|\xi - z|^{2(n+1)}}$$
$$-\frac{1}{(n+1)W(n+1)} \int_{|\xi|<1} \frac{\widetilde{\beta}(\xi) \wedge \eta(\overline{\xi} - \overline{z}) \wedge \hat{\omega}(\xi)}{|\xi - z|^{2(n+1)}},$$

where $W(n+1)$ is a constant depending only on $n+1$, $\hat{\omega}(z) = dz_1 \wedge \cdots \wedge dz_{n+1}$ and

$$\eta(z) = \sum_{j=1}^{n+1} (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge dz_{n+1}.$$

Apparently, the first integral is C^∞ for $|z| < 1$. We need only to explain that

$$\int_{|\xi|<1} \frac{\widetilde{\beta}(\xi)\eta(\overline{\xi} - \overline{z}) \wedge \hat{\omega}(\xi)}{|\xi - z|^{2(n+1)}}$$

defines a continuous function over $B_{n+1}(1) \cap \overline{\widehat{X_\epsilon}}$ whose restriction to $B_{n+1}(1) \cap X_\epsilon$ is smooth.

From the way $\widetilde{\beta}$ was constructed and after a smooth change of coordinates, it then suffices to prove the following fact:

Fact: Assume that $h(x)$ is a function defined over $\mathbf{R}^n$ with compact support. Suppose that h is C^∞-smooth for $x_n < 0$, extends smoothly up to $x_n \le 0$. Also suppose that h is also C^∞-smooth for $x_n > 0$ and extends smoothly up to $x_n \ge 0$ from the upper half-space. Let

$$J_h(x) = \int_{\mathbf{R}^n} \frac{h(\xi)(\xi_1 - x_1)}{|x-\xi|^n} \mathrm{d}(\mathrm{Vol})(\xi).$$

Then $J_h(x)$ is continuous over $\{x_n \le 0\}$ and the boundary value of J_h to the hyperplane defined by $\{x_n = 0\}$ from $\{x_n < 0\}$ is C^∞-smooth.

Indeed, use the polar coordinates (r, τ) centered at $x \in \mathbf{R}^n$. Here $\tau = (\tau_1, \cdots, \tau_{n-1}, \tau_n)$ is in the unit sphere in $\mathbf{R}^n$, r is the distance from ξ to x and $\xi - x = r\tau$. Write $dS(\tau)$ for the volume element of the unit sphere. Then we have

$$\begin{aligned} J_h(x) &= \int_0^\infty dr \int_{|\tau|=1} \frac{h(x+r\tau) r\tau_1 r^{n-1}}{r^n} dS(\tau) \\ &= \int_0^\infty dr \int_{|\tau|=1} h(x+r\tau)\tau_1 dS(\tau). \end{aligned}$$

Hence, it follows easily that $J_h(x)$ is C^∞-smooth if h is smooth over $\mathbf{R}^n$. Also, under the assumptions in the Fact, it immediately implies that J_h is smooth at any point with $x_n \neq 0$. Now, extend the function h on the lower half space to an element $\widetilde{h} \in C_0^\infty(\mathbf{R}^n)$. Considering $J_{h-\widetilde{h}}$ instead of J_h, we can assume without loss of generality that $h(x) = 0$ for $x = (x', x_n)$ with $x_n < 0$. Also, we can assume $x_n \approx 0$.

Next, for $x_n < 0$ with $-x_n \le r$, write $\theta(x_n, r) \in [0, \pi/2]$ with $r\cos(\theta(x_n, r)) = -x_n$. Use the spherical coordinates

$$\begin{aligned} &\tau_n = \cos\theta_{n-1},\ \tau_{n-1} = \cos\theta_{n-2}\sin\theta_{n-1}, \cdots, \tau_2 \\ &= \cos\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}, \tau_1 = \sin\theta_{n-1} \cdots \sin\theta_1, \end{aligned}$$

with $\theta_1 \in [0, 2\pi],\ \theta_2, \cdots, \theta_{n-1} \in [0, \pi]$. Notice that $h(x + r\tau) = 0$ when $\theta_{n-1} \notin [0, \theta(x_n, r)]$ or when $r < -x_n$. Hence, we can easily see the following expression for J_h:

$$J_h = \int_{-x_n}^\infty \mathrm{d}r \int_0^{\theta(x_n, r)} G(\theta_{n-1}, r, x) \mathrm{d}\theta_{n-1}.$$

Here $G(\theta_{n-1}, r, x)$ is computed by the iterated integral with respect to $\theta_1, \cdots, \theta_{n-2}$ in the procedure of applying the Fobius theorem to the multiple integral $\int_{|\tau|=1} h(x+r\tau)\tau_1 dS(\tau)$. Apparently, we can view $G(\theta_{n-1}, r, x)$ as a smooth function in (θ_{n-1}, r, x') with parameter x_n for

$$r \ge -x_n, |x'| \le 1, \theta_{n-1} \in [0, \theta(x_n, r)].$$

As $x_n \to 0^-$, $G(\theta_{n-1}, r, x)$ is uniformly bounded and approaches (uniformly on compacta) to a function that is smooth over the region given by

$$\theta_{n-1} \in [0, \pi/2],\ r \geq 0,\ |x'| \leq 1.$$

Notice that the limit is $G(\theta_{n-1}, r, (x_1, \cdots, x_{n-1}, 0))$. Also $G(\theta_{n-1}, r, x) \equiv 0$ when $r \geq r_0 >> 1$. Therefore, we see that

$$\lim_{x_n<0, x_n \to 0} J_h(x_1, \cdots, x_{n-1}, x_n)$$

$$= \int_0^{r_0} \mathrm{d}r \int_0^{\pi/2} G(\theta_{n-1}, r, (x_1, \cdots, x_{n-1}, 0)) \mathrm{d}\theta_{n-1} = J_h(x', 0)$$

Thus, we see that J_h is continuous on $\{x_n \leq 0\}$ and has boundary value smooth over $\{x_n = 0\}$. The proof of the Fact is complete. The proof of Lemma 3.4 and thus the proof of Proposition 3.2 are complete, too. This then finally completes the proof of Theorem 3.1 ∎.

Completion of the Proof of the Main Theorem: With Theorem 3.1 at our disposal, the proof of the rest statements in the Main Theorem can be easily achieved: Let f be the smooth CR embedding of M_0 into $\mathbf{C}^m$ as in Main Theorem (IV). By Theorem 3.1, we can find a holomorphic extension F of f to $\widehat{X_{\epsilon'}}$ ($|\epsilon'| << 1$) with F smooth up to $X_{\epsilon'}$. Then the map $T = (F, \pi)$ embeds both X_ϵ and $\widehat{X_\epsilon}$ into $\mathbf{C}^m \times \Delta_\epsilon$ for $|\epsilon| << \epsilon'$. Notice that T must be proper from $\widehat{X_\epsilon}$ into $\mathbf{C}^m \times \Delta_\epsilon \setminus T(X_\epsilon)$, by the asumption. Write $\widehat{X'}_\epsilon = T(\widehat{X_\epsilon})$. We conclude easily that $\widehat{X'}_\epsilon$ must be a Stein space with properties stated in (IV1)-(IV3). The proof of the Main Theorem is complete. ∎.

Remark 3.5: Fix a distance function *dist* over $\overline{\widehat{X_\epsilon}}$. Fix certain C^k-norms $\|\cdot\|_k^0$ over M_0 ($k = 1, 2, \cdots$). Let Φ be an extension of ϕ as constructed in the proof of Theorem 3.1. From our proof of Theorem 3.1, Φ can be written as $\Phi_1 + t\Phi_2$, (see (3.1)), with Φ_1 a certain smooth extension of ϕ to $\overline{\widehat{X_\epsilon}}$ and Φ_2 a certain correction function from solving the $\overline{\partial}$-equation. We can make use of the estimates in [6] to handle Φ_1 and those in [10] to handle Φ_2 over Ω_ϵ. Meanwhile, we can apply Siu's version of Cartan's Theorem A and B with bounds to handle the bounds for solutions from solving the Cousin problem [§9, Siu1]. One can then conclude the following statement: For any $p_1 \in M_{t_1},\ p_2 \in M_{t_2}$, $\delta > 0$, $C^* > 0$, there exist an $\epsilon'(\delta, C^*) > 0$, depending only on δ and C^*, and a certain fixed positive integer k_0, such that for any $|t_1|, |t_2| < \frac{1}{2}\epsilon$, $\|\phi\|_{k_0}^0 < C^*$ with ϕ as described in Theorem 3.1, when $\mathrm{dist}(p_1, p_2) < \epsilon'(\delta, C^*)$, one has $|\Phi(p_1) - \Phi(p_2)| < \delta$ for a certain holomorphic extension Φ of ϕ with properties described in Theorem 3.1.

4. Extension of holomorphic functions and simultaneous embeddings

Let (X, π, Δ) be the strongly pseudoconvex CR family as in Main Theorem. Let $(\widehat{X}, \widehat{\pi}, \Delta)$ be the Siu-Ling completion as in the Main Theorem. Still write $M_t := \pi^{-1}(t)$ for the connected strongly pseudoconvex manifold of dimension at least 5. In this section, we discuss the question when a smooth CR function defined over M_0 can be extended holomorphically to $\widehat{M_0} := \widehat{\pi}^{-1}(0)$. This can then be applied with the Main Theorem to study the simultaneous embedding and blowing-down problems. We first briefly recall the definition of the Kohn-Rossi cohomology group.

Let M be a strongly pseudoconvex CR manifold of real dimension $2n-1$ with contact form θ and holomorphic complex tangent bundle $\mathrm{T}^{(1,0)}M$. Assign the Hermitian metric in $T_p^{(1,0)}M$ for each $p \in M$ to be the Levi-form defined there. (See §2). Let T be the Reeb vector field associated with θ in the sense that $<\theta, T>=1$ and the contraction of $d\theta$ along T is zero. For each $p \in M$, let $\{L_j(p)\}_{j=1}^{n-1}$ be an orthonormal basis of $T_p^{(1,0)}M$ with dual frame $\{\omega_j(p)\}$. ($<\omega_j, T>=0$ and ω_j annihilates any vector of type $(0,1)$). Then $\{\omega_1, \cdots, \omega_{n-1}, \overline{\omega}_1, \cdots, \overline{\omega}_{n-1}, \theta\}$ forms a basis of CT^*M and any k-form α at p has a unique representation:

$$\alpha = \sum a_{j_1,\cdots,j_l\overline{i_1},\cdots,\overline{i_q}} \omega_{j_1}(p) \wedge \cdots \wedge \omega_{j_l}(p) \wedge \overline{\omega_{i_1}(p)} \wedge \cdots \wedge \overline{\omega_{i_q}(p)}$$
$$+ \sum b_{j_1,\cdots,j_l\overline{i_1},\cdots,\overline{i_q}}(p)\theta \wedge \omega_{j_1}(p) \wedge \cdots \wedge \omega_{j_l}(p) \wedge \overline{\omega_{i_1}(p)} \wedge \cdots \wedge \overline{\omega_{i_q}(p)},$$

where in the first summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l+q=k$ and in the second summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l+q+1=k$.

A form at p is called of type $(0,k)$ if it can be expressed as

$$\sum_{i_1<\cdots<i_q} a_{0\overline{I}} \overline{\omega_{i_1}(p)} \wedge \cdots \wedge \overline{\omega_{i_q}(p)}.$$

Namely, in the above representation,

$$b_{j_1,\cdots,j_l\overline{i_1},\cdots,\overline{i_q}}(p) = 0,\ a_{j_1,\cdots,j_l\overline{i_1},\cdots,\overline{i_q}}(p) = a_{\overline{i_1},\cdots,\overline{i_k}}(p).$$

Let $\pi_{(0,q)}$ be the projection from the space of q-forms to the space of $(0,q)$-forms $\Lambda^{(0,q)}$ over M. Then we define $\overline{\partial}_b = \pi_{(0,q)} \circ d_{q-1}$, where d_{q-1} is the regular De Rham differential operator at the degree $(q-1)$. The Kohn-Rossi cohomology group $H_{KR}^{(0,q)}(M)$ of order $(0,q)$ is defined as the quotient of the space of $\overline{\partial}_b$-closed $(0,q)$-forms with the space of all $\overline{\partial}_b$-exact $(0,q)$-forms. Our definition of the Kohn-Rossi cohomology group $H_{KR}^{(0,q)}(M)$ is taken from Tanaka [35] and is isomorphic to the intrinsic definition given by [13] in the strongly pseudoconvex case. It is well-known that $H_{KR}^{(0,q)}(M)$ is a pure CR invariant, independent of the choice of the contact form θ. (See [35] [13] or the following Theorem 4.1 (4.II) and [38]).

Now, suppose that M is a connected compact strongly pseudoconvex CR manifold of real dimension at least 5. Suppose M bounds a complex space $\widehat{M}$, that has M as its smooth boundary. Let ρ_0 be a smooth function defined over $\widehat{M} \cup M$ such that $\rho < 0$ in $\widehat{M}$, $\rho \equiv 0$ along M, and $d\rho_0 \neq 0$ along M. Moreover, we assume that ρ_0 is strongly plurisubharmonic in a small neighborhood of M in $\widehat{M} \cup M$. Let $\mathrm{codh}_x(\widehat{M}) := \mathrm{codh}_{\mathcal{O}_x(\widehat{M})}\mathcal{O}_x(\widehat{M})$ be the homological codimension of $\widehat{M}$ at $x \in \widehat{M}$. Then the following statements are well-known.

Theorem 4.1: With the above notation, we have
(4.I): $\mathrm{codh}_x(\widehat{M}) \geq 3$ for any $x \in \widehat{M}$, if and only if $\widehat{M}$ is normal and $H^{(0,1)}_{KR}(M) = 0$.
(4.II): Let $\widehat{M}_\epsilon := \{x \in \widehat{M}, 0 > \rho_0(x) > -\epsilon\}$. Assume that $0 < \epsilon << 1$. Then $H^{(0,1)}_{KR}(M)$ is isomorphic to $H^1(\widehat{M}_\epsilon, \mathcal{O})$.
(4.III): ([Corollary 3.3.5, 13]) Any smooth CR function defined over M extends holomorphically to $\widehat{M}$ if either $\widehat{M}$ is smooth or only has isolated normal singularities.

Proof: (4.I) and (4.II) follow from the arguments in [38] as follows.

Let $\{y_1, \ldots, y_m\}$ be the set of singular points of $\widehat{M}$. It is well known that $\mathrm{codh}_x(\widehat{M}) \geq 3$ for any $x \in \widehat{M}$ is equivalent to $H^k_{\{y_i\}}(\widehat{M}, \mathcal{O}) = 0$ for $0 \leq k < 3$ and $0 \leq i \leq m$ (cf. Theorem 3.3 of [33]). In particular, $y_1, \ldots, y_m$ are normal singularities. On the other hand, $\dim H^{(0,1)}_{KR}(M) = \sum_{i=1}^{m} \dim H^2_{\{y_i\}}(\widehat{M}, \mathcal{O})$ by Theorem B of [38]. Hence (4.I) is proved.

Let $\widehat{M}_{res}$ be a resolution of singularity of $\widehat{M}$. By definition, $H^i_\infty(\widehat{M})$ is the i-th cohomology of the quotient complex

$$C^\infty(\widehat{M}_{res}, \Lambda^{0,*})/C^\infty_c(\widehat{M}_{res}, \Lambda^{0,*}).$$

Here $C^\infty(\widehat{M}_{res}, \Lambda^{0,*})$ is the C^∞-Dolbeault complex, and $C^\infty_c(\widehat{M}_{res}, \Lambda^{0,*})$ is the subcomplex of smooth compactly supported $(0,*)$-forms. Then by Laufer [21], $\lim_{\epsilon \to 0} H^i(\widehat{M}_\epsilon, \mathcal{O}) \cong H^i_\infty(\widehat{M}, \mathcal{O})$. On the other hand, by Andreotti and Grauert (Théoréme 15 of [1]), $H^i(\widehat{M} - \{y_1, \ldots, y_m\}, \mathcal{O}) \cong H^i(\widehat{M}_\epsilon, \mathcal{O})$ for $i \leq n-2$ and

$$H^{n-1}(\widehat{M} - \{y_1, \ldots, y_m\}, \mathcal{O}) \to H^{n-1}(\widehat{M}_\epsilon, \mathcal{O})$$

is injective. By (3.19) and (3.20) in [38], one sees that $H^i_\infty(\widehat{M}, \mathcal{O}) \cong H^{(0,i)}_{KR}(M)$. (4.II) follows immediately. ∎

Again, let (X, π, Δ) be the strongly pseudoconvex family as before with $(\widehat{X}, \widehat{\pi}, \Delta)$ as its Siu-Ling completion. Let ρ^{ϵ_0} be as in the Main Theorem (II) with $\epsilon_0 < 1$. We write $\Omega_{\epsilon_0} := \{p \in \widehat{X}_{\epsilon_0} : -\epsilon_0' < \rho(p) < 0\}$ with $\epsilon_0' << 1$. Write $\mathcal{G}_q := \widehat{\pi}_{(q)}(\mathcal{O}|_{\Omega_{\epsilon_0}})$ for the q-th direct image sheaf of the structure sheaf $\mathcal{O}|_{\Omega_{\epsilon_0}}$ over Δ_{ϵ_0}. (See [15]). By definition, for any open subset U in Δ_{ϵ_0}, $\Gamma(U, \mathcal{G}_q) = H^q(\widehat{\pi}^{-1}(U) \cap \Omega_{\epsilon_0}, \mathcal{O})$. By a theorem of Ling ([Theorem 5.3.4, 22]), $\mathcal{G}_q$ is coherent if $q < N - 2$, where N is the complex dimension of $\widehat{X}$. In particular, when $N \geq 5$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are coherent analytic sheaves. When $N \geq 4$, $\mathcal{G}_1$ is coherent.

Theorem 4.2: Let (X, π, Δ) be the strongly pseudoconvex CR family as in the Main Theorem. Let $(\widehat{X}_{\epsilon_0}, \widehat{\pi}, \Delta_{\epsilon_0})$ be the Siu-Ling completion of $(X_{\epsilon_0}, \pi, \Delta_{\epsilon_0})$ with $\epsilon_0 < 1$. Assume that $\dim_{\mathbf{R}} M_0 \geq 5$ and t is not a zero divisor of the germ of the first direct image sheaf $\mathcal{G}_1$ at $t = 0$. Let ϕ be a smooth CR function over $M_0 = \pi^{-1}(0)$. Then it admits an extension that is holomorphic over $\widehat{X}_\epsilon$ and smooth up to the strongly pseudoconvex manifold X_ϵ, where $0 < \epsilon < \epsilon_0$.

Proof of Theorem 4.2: By Theorem 3.1, to prove Theorem 4.2, it suffices to explain the ϕ defined above admits a holomorphic extension to $\widehat{M_0}$.

Let $0 < \eta_1 << \epsilon_0$ be such that $H^1(\Omega_{\epsilon_0} \cap \widehat{\pi}^{-1}(\Delta_{\eta_1}), \mathcal{O})$ has a finite set of generators $\{\xi_j\}$, whose restrictions to $(\mathcal{G}_1)_x$ also generate $(\mathcal{G}_1)_x$ for any $x \in \Delta_{\eta_1}$. Pick an $\eta_2 << \eta_1$ and a Stein open covering $\{V_j\}$ of $\widehat{\pi}^{-1}(\Delta_{\eta_2}) \cap \Omega_{\epsilon_0}$ such that ϕ has a holomorphic extension ϕ_j to each V_j. Define $\phi_{jl} = \frac{\phi_j - \phi_l}{t}$ over $V_j \cap V_l$. Then $E := \{\phi_{jl}\}$ is a closed 1-cochain and thus defines an element in $H^1(\Omega_{\epsilon_0} \cap \widehat{\pi}^{-1}(\Delta_{\eta_2}), \mathcal{O})$. By our choices and after shrinking η_2 if necessary, we have holomorphic functions a_j over Δ_{η_2} such that $E = \sum_j a_j(t)\xi_j$ in $H^1(\Omega_{\epsilon_0} \cap \widehat{\pi}^{-1}(\Delta_{\eta_2}), \mathcal{O})$. (If we need to shrink η_2, the new E is taken as the naturally restricted element and $V_j's$ will be naturally restricted too. For simplicity, we do not use new notation).

Hence there is a holomorphic function ψ_j over V_j for each j such that $\left(E - \sum_j a_j(t)\xi_j\right)(V_j \cap V_l) = \psi_j - \psi_l$. It thus follows that

$$\left(\sum_j t a_j(t)\xi_j\right)(V_j \cap V_l) = (-t\psi_j + \phi_j) - (-t\psi_l + \phi_l).$$

Hence, $\sum_j t a_j(t)\xi_j = 0$. Since under the assumption of Theorem 4.2, t is not a zero divisor of $(\mathcal{G}_1)_0$, it follows that at the very beginning, we can already choose $a_j's$ to be 0. Hence, we have that $(t\psi_j - \phi_j) = (t\psi_l - \phi_l)$ on $V_j \cap V_l$. Hence $\Phi := \phi_j - t\psi_j$ over V_j for each j, well defines a holomorphic extension of ϕ to $\widehat{\pi}^{-1}(\Delta_{\eta_2}) \cap \Omega_{\epsilon_0}$. Now by the way $\widehat{X_{\eta_2}}$

was constructed, one sees that Φ extends holomorphically to $\widehat{X_{\eta_2}}$. (See [Proposition 4.3.4, Lin]). Applying Theorem 3.1, we see the proof of Theorem 4.2. ■

Remark 4.3: By a result of Siu, if $H^1(\Omega_{\epsilon_0} \cap \widehat{\pi}^{-1}(t), \mathcal{O})$ has a fixed dimension and $N = \dim \Omega_{\epsilon_0} \geq 5$ for each t, then $\mathcal{G}_1$ is a locally free coherent sheaf. In particular, t is not a zero divisor of $\mathcal{G}_1$. ([Theorem 2, 31]). Combining this fact with the above mentioned Theorem 4.1 (4.II), we see that the hypothesis in Theorem 4.2 holds if $H^{(0,1)}_{KR}(M_t)$ has a constant dimension for $|t| << 1$ and $\dim_{\mathbf{R}}(M_t) \geq 7$.

As an immediate application of the Main Theorem and Theorem 4.2, we obtain:

Corollary 4.4: Let $(X := \cup_{|t|<1} M_t, \pi, \Delta)$ be a CR family of compact strongly pseudoconvex CR manifolds. Assume that $\dim_{\mathbf{R}} M_0 \geq 5$ and t is not a zero divisor of the germ of the first direct image sheaf $\mathcal{G}_1$ defined above at $t = 0$. Suppose that f_0 is a smooth CR embedding from M_0 into $\mathbf{C}^m$. Then there is a smooth CR diffeomorphism $F = (\widetilde{f}, \pi)$ from $X_\epsilon := \cup_{|t|<\epsilon} M_t$ into $\mathbf{C}^m \times \Delta_\epsilon$ with $\epsilon << 1$ such that $\widetilde{f}|_{M_0} = f$.

Remark 4.5: Corollary 4.4, in particular, implies Corollary 1.6 when $\dim_{\mathbf{R}}(M_t) \geq 7$ by the above mentioned theorem of Siu.

Next by applying Theorem 3.1 (Remark 3.5), Theorem 4.2 and Tanaka's theorem, we will complete the proof of Corollary 1.6 when $\dim_{\mathbf{R}}(M_t) \geq 5$ as follows:

Proof of Corollary 1.6: Let (X, π, Δ) be as in Corollary 1.6 with $(\widehat{X}, \widehat{\pi}, \Delta)$ as its Siu-Ling completion. Fix $\epsilon_0 < 1$ and define $\mathcal{G}_1 = \pi_{(1)}(\mathcal{O}(\Omega_{\epsilon_0}))$ as before. Since $N = \dim \widehat{X} \geq 4$, by Ling's theorem, $\mathcal{G}_1$ is a coherent sheaf over Δ_{ϵ_0}. Without loss of generality, by what we did above, we can assume that $t - t_0$ is not a zero divisor of $(\mathcal{G}_1)_{t=t_0}$ except at $t_0 = 0$. (See [3.10, 34]). Let f_0 be a CR diffeomorphism from M_0 into $\mathbf{C}^m$. Then, by the Tanaka theorem [35], there is a certain small $0 < \epsilon_0' < \epsilon_0$ such that f_0 extends to a smooth family $\{f_t\}$ with f_t a CR embedding from M_t into $\mathbf{C}^m$ for $|t| < \epsilon_0'$. Now, by the assumption and making use of Theorem 4.2, f_t extends holomorphically to $(\widehat{X}_{\epsilon_1}, \widehat{\pi}, \Delta_{\epsilon_1})$ for a certain fixed ϵ_1 with $\epsilon_1 < 1$ for all t with $0 < |t| < \epsilon_1$. Notice that f_t is a smooth function over X_{ϵ_1}. Write the holomorphic extension $f_t|_{M_t}$ to $(\widehat{X}_{\epsilon_1}, \widehat{\pi}, \Delta_{\epsilon_1})$ as F^t. By Remark 3.5, we can choose F^t such that $\|F^t|_{M_0} - f_0\|_{C^0(M_0)} \to 0$ as $t \to 0$. By the maximum principle in complex spaces, $\{F^t|_{\widehat{M_0}}\}$ converges uniformly over $\overline{\widehat{M_0}}$. By the normal family for holomorphic functions over complex spaces (See [Theorem 8, pp171, 15]), we conclude that $F^t \to F^0$, that is holomorphic over

$\widehat{M_0}$ and has boundary value f_0 over M_0. Namely, we proved that f_0 extends holomorphically to $\widehat{M_0}$. The rest of the argument now follows easily from Theorem 3.1. ■

We call $\widetilde{M}$ a smooth strongly pseudoconvex complex manifold if $\widetilde{M}$ is a complex manifold with smooth boundary $\partial\widetilde{M}$, that is strongly pseudoconvex with respect to $\widetilde{M}$. Let $\widetilde{X}$ be a complex manifold with X as part of its strongly pseudoconvex boundary. We call $(\widetilde{X}, \widetilde{\pi}, \Delta)$ a family of smooth strongly pseudoconvex complex manifolds if (I): $\widetilde{\pi}$ is a surjective holomorphic map from $\widetilde{X}$ to Δ, which extends smoothly to $X = \cup_t \partial\widetilde{\pi}^{-1}(t) = \cup_t \partial\widetilde{X}_t$, where $\widetilde{X}_t = \widetilde{\pi}^{-1}(t)$; (II) (X, π, Δ) is a CR family of strongly pseudoconvex manifolds. Now, let f be a holomorphic map from $\widetilde{X}_0 := \widetilde{\pi}^{-1}(0)$ into $\mathbf{C}^m$, that is biholomorphic near $\partial\widetilde{X}_0$ and extends to a smooth CR diffeomorphism from the boundary to its image. Also, assume that $f(\partial\widetilde{X}_0)$ bounds a complex space, denoted by Y, with at most isolated singularities and has $f(\partial\widetilde{X}_0)$ as its smooth boundary. We say $\widetilde{X}_0$ resolves the singularities of Y through f when Y does have isolated singularities and f is proper from $\widetilde{X}$ to Y. Notice that f then must be biholomorphic from $\widetilde{X_0} \setminus E$ into $Y \setminus \mathrm{Sing}(Y)$, where $\mathrm{Sing}(Y)$ is the singular set of Y and $E = f^{-1}(\mathrm{Sing}(Y))$ is the exceptional set of $\widetilde{X}$. In this setting, we also call f a blowing-down map from $\widetilde{X}_0$ to its image. There had been many papers in the past when a family of strongly pseudoconvex complex manifolds with exceptional sets can be simultaneously blowed-down. (See the papers [26. 27, 28] [21] and the references therein). As an immediate application of Corollaries 1.6, 4.4, we have the following result, a certain local version of which was already proved by Riemenschneider by different methods in [26-27].

Corollary 4.6: Let $(\widetilde{X}, \widetilde{\pi}, \Delta)$ be a smooth holomorphic family of strongly pseudoconvex complex manifolds. Assume that $X = \cup_{t\in\Delta}\partial\widetilde{X_t}$ can be CR embedded into a complex manifold. Suppose that

$$\dim H^{(0,1)}_{KR}(\partial\widetilde{X}_t) \equiv \text{constant}$$

and $\widetilde{X}_0$ is at least of complex dimension 3. Suppose that f_0 is a blowing-down map from $\widetilde{X}_0$ to $\mathbf{C}^m$. Then there is a map $F = (\widetilde{f}, \widetilde{\pi})$ from $\widetilde{X}_\epsilon := \widetilde{\pi}^{-1}(\Delta_\epsilon)$ to $\mathbf{C}^m \times \mathbf{C}$, which extends smoothly over $\cup_{|t|<\epsilon<<1}\partial\widetilde{X}_t$ such that $\widetilde{f}|_{\widetilde{X}_t}$ is a (holomorphic) blowing-down map from $\widetilde{X}_t$ to $\mathbf{C}^m$ with $\widetilde{f}|_{\widetilde{X}_0} = f_0$.

Proof of Corollary 4.6: Applying Corollary 1.6 to f_0, we conclude that f_0 extends to a smooth CR diffeomorphism to a neighborhood of $\partial(\widetilde{X}_0)$ in $\cup_{|t|<<1}\partial\widetilde{X_t}$. Then applying the Kohn-Rossi extension theorem

((4.III)), the CR diffeomorphism extends to the required map as in the corollary. ■

Finally, let (X, π, Δ) be as in Corollary 1.5 with $(\widehat{X}, \widehat{\pi}, \Delta)$ as its Siu-Ling completion. On the other hand, by the work of Rossi, Andreotti-Siu [3], for each $t \in \Delta$, the fiber X_t itself admits a normal Stein filling $\widehat{X_{t,nor}}$ with $codh(\widehat{X_{t,nor}}) \geq 2$. Assume that $\mathrm{codh}(\widehat{X_{t,nor}}) \geq 3$, which is equivalent to $H^{(0,1)}_{KR}(M_t) = 0$ by Theorem 4.1 (4.I). Fujiki then showed in [14] that $(\cup_{t\in\Delta}\widehat{X_{t,nor}}, \pi, \Delta)$ carries a normal Stein complex structure. In view of our Main Theorem, we see that the completion by Fujiki must then be isomorphic to the Siu-Ling completion of (X, π, Δ) by the uniqueness of both fillings.

Proof of Corollary 1.5: Indeed, we need only to show that $\widehat{M_0}$ is normal under the assumption of Corollary 1.5. By the theorems of Boutet de Monvel and Harvey-Lawson [8] [19], there is a CR diffeomorphism f from M_0 to $M_0' = f(M_0)$, that bounds a Stein space V_0 with only isolated normal singularities. Moreover V_0 is smooth near M_0'. From the proof in Corollary 1.6, we see that f extends holomorphically to $\widehat{M_0}$. Let g be the inverse of f near M_0 in $\overline{\widehat{M_0}}$. Since $\widehat{M_0}$ is also Stein, we see easily that g extends to a holomorphic map, still denoted by g, from V_0 into $\widehat{M_0}$. Apparently, from the uniqueness property of holomorphic functions, f and g are the holomorphic inverse of each other. Hence, $\widehat{M_0}$ is biholomorphically equivalent to V_0 and thus must be normal, too. ■

Remark 4.7: From the proof of Corollary 1.6, it is clear that in Corollary 1.6, one can weaken the assumption: $\dim(H^{(0,1)}_{KR}(M_t)) = const$ for each t by that of $\dim(H^{(0,1)}_{KR}(M_{\gamma(t)})) = const$ for a certain smooth function $\gamma(t)$ with $\gamma(0) = 0$ and $d(\gamma)(0) \neq 0$. Also, if the last condition holds, then $\widehat{M_0}$ in the Siu-Ling completion must be normal. However, this observation only makes sense when M_t has real dimension 5, for otherwise, by a result of Siu [31, Theorem 1], the set $\{t \in \Delta : d_1(t) = \dim(H^{(0,1)}_{KR}(M_t)) = d_1(0)\}$ is either the whole Δ or has 0 as an isolated point. Also, one can similarly define the notion of the CR family of compact strongly pseudoconvex CR manifolds with the parameter space being polydisks in any complex space and establish the similar results in this setting.

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X. Huang: School of Mathematics, Wuhan University, Wuhan, Hubei 430072, P. R. China; Current address: Department of Mathematics, Rutgers University at New Brunswick, NJ 08903, USA

E-mail address: huangx@math.rutgers.edu

H. Luk: Department of Mathematics, The Chinese University of Hongkong, Shatin, N. T., Hongkong

E-mail address: hsluk@math.cuhk.edu.hk

Stephen Yau: Institute of Mathematics, East China Normal University, Current Address: Department of Mathematics, University of Illinios, Chicago, IL 60607, USA

E-mail address: yau@uic.edu.