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Boundary characterization of holomorphic isometric embeddings between indefinite hyperbolic spaces

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We provide in this paper a boundary characterization in terms of the boundary CR invariants (or pseudo-Hermitian invariants) for holomorphic isometric embeddings between indefinite hyperbolic spaces of general codimensions.

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1. Introduction

For two integers $n \geq 2$ and $0 \leq l < n$, we equip $\mathbb{C}^{n+1}$ with a Hermitian form H_{l+1} with $l+1$ negative and $n-l$ positive eigenvalues. More precisely, write $I_{k,m}$ for the $m \times m$ diagonal matrix where its first k diagonal elements equal -1 and the rest equal $+1$. Then the Hermitian form H_{l+1} is given by $H_{l+1}(z, \overline{z}) = zI_{l+1,n+1}\overline{z}^t$ for $z \in \mathbb{C}^{n+1}$. This naturally leads to the definition of the generalized ball $\mathbb{B}_l^n$, which is a domain in the projective space $\mathbb{P}^n$:

$$\mathbb{B}_l^n = \{[z_0, \cdots, z_n] \in \mathbb{P}^n : |z_0|^2 + \cdots + |z_l|^2 > |z_{l+1}|^2 + \cdots + |z_n|^2\}.$$

Note when $l = 0, \mathbb{B}_0^n$ becomes the standard unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ (embedded into $\mathbb{P}^n$). The generalized ball $\mathbb{B}_l^n$ is indeed an open orbit of the real form $SU(l+1, n+1)$ of the complex simple Lie group $SL(n+1, \mathbb{C})$ when acting on $\mathbb{P}^n$. Here $SU(l+1, n+1)$ is the special unitary group which consists of matrices preserving the Hermitian form H_{l+1} on $\mathbb{C}^{n+1}$:

$$SU(l+1, n+1) = \{A \in SL(n+1, \mathbb{C}) : AI_{l+1,n+1}\overline{A}^t = I_{l+1,n+1}\}.$$

The topological boundary $\partial\mathbb{B}_l^n$ of $\mathbb{B}_l^n$, sometimes called the generalized sphere of signature l, is the unique closed orbit under the action of $SU(l+1, n+1)$ on $\mathbb{P}^n$. The generalized sphere $\partial\mathbb{B}_l^n$ or its local realization, the real hyperquadric

$$\mathbb{H}_l^n = \{(z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : \mathrm{Im} w = -\sum_{j=1}^{l} |z_j|^2 + \sum_{j=l+1}^{n-1} |z_j|^2\}$$

serves as a basic model for Levi-nondegenerate hypersurfaces (see [5]) and plays a fundamental role in CR geometry. Note that when $l = 0$, $\mathbb{H}_0^n$ is the standard Heisenberg hypersurface. Due to the special geometric structure of $\partial\mathbb{B}_l^n$ or $\mathbb{H}_l^n$, many striking rigidity phenomena have been discovered for mappings into the real hyperquadrics. Results along these lines can be found for instance in [29,11,14,15,9,8,7,13] and references therein. In particular, in [14] the first author defined a useful geometric invariant for a nonconstant C^2-smooth CR map F from $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^N (N \geq n \geq 2)$, called the geometric rank of F. He proved that if $n \geq 2$ and $N \leq 2n-2$, then the geometric rank must be identically zero and furthermore F extends to a linear fractional holomorphic proper map from $\mathbb{B}^n$ to $\mathbb{B}^N$.

The mapping property is of different flavor when $l > 0$. By studying local holomorphic mappings from $\mathbb{H}_l^n$ to $\mathbb{H}_l^N$, Baouendi-Huang [4] proved that any proper holomorphic map from $\mathbb{B}_l^n$ to $\mathbb{B}_l^N$ extends to a totally geodesic embedding from $\mathbb{P}^n$ to $\mathbb{P}^N$ whenever $0 < l < n-1$ and $N \geq n$. After their work, many interesting results have been established. Here we mention [1,2,10,26,13,12]. It remains an open problem to study the analytic property of mappings into hyperquadrics in the general setting.

On the other hand, the generalized ball has distinguished geometric feature as well. Recall the automorphism group of $\mathbb{B}^n_l$ is given by $SU(l+1, n+1)$ (see, for example, section 1 in [5]). The generalized ball $\mathbb{B}^n_l$ possesses a canonical indefinite metric $\omega_{\mathbb{B}^n_l}$ that is invariant under the action of its automorphisms:

$$\omega_{\mathbb{B}^n_l} = -\sqrt{-1}\partial\bar{\partial}\log\Big(\sum_{j=0}^{l} |z_j|^2 - \sum_{j=l+1}^{n-1} |z_j|^2\Big).$$

When $l = 0$, the metric is identical with the (normalized) Poincaré metric on the unit ball. A generalized ball equipped with the indefinite metric $\omega_{\mathbb{B}^n_l}$ is often called an indefinite hyperbolic space form.

In this paper, we give a complete characterization for local holomorphic isometric embeddings between indefinite hyperbolic spaces in terms of a boundary CR invariant of the maps. Let Ω be a connected open set of $\mathbb{B}^n_l$ and F a holomorphic map from Ω to $\mathbb{B}^N_{l'}$. We say F is isometric if $F^*(\omega_{\mathbb{B}^N_{l'}}) = \omega_{\mathbb{B}^n_l}$ on Ω.

Theorem 1.1. *Let $N \geq n \geq 3, 0 \leq l \leq n-1, l \leq l' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}^n_l$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $U \cap \mathbb{B}^n_l$ is connected and $F(U \cap \mathbb{B}^n_l) \subset \mathbb{B}^N_{l'}$, $F(U \cap \partial\mathbb{B}^n_l) \subset \partial\mathbb{B}^N_{l'}$. Then the following are equivalent.*
(1) F is CR transversal and has geometric rank zero at generic points on $U \cap \partial\mathbb{B}^n_l$ near p.
(2) F is an isometric embedding from $(U \cap \mathbb{B}^n_l, \omega_{\mathbb{B}^n_l})$ to $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$.

We recall that F is called CR transversal at $p \in \partial\mathbb{B}^n_l$ if $T_{F(p)}(\partial\mathbb{B}^N_{l'}) + dF(T_p\mathbb{P}^n) = T_{F(p)}\mathbb{P}^N$. We remark that if a map F as in the assumption of Theorem 1.1 exists and F is CR transversal at some point near p, then we must have $l' \geq l$ and $N - l' \geq n - l$. The definition of geometric rank, which serves as a crucial invariant for holomorphic maps between open pieces of the generalized spheres, will be given in Definition 3.3 of §3. It can be routinely computed through the fourth order jets of the map. In the language of pseudo-Hermitian geometry, the zero geometric rank at a point $\hat{q} \in F(U \cap \partial\mathbb{B}^n_l)$ is equivalent to the condition that for any $X_{\hat{q}} \in T^{(1,0)}_{\hat{q}} F(\partial\mathbb{B}^n_l)$, the value at $X_{\hat{q}}$ of the CR second fundamental form $\prod(X_{\hat{q}}, X_{\hat{q}}) \in T^{(1,0)}_{\hat{q}}(\partial\mathbb{B}^N_{l'})/dF(T^{(1,0)}(\partial\mathbb{B}^n_l))$ of $F(\partial\mathbb{B}^n_l) \subset \partial\mathbb{B}^N_{l'}$ stays in the null cone of the Levi form $\mathcal{L}_{\hat{q}}$ of $\partial\mathbb{B}^N_{l'}$ at $\hat{q}$, namely, $\hat{\mathcal{L}}_{\hat{q}}(\prod(X_{\hat{q}}, X_{\hat{q}}), \prod(X_{\hat{q}}, X_{\hat{q}})) = 0$. See Proposition 3.5 for more discussions on this matter.

In general, in the zero geometric rank case, the second fundamental form does not vanish identically. Instead, its image can have the largest possible real dimension $2(l'-l)$, that is nonzero unless $l' = l$. This is indeed the main difficulty we will encounter in the course of the proof of Theorem 1.1. The vanishing of CR second fundamental form is linked with the linearity (or the total geodesy) of the map, while our main theorem shows that the zero geometric rank condition, or equivalently the condition that the CR second fundamental form stays in the null cone of the Levi form, is precisely the one to characterize holomorphic isometric embeddings. For results related to the vanishing of the CR second fundamental form, we refer the reader to [33] and many references therein.

Remark 1.2.

1. The assumptions that $F(U \cap \mathbb{B}^n_l) \subset \mathbb{B}^N_{l'}$ and $F(U \cap \partial\mathbb{B}^n_l) \subset \partial\mathbb{B}^N_{l'}$ do not guarantee F to be CR transversal in general even at a single boundary point, as the following Example 1.3 shows.
2. Note the Levi form of the boundary of $\mathbb{B}^n_l, l > 0$, has at least one negative eigenvalue at each point. By applying the Lewy extension type theorem for mappings into compact Kähler manifolds of Siu and Ivashkovich (see references in [4]), we see that if F is holomorphic from $U \cap \mathbb{B}^n_l, l > 0$, into $\mathbb{P}^N$, then F extends to a holomorphic mapping from a neighborhood of $\partial\mathbb{B}^n_l \cap U$ to $\mathbb{P}^N$.
3. Let H be a holomorphic map in U such that $F(U \cap \mathbb{B}^n_l) \subset \mathbb{P}^n \setminus \overline{\mathbb{B}^N_{l'}}$. Note $\mathbb{P}^n \setminus \overline{\mathbb{B}^N_{l'}} \approx \mathbb{B}^N_{N-l'-1}$. We can thus regard H as a map from $U \cap \mathbb{B}^n_l$ to $\mathbb{B}^N_{N-l'-1}$.

Example 1.3. Let $F(z)$ be the polynomial map from $\mathbb{P}^5$ to $\mathbb{P}^N (N \geq 20)$ given by $F([z_0, \cdots, z_5]) = [f, g, 0, \cdots, 0]$. Here

$$f = (z_0^2, \sqrt{2}z_0z_1, \sqrt{2}z_0z_2, z_1^2, z_2^2, z_3^2, z_4^2, z_5^2, \sqrt{2}z_1z_2, \sqrt{2}z_3z_4, \sqrt{2}z_3z_5, \sqrt{2}z_4z_5);$$
$$g = (\sqrt{2}z_0z_3, \sqrt{2}z_0z_4, \sqrt{2}z_0z_5, \sqrt{2}z_1z_3, \sqrt{2}z_1z_4, \sqrt{2}z_1z_5, \sqrt{2}z_2z_4, \sqrt{2}z_2z_4, \sqrt{2}z_2z_5).$$

Note

$$\|f\|^2 - \|g\|^2 = (|z_0|^2 + |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 - |z_5|^2)^2.$$

Hence $F(\mathbb{B}^5_2) \subset \mathbb{B}^N_{11}$ and $F(\partial\mathbb{B}^5_2) \subset \partial\mathbb{B}^N_{11}$. The map F, however, is not CR transversal at any boundary point of $\mathbb{B}^5_2$.

When $1 \leq l' < 2l \leq n-1$, the CR transversality automatically holds at $F(q)$ for a generic point $q \in U \cap \partial\mathbb{B}^n_l$ (see [4] and [3]), and the geometric rank of F is always zero at such a point q. Hence our main theorem gives, in this special case, a different proof of the following theorem obtained in [2] (see also [26] for a different approach for a global version of this theorem).

Theorem 1.4 *(Baouendi-Ebenfelt-Huang [2]). Let $N \geq n$, $1 \leq l \leq \frac{n-1}{2}$, $1 \leq l' \leq \frac{N-1}{2}$ and $1 \leq l \leq l' < 2l$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}^n_l$ with $U \cap \mathbb{B}^n_l$ being connected, and F a holomorphic map from $U \cap \mathbb{B}^n_l$ into $\mathbb{B}^N_{l'}$. Assume that for any sequence $\{q_j\}_{j=1}^\infty \subset U \cap \mathbb{B}^n_l$ that converges to $\partial\mathbb{B}^n_l$, the limit set of $\{F(q_j)\}_{j=1}^\infty$ is contained in $\partial\mathbb{B}^N_{l'}$. Then F is an isometric embedding from $(U \cap \mathbb{B}^n_l, \omega_{\mathbb{B}^n_l})$ into $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$.*

We remark that in [15], the first author proved a semi-rigidity theorem for proper holomorphic map as the codimension increases. For holomorphic maps between generalized balls, the rigidity gradually disappears as the difference of the signature increases. Our main theorem will provide a useful tool for such a study in the future as the rank zero maps always appear in any signature difference case.

Moreover, the mapping problem between indefinite hyperbolic spaces has been recently discovered to be of critical importance in the study of mappings between bounded symmetric domains. By using holomorphic double fibration, Ng [27] applied the results for mappings between generalized balls in [4] to prove rigidity properties for proper maps between the type I domains. Xiao-Yuan [31,30] established rigidity results for proper maps from the unit ball to the type IV domain D_m^{IV} by regarding D_m^{IV} as an isometric submanifold of the indefinite hyperbolic space. We remark that every irreducible bounded symmetric domain (equipped with Kähler-Einstein metric) can be isometrically embedded into an indefinite hyperbolic space (with a normalizing constant) in a canonical way. For instance, let $D^I_{p,q}(p \leq q)$ be the type I classical domain. Recall the Borel embedding realizes $D^I_{p,q}(p \leq q)$ as an open subset in its compact dual $G_{p,q}$ (the Grassmannian of p-plane in $\mathbb{C}^{p+q}$). And $G_{p,q}$ can be holomorphically embedded into a projective space $\mathbb{P}^N$ for some appropriate N by the Plüker embedding P. Then P maps $D^I_{p,q}(p \leq q)$ isometrically into $\mathbb{B}^N_l \subset \mathbb{P}^N$ for some appropriate l. We should also mention closely related studies on various rigidity properties for holomorphic proper or isometric maps, and CR mappings. To name a few, the readers are referred to the work by Ebenfelt [8], Ji [16], Kim-Zaitsev [18,19], Kim [17], Kossovskiy-Lamel [20], Lamel-Mir [22,21], Mok [23,24], Mok-Ng [25], Yuan-Zhang [32] and references therein.

The paper is organized as follows. We analyze holomorphic isometries between indefinite hyperbolic spaces in Section 2. In Section 3, we recall the notion of geometric rank for mappings between real hyperquadrics and set up notations and definitions needed later. We also prove the equivalence of the geometric rank zero and Levi null cone condition. Section 4 and Section 5 are devoted to proving the main theorem. The proof is based on an induction argument. Compared with the method employed in [1] and [2], a crucial lemma (Lemma 3.2, [14]) due to the first author cannot be applied anymore for τ could be arbitrarily large, and this poses a major difficulty. Our main tools for the proof include methods from CR geometry, normal form theory and most importantly the moving point trick introduced for studying maps between manifolds with huge group actions in [14].

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2. Isometries between indefinite hyperbolic spaces

We establish the following theorem in this section:

Theorem 2.1. *Let F be a holomorphic map from an open connected subset Ω of $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$. Assume Ω is contained in the affine cell $U_0 = \{[z_0, \cdots, z_n] \in \mathbb{P}^n : z_0 \neq 0\}$ and $F(\Omega)$ is contained in the affine cell $V_0 = \{[w_0, \cdots, w_N] \in \mathbb{P}^N : w_0 \neq 0\}$. Then the following are equivalent:*
(a). F is an isometric embedding from $(\Omega, \omega_{\mathbb{B}^n_l})$ to $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$.
(b). After composing with automorphisms of $\mathbb{B}^n_l$ and $\mathbb{B}^N_{l'}$ from the right and the left, respectively, F equals to the following map in the local affine coordinates on U_0 and V_0:

$$(z_1, \cdots, z_l, \phi, z_{l+1}, \cdots, z_n, \psi).$$

Here ϕ, ψ are holomorphic maps with $l' - l$ and $N - n - l' + l$ components, respectively, and satisfy $\|\phi\| \equiv \|\psi\|$.

Here $\|\cdot\|$ denotes the usual Euclidean norm. We remark that if a map F as in (a) exists, then we must have $l' \geq l$, $N - l' \geq n - l$. Before we prove the above theorem, we fix some notations. We denote by $\delta_{j,l}$ the symbol which takes value -1 when $1 \leq j \leq l$ and 1 otherwise. If $l = 0$, $\delta_{j,0}$ is identically one for all $j \geq 1$. We also denote by $\delta_{j,l,l',n}$ the symbol which takes value -1 when $1 \leq j \leq l$ or $n \leq j \leq n + l' - l - 1$ and 1 otherwise. When $l' = l$, $\delta_{j,l,l,n}$ is the same as $\delta_{j,l}$.

Let $m \geq 1$. For two m-tuples $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ of complex numbers, we write $\langle x, y\rangle_l = \sum_{j=1}^m \delta_{j,l} x_j y_j$ and $|x|^2_l = \langle x, \bar{x}\rangle_l$. Also write $\langle x, y\rangle_{l,l',n} = \sum_{j=1}^m \delta_{j,l,l',n} x_j y_j$ and $|x|^2_{l,l',n} = \langle x, \bar{x}\rangle_{l,l',n}$. Note if $m \leq n-1$, the two symbols $\langle \cdot, \cdot\rangle_l$ and $\langle \cdot, \cdot\rangle_{l,l',n}$ are identical. We use $\langle \cdot, \cdot\rangle$ to denote the usual inner product: $\langle x, y\rangle = \sum_{j=1}^m x_j y_j$. Denote by $I_{l,m}$ the $m \times m$ diagonal matrix whose jth diagonal element equals to $\delta_{j,l}, 1 \leq j \leq m$. Similarly we define $I_{l',m}$. Write $I_{l,l',n,m}$ for the $m \times m$ diagonal matrix whose jth diagonal element equals to $\delta_{j,l,l',n}, 1 \leq j \leq m$.

Proof of Theorem 2.1. It is easy to see (b) implies (a). We will show (a) implies (b). Let $F : \Omega \to \mathbb{B}^N_{l'}$ be as in the theorem. Write $p_0 = [1, 0, \cdots, 0] \in U_0$. By composing F with automorphisms of $\mathbb{B}^n_l$ and $\mathbb{B}^N_{l'}$ and shrinking Ω if necessary, we can assume that $p_0 \in \Omega$, $F(p_0) = [1, 0, \cdots, 0] \in V_0$, and $F(\Omega) \subset V_0$. Write $F(z_1, \cdots, z_n) = (F_1(z_1, \cdots, z_n), \cdots, F_N(z_1, \cdots, z_n))$ in the local affine coordinates of U_0 and V_0, which are identified as complex Euclidean spaces. By the isometry assumption, we have $\partial\bar{\partial}\log(1 - |F|^2_{l'}) = \partial\bar{\partial}\log(1 - |z|^2_l)$. Since now $F(0) = 0$, by a standard reduction, we get $1 - |F|^2_{l'} = 1 - |z|^2_l$ or $|F|^2_{l'} = |z|^2_l$. The conclusion then follows from the following Proposition 2.2. $\square$

Recall $U(l', N) = \{A \in GL(N, \mathbb{C}) : A I_{l',N} \overline{A}^t = I_{l',N}\}$.

Proposition 2.2. *Let $f = (f_1, \cdots, f_N)$ and $g = (g_1, \cdots, g_n)$ be two holomorphic maps on an open connected set $V \subset \mathbb{C}^m$. Assume that $l \leq n, l' \leq N$, and $|f|^2_{l'} = |g|^2_l$ on V, and $\{g_1, \cdots, g_n\}$ is a linearly independent set over $\mathbb{C}$. Then $l' \geq l$ and $N - l' \geq n - l$. Moreover, there exists a matrix $T \in U(l', N)$ and two holomorphic maps ϕ, ψ with $l' - l$ and $N - n - l' + l$ components, respectively, such that*
(1): $(f_1, \cdots, f_N) = (g_1, \cdots, g_l, \phi, g_{l+1} \cdots, g_n, \psi)T$.
(2): $\|\phi\| \equiv \|\psi\|$.

Proof. It follows from [28] that $l' \geq l$ and $N - l' \geq n - l$. We will thus prove only the latter part of the conclusion. To make notations simple, by reordering the components of f, we assume $|f|^2_{l,l',n+1} = |g|^2_l$.

We write the vector space $W = \mathrm{Span}_{\mathbb{C}}\{g_1, \cdots, g_n, f_1, \cdots, f_N\}$. Since $\{g_1, \cdots, g_n\}$ is a linearly independent set over $\mathbb{C}$, we can extend it a basis of $W : \{g_1, \cdots, g_n, \varphi_1, \cdots, \varphi_k\}$. Here $k \geq 0$ ($k = 0$ means no φ_j's appear) and $\varphi_1, \cdots, \varphi_k$ are holomorphic functions on V.

Note there is an $(n + k) \times N$ matrix B such that $(f_1, \cdots, f_N) = (g_1, \cdots, g_n, \varphi_1, \cdots, \varphi_k)B$. Then it yields that

$$
\begin{aligned}
|f|^2_{l,l',n+1} &= f I_{l,l',n+1,N} \bar{f}^t \\
&= (g_1, \cdots, g_n, \varphi_1, \cdots, \varphi_k) B I_{l,l',n+1,N} \bar{B}^t (\bar{g}_1, \cdots, \bar{g}_n, \bar{\varphi}_1, \cdots, \bar{\varphi}_k)^t.
\end{aligned} \tag{2.1}
$$

By assumption, (2.1) equals to $|g|^2_l$. We will need the following lemma.

Lemma 2.3. *Let $h_1, ..., h_m$ be m linearly independent holomorphic functions in an open connected set V. Assume that $(h_1, ..., h_m)C(\overline{h}_1, ..., \overline{h}_m)^t = 0$, where C is a Hermitian matrix, then $C = 0$. Consequently, if*

$$(h_1, ..., h_m)C_1(\overline{h}_1, ..., \overline{h}_m)^t = (h_1, ..., h_m)C_2(\overline{h}_1, ..., \overline{h}_m)^t,$$

where C_1, C_2 are Hermitian matrices, then $C_1 = C_2$.

Proof of Lemma 2.3. We only prove the first part of the lemma. The second part of the statement is an easy consequence. We will prove by seeking a contradiction. Suppose $C \neq 0$. First write $C = PD\overline{P}^t$, where $D = \mathrm{diag}(\lambda_1, ..., \lambda_s, 0, ..., 0), s > 0$, is a diagonal matrix, with all $\lambda_i \neq 0, \lambda_1 \geq ... \geq \lambda_s$, and P is a unitary matrix. We write $(g_1, ..., g_m) := (h_1, ..., h_m)P$, which are also linearly independent. By hypotheses we have $\sum_{i=1}^s \lambda_i |g_i|^2 = 0$. Clearly, λ_i cannot be all positive. Assume $\lambda_1 \geq ... \geq \lambda_\nu > 0 > \lambda_{\nu+1} \geq ... \geq \lambda_s$. It yields that,

$$\sum_{i=1}^{\nu} \lambda_i |g_i|^2 = \sum_{i=\nu+1}^{s} (-\lambda_i)|g_i|^2.$$

It then follows from a lemma of D'Angelo ([6]) that the functions $g_1, ..., g_m$ are linearly dependent. We thus get a contradiction. □

Since (2.1) equals $|g|_l^2$, it follows from Lemma 2.3 that

$$BI_{l,l',n+1,N}\bar{B}^t = \mathrm{diag}(-1,\cdots,-1,1,\cdots,1,0,\cdots,0) \tag{2.2}$$

where on the right hand side of (2.2), there are l negative and $n-l$ positive ones, and k zeros on the diagonal. Now denote the rows of B by $\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_k$, where $\alpha_i,\beta_j, 1\le i\le n, 1\le j\le k$ are N-dimensional row vectors. As a consequence of (2.2), we have if we write $D=\begin{pmatrix}\alpha_1\\ \cdots\\ \alpha_n\end{pmatrix}$, then $DI_{l,l',n+1,N}\bar{D}^t = I_{l,n}$. By page 386 of [4], we are able to extend $\{\alpha_1,\cdots,\alpha_n\}$ to $\{\alpha_1,\cdots\alpha_n,\alpha_{n+1},\cdots,\alpha_N\}$ such that

$$AI_{l,l',n+1,N}\bar{A}^t = I_{l,l',n+1,N},$$

where A is the $N\times N$ matrix whose jth row is $\alpha_j, 1\le j\le N$. Thus $A\in U(l,l',n+1,N)$. Here $U(l,l',n+1,m)=\{T\in GL(m,\mathbb{C}): TI_{l,l',n+1,m}\bar{T}^t = I_{l,l',n+1,m}\}$. Consequently, $I_{l,l',n+1,N}\bar{A}^t$ and $C:=(I_{l,l',n+1,N}\bar{A}^t)^{-1}$ are also in $U(l,l',n+1,N)$. Now set

$$\hat{f} = fI_{l,l',n+1,N}\bar{A}^t = (g_1,\cdots,g_n,\varphi_1,\cdots,\varphi_k)BI_{l,l',n+1,N}\bar{A}^t.$$

Note by (2.2), $\langle\alpha_i,\overline{\beta_j}\rangle_{l,l',n+1}=0$ for $1\le i\le n, 1\le j\le k$. A direct computation verifies that

$$BI_{l,l',n+1,N}\bar{A}^t = \begin{pmatrix} I_{l,n} & O_{n\times(N-n)}\\ O_{k\times n} & M\end{pmatrix}.$$

Here $O_{p\times q}$ denotes the $p\times q$ zero matrix, M is a certain $k\times(N-n)$ matrix. Consequently,

$$\hat{f} = (-g_1,\cdots,-g_l,g_{l+1},\cdots,g_n,h_1,\cdots,h_{N-n}).$$

Write $\tilde{f}=\hat{f}I_{l,N} = fI_{l,l',n+1,N}\bar{A}^tI_{l,N}$. Then

$$\tilde{f} = (g_1,\cdots,g_l,g_{l+1},\cdots,g_n,h_1,\cdots,h_{N-n}).$$

Since $I_{l,l',n+1,N}\bar{A}^tI_{l,N}\in U(l,l',n+1,N)$, we have $|\tilde{f}|^2_{l,l',n+1} = |f|^2_{l,l',n+1} = |g|^2_l$. This yields that $\sum_{j=1}^{l'-l}|h_j|^2 = \sum_{j=l'-l+1}^{N-n}|h_j|^2$. Writing $\phi=(h_1,\cdots,h_{l'-l})$ and $\psi=(h_{l'-l+1},\cdots,h_{N-n})$, we have $f=\tilde{f}T=(g_1,\cdots,g_l,g_{l+1},\cdots,g_n,\phi,\psi)T$ and $\|\phi\|^2\equiv\|\psi\|^2$. Here T is the inverse of $I_{l,l',n+1,N}\bar{A}^tI_{l,N}$, which is still in $U(l,l',n+1,N)$.

Then we reorder the components of f and $\tilde{f}$ back to obtain Proposition 2.2. □

3. Geometric rank, second fundamental form and Levi null cone

In this section, we give the definition of geometric rank of CR transversal maps in the positive signature case and justify its invariant property. We also show the equivalence between geometric rank zero condition and the null cone condition of the values of CR second fundamental form associated with the image manifold of the mapping.

We first set up certain notations and terminologies which will also be needed in §4 and §5 for the proof of our main theorem. For $0 \le l \le n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}_l^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > -\sum_{j=1}^{l} |z_j|^2 + \sum_{j=l+1}^{n-1} |z_j|^2\}.$$

The boundary of $\mathbb{S}_l^n$ is the standard hyperquadrics: $\mathbb{H}_l^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) = \sum_{j=1}^{n-1} \delta_{j,l} |z_j|^2\}$. We also define for $l \le l' \le N-1$

$$\mathbb{S}_{l,l',n}^N = \{(z,w) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(w) > \sum_{j=1}^{N-1} \delta_{j,l,l',n} |z_j|^2\}.$$

We similarly define $\mathbb{S}_{l'}^N, \mathbb{H}_{l'}^N, \mathbb{H}_{l,l',n}^N$. Now for $(z,w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n$, let $\Psi_n(z,w) = [i+w, 2z, i-w] \in \mathbb{P}^n$. Then Ψ_n is the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}_l^n$ and its boundary $\mathbb{H}_l^n$ onto $\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$ and $\partial\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$, respectively.

Note $\mathbb{H}_{l,l',n}^N$ is identical to $\mathbb{H}_{l'}^N$ if $l' = l$. When $l' > l$, $\mathbb{H}_{l'}^N$ is holomorphically equivalent to $\mathbb{H}_{l,l',n}^N$ by a permutation of coordinates in $\mathbb{C}^N$. We will more often work with $\mathbb{H}_{l,l',n}^N$ instead of $\mathbb{H}_{l'}^N$, as it makes notations simpler.

We will write $\mathrm{Aut}(\mathbb{H}_l^n)$ and $\mathrm{Aut}_0(\mathbb{H}_l^n)$ for the (holomorphic) automorphism group of $\mathbb{H}_l^n$ and the isotropy group of $\mathbb{H}_l^n$ at 0, respectively. Write $\mathrm{Aut}^+(\mathbb{H}_l^n)$ and $\mathrm{Aut}_0^+(\mathbb{H}_l^n)$ for the automorphisms in $\mathrm{Aut}(\mathbb{H}_l^n)$ and $\mathrm{Aut}_0(\mathbb{H}_l^n)$, respectively, that in addition preserves sides (that is, maps $\mathbb{S}_l^n$ to $\mathbb{S}_l^n$). Clearly they are subgroups of $\mathrm{Aut}(\mathbb{H}_l^n)$ and $\mathrm{Aut}_0(\mathbb{H}_l^n)$, respectively. We define $\mathrm{Aut}(\mathbb{H}_{l,l',n}^N)$, $\mathrm{Aut}_0(\mathbb{H}_{l,l',n}^N)$ and $\mathrm{Aut}^+(\mathbb{H}_{l,l',n}^N)$ and $\mathrm{Aut}_0^+(\mathbb{H}_{l,l',n}^N)$ similarly.

Recall we denote by $(z,w) = (z_1, \cdots, z_{n-1}, w)$ the coordinates of $\mathbb{C}^n$. Write u for the real part of w and write

$$L_j := 2i\delta_{j,l}\bar{z}_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j},\ 1 \le j \le n-1, \quad T := \frac{\partial}{\partial u}. \tag{3.1}$$

Then $\{L_1, \cdots, L_{n-1}\}$ forms a global basis for the CR tangent bundle $T^{(1,0)}\mathbb{H}_l^n$ of $\mathbb{H}_l^n$, where T is a tangent vector field of $\mathbb{H}_l^n$ transversal to $T^{(1,0)}\mathbb{H}_l^n \oplus T^{(0,1)}\mathbb{H}_l^n$.

Let $F = (\tilde{f}, g) = (f, \phi, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a holomorphic map from a neighborhood U of $p_0 \in \mathbb{H}_l^n$ to $\mathbb{C}^N$, satisfying $F(U \cap \mathbb{S}_l^n) \subset \mathbb{S}_{l,l',n}^N$ and $F(U \cap \mathbb{H}_l^n) \subset \mathbb{H}_{l,l',n}^N$. We additionally assume $M_1 := U \cap \mathbb{H}_l^n$ is connected and F is CR transversal on M_1. We will define the geometric rank for such a map F.

First for each $p \in M_1$, we associate it with a map F_p defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (\tilde{f}_p, g_p) = (f_p, \phi_p, g_p). \tag{3.2}$$

Here for each $p = (z_0, w_0) \in M_1$, we write $\sigma^0_{(z_0,w_0)} \in \mathrm{Aut}^+(\mathbb{H}^n_l)$ for the map

$$\sigma^0_{(z_0,w_0)}(z, w) = (z + z_0, w + w_0 + 2i\langle z, \bar{z_0}\rangle_l),$$

and define $\tau^0_{(z_0,w_0)} \in \mathrm{Aut}^+(\mathbb{H}^n_{l,l',n})$ by

$$\tau^F_{(z_0,w_0)}(\xi, \eta) = (\xi - \tilde{f}(z_0, w_0), \eta - \overline{g(z_0, w_0)} - 2i\langle \xi, \overline{\tilde{f}(z_0, w_0)}\rangle_{l,l',n}).$$

Then F_p is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}^n_l$ into $\mathbb{H}^N_{l,l',n}$ with $F_p(0) = 0$. Moreover, $F(U \cap \mathbb{S}^n_l) \subset \mathbb{S}^N_{l,l',n}$.

Note the fundamental commutator identities hold:

$$\begin{aligned} &[\bar{L}_j, L_j] = 2i\delta_{j,l}(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}) = 2i\delta_{j,l}\frac{\partial}{\partial u}, \quad 1 \le j \le n-1; \\ &[\bar{L}_j, L_k],\ [T, L_k],\ [L_j, L_k],\ [L_k, L_k] = 0, \quad \text{if} \quad 1 \le j \ne k \le n-1. \end{aligned} \tag{3.3}$$

By the assumption that $F(U \cap M_1) \subset \mathbb{H}^N_{l,l',n}$, we have

$$\mathrm{Im}\, g = \langle \tilde{f}, \bar{\tilde{f}}\rangle_{l,l',n} \quad \text{on} \quad M_1. \tag{3.4}$$

In the following, for a holomorphic map $h = (h_1, \cdots, h_K)$ from $\mathbb{C}^n$ to $\mathbb{C}^K$, we write $h'_{z_j} = (\frac{\partial h_1}{\partial z_j}, \cdots, \frac{\partial h_K}{\partial z_j}), h''_{wz_j} = h''_{z_j w} = (\frac{\partial^2 h_1}{\partial w \partial z_j}, \cdots, \frac{\partial^2 h_K}{\partial w \partial z_j}), 1 \le j \le n-1$. The notations $h'_w, h''_{z_j z_k}, h''_{ww}$ are understood similarly. We apply $\bar{L}_j L_j$ to (3.4) and obtain

$$\lambda(p) := (g_p)_w(0) = g_w(p) - 2i\langle \tilde{f}'_w(p), \overline{\tilde{f}(p)}\rangle_{l,l',n} = \delta_{j,l}\langle L_j(\tilde{f}), \overline{L_j(\tilde{f})}\rangle_{l,l',n}(p). \tag{3.5}$$

Note this implies $\lambda(p)$ is a real number. Recall the CR-transversality assumption is equivalent to $\lambda(p) \ne 0$ (see for example, [4]). Furthermore, since F_p preserves the sides, we have $\lambda(p) > 0$ (see e.g. page 396 in [4]).

We apply $\bar{L}_k, L_j, j \ne k$ to (3.4) and get $\langle L_j(\tilde{f}), \overline{L_k(\tilde{f})}\rangle_{l,l',n} \mid_p = 0$. Let for $1 \le j \le n-1$,

$$E_j(p) := (\frac{\partial \tilde{f}_p}{\partial z_j}) \mid_0 = (\frac{\partial f_{p,1}}{\partial z_j}, \cdots, \frac{\partial f_{p,n-1}}{\partial z_j}, \frac{\partial \phi_{p,1}}{\partial z_j} \cdots, \frac{\partial \phi_{p,N-n}}{\partial z_j}) \mid_0 = L_j(\tilde{f})(p);$$

$$E_w(p) := (\frac{\partial \tilde{f}_p}{\partial w}) \mid_0 = (\frac{\partial f_{p,1}}{\partial w}, \cdots, \frac{\partial f_{p,n-1}}{\partial w}, \frac{\partial \phi_{p,1}}{\partial w} \cdots, \frac{\partial \phi_{p,N-n}}{\partial w}) \mid_0 = T(\tilde{f})(p).$$

Then

$$\langle E_j(p), \overline{E_j(p)}\rangle_{l,l',n} = \delta_{j,l}\lambda(p), \quad \langle E_j(p), \overline{E_k(p)}\rangle_{l,l',n} = 0, \quad 1 \le j \ne k \le n-1. \tag{3.6}$$

Write E for the $(n-1) \times (N-1)$ matrix whose jth row is $\frac{E_j(p)}{\sqrt{\lambda(p)}}, 1 \le j \le n-1$. Then E satisfies $EI_{l,l',n,N-1}\bar{E}^t = I_{l,n-1}$. Here $I_{l,n-1}$ and $I_{l,l',n,N-1}$ are as defined in §3.

As in [4], we can choose $(N-1)$-dimensional row vectors $C_1(p), \cdots, C_{N-n}(p)$ such that if we write

$$A(p) = \begin{pmatrix} \frac{E_1(p)}{\sqrt{\lambda(p)}} \\ \cdots \\ \frac{E_{n-1}(p)}{\sqrt{\lambda(p)}} \\ C_1(p) \\ \cdots \\ C_{N-n}(p) \end{pmatrix}$$

then

$$A(p) I_{l,l',n,N-1} \overline{A(p)}^t = I_{l,l',n,N-1}, \text{ i.e., } A(p) \in U(l,l',n,N-1). \tag{3.7}$$

Here recall $U(l,l',n,m) = \{T \in GL(m,\mathbb{C}) : T I_{l,l',n,m} \bar{T}^t = I_{l,l',n,m}\}$. Note that one can choose $C_j(p)$'s in such a way that $A(p)$ is smooth in p for $p \approx p_0$ by the standard Gram-Schmidt process.

Next note $B(p) := A^{-1}(p) = I_{l,l',n,N-1} \overline{A(p)}^t I_{l,l',n,N-1}$ is also in $U(l,l',n,N-1)$. Write

$$B(p) = (B_1(p), \cdots, B_{n-1}(p), \hat{B}_n(p), \cdots, \hat{B}_{N-1}(p)),$$

where $B_j(p)'$s and $\hat{B}_i(p)'$s are $(N-1)$-dimensional column vectors. Note $B_1(p), \cdots,$ $B_{n-1}(p)$ only depend on $E_1(p), \cdots, E_{n-1}(p)$. Indeed, we have

$$(B_1(p), \cdots, B_{n-1}(p)) = I_{l,l',n,N-1} \left(\frac{\overline{E_1(p)}^t}{\sqrt{\lambda(p)}}, \cdots, \frac{\overline{E_{n-1}(p)}^t}{\sqrt{\lambda(p)}} \right) I_{l,n-1}. \tag{3.8}$$

Define $F_p^* = (\tilde{f}_p^*, g_p^*) = ((f_p^*)_1, \cdots, (f_p^*)_{n-1}, (\phi_p^*)_1, \cdots, (\phi_p^*)_{N-n}, g_p^*)$ by

$$F_p^* = \frac{1}{\sqrt{\lambda(p)}} F_p \begin{pmatrix} B(p) & 0 \\ 0 & \frac{1}{\sqrt{\lambda(p)}} \end{pmatrix}. \tag{3.9}$$

Then F_p^* is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}_l^n$ into $\mathbb{H}_{l,l',n}^N$ with $F_p^*(0) = 0$ and the following holds (see [4], [2] for more details).

$$\begin{cases} f_p^* = z + O(|w| + |(z,w)|^2) \\ \phi_p^* = O(|w| + |(z,w)|^2) \\ g_p^* = w + O(|(z,w)|^2). \end{cases}$$

Let

$$a(p) = (a_1(p), \cdots, a_{n-1}(p), a_n(p), \cdots, a_{N-1}(p)) := \frac{\partial \tilde{f}_p^*}{\partial w}(0) = \frac{1}{\sqrt{\lambda(p)}} E_w(p) B(p). \tag{3.10}$$

Note

$$a_k(p) = \frac{1}{\sqrt{\lambda(p)}} E_w(p) B_k(p) \quad \text{for} \quad 1 \le k \le n-1, \ \text{and} \ |a(p)|^2_{l,l',n} = \frac{1}{\lambda(p)} |E_w(p)|^2_{l,l',n}. \tag{3.11}$$

Set for $1 \le k, j \le n-1$,

$$\begin{aligned}
d_{kj}(p) &:= \frac{\partial^2 (f_p^*)_k}{\partial z_j \partial w} \mid_0 = \frac{1}{\sqrt{\lambda(p)}} (\tilde{f}_p)''_{wz_j}(0) B_k(p) = \frac{1}{\sqrt{\lambda(p)}} L_j(\tilde{f}'_w)(p) B_k(p), \\
c_k(p) &:= \frac{\partial^2 g_p^*}{\partial z_k \partial w} \mid_0 = \frac{1}{\lambda(p)} (g_p)''_{wz_k}(0) = \frac{1}{\lambda(p)} L_k(g'_w - 2i \langle \tilde{f}'_w, \overline{\tilde{f}(p)} \rangle_{l,l',n,N}) \mid_p, \\
r(p) &:= \frac{1}{2} \mathrm{Re}\left(\frac{\partial^2 g_p^*}{\partial w^2} \right) \mid_0 = \frac{1}{2\lambda(p)} \mathrm{Re}\big((g_p)''_{ww}(0) \big) \\
&= \frac{1}{2\lambda(p)} \mathrm{Re}\big(g''_{ww} - 2i \langle \tilde{f}''_{ww}, \overline{\tilde{f}(p)} \rangle_{l,l',n,N} \big) \mid_p .
\end{aligned}$$

Write $(\xi, \eta) = (\xi_1, \cdots, \xi_{N-1}, \eta)$ for the coordinates of $\mathbb{C}^N$ and define

$$G_p(\xi, \eta) = \left(\frac{\xi - a(p)\eta}{Q_p(\xi, \eta)}, \ \frac{\eta}{Q_p(\xi, \eta)} \right), \tag{3.12}$$

where $Q_p(\xi, \eta) = 1 + 2i \langle \xi, \overline{a(p)} \rangle_{l,l',n} + \big(r(p) - i \langle a(p), \overline{a(p)} \rangle_{l,l',n} \big) \eta$. Then $G_p \in \mathrm{Aut}_0^+(\mathbb{H}^N_{l,l',n})$. Let F_p^{**} be the composition of F_p^* with G_p:

$$F_p^{**} = (\tilde{f}_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) := G_p \circ F_p^*. \tag{3.13}$$

We recall some notations (from [14,15] and [4]) for functions of weighted degree that will be used in the remaining context of the paper. We assign the weight of z to be 1, and assign the weight of u and w to be 2. We say a smooth function $h(z, \bar{z}, u)$ on $U \cap \mathbb{H}^n_l$ is of quantity $O_{wt}(s)$ for $0 \le s \in \mathbb{N}$, if $\left| \frac{h(tz, t\bar{z}, t^2 u)}{t^s} \right|$ is bounded for (z, u) on any compact subset of $U \cap \mathbb{H}^n_l$ and t close to 0. Similarly, we say h is of quantity $o_{wt}(s)$ for $0 \le s \in \mathbb{N}$, if $\left| \frac{h(tz, t\bar{z}, t^2 u)}{t^s} \right|$ converges to 0 uniformly for (z, u) on any compact subset of $U \cap \mathbb{H}^n_l$ as t goes to 0.

In general, for a smooth function $h(z, \bar{z}, u)$ on $U \cap \mathbb{H}^n_l$, we denote $h^{(k)}(z, \bar{z}, u)$ the sum of terms-weighted degree k in the Taylor expansion of h at 0. And $h^{(k)}(z, \bar{z}, u)$ also sometimes denotes a weighted homogeneous polynomial of degree k, if h is not specified. When $h^{(k)}(z, \bar{z}, u)$ extends to a holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z, w)$ or $h^{(k)}(z)$ if it depends only on z.

By Lemma 2.2 in [4], we have the following normalization and CR Gauss-Codazzi equation:

Lemma 3.1. *For each $p \in M$, F_p^{**} satisfies the normalization condition:*

$$\begin{cases} f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z)w + O_{wt}(4) \\ \phi_p^{**} = \phi_p^{**(2)}(z) + O_{wt}(3) \\ g_p^{**} = w + O_{wt}(5), \end{cases}$$

with

$$\langle \bar{z}, a_p^{**(1)}(z) \rangle_l |z|_l^2 = |\phi_p^{**(2)}(z)|_\tau, \;\; \tau = l' - l. \tag{3.14}$$

Remark 3.2. As mentioned in [4], there exists $\tau_p^{**} \in \mathrm{Aut}_0^+(\mathbb{H}^N_{l,l',n})$ such that $F_p^{**} = \tau_p^{**} \circ F_p$. From (3.14), we see, if we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$, then $\mathcal{A}(p)I_{l,n-1}$ is a $(n-1)\times(n-1)$ Hermitian matrix.

We next claim that $\mathcal{A}(p)$ is independent of the choice of $C_j(p)$. To see this, we first recall

$$\tilde{f}_p^{**} = \frac{\tilde{f}_p^{*} - a(p)g_p^*}{1 + 2i\langle \tilde{f}_p^*, \overline{a(p)} \rangle_{l,l',n} + \big(r(p) - i\langle a(p), \overline{a(p)}\rangle_{l,l',n}\big) g_p^*}.$$

Then,

$$P_j^k = \frac{\partial^2 (f_p^{**})_k}{\partial z_j \partial w} |_0 = d_{kj}(p) - a_k(p)c_j(p) - \delta_j^k (i\langle a(p), \overline{a(p)}\rangle_{l,l',n} + r(p)). \tag{3.15}$$

Here δ_j^k is the Kronecker symbol. Note each term in (3.15) is independent of $C_j(p)$.

Definition 3.3. The rank of the $(n-1)\times(n-1)$ matrix $\mathcal{A}(p) = -2i(P_j^k)_{1\le j,k\le(n-1)}$, denoted by $Rk_F(p)$, is called the geometric rank of F at p. In particular, F is said to have geometric rank zero at p if $Rk_F(p) = 0$, which occurs if and only if $\mathcal{A}(p) = 0$.

Since $\mathcal{A}(p)$ is smooth on p, we see that $Rk_F(p)$ is a lower semi-continuous function in $p \in U \cap \mathbb{H}_l^n$. Furthermore, with the same proof as that for Lemma 2.2 (A), (B) in [15], we have the following invariant property of geometric rank:

Proposition 3.4. *Let F_1 be holomorphic in a small neighborhood $U \subset \mathbb{C}^n$ of $p \in \mathbb{H}_l^n$ as above. That is $F_1(U \cap \mathbb{H}_l^n) \subset \mathbb{H}^N_{l,l',n}$ and $F_1(U \cap \mathbb{S}_l^n) \subset \mathbb{S}^N_{l,l',n}$. Moreover, F_1 is CR-transversal along $U \cap \mathbb{H}_l^n$. Assume that $F_2 = \tau \circ F_1 \circ \sigma$ with $\sigma \in \mathrm{Aut}^+(\mathbb{H}_l^n)$ and $\tau \in \mathrm{Aut}^+(\mathbb{H}^N_{l,l',n})$. Then F_2 is CR-transversal and side-preserving map from $\mathbb{H}_l^n$ to $\mathbb{H}^N_{l,l',n}$, and $Rk_{F_2}(p) = Rk_{F_1}(\sigma(p))$.*

We next define the geometric ranks for maps between generalized spheres. Let F be a holomorphic map from a small neighborhood U of $q \in \partial\mathbb{B}^n_l$ to $\mathbb{C}^N$. Assume $F(U \cap \mathbb{B}^n_l) \subset \mathbb{B}^N_{l'}$ and $F(U \cap \partial\mathbb{B}^n_l) \subset \partial\mathbb{B}^N_{l'}$, and in addition F is CR-transversal along $U \cap \partial\mathbb{B}^n_l$. We can find some Cayley transformations Φ_q that biholomorphically maps $\mathbb{S}^n_l$ and $\mathbb{H}^n_l$ to $\mathbb{B}^n_l \setminus V$ and $\partial\mathbb{B}^n_l \setminus V$ for some variety V with $q \notin V$. Write $p = \Phi_q^{-1}(q) \in \mathbb{H}^n_l$.

Similarly, we can find some Cayley transformation $\Psi_{F(q)}$ that biholomorphically maps $\mathbb{S}^N_{l,l',n}$ and $\mathbb{H}^N_{l,l',n}$ to $\mathbb{B}^N_{l'} \setminus W$ and $\partial\mathbb{B}^N_{l'} \setminus W$ for some variety W with $F(q) \notin W$. Set $\hat{F} = \Psi^{-1}_{F(q)} \circ F \circ \Phi_q$ and regard it as a germ of map at $p \in \mathbb{H}^n_l$. We then define the geometric rank of F at q, denoted by $Rk_F(q)$, to be the geometric rank $Rk_{\hat{F}}(p)$ of $\hat{F}$ at p. By the above proposition, $Rk_F(q)$ is independent of the choices of Φ_q and $\Psi_{F(q)}$, and thus it is well-defined. Note $Rk_F(q)$ is a lower semi-continuous function in $q \in U \cap \partial\mathbb{B}^n_l$.

We next give a description of the geometric rank zero condition in terms of the CR second fundamental form and the Levi null cone. The reader is refereed to [10] for many notations and background on this matter.

Let $\hat{M} \subset \mathbb{C}^N$ be a Levi non-degenerate hypersurface with signature l'. Let $M \subset \hat{M}$ be a Levi non-degenerate submanifold of hypersurface type of signature l and of CR dimension $n-1$. Let $\hat{\theta}$ be a contact form of $\hat{M}$ and $\hat{T}$ be its corresponding Reeb vector field. Let $\{\hat{L}_1, \cdots, \hat{L}_{N-1}\}$ be a frame of $T^{(1,0)}\hat{M}$ near $\hat{q} \in M \subset \hat{M}$. We can assume that $\{\hat{L}_1, \cdots, \hat{L}_{n-1}\}$ are tangent to M when restricted to M. Let $\{\hat{\theta}^1, \cdots, \hat{\theta}^{N-1}, \hat{\theta}\}$ be the dual frame of $\{\hat{L}_1, \cdots, \hat{L}_{N-1}, \hat{T}\}$. Then the Levi matrix $(\hat{g}_{\alpha\overline{\beta}})$ is given by $d\hat{\theta} = i\hat{g}_{\alpha\overline{\beta}}\hat{\theta}^\alpha \wedge \overline{\hat{\theta}^\beta}$. We normalize the frames such that $(\hat{g}_{\alpha\overline{\beta}}) = I_{l,l',n,N-1}$. Let $\hat{\omega}^j_k$ be the Webster connection with respect to this frame (see [2]). Identify the CR normal bundle $\mathcal{N} = T^{(1,0)}\hat{M}/T^{(1,0)}M$ along M as the orthogonal complement (with respect to the Levi form of $\hat{M}$) of $T^{(1,0)}M$ in $T^{(1,0)}\hat{M}$ restricted to M. Then $\mathcal{N}$ has a frame $\{\hat{L}_a\}_{a=n}^{N-1}$. Write $\hat{\omega}^j_k = \hat{\omega}^j_{k\sigma}\hat{\theta}^\sigma$. Then the CR second fundamental form $\prod : T^{(1,0)}_{\hat{q}}M \times T^{(1,0)}_{\hat{q}}M \to T^{(1,0)}_{\hat{q}}\hat{M}/T^{(1,0)}_{\hat{q}}M$ of M in its ambient space $\hat{M}$ is given by $\prod(\hat{L}_\alpha, \hat{L}_\beta) = \sum_{a=n}^{N-1} \hat{\omega}^a_{\alpha\beta}\hat{L}_a$. Notice the concept of CR second fundamental form is invariant under holomorphic change of coordinates.

Now, we take $\hat{M} = \mathbb{H}^N_{l,l',n}$ and $M = F(U \cap \mathbb{H}^n_l)$. For any $p \in M \subset \hat{M}$, as in Remark 3.2, after a holomorphic change of coordinates, we can assume that $p = 0$ and M is the image of $\mathbb{H}^n_l$ under F which satisfies the normalization in Lemma 3.1. Then by the computation in [9], writing $\phi^{(2)} = (\phi^{(2)}_1, \cdots, \phi^{(2)}_{N-n})$, we have $\prod(\sum_{\alpha=1}^{n-1} b_\alpha \frac{\partial}{\partial z_\alpha}|_0, \sum_{\alpha=1}^{n-1} b_\alpha \frac{\partial}{\partial z_\alpha}|_0) = \sum_{j=1}^{N-n} \phi^{(2)}_j(b)\frac{\partial}{\partial \xi_{j+n-1}}|_0$. Here $(\xi, \eta) = (\xi_1, \cdots, \xi_N, \eta)$ is the coordinates of the target Euclidean space $\mathbb{C}^N$. Recall that the rank zero condition is equivalent to $|\phi^{(2)}|_\tau \equiv 0$. With these set-ups, we have the following:

Proposition 3.5. *Let F be a holomorphic map in an open set U and maps $M := U \cap \mathbb{H}^n_l$ into $\mathbb{H}^N_{l,l',n}$. Assume F is CR transversal along M. Then the following two statements are equivalent:*

(1): F has geometric rank zero at every $q \in M$.

(2): For any $\hat{q} \in F(M) \subset \mathbb{H}^N_{l'}$, and any $X_{\hat{q}} \in T^{(1,0)}M$, then $\hat{\mathcal{L}}_{\hat{q}}(\prod(X_{\hat{q}}, X_{\hat{q}}), \prod(X_{\hat{q}}, X_{\hat{q}}))$ $= 0$. Here $\hat{\mathcal{L}}_{\hat{q}}$ is the Levi form of the ambient space at $\hat{q}$. Namely, the value of the second

fundamental form of M in its ambient space is in the null cone of the Levi form of the ambient space.

Claim. *A holomorphic map $F = (\widetilde{f}, g) = (z, \phi, \psi, w) : \mathbb{H}_l^n \to \mathbb{H}_{l,l',n}^N$ with $\|\phi\| \equiv \|\psi\|$ has geometric rank zero (here ϕ, ψ have $l' - l$ and $N - n - l' + l$ components, respectively).*

Proof. We first prove it is the case at $p = 0$. Note $\langle \phi, \overline{\phi(0)} \rangle = \langle \psi, \overline{\psi(0)} \rangle$. Consequently, $\langle \widetilde{f}, \widetilde{f}(0) \rangle_{l,l',n} = 0$. Write $F(0) = (\widetilde{q}, q_n) = (\widetilde{q}, 0)$ and take $\tau_0 \in \mathrm{Aut}^+(\mathbb{H}_{l,l',n}^N)$ to be $\tau_0(\xi, \eta) = (\xi - \widetilde{q}, \eta - 2i\langle \xi, \overline{\widetilde{q}} \rangle_{l,l',n})$. Set $F_1 = \tau_0 \circ F$. Then $F_1(0) = 0$ and $F_1 = (z, \phi_1, \psi_1, w)$ with $\|\phi_1\| \equiv \|\psi_1\|$. We replace F by F_1 and still write the new map as F. This will not change the geometric rank at $p = 0$. (See Proposition 3.4.) Then notice there exist holomorphic functions $\varphi_1, \cdots, \varphi_k$ which has no constant terms or linear terms in z such that $\mathrm{Span}_{\mathbb{C}}\{z_1, \cdots, z_{n-1}, \phi, \psi\} = \mathrm{Span}_{\mathbb{C}}\{z_1, \cdots, z_{n-1}, \varphi_1, \cdots, \varphi_k\}$. By the proof of Proposition 2.2, we can find some matrix $T \in U(l, l', n-1, N-1)$ such that $F_2 = (\widetilde{f}T, w) = (z, \hat{\phi}, \hat{\psi}, w)$, and the components of $\hat{\phi}, \hat{\psi}$ are linear combinations of φ_j's. In particular, they have no linear terms in z. Then it is easy to verify by definition that the geometric rank of F_2 is zero at $p = 0$. By Proposition 3.4, F also has geometric rank zero at $p = 0$. To study the geometric rank at a point $p \approx 0$, we note there exists $\sigma \in \mathrm{Aut}^+(\mathbb{H}_l^n)$ such that $\sigma(0) = p$. Moreover, there exists $\tau \in \mathrm{Aut}^+(\mathbb{H}_{l,l',n}^N)$ such that $G := \tau \circ F \circ \sigma = (z, \phi, \psi, w)$. By the preceding argument, $\mathrm{Rk}_G(0) = 0$. By Proposition 3.4, we have $\mathrm{Rk}_F(p) = \mathrm{Rk}_F(\sigma(0)) = \mathrm{Rk}_G(0) = 0$. □

4. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. We will first work with maps between hyperquadrics instead of generalized spheres. This makes it easier to apply techniques from CR geometry. The following result is crucial to establish Theorem 1.1.

Theorem 4.1. *Let $U \subset \mathbb{C}^n$ a small (connected) neighborhood of 0. Let F be a holomorphic map from U to $\mathbb{C}^N$ such that $F(0) = 0$ and $F(U \cap \mathbb{H}_l^n) \subset \mathbb{H}_{l,l',n}^N$, $F(U \cap \mathbb{S}_l^n) \subset \mathbb{S}_{l,l',n}^N$. Then the following statements are equivalent.*
(1) F is CR transversal at 0 and F has geometric rank zero near 0 along H_l^n.
(2) There exists some $\tau \in \mathrm{Aut}_0^+(\mathbb{H}_{l,l',n}^N)$ such that the following holds near 0:

$$\tau \circ F = (z, \phi, \psi, w),$$

where ϕ and ψ are holomorphic maps near 0 with $l' - l$ and $N - n - l' + l$ components, respectively, satisfying $\|\phi\| \equiv \|\psi\|$.

At the end of §3, we have shown that (2) implies (1). We will therefore prove only the converse implication in this section.

4.1. Some preliminaries

Assume F satisfies the assumption in (1). By Lemma 3.1 and the zero geometric rank condition, we can compose F with some element in $\mathrm{Aut}_0^+(\mathbb{H}^N_{l,l',n})$ to make $F = (\tilde{f}, g) = (f, \varphi, g)$ satisfy the following normalization:

$$\begin{cases} f = z + O_{wt}(4) \\ \phi = \phi^{(2)}(z) + O_{wt}(3) \\ g = w + O_{wt}(5). \end{cases} \tag{4.1}$$

For $p \in M$ near 0, let F_p be as in (3.2) and $\lambda(p)$ as in (3.5). Recall $\lambda(p) \approx 1$ is a real number for $p \approx 0$. Let $F_p^* = (\frac{1}{\sqrt{\lambda}} \tilde{f}_p B(p), \frac{1}{\lambda} g_p)$ be as in (3.9). We next let $B(p) = (B_1(p), \cdots, B_{n-1}(p), \hat{B}_n(p), \cdots, \hat{B}_{N-1}(p)) \in U(l, l', n, N-1)$ be as in §3. In particular, (3.8) holds:

$$(B_1(p), \cdots, B_{n-1}(p)) = I_{l,l',n,N-1} \left(\frac{\overline{E_1(p)}^t}{\sqrt{\lambda(p)}}, \cdots, \frac{\overline{E_{n-1}(p)}^t}{\sqrt{\lambda(p)}} \right) I_{l,n-1}. \tag{4.2}$$

Let $a(p)$ be as in (3.10) and (3.11). Then we have

$$a_j(p) = \frac{1}{\lambda(p)} \langle T(\tilde{f}), \delta_{j,l} \overline{L_j(\tilde{f})} \rangle_{l,l',n}, \quad \text{if} \quad 1 \le j \le n-1; \tag{4.3}$$

$$a_j(p) = \frac{1}{\sqrt{\lambda(p)}} \langle T(\tilde{f}), \hat{B}_j(p) \rangle, \quad \text{if} \quad n \le j \le N-1. \tag{4.4}$$

Let $r(p) = \frac{1}{2} \mathrm{Re} \left(\frac{\partial^2 g_p^*}{\partial w^2} \right) |_0$ be as in §3 and write

$$\triangle(f_p^*, g_p^*) := 1 + 2i \langle \tilde{f}_p^*, \overline{a(p)} \rangle_{l,l',n} + \big(r(p) - i \langle a(p), \overline{a(p)} \rangle_{l,l',n} \big) g_p^*. \tag{4.5}$$

Let $F_p^{**} = (\tilde{f}_p^{**}, g_p^{**})$ be as in (3.13). That is,

$$\tilde{f}_p^{**} = \frac{\tilde{f}_p^* - a(p) g_p^*}{\triangle(f_p^*, g_p^*)}, \quad g_p^{**} = \frac{g_p^*}{\triangle(f_p^*, g_p^*)}. \tag{4.6}$$

Then the normalization in Lemma 3.1 holds.

We pause to fix some notations. Write $\mathbb{N}$ for the set of non-negative integers. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$ and $\beta \in \mathbb{N}$, we write $D_z^\alpha D_w^\beta = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1} \cdots \partial z_{n-1}^{\alpha_{n-1}} \partial w^\beta}$. In the following context, we will introduce a notion of weighted degree in $p \in \mathbb{H}^n_l$. Let $h(z, w, p)$ be a smooth function in $W \times V$. Here $(z, w) \in W$ and W is an open set in $\mathbb{C}^{n-1} \times \mathbb{C}$, while $p \in V$ and V is an open subset of $\mathbb{H}^n_l$ containing 0. We say $h \in O_{wt,p}(k)$ with $k \ge 0$ if for every $\alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N}$ and $(z_0, w_0) \in W$, it holds that $\mathcal{H}(p) :=$

$D_z^\alpha D_w^\beta h(z_0, w_0, p)$ is in $O_{wt}(k)$. That is, writing $p = (\hat{p}, p_n) = (p_1, \cdots, p_{n-1}, p_n = u + iv)$, we have $\left|\frac{\mathcal{H}(t\hat{p}, t\bar{\hat{p}}, t^2 u)}{t^k}\right|$ is bounded for p on any compact subset of V and t close to 0. Sometimes even h is independent of (z, w), we will still use the notion $O_{wt,p}(k)$ to distinguish the variables (z, w) and p. We next prove the following proposition.

Proposition 4.2.

$$\lambda(p) = 1 + O_{wt,p}(3), \tag{4.7}$$

$$a_j(p) = O_{wt,p}(2) \quad \textit{if} \quad 1 \le j \le n-1, \tag{4.8}$$

$$a_j(p) = O_{wt,p}(1) \quad \textit{if} \quad n \le j \le N-1, \tag{4.9}$$

$$r(p) = O_{wt,p}(1). \tag{4.10}$$

Proof of Proposition 4.2. Note by (4.1), $Tg(p) = g'_w(p) = 1 + O_{wt,p}(3)$, $f(p) = O_{wt,p}(1)$, $Tf(p) = f'_w(p) = O_{wt,p}(2)$, and also $\varphi(p) = O_{wt,p}(2)$, $T\varphi(p) = \varphi'_w(p) = O_{wt,p}(1)$. And $L_j\varphi(p) = O_{wt,p}(1)$ for $1 \le j \le n-1$. We have by (3.5), $\lambda(p) = 1 + O_{wt,p}(3)$. It follows from (4.3) and (4.7) that

$$a_j(p) = O_{wt,p}(2) \quad \text{if} \quad 1 \le j \le n-1.$$

When $n \le j \le N-1$, since $T\tilde{f}(p) = O_{wt,p}(1)$, we conclude by (4.4) that $a_j(p) = O_{wt,p}(1)$. Note $g''_{ww}(p) = O_{wt,p}(1)$, $\tilde{f}(p) = O_{wt,p}(1)$. Using (4.7), we have $r(p) = O_{wt,p}(1)$. □

4.2. A crucial proposition

The key step to prove Theorem 4.1 is to establish the following Proposition 4.3. The proof of the proposition heavily relies on the moving point trick (see [14]). Recall $\tau = l' - l$.

Proposition 4.3. *Let F be as above. Fix an integer $s \ge 5$. Assume*

$$\begin{cases} (f_p^{**})^{(t-1)} \equiv 0, \\ (g_p^{**})^{(t)} \equiv 0, \\ \langle (\varphi_p^{**})^{(s_1)}, (\varphi_p^{**})^{(s_2)} \rangle_\tau \equiv 0, \end{cases} \tag{4.11}$$

for all $p \in \mathbb{H}_l^n$ close to 0 and all $4 \le s_1 + s_2 = t < s$. Then (4.11) holds at $p = 0$ for $s_1 + s_2 = t = s$.

Proof of Proposition 4.3. We split the proof into several lemmas (Lemma 4.4–4.14). Recall we denote by $\mathbb{N}$ the set of non-negative integers. Let $\alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N}$. We say $(\alpha, \beta) \in \mathcal{E}$ if $|\alpha| + \beta = 1$. Fix an integer $\hat{s} \ge 2$. Write $I_{\hat{s}}$ for the collection of indices $(\alpha, \beta) \in \mathbb{N}^{n-1} \times \mathbb{N}$ that satisfies $|\alpha| + 2|\beta| = \hat{s}$ and $(\alpha, \beta) \notin \mathcal{E}$.

Lemma 4.4. *Let $s \geq 5$ be as in Proposition 4.3. Fix $s_1 \geq 2, s_2 \geq 2$ with $s_1 + s_2 < s$. Then for any $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$, and any $1 \leq j \leq n-1$, the following hold:*

$$\langle L_j L^{\alpha_1} T^{\beta_1} \varphi^{(s_1+1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau(p) = -\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{\bar{L}_j L^{\alpha_2} T^{\beta_2} \varphi^{(s_2+1)}} \rangle_\tau(p); \quad (4.12)$$
$$\langle \bar{L}_j L^{\alpha_1} T^{\beta_1} \varphi^{(s_1+1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau(p) = -\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{L_j L^{\alpha_2} T^{\beta_2} \varphi^{(s_2+1)}} \rangle_\tau(p). \quad (4.13)$$

Proof of Lemma 4.4. We start with the hypothesis (4.11), which implies that

$$\langle D_z^{\alpha_1} D_w^{\beta_1} \tilde{f}_p^{**}, \overline{D_z^{\alpha_2} D_w^{\beta_2} \tilde{f}_p^{**}} \rangle_{l,l',n}(0) = 0, \quad (4.14)$$

where $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. Note that by (4.5), (4.6), we have

$$\tilde{f}_p^{**} = \tilde{f}_p^*[1 - 2i\langle \tilde{f}_p^*, \overline{a(p)} \rangle_{l,l',n} - r(p) g_p^*] - a(p) g_p^* + O_{wt,p}(2). \quad (4.15)$$

Recall $g_p^* = \frac{g_p}{\lambda(p)}$. Also $\lambda(p) = 1 + O_{wt,p}(3)$ and

$$D_z^\alpha D_w^\beta g_p(0) = L^\alpha T^\beta g(p) - 2i\langle L^\alpha T^\beta \tilde{f}(p), \overline{\tilde{f}(p)} \rangle_{l,l',n}. \quad (4.16)$$

It then follows from the assumption that whenever $|\alpha| + 2\beta \leq s - 2$, and $(\alpha, \beta) \notin \mathcal{E}$, we have

$$D_z^\alpha D_w^\beta g_p^*(0) = O_{wt,p}(2).$$

Recall by Proposition 4.2, $a(p) = O_{wt,p}(1)$ and $r(p) = O_{wt,p}(1)$. We then obtain from (4.14) and (4.15) that

$$\begin{aligned}
&\langle D_z^{\alpha_1} D_w^{\beta_1} \tilde{f}_p^*, \overline{D_z^{\alpha_2} D_w^{\beta_2} \tilde{f}_p^*} \rangle_{l,l',n}(0) \\
&+\langle D_z^{\alpha_1} D_w^{\beta_1} \tilde{f}_p^*, \overline{-2i D_z^{\alpha_2} D_w^{\beta_2} (\tilde{f}_p^* \langle \tilde{f}_p^*, \bar{a} \rangle_{l,l',n})} \rangle_{l,l',n}(0) \\
&+\langle -2i D_z^{\alpha_1} D_w^{\beta_1} (\tilde{f}_p^* \langle \tilde{f}_p^*, \bar{a} \rangle_{l,l',n}), \overline{D_z^{\alpha_2} D_w^{\beta_2} \tilde{f}_p^*} \rangle_{l,l',n}(0) \\
&+\langle D_z^{\alpha_1} D_w^{\beta_1} \tilde{f}_p^*, \overline{-D_z^{\alpha_2} D_w^{\beta_2} (r(p) \tilde{f}_p^* g_p^*)} \rangle_{l,l',n}(0) \\
&+\langle -D_z^{\alpha_1} D_w^{\beta_1} (r(p) \tilde{f}_p^* g_p^*), \overline{D_z^{\alpha_2} D_w^{\beta_2} \tilde{f}_p^*} \rangle_{l,l',n}(0) = O_{wt,p}(2).
\end{aligned} \quad (4.17)$$

On the other hand, recall $\tilde{f}_p^* = \frac{1}{\sqrt{\lambda(p)}} \tilde{f}_p B(p)$, and $B(p) I_{l,l',n,N-1} \overline{B(p)}^t = I_{l,l',n,N-1}$. This implies for any $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathbb{N}^{n-1}$, and $\hat{\beta}_1, \hat{\beta}_2 \in \mathbb{N}$,

$$\begin{aligned}
&\langle D_z^{\hat{\alpha}_1} D_w^{\hat{\beta}_1} \tilde{f}_p^*, \overline{D_z^{\hat{\alpha}_2} D_w^{\hat{\beta}_2} \tilde{f}_p^*} \rangle_{l,l',n}(0) \\
=&\frac{1}{\lambda(p)} \langle D_z^{\hat{\alpha}_1} D_w^{\hat{\beta}_1} \tilde{f}_p, \overline{D_z^{\hat{\alpha}_2} D_w^{\hat{\beta}_2} \tilde{f}_p} \rangle_{l,l',n}(0) = \frac{1}{\lambda(p)} \langle L^{\hat{\alpha}_1} T^{\hat{\beta}_1} \tilde{f}(p), \overline{L^{\hat{\alpha}_2} T^{\hat{\beta}_2} \tilde{f}(p)} \rangle_{l,l',n}.
\end{aligned} \quad (4.18)$$

Now letting $|\hat{\alpha_1}|+2\hat{\beta}_1 \geq 2$ or $|\hat{\alpha_2}|+2\hat{\beta}_2 \geq 2$ and $|\hat{\alpha_1}|+2\hat{\beta}_1+|\hat{\alpha_2}|+2\hat{\beta}_2 \leq s-1$, we have $L^{\hat{\alpha}_1}T^{\hat{\beta}_1}f(p) = O_{wt,p}(1)$ or $L^{\hat{\alpha}_2}T^{\hat{\beta}_2}f(p) = O_{wt,p}(1)$, and $\langle L^{\hat{\alpha}_1}T^{\hat{\beta}_1}\varphi(p), \overline{L^{\hat{\alpha}_2}T^{\hat{\beta}_2}\varphi(p)}\rangle_\tau = O_{wt,p}(1)$. Thus the quantity in (4.18) belongs to $O_{wt,p}(1)$. But $r(p) = O_{wt,p}(1)$, and $a_j(p) = O_{wt,p}(1)$ for all j. This implies the last four terms on the left hand side of (4.17) belong to $O_{wt,p}(2)$. Hence we obtain from (4.17) that the weighted degree one part (in p) in expansion of $\langle D_z^{\alpha_1}D_w^{\beta_1}\tilde{f}_p^*, \overline{D_z^{\alpha_2}D_w^{\beta_2}\tilde{f}_p^*}\rangle_{l,l',n}(0)$ equals 0. By (4.18) and (4.7), we conclude that the weighted degree one part in expansion of $\langle L^{\alpha_1}T^{\beta_1}\tilde{f}(p), \overline{L^{\alpha_2}T^{\beta_2}\tilde{f}(p)}\rangle_{l,l',n}$ equals zero.

Using again the fact

$$D_z^{\hat{\alpha}}D_w^{\hat{\beta}}f_p(0) = L^{\hat{\alpha}}T^{\hat{\beta}}f(p) = O_{wt,p}(1),$$

whenever $2 \leq |\hat{\alpha}| + 2\hat{\beta} \leq s-2$, we have $\langle L^{\alpha_1}T^{\beta_1}f(p), \overline{L^{\alpha_2}T^{\beta_2}f(p)}\rangle_l$ belongs to $O_{wt,p}(2)$. Hence we conclude the weighted degree one part in the expansion of $\langle L^{\alpha_1}T^{\beta_1}\varphi(p), \overline{L^{\alpha_2}T^{\beta_2}\varphi(p)}\rangle_\tau$ equals 0. This means

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1+1)}(p), \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}(p)}\rangle_\tau + \langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}(p), \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+1)}(p)}\rangle_\tau = 0. \tag{4.19}$$

We finally apply L_j and $\overline{L_j}$ to (4.19) and obtain the two equations in Lemma 4.4. □

Write $e_j \in \mathbb{N}^{n-1}$, $1 \leq j \leq n-1$, for the $(n-1)$-tuple whose jth component equals 1 and all other components equal 0. Write $\alpha_1 = \sum_{j=1}^{n-1} k_j^1 e_j$, i.e. $\alpha_1 = (k_1^1, \cdots, k_{n-1}^1)$. Similarly, write $\alpha_2 = \sum_{j=1}^{n-1} k_j^2 e_j$. We have the following lemmas.

Lemma 4.5. *Let $s_1 \geq 2, s_2 \geq 2$ and $s_1+s_2 = s \geq 5$. Assume $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$, $\beta_1 \geq 1$, and $(\alpha_1, \beta_1-1) \notin \mathcal{E}$. If there is some $1 \leq j_0 \leq n-1$, such that $k_{j_0}^1 = 0$ and $k_{j_0}^2 = 0$, then*

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau(p) = -\langle L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1-2)}, \overline{L^{\alpha_2}T^{\beta_2+1}\varphi^{(s_2+2)}}\rangle_\tau(p).$$

Proof of Lemma 4.5. We have

$$\begin{aligned} &\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle TL^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau \\ =&\frac{\delta_{j_0,l}}{2i}\langle(\bar{L}_{j_0}L_{j_0} - L_{j_0}\bar{L}_{j_0})L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau. \end{aligned} \tag{4.20}$$

Here we have used the identity $[\bar{L}_j, L_j] = 2i\delta_{j,l}T$. By the assumption $k_{j_0}^1 = 0$,

$$L_{j_0}\bar{L}_{j_0}L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1)} = L_{j_0}L^{\alpha_1}T^{\beta_1-1}\bar{L}_{j_0}\varphi^{(s_1)} = 0.$$

Furthermore, note $(\alpha_1 + e_{j_0}, \beta_1 - 1) \notin \mathcal{E}$. We apply Lemma 4.4 twice to obtain (4.20) equals

$$\begin{aligned} &-\frac{\delta_{j_0,l}}{2i}\langle L_{j_0}L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1-1)}, \overline{L_{j_0}L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+1)}}\rangle_\tau \\ =&\frac{\delta_{j_0,l}}{2i}\langle L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1-2)}, \overline{\bar{L}_{j_0}L_{j_0}L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+2)}}\rangle_\tau. \end{aligned} \tag{4.21}$$

Again noting

$$\bar{L}_{j_0}L_{j_0} = L_{j_0}\bar{L}_{j_0} + 2i\delta_{j_0,l}T \quad \text{and} \quad L_{j_0}\bar{L}_{j_0}L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+1)} = L_{j_0}L^{\alpha_2}T^{\beta_2}\bar{L}_{j_0}\varphi^{(s_2+1)} = 0,$$

we have (4.21) equals to the following

$$\begin{aligned} &\frac{\delta_{j_0,l}}{2i}(-2i\delta_{j_0,l})\langle L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1-2)}, \overline{TL^{\alpha_2}T^{\beta_2}\varphi^{(s_2+2)}}\rangle_\tau \\ =&-\langle L^{\alpha_1}T^{\beta_1-1}\varphi^{(s_1-2)}, \overline{L^{\alpha_2}T^{\beta_2+1}\varphi^{(s_2+2)}}\rangle_\tau. \end{aligned}$$

This proves Lemma 4.5. □

Lemma 4.6. *Let s, s_1, s_2 be as in Lemma 4.5 and $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. Then*

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau(p) = C\langle L^{\hat{\alpha}_1}T^{\hat{\beta}_1}\varphi^{(s_1^*)}, \overline{L^{\hat{\alpha}_2}T^{\hat{\beta}_2}\varphi^{(s_2^*)}}\rangle_\tau(p).$$

Here C is a nonzero constant. Moreover, if $s = 2s^*$ is even, then $s_1^* = s_2^* = s^*$. If $s = 2s^* + 1$ is odd, then $s_1^* = s^*, s_2^* = s^* + 1$. And $(\hat{\alpha}_1, \hat{\beta}_1) \in I_{s_1^*}, (\hat{\alpha}_2, \hat{\beta}_2) \in I_{s_2^*}$.

Proof of Lemma 4.6. We first notice that the following equations follow from (3.3) and an induction argument:

$$\bar{L}_jL_j^k = L_j^k\bar{L}_j + 2ik\delta_{j,l}TL_j^{k-1}; \quad \bar{L}_j^kL_j = L_j\bar{L}_j^k + 2ik\delta_{j,l}T\bar{L}_j^{k-1}. \tag{4.22}$$

The proof of Lemma 4.6 for $s = 5$ is slightly different and we will leave it to §5. We will therefore assume $s \geq 6$ in the following context of proof. Furthermore, since $\langle A, \bar{B}\rangle_\tau = \overline{\langle B, \bar{A}\rangle_\tau}$ for two vectors A, B, we will assume $s_1 \geq s_2 - 1$.

We prove by induction on $m = s_1 - s_2 \geq 0$. If $m = -1$ or 0, i.e. $s_1 = s_2 - 1$ or $s_1 = s_2$, the conclusion is trivial. Now suppose the conclusion holds for $-1 \leq m \leq k$ with $k \geq 0$ and consider the case where $m = k + 1$. In this case, $s_1 - s_2 \geq 1$ and thus $s_1 \geq 4$. We have two different cases (A) and (B).

(A) If $\alpha_1 \neq 0$, we let i_0 be the smallest integer such that $k_{i_0}^1 \neq 0$ and write $\tilde{\alpha}_1 = \alpha_1 - e_{i_0}$. Then we have by Lemma 4.4,

$$\begin{aligned} &\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau \\ =&\langle L_{i_0}L^{\tilde{\alpha}_1}T^{\beta_1}\varphi^{(s_1-1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau \\ =&-\langle L^{\tilde{\alpha}_1}T^{\beta_1}\varphi^{(s_1-1)}, \overline{\bar{L}_{i_0}L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+1)}}\rangle_\tau. \end{aligned} \tag{4.23}$$

Write $\alpha_2 = \sum_{j=1}^{n-1} k_j^2 e_j$. Note if $k_{i_0}^2 = 0$, then $\bar{L}_{i_0} L^{\alpha_2} T^{\beta_2} \varphi^{(s_2+1)} = 0$, and the conclusion is trivially true. Now assume $k_{i_0}^2 \neq 0$ and write $\tilde{\alpha}_2 = \alpha_2 - k_{i_0}^2 e_{i_0}$. Then by (4.22), we have (4.23) equals

$$\begin{aligned}
& - \langle L^{\tilde{\alpha}_1} T^{\beta_1} \varphi^{(s_1-1)}, \overline{\bar{L}_{i_0} L_{i_0}^{k_{i_0}^2} L^{\tilde{\alpha}_2} T^{\beta_2} \varphi^{(s_2+1)}} \rangle_\tau \\
= & - \langle L^{\tilde{\alpha}_1} T^{\beta_1} \varphi^{(s_1-1)}, \overline{(L_{i_0}^{k_{i_0}^2} \bar{L}_{i_0} + 2ik_{i_0}^2 \delta_{i_0,l} T L_{i_0}^{k_{i_0}^2-1}) L^{\tilde{\alpha}_2} T^{\beta_2} \varphi^{(s_2+1)}} \rangle_\tau \\
= & 2ik_{i_0}^2 \delta_{i_0,l} \langle L^{\tilde{\alpha}_1} T^{\beta_1} \varphi^{(s_1-1)}, \overline{L_{i_0}^{k_{i_0}^2-1} L^{\tilde{\alpha}_2} T^{\beta_2+1} \varphi^{(s_2+1)}} \rangle_\tau .
\end{aligned}$$

Now $(s_1-1)-(s_2+1) = k-1$ and thus the proof is finished by the inductive hypothesis.

(B) If $\alpha_1 = 0$, then $\beta_1 \geq 2$. We have by (3.3) and Lemma 4.4,

$$\begin{aligned}
& \langle T^{\beta_1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau \\
= & \frac{\delta_{1,l}}{2i} \langle (\bar{L}_1 L_1 - L_1 \bar{L}_1) T^{\beta_1-1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau \\
= & \frac{\delta_{1,l}}{2i} \langle \bar{L}_1 L_1 T^{\beta_1-1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau \\
= & \frac{-\delta_{1,l}}{2i} \langle L_1 T^{\beta_1-1} \varphi^{(s_1-1)}, \overline{L_1 L^{\alpha_2} T^{\beta_2} \varphi^{(s_2+1)}} \rangle_\tau .
\end{aligned}$$

Again now $(s_1-1)-(s_2+1) = k-1$ and the proof is done by the inductive hypothesis. This establishes Lemma 4.6. □

Lemma 4.7. *(a). Let s_1, s_2, s be as in Lemma 4.5. Let $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. If $\alpha_1 \neq \alpha_2$, then*

$$\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau (p) = 0.$$

(b). Let s_1, s_2, s be as in Lemma 4.5 and assume s is odd. Let $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. Then

$$\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau (p) = 0.$$

Proof of Lemma 4.7. Again we will treat the case for $s = 5$ separately and leave its proof to §5. We therefore assume here $s \geq 6$. Writing $\alpha_i = \sum_{j=1}^{n-1} k_j^i e_j, 1 \leq i \leq 2$, by assumption there is some $1 \leq j \leq n-1$ such that $k_j^1 \neq k_j^2$. To make the notation simple, we assume, without loss of generality, that $k_j^1 > k_j^2$, and will further assume $j = 1$. We will first need the following claim.

Claim 4.8. *If $L^{\alpha_1} T^{\beta_1} = L_1^2$, or $L^{\alpha_1} T^{\beta_1} = L_1 L_j, j \neq 1$, or $L^{\alpha_1} T^{\beta_1} = L_1 T$, then*

$$\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau = 0.$$

Proof. We prove the three cases separately.

(I). If $L^{\alpha_1}T^{\beta_1} = L_1^2$, we have $s_1 = 2$. Since $s \geq 6$, we have $s_2 \geq 4$, i.e. $|\alpha_2| + 2\beta_2 \geq 4$. Note $k_1^2 = 0$ or 1. First assume $k_1^2 = 0$.

(1). If $\alpha_2 \neq 0$, then there is some $2 \leq j \leq n-1$ such that $k_j^2 \neq 0$. Write $\tilde{\alpha}_2 = \alpha_2 - e_j$. We have by Lemma 4.4,

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1^2\varphi^{(2)}, \overline{L_jL^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau$$
$$= -\langle \bar{L}_jL_1^2\varphi^{(3)}, \overline{L^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau = 0.$$

(2). If $\alpha_2 = 0$, then $\beta_2 \geq 2$. We have by (3.3) and Lemma 4.4,

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1^2\varphi^{(2)}, \overline{TT^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau$$
$$=\frac{-\delta_{2,l}}{2i}\langle L_1^2\varphi^{(2)}, \overline{(\bar{L}_2L_2 - L_2\bar{L}_2)T^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau = \frac{-\delta_{2,l}}{2i}\langle L_1^2\varphi^{(2)}, \overline{\bar{L}_2L_2T^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau$$
$$=\frac{\delta_{2,l}}{2i}\langle L_2L_1^2\varphi^{(3)}, \overline{L_2T^{\beta_2-1}\varphi^{(s_2-1)}}\rangle_\tau = \frac{-\delta_{2,l}}{2i}\langle L_1L_2\varphi^{(2)}, \overline{\bar{L}_1L_2T^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau = 0.$$

Now consider the case $k_1^2 = 1$, write $\tilde{\alpha}_2 = \alpha_2 - e_1$. Then by Lemma 4.4 and (4.22),

$$\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1^2\varphi^{(2)}, \overline{L_1L^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau$$
$$= -\langle \bar{L}_1L_1^2\varphi^{(3)}, \overline{L^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau = -4i\delta_{1,l}\langle L_1T\varphi^{(3)}, \overline{L^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau.$$

It is reduced to case (III).

(II). If $L^{\alpha_1}T^{\beta_1} = L_1L_j, j \neq 1$. We again have $s_1 = 2$, $s_2 \geq 4$. Note also we must have $k_1^2 = 0$. If there exist $i \neq j$ such that $k_i^2 \neq 0$, then write $\tilde{\alpha}_2 = \alpha_2 - e_i$ and by Lemma 4.4,

$$\langle L_1L_j\varphi^{(2)}, \overline{L_iL^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = -\langle \bar{L}_iL_1L_j\varphi^{(3)}, \overline{L^{\tilde{\alpha}_2}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau = 0.$$

Now assume $k_i^2 = 0$ for all $i \neq j$ and $k_j^2 \neq 0$. Then by Lemma 4.4 and (3.3),

$$\langle L_1L_j\varphi^{(2)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1L_j\varphi^{(2)}, \overline{L_jL^{\alpha_2-e_j}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau$$
$$= -\langle L_1\bar{L}_jL_j\varphi^{(3)}, \overline{L^{\alpha_2-e_j}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau = -2i\delta_{j,l}\langle L_1T\varphi^{(3)}, \overline{L^{\alpha_2-e_j}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau.$$

Again it is reduced to case (III).

Finally consider $\alpha_2 = 0$. Then $\beta_2 \geq 2$. We have again by (3.3) and Lemma 4.4,

$$\langle L_1L_j\varphi^{(2)}, \overline{T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \frac{-\delta_{1,l}}{2i}\langle L_1L_j\varphi^{(2)}, \overline{\bar{L}_1L_1T^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau$$
$$=\frac{\delta_{1,l}}{2i}\langle L_1^2L_j\varphi^{(3)}, \overline{L_1T^{\beta_2-1}\varphi^{(s_2-1)}}\rangle_\tau = -\frac{\delta_{1,l}}{2i}\langle L_1^2\varphi^{(2)}, \overline{\bar{L}_jL_1T^{\beta_2-1}\varphi^{(s_2)}}\rangle_\tau = 0.$$

(III). If $L^{\alpha_1}T^{\beta_1} = L_1T$, then $s_1 = 3$, $s_2 \geq 3$ and $k_1^2 = 0$. We have several subcases:

(1). If $|\alpha_2| \geq 2$, then there exist $j \neq 1$, such that $k_j^2 \neq 0$. By Lemma 4.4,

$$\begin{aligned}&\langle L_1 T\varphi^{(3)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1 T\varphi^{(3)}, \overline{L_j L^{\alpha_2 - e_j}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau \\ =& - \langle \bar{L}_j L_1 T\varphi^{(4)}, \overline{L^{\alpha_2 - e_j}T^{\beta_2}\varphi^{(s_2-1)}}\rangle_\tau = 0.\end{aligned}$$

(2). If $|\alpha_2| = 1$ and $\beta_2 \geq 2$, then the same computation as in the preceding case (1) yields

$$\langle L_1 T\varphi^{(3)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = 0.$$

(3). Next consider the case $|\alpha_2| = 1$, and $\beta_2 = 1$. Writing $L^{\alpha_2} = L_j, j \neq 1$, we have

$$\begin{aligned}&\langle L_1 T\varphi^{(3)}, \overline{L_j T\varphi^{(3)}}\rangle_\tau \\ =&\frac{-\delta_{j,l}}{2i}\langle L_1 T\varphi^{(3)}, \overline{(\bar{L}_j L_j - L_j\bar{L}_j)L_j\varphi^{(3)}}\rangle_\tau \\ =&\frac{-\delta_{j,l}}{2i}\langle L_1 T\varphi^{(3)}, \overline{\bar{L}_j L_j^2\varphi^{(3)}}\rangle_\tau + \frac{\delta_{j,l}}{2i}\langle L_1 T\varphi^{(3)}, \overline{L_j\bar{L}_j L_j\varphi^{(3)}}\rangle_\tau.\end{aligned} \tag{4.24}$$

Note by Lemma 4.4 and (3.3) respectively, we have

$$\begin{aligned}\langle L_1 T\varphi^{(3)}, \overline{\bar{L}_j L_j^2\varphi^{(3)}}\rangle_\tau = -\langle L_j L_1 T\varphi^{(4)}, \overline{L_j^2\varphi^{(2)}}\rangle_\tau = \langle L_j T\varphi^{(3)}, \overline{\bar{L}_1 L_j^2\varphi^{(3)}}\rangle_\tau = 0;\\ \langle L_1 T\varphi^{(3)}, \overline{L_j(\bar{L}_j L_j)\varphi^{(3)}}\rangle_\tau = -2i\delta_{j,l}\langle L_1 T\varphi^{(3)}, \overline{L_j T\varphi^{(3)}}\rangle_\tau.\end{aligned}$$

We substitute the above two equations into (4.24) to get

$$\langle L_1 T\varphi^{(3)}, \overline{L_j T\varphi^{(3)}}\rangle_\tau = -\langle L_1 T\varphi^{(3)}, \overline{L_j T\varphi^{(3)}}\rangle_\tau.$$

Hence $\langle L_1 T\varphi^{(3)}, \overline{L_j T\varphi^{(3)}}\rangle_\tau = 0$.
(4). We finally consider the case where $|\alpha_2| = 0$. Then we must have $\beta_2 \geq 2$. In this case, the proof is similar to the case (I).(2).

This proves Claim 4.8. □

Now we come back to the proof of part (a), Lemma 4.7. Recall $k_1^1 > k_1^2$. Suppose $k_1^2 = 0$. If $L^{\alpha_1}T^{\beta_1} \in \{L_1^2, L_1L_2, \cdots, L_1L_{n-1}, L_1T\}$, then we are done by Claim 4.8. Otherwise, i.e. $L^{\alpha_1}T^{\beta_1} \notin \{L_1^2, L_1L_2, \cdots, L_1L_{n-1}, L_1T\}$, we have $(\alpha_1 - e_1, \beta_1) \notin \mathcal{E}$ and by Lemma 4.4,

$$\begin{aligned}&\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = \langle L_1 L^{\alpha_1 - e_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau \\ =& - \langle L^{\alpha_1 - e_1}T^{\beta_1}\varphi^{(s_1-1)}, \overline{\bar{L}_1 L^{\alpha_2}T^{\beta_2}\varphi^{(s_2+1)}}\rangle_\tau = 0.\end{aligned}$$

Now we assume $k_1^2 \geq 1$. Note $k_1^1 - (k_1^2 - 1) \geq 2$. We can keep applying Lemma 4.4 and (4.22) to move $(k_1^2 - 1)$ L_1's to $L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}$, and annihilate $(k_1^2 - 1)$ L_1's in L^{α_2}.

Then we get a new inner product $\langle L^{\tilde{\alpha}_1}T^{\tilde{\beta}_1}\varphi^{(\tilde{s}_1)}, \overline{L^{\tilde{\alpha}_2}T^{\tilde{\beta}_2}\varphi^{(\tilde{s}_2)}}\rangle_\tau$. Here $L^{\tilde{\alpha}_2}$ contains only one L_1 and $L^{\tilde{\alpha}_1}$ has at least two L_1's. If $L^{\tilde{\alpha}_1}T^{\tilde{\beta}_1} = L_1^2$, then the conclusion follows from Claim 4.8. Otherwise, we can apply Lemma 4.4 again and move one more L_1 from $L^{\tilde{\alpha}_1}T^{\tilde{\beta}_1}\varphi^{(\tilde{s}_1)}$ to $L^{\tilde{\alpha}_2}T^{\tilde{\beta}_2}\varphi^{(\tilde{s}_2)}$, and get new inner product $\langle L^{\hat{\alpha}_1}T^{\hat{\beta}_1}\varphi^{(\hat{s}_1)}, \overline{L^{\hat{\alpha}_2}T^{\hat{\beta}_2}\varphi^{(\hat{s}_2)}}\rangle_\tau$, where $L^{\hat{\alpha}_2}$ contains no L_1 and $L^{\hat{\alpha}_1}$ has at least one L_1. This is reduced to the known case we considered before ($k_1^2 = 0$).

For part (b) of Lemma 4.7, note if $s = s_1 + s_2$ is odd, we must have $\alpha_1 \neq \alpha_2$. Thus the conclusion follows from part (a). □

Lemma 4.9. *Let s_1, s_2, s be as in Lemma 4.5. Let $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. Then one of the following must hold.*

1. $\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = 0.$
2. $\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = C\langle T^{\hat{s}}\varphi^{(s^*)}, \overline{T^{\hat{s}}\varphi^{(s^*)}}\rangle_\tau,$ *where $s = 2s^* = 4\hat{s}$, with $\hat{s} \geq 2, s^* \geq 4$.*
3. $\langle L^{\alpha_1}T^{\beta_1}\varphi^{(s_1)}, \overline{L^{\alpha_2}T^{\beta_2}\varphi^{(s_2)}}\rangle_\tau = C\langle L_jT^{\hat{s}}\varphi^{(s^*+1)}, \overline{L_jT^{\hat{s}}\varphi^{(s^*+1)}}\rangle_\tau,$

where $s = 2s^* + 2 = 4\hat{s} + 2$, with $s^* = 2\hat{s} \geq 2$, and $1 \leq j \leq n-1$. Here, as before, C denotes a nonzero constant which may be different in different contexts.

Proof of Lemma 4.9. If s is odd, then the inner product equals 0 by part (b) of Lemma 4.7. Now assume s is even and thus $s \geq 6$. By Lemma 4.6, we can assume $s_1 = s_2 = s \geq 3$. And by Lemma 4.7, we can assume $\alpha_1 = \alpha_2 = \alpha$, for otherwise the inner product again equals 0. Consequently, $\beta_1 = \beta_2 = \beta$. We will prove by induction on $m = |\alpha|$. If $m = 0, 1$, the conclusion (2) or (3) holds trivially. Now suppose the conclusion holds for $0 \leq m \leq k$ for some $k \geq 1$, and consider the case $m = k+1 \geq 2$ (note in the case $m = 2$, we must have $\beta \geq 1$ due to the fact $s_1 = s_2 \geq 3$). Pick $1 \leq j_0 \leq n-1$ such that, writing $\alpha = \sum_{j=1}^{n-1} k_je_j$, we have $j_0 \neq 0$. If $\alpha = k_{j_0}e_{j_0}$, then $k_{j_0} = k+1 \geq 2$. In this case by Lemma 4.4 and (4.22),

$$
\begin{aligned}
&\langle L^{\alpha}T^{\beta}\varphi^{(s)}, \overline{L^{\alpha}T^{\beta}\varphi^{(s)}}\rangle_\tau = \langle L_{j_0}^{k_{j_0}}T^{\beta}\varphi^{(s)}, \overline{L_{j_0}^{k_{j_0}}T^{\beta}\varphi^{(s)}}\rangle_\tau \\
=& - \langle \bar{L}_{j_0}L_{j_0}^{k_{j_0}}T^{\beta}\varphi^{(s+1)}, \overline{L_{j_0}^{k_{j_0}-1}T^{\beta}\varphi^{(s-1)}}\rangle_\tau \\
=& - 2ik_{j_0}\delta_{j_0,l}\langle L_{j_0}^{k_{j_0}-1}T^{\beta+1}\varphi^{(s+1)}, \overline{L_{j_0}^{k_{j_0}-1}T^{\beta}\varphi^{(s-1)}}\rangle_\tau \\
=& 2ik_{j_0}\delta_{j_0,l}\langle L_{j_0}^{k_{j_0}-2}T^{\beta+1}\varphi^{(s)}, \overline{\bar{L}_{j_0}L_{j_0}^{k_{j_0}-1}T^{\beta}\varphi^{(s)}}\rangle_\tau \\
=& 4k_{j_0}(k_{j_0}-1)\langle L_{j_0}^{k_{j_0}-2}T^{\beta+1}\varphi^{(s)}, \overline{L_{j_0}^{k_{j_0}-2}T^{\beta+1}\varphi^{(s)}}\rangle_\tau.
\end{aligned}
$$

This is reduced to the case $m \leq k$.

If there is $1 \leq i_0 \neq j_0 \leq n-1$ such that $k_{i_0} \neq 0, k_{j_0} \neq 0$. We write $\tilde{\alpha} = \alpha - k_{i_0}e_{i_0} - k_{j_0}e_{j_0}$ and compute, again by using Lemma 4.4 and (4.22),

$$
\begin{aligned}
&\langle L^\alpha T^\beta \varphi^{(s)}, \overline{L^\alpha T^\beta \varphi^{(s)}} \rangle_\tau = \langle L_{i_0}^{k_{i_0}} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s)}, \overline{L_{i_0}^{k_{i_0}} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s)}} \rangle_\tau \\
=& - \langle \bar L_{i_0} L_{i_0}^{k_{i_0}} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s+1)}, \overline{L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s-1)}} \rangle_\tau \\
=& - 2ik_{i_0}\delta_{i_0,l} \langle L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^{\beta+1} \varphi^{(s+1)}, \overline{L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s-1)}} \rangle_\tau \\
=& 2ik_{i_0}\delta_{i_0,l} \langle L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}-1} L^{\tilde\alpha} T^{\beta+1} \varphi^{(s)}, \overline{L_{i_0}^{k_{i_0}-1} \bar L_{j_0} L_{j_0}^{k_{j_0}} L^{\tilde\alpha} T^\beta \varphi^{(s)}} \rangle_\tau \\
=& 4k_{i_0}k_{j_0}\delta_{i_0,l}\delta_{j_0,l} \langle L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}-1} L^{\tilde\alpha} T^{\beta+1} \varphi^{(s)}, \overline{L_{i_0}^{k_{i_0}-1} L_{j_0}^{k_{j_0}-1} L^{\tilde\alpha} T^{\beta+1} \varphi^{(s)}} \rangle_\tau .
\end{aligned}
$$

This is again reduced to the case $m \le k$. By induction, we see the conclusion holds. □

We now continue to prove Proposition 4.3.

Lemma 4.10. $f^{(s-1)} \equiv 0, g^{(s)} \equiv 0$.

Proof of Lemma 4.10. We split the proof into two parts, depending on whether s is odd or even.

(I). First assume s is odd. Write $s = 2s^* - 1, s^* \ge 3$.

(I.a). We first prove that $f^{(s-1)}(z,w) \equiv 0$. Fix some j_0 with $1 \le j_0 \le n-1$. Write

$$
f_{j_0}^{(2s^*-2)} = \sum_{k=0}^{s^*-1} a^{(2k)}(z) w^{s^*-k-1}.
$$

Here $a^{(2k)}(z)$ depends on j_0. But we will not write it as a subscript to simplify the notation. By the assumption of Proposition 4.3, for $1 \le j,k \le n-1$, it holds for $p \approx 0$ on $\mathbb{H}_l^n$ that

$$
\frac{\partial}{\partial z_k} \frac{\partial^{s^*-2}}{\partial w^{s^*-2}} (f_p^{**})_j(0) = 0.
$$

By (4.8), (4.15), we have if $1 \le j \le n-1$,

$$
(f_p^{**})_j = (f_p^*)_j \left[1 - 2i \langle \tilde f_p^*, \overline{a(p)} \rangle_{l,l',n} - r(p) g_p^* \right] + O_{wt,p}(2). \tag{4.25}
$$

Moreover, if $1 \le j \le n-1$,

$$
(f_p^*)_j = \frac{1}{\sqrt{\lambda(p)}} \tilde f_p B_j(p) = \frac{1}{\lambda(p)} \langle \tilde f_p, \delta_{j,l} \overline{L_j \tilde f(p)} \rangle_{l,l',n}. \tag{4.26}
$$

For $1 \le j \le N-n, (\phi_p^*)_j = \frac{1}{\sqrt{\lambda(p)}} \langle \tilde f_p, \hat B_j(p) \rangle_{l,l',n}$. And $\lambda(p) = 1 + O_{wt,p}(3)$. Recall if $1 \le j \le n-1, a_j(p) = O_{wt,p}(2)$ and for $j \ge n, a_j(p) = \langle T(\tilde f)(p), \hat B_j(p) \rangle + O_{wt,p}(2) = \langle \varphi_w^{(3)}(p), b_j(p) \rangle + O_{wt,p}(2)$. Here $b_j(p)$ are the last $(N-n)$ component of $\hat B_j(p)$. Hence we have

$$\frac{\partial}{\partial z_k}\frac{\partial^{s^*-2}}{\partial w^{s^*-2}}(f_p^{**})_{j_0}(0) = \langle L_k T^{s^*-2}\tilde{f}(p), \delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}$$
$$+ \langle L_k\tilde{f}(p), \delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}\Big[-2i\sum_{j=n}^{N-n-1}\langle T^{s^*-2}\tilde{f}_p, \hat{B}_j(p)\rangle$$
$$\cdot\, \delta_{j,l,l',n}\langle\overline{\varphi_w^{(3)}(p)}, \overline{b_j(p)}\rangle - r(p)T^{s^*-2}g(p)\Big] + O_{wt,p}(2).$$

We choose $k \neq j_0$. Note then $\langle L_k\tilde{f}(p), \overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n} = O_{wt,p}(2)$. We thus have the weighted degree one part in p in the expansion of $\langle L_k T^{s^*-2}\tilde{f}(p), \delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}$ must be zero. This implies, writing $p = (\hat{p}, p_n) = (p_1, \cdots, p_{n-1}, p_n) \in \mathbb{H}_l^n$,

$$2i\delta_{k,l}\delta_{j_0,l}(s^*-1)!\bar{p}_k a^{(0)} + \delta_{j_0,l}a_{z_k}^{(2)}(\hat{p}) + \delta_{j_0,l}\langle L_k T^{s^*-2}\varphi^{(2s^*-3)}(p), \overline{\varphi_{z_{j_0}}^{(2)}(\hat{p})}\rangle_\tau \equiv 0. \quad (4.27)$$

Collecting the anti-holomorphic part in $\hat{p}$ in (4.27), we have

$$2i\delta_{k,l}\delta_{j_0,l}(s^*-1)!\bar{p}_k a^{(0)} + \delta_{j_0,l}\langle L_k T^{s^*-2}\varphi^{(2s^*-3)}, \overline{\varphi_{z_{j_0}}^{(2)}(\hat{p})}\rangle_\tau \equiv 0.$$

We apply $\bar{L}_k$ to this equation to get

$$2i\delta_{k,l}\delta_{j_0,l}(s^*-1)!a^{(0)} + \delta_{j_0,l}\langle L_k T^{s^*-2}\varphi^{(2s^*-3)}, \overline{L_k L_{j_0}\varphi^{(2)}}\rangle_\tau \equiv 0.$$

By Lemma 4.7, $\langle L_k T^{s^*-2}\varphi^{(2s^*-3)}, \overline{L_k L_{j_0}\varphi^{(2)}}\rangle_\tau \equiv 0$. Thus we conclude $a^{(0)} = 0$. Since $1 \leq j_0 \leq n-1$ is arbitrary, this conclusion in the case $s^* = 3$ implies the following fact.

Corollary 4.11. $f''_{ww}(p) = O_{wt,p}(1)$.

Now we prove by induction that all $a^{(2k)} \equiv 0$ for $1 \leq k \leq s^*-1$. Suppose we have already proved $a^{(2i)} \equiv 0$ for $1 \leq i < k$, with some $1 \leq k \leq s^*-1$. Next we aim to prove $a^{(2k)}(z) \equiv 0$. For that, note by assumption, for any $\alpha \in \mathbb{N}^{n-1}$ with $|\alpha| = 2k-1$, we have $D_{z^\alpha}D_w^{(s^*-k-1)}(f_p^{**})_j(0) = 0$.

As before, we have, writing $\alpha = \sum_{\mu=1}^{n-1} k_\mu e_\mu$,

$$0 = D_{z^\alpha}D_w^{(s^*-k-1)}(f_p^{**})_{j_0}(0) = \langle L^\alpha T^{s^*-k-1}\tilde{f}(p), \delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}$$
$$+ \sum_{k_\mu \geq 1}\tilde{c}(k_\mu)\langle L_\mu\tilde{f}(p), \delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}\Big(-2i\sum_{j=n}^{N-n+1}\langle L^{\alpha-e_\mu}T^{s^*-k-1}\tilde{f}_p, \hat{B}_j(p)\rangle \quad (4.28)$$
$$\cdot\, \delta_{j,l,l',n}\langle\overline{\varphi_w^{(3)}(p)}, \overline{b_j(p)}\rangle - r(p)L^{\alpha-e_\mu}T^{s^*-k-1}g(p)\Big) + O_{wt,p}(2)$$

Here the sum $\sum_{k_\mu \geq 1}$ is taken over those μ satisfying $k_\mu \geq 1$, and $\tilde{c}(k_\mu)$ is some integer depending on k_μ (the value of $\tilde{c}(k_\mu)$ is determined by the Leibniz rule).

We will need to use the following facts.

Claim 4.12. *Let F be as in Proposition 4.3. Then $g''_{ww}(p) = O_{wt,p}(2)$.*

Proof. This is trivial if $s > 5$ in Proposition 4.3 and therefore we only need to prove for the case $s = 5$. We will postpone the proof to (I.b). □

Corollary 4.13. $r(p) = O_{wt,p}(2)$.

Proof of Corollary 4.13. Recall

$$r(p) = \frac{1}{2\lambda(p)}\mathrm{Re}\big(g''_{ww}(p) - 2i\langle \tilde{f}''_{ww}(p), \overline{\tilde{f}(p)}\rangle_{l,l',n,N}\big).$$

Then the conclusion follows easily from Corollary 4.11 and Claim 4.12. □

Write $\varphi^{(3)}_w(z,w) = d^{(1)}(z)$. Also recall $a^{(2j)}(z) \equiv 0$ for $1 \le j < k$. We conclude by collecting the weighted degree one terms in (4.28) that

$$(s^* - k - 1)! D_{z^\alpha} a^{(2k)}(\hat{p}) + \sum_{j=n}^{N-n+1} \hat{c}_j \langle \overline{d^{(1)}(\hat{p})}, \overline{b_j(0)}\rangle \equiv 0. \tag{4.29}$$

Here $\hat{c}_j$'s are constants, which may be 0. For instance, if $k_{j_0} = 0$, then all $\hat{c}_j = 0$. We further collect holomorphic terms in (4.29) to see $(s^* - k - 1)! D_{z^\alpha} a^{(2k)}(\hat{p}) \equiv 0$. Thus we conclude $D_{z^\alpha} a^{(2k)}(\hat{p}) \equiv 0$ for every $\alpha \in \mathbb{N}^{n-1}$ with $|\alpha| = 2k - 1$. This yields that $a^{(2k)}(z) \equiv 0$. By induction, we see $f^{(s-1)}(z,w) \equiv 0$.

(I.b). Next we will prove $g^{(s)} \equiv 0$. Before that we first prove Claim 4.12. For that, we write

$$g^{(5)}(z,w) = c^{(1)}(z)w^2 + c^{(3)}(z)w + c^{(5)}(z).$$

Recall $g^{**}_p = \frac{g^*_p}{\triangle(f^*_p, g^*_p)}$ (see (4.5), (4.6)), and $g^*_p = \frac{1}{\lambda(p)} g_p$. We then have (using (4.7))

$$g^{**}_p = g_p + g_p\big(-2i\langle \tilde{f}^*_p, \overline{a(p)}\rangle_{l,l',n} - r(p) g_p\big) + O_{wt,p}(2). \tag{4.30}$$

Proof of Claim 4.12. By the assumption of Proposition 4.3, $(g^{**}_p)^{(4)} \equiv 0$, and thus

$$D^2_w(g^{**}_p)(0) = 0. \tag{4.31}$$

Recall by (4.16), $D^\beta_w g_p(0) = T^\beta g(p) - 2i\langle T^\beta \tilde{f}(p), \overline{\tilde{f}(p)}\rangle_{l,l',n}$. Then it follows from Corollary 4.11 that if $\beta \le 2$, $D^\beta_w g_p(0) = T^\beta g(p) + O_{wt,p}(2)$. Consequently, equations (4.30) and (4.31) yield that

$$T^2 g(p) + Tg(p)\big(-2i\langle T\tilde{f}^*_p(0), \overline{a(p)}\rangle_{l,l',n} - r(p) Tg(p)\big) + O_{wt,p}(2) = 0.$$

By the same argument as before, we obtain

$$T^2 g(p) + Tg(p)\Big(-2i \sum_{j=n}^{N-n+1} \langle T\tilde{f}_p(0), \hat{B}_j(p)\rangle \cdot \delta_{j,l,l',n} \cdot \langle \overline{\varphi_w^{(3)}(p)}, \overline{b_j(p)}\rangle - r(p)Tg(p)\Big) + O_{wt,p}(2) = 0. \tag{4.32}$$

Recall $p = (\hat{p}, p_n) = (p_1, \cdots, p_{n-1}, p_n)$ and note

$$\begin{aligned} r(p) &= \frac{1}{2\lambda(p)} \mathrm{Re}\big(g''_{ww}(p) - 2i\langle \tilde{f}''_{ww}(p), \overline{\tilde{f}(p)}\rangle_{l,l',n,N}\big) \\ &= \frac{1}{2}\mathrm{Re}\{g''_{ww}(p)\} + O_{wt,p}(2) \\ &= \frac{1}{2}\big(c^{(1)}(\hat{p}) + \overline{c^{(1)}(\hat{p})}\big) + O_{wt,p}(2). \end{aligned}$$

Here we have used (4.7) and Corollary 4.11.

Collecting the weighted degree one terms in (4.32), we have

$$2c^{(1)}(\hat{p}) + \sum_{j=n}^{N-n-1} \hat{c}_j \langle \overline{d^{(1)}(\hat{p})}, \overline{b_j(0)}\rangle - \frac{1}{2}\big(c^{(1)}(\hat{p}) + \overline{c^{(1)}(\hat{p})}\big) \equiv 0.$$

Here $\hat{c}_j$'s are some constants. We further collect holomorphic terms to get $\frac{3}{2}c^{(1)}(\hat{p}) = 0$. We thus have $c^{(1)}(z) = 0$ and this proves Claim 4.12. $\square$

We are now at the position to prove $g^{(s)} \equiv 0$. For that we write

$$g^{(s)}(z, w) = \sum_{j=0}^{s^*-1} c^{(2j+1)}(z) w^{s^*-j-1}.$$

We will first prove $c^{(1)}(z) = 0$ (this is already established for $s = 5$). By assumption, $D_w^{s^*-1}(g_p^{**})(0) = 0$. Recall $D_w^\beta g_p(0) = T^\beta g(p) - 2i\langle T^\beta \tilde{f}(p), \overline{\tilde{f}(p)}\rangle_{l,l',n}$. It follows from (I.a) that $T^{s^*-1} f(p) = O_{wt,p}(1)$ and thus $D_w^{s^*-1} g_p(0) = T^{s^*-1} g(p) + O_{wt,p}(2)$. Then by (4.30), Proposition 4.2, and Corollary 4.13.

$$\begin{aligned} 0 &= D_w^{s^*-1}(g_p^{**})(0) \\ &= T^{s^*-1} g(p) + Tg(p)\Big(-2i \sum_{j=n}^{N-n+1} \langle T^{s^*-2} \tilde{f}^*(p), \hat{B}_j(p)\rangle \cdot \delta_{j,l,l',n} \cdot \langle \overline{\varphi_w^{(3)}(p)}, \overline{b_j(p)}\rangle\Big) \\ &\quad + O_{wt,p}(2). \end{aligned}$$

Recall $\varphi_w^{(3)}(z, w) = d^{(1)}(z)$. Collecting weighted degree one terms in the above equation, we get

$$(s^*-1)!c^{(1)}(\hat{p})+\sum_{j=n}^{N-n-1}\hat{c}_j\langle\overline{d^{(1)}(\hat{p})},\overline{b_j(0)}\rangle=0 \text{ for some constants } \hat{c}_j's.$$

We then collect holomorphic terms to see $c^{(1)}(z)=0$.

Now suppose we have already proved $c^{(2j+1)}(z)\equiv 0$ for $0\le j<k$ with $1\le k\le s^*-1$, and aim to prove $c^{(2k+1)}(z)\equiv 0$. For that we use the fact that $D_{z^\alpha}D_w^{s^*-1}(g_p^{**})(0)=0$ for any $\alpha\in\mathbb{N}^{n-1}$ with $|\alpha|=2k$. By a similar argument as above, using (4.30) and (I.a), and collecting holomorphic weighted degree one terms, we see $D_{z^\alpha}c^{(2k+1)}(z)\equiv 0$. This implies $c^{(2k+1)}(z)\equiv 0$. By induction, we see all $c^{(2k+1)}(z)=0$ and thus $g^{(s)}\equiv 0$.

(II). Finally we consider the case $s=2s^*$ is even. Here $s^*\ge 3$.

(II.a). We start with the proof of $f^{(s-1)}\equiv 0$. For that, we fix some $1\le j_0\le n-1$ and write

$$f_{j_0}^{(s-1)}(z,w)=\sum_{j=0}^{s^*-1}a^{(2j+1)}(z)w^{s^*-j-1}.$$

To prove $a^{(1)}(z)=0$, we notice, by assumption, that $D_w^{s^*-1}(f_p^{**})_{j_0}(0)=0$. By (4.25), Proposition 4.2, and Corollary 4.13, we have

$$\begin{aligned}D_w^{s^*-1}(f_p^{**})_{j_0}(0)&=D_w^{s^*-1}(f_p^{*})_{j_0}(0)+O_{wt,p}(2)\\&=\langle T^{s^*-1}\tilde{f}(p),\delta_{j_0,l}\overline{L_{j_0}\tilde{f}(p)}\rangle_{l,l',n}+O_{wt,p}(2)\equiv 0.\end{aligned}$$

This implies $\delta_{j_0,l}(s^*-1)!a^{(1)}(\hat{p})\equiv 0$. We thus have $a^{(1)}(z)\equiv 0$.

Now suppose we have proved $a^{(2j+1)}(z)\equiv 0$ for $1\le j<k$ with $1\le k\le s^*-1$, and consider the case $j=k$. By assumption, $D_{z^\alpha}D_w^{s^*-1}(f_p^{**})(0)\equiv 0$ with $|\alpha|=2k$. We use the same argument as before to derive $D_{z^\alpha}a^{(2k+1)}(\hat{p})\equiv 0$ for any $|\alpha|=2k$. Thus we have $a^{(2k+1)}(z)\equiv 0$. By induction, we have $f^{(s-1)}\equiv 0$.

(II.b). It remains to prove $g^{(s)}\equiv 0$. Write

$$g^{(s)}(z,w)=\sum_{k=0}^{s^*}c^{(2k)}(z)w^{s^*-k}.$$

We first prove that $c^{(0)}(z)\equiv 0$. By assumption, $D_{z_j}D_w^{s^*-1}(g_p^{**})(0)\equiv 0$, for any $1\le j\le n-1$. By (4.16), $D_{z_j}D_w^{s^*-1}g_p(0)=L_jT^{s^*-1}g(p)-2i\langle L_jT^{s^*-1}\tilde{f}(p),\overline{\tilde{f}(p)}\rangle_{l,l',n}$. It follows from (II.a) that $L_jT^{s^*-1}f(p)=O_{wt,p}(1)$ and thus $D_{z_j}D_w^{s^*-1}g_p(0)=L_jT^{s^*-1}g(p)+O_{wt,p}(2)$. Then by (4.30) and Proposition 4.2, we get

$$\begin{aligned}D_{z_j}D_w^{s^*-1}(g_p^{**})(0)=L_jT^{s^*-1}g(p)+Tg(p)\big(-2i\langle D_{z_j}D_w^{s^*-2}\tilde{f}_p^{*}(0),\overline{a(p)}\rangle_{l,l',n}\big)\\+O_{wt,p}(2)\equiv 0.\end{aligned}\tag{4.33}$$

Note for any $\alpha\in\mathbb{N}^{n-1}$ and $\beta\in\mathbb{N}$,

$$
\begin{aligned}
&\langle D_{z^\alpha} D_w^\beta \tilde{f}_p^*(0), \overline{a(p)}\rangle_{l,l',n} \\
=&\frac{1}{\sqrt{\lambda}}\big(D_{z^\alpha} D_w^\beta \tilde{f}_p(0)\big)\cdot B(p)\cdot I_{l,l',n,N}\cdot\big(\frac{1}{\sqrt{\lambda}}T\overline{\tilde{f}}\cdot\overline{B(p)}\big)^t \\
=&\frac{1}{\lambda}\big(D_{z^\alpha} D_w^\beta \tilde{f}_p(0)\big)\cdot B(p)\cdot I_{l,l',n,N}\cdot\overline{B(p)}^t T\overline{\tilde{f}}^t \\
=&\langle D_{z^\alpha} D_w^\beta \tilde{f}_p(0), T\overline{\tilde{f}}\rangle_{l,l',n}+O_{wt,p}(2) \\
=&\langle L^\alpha T^\beta \tilde{f}(p), T\overline{\tilde{f}}\rangle_{l,l',n}+O_{wt,p}(2) \\
=&\langle L^\alpha T^\beta \varphi^{(s-3)}(p), T\overline{\varphi^{(3)}}\rangle_\tau+O_{wt,p}(2);\ \text{and}
\end{aligned}
\tag{4.34}
$$

$$
L_jT^{s^*-1}g(p)=2i\delta_{j,l}s^*!\overline{p_j}c^{(0)}+(s^*-1)!c^{(2)}_{z_j}(\hat{p}). \tag{4.35}
$$

Collecting weighted degree one terms in (4.33) and using (4.34), (4.35), we obtain

$$
2i\delta_{j,l}s^*!\bar{p}_jc^{(0)}+(s^*-1)!c^{(2)}_{z_j}(\hat{p})=2i\langle L_jT^{s^*-2}\varphi^{(s-3)}(p), T\overline{\varphi^{(3)}(p)}\rangle_\tau. \tag{4.36}
$$

We apply $\bar{L}_j$ to (4.36) and get

$$
\delta_{j,l}s^*!c^{(0)}=\langle L_jT^{s^*-2}\varphi^{(s-3)}(p), \overline{L_jT\varphi^{(3)}(p)}\rangle_\tau. \tag{4.37}
$$

We then have two cases:

Case I. If $s^*=2\sigma$ is even, $\sigma\geq 2$. Then by Lemma 4.5, we have

$$
c^{(0)}=\frac{\delta_{j,l}}{s^*!}\langle L_jT^{s^*-2}\varphi^{(s-3)}, \overline{L_jT\varphi^{(3)}}\rangle_\tau=\frac{\delta_{j,l}(-1)^{\sigma-2}}{s^*!}\langle L_jT^\sigma\varphi^{(2\sigma+1)}, \overline{L_jT^{\sigma-1}\varphi^{(2\sigma-1)}}\rangle_\tau. \tag{4.38}
$$

Note $\sigma\geq 2$. By Lemma 4.4 and (3.3), we obtain

$$
\begin{aligned}
c^{(0)}&=\frac{\delta_{j,l}(-1)^{\sigma-1}}{s^*!}\langle T^\sigma\varphi^{(2\sigma)}, \overline{\bar{L}_jL_jT^{\sigma-1}\varphi^{(2\sigma)}}\rangle_{l,l',n} \\
&=\frac{\delta_{j,l}(-1)^{\sigma-1}}{s^*!}(-2i\delta_{j,l})\langle T^\sigma\varphi^{(2\sigma)}, \overline{T^\sigma\varphi^{(2\sigma)}}\rangle_{l,l',n} \\
&=\frac{2i(-1)^\sigma}{s^*!}\mu.
\end{aligned}
\tag{4.39}
$$

Here $\mu=\langle T^\sigma\varphi^{(2\sigma)}, \overline{T^\sigma\varphi^{(2\sigma)}}\rangle_\tau\in\mathbb{R}$.

Now we use the following fundamental identity:

$$
\operatorname{Im}\{g\}=\langle\tilde{f},\ \overline{\tilde{f}}\rangle_{l,l',n},\qquad \text{on}\qquad \operatorname{Im} w=\langle z,\bar{z}\rangle_l.
$$

We collect the weighted degree s terms in the above equation and recall we already know $f^{(t)}(z,w)\equiv 0$ for $t\leq s-1$. We then see

$$\mathrm{Im}\{g^{(s)}\} = \sum_{s_1,s_2\geq 2,\ s_1+s_2=s} \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau, \quad \text{on} \quad \mathrm{Im}\, w = \langle z, \overline{z}\rangle_l; \quad \text{or} \tag{4.40}$$

$$\mathrm{Im}\{\sum_{k=0}^{s^*} c^{(2k)}(z) w^{s^*-k}\} = \sum_{s_1,s_2\geq 2,\ s_1+s_2=s} \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau, \quad \text{on} \quad \mathrm{Im}\, w = \langle z, \overline{z}\rangle_l. \tag{4.41}$$

Writing $w = u + iv$, we collect terms of u^{s^*} in equation (4.41). Since by (4.39), $c^{(0)}$ is an imaginary number, we get

$$\begin{aligned} \frac{c^{(0)}}{i} &= \sum_{j=2}^{s^*-2} \langle \frac{1}{j!} T^j \varphi^{(2j)}, \frac{1}{(s^*-j)!} \overline{T^{s^*-j}\varphi^{(s-2j)}}\rangle_\tau \\ &= \sum_{j=2}^{s^*-2} \frac{1}{j!(s^*-j)!} \langle T^j \varphi^{(2j)}, \overline{T^{s^*-j}\varphi^{(s-2j)}}\rangle_\tau. \end{aligned} \tag{4.42}$$

By Lemma 4.5, we have $\langle T^j \varphi^{(2j)}, \overline{T^{s^*-j}\varphi^{(s-2j)}}\rangle_\tau = (-1)^{\sigma-j}\mu$, for all $2 \leq j \leq s^* - 2$. Then (4.42) is reduced to

$$\frac{c^{(0)}}{i} = \sum_{j=2}^{s^*-2} \frac{(-1)^{\sigma-j}\mu}{j!(s^*-j)!}. \tag{4.43}$$

We combine (4.39) and (4.43) to get

$$2\mu = \sum_{j=2}^{s^*-2} \frac{(-1)^j \mu s^*!}{j!(s^*-j)!} = \sum_{j=2}^{s^*-2} (-1)^j \mu \binom{s^*}{j}. \tag{4.44}$$

But

$$\sum_{j=2}^{s^*-2} (-1)^j \binom{s^*}{j} = 2s^* - 2.$$

We thus obtain from (4.44) that $2\mu = (2s^* - 2)\mu$. This implies $\mu = 0$. Consequently, by (4.39), $c^{(0)} = 0$.

Case II. If $s^* = 2\sigma + 1$ is odd, here $\sigma \geq 1$, then by (4.37) and Lemma 4.5, we have

$$c^{(0)} = \frac{\delta_{j,l}}{s^*!} \langle L_j T^{s^*-2} \varphi^{(s-3)}, \overline{L_j T \varphi^{(3)}}\rangle_\tau = \frac{\delta_{j,l}(-1)^{\sigma-1}}{s^*!} \langle L_j T^\sigma \varphi^{(2\sigma+1)}, \overline{L_j T^\sigma \varphi^{(2\sigma+1)}}\rangle_\tau. \tag{4.45}$$

Consequently, $c^{(0)}$ is real, and we cannot use the method in Case I. We instead apply $T^{2\sigma} \bar{L}_1 L_1$ to (4.40) and evaluate at 0 to obtain

$$\frac{1}{2i} T^{2\sigma} \bar{L}_1 L_1 g^{(s)}(0) = \sum_{s_1,s_2\geq 2, s_1+s_2=s} T^{2\sigma} \bar{L}_1 L_1 \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau(0). \tag{4.46}$$

Note $\bar{L}_1 L_1 g^{(s)} = 2i\delta_{1,l} T g^{(s)}$ and $\bar{L}_1 L_1 \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_\tau = \langle L_1 \varphi^{(s_1)}, \overline{L_1 \varphi^{(s_2)}} \rangle_\tau + 2i\delta_{1,l} \langle T\varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_\tau$. We thus derive from (4.46) that

$$\begin{aligned}\delta_{1,l} T^{2\sigma+1} g^{(s)}(0) &= \sum_{j=0}^{2\sigma} \sum_{s_1, s_2 \geq 2, s_1+s_2=s} \binom{2\sigma}{j} \langle L_1 T^j \varphi^{(s_1)}, \overline{L_1 T^{2\sigma-j} \varphi^{(s_2)}} \rangle_\tau (0) \\ &\quad + 2i\delta_{1,l} \sum_{j=0}^{2\sigma} \sum_{s_1, s_2 \geq 2, s_1+s_2=s} \binom{2\sigma}{j} \langle T^{j+1} \varphi^{(s_1)}, \overline{T^{2\sigma-j} \varphi^{(s_2)}} \rangle_\tau (0) \\ &= \sum_{j=1}^{2\sigma-1} \binom{2\sigma}{j} \langle L_1 T^j \varphi^{(2j+1)}, \overline{L_1 T^{2\sigma-j} \varphi^{(s-2j-1)}} \rangle_\tau \\ &\quad + 2i\delta_{1,l} \sum_{j=1}^{2\sigma-2} \binom{2\sigma}{j} \langle T^{j+1} \varphi^{(2j+2)}, \overline{T^{2\sigma-j} \varphi^{(s-2j-2)}} \rangle_\tau .\end{aligned} \tag{4.47}$$

Note the left hand side $\delta_{1,l} T^{2\sigma+1} g^{(s)}(0)$ equals $\delta_{1,l} c^{(0)} (s^*!)$. If $\sigma = 1$, then the right hand side equals $2\mu^*$, where

$$\mu^* = \langle L_1 T \varphi^{(3)}, \overline{L_1 T \varphi^{(3)}} \rangle_\tau .$$

By (4.45), we have $c^{(0)} = \frac{\delta_{1,l}}{s^*!} \mu^*$. Then (4.47) is reduced to $\mu^* = 2\mu^*$. Thus $\mu^* = c^{(0)} = 0$.

Now assume $\sigma \geq 2$. Write

$$\begin{aligned}\mu_1 &:= \langle L_1 T^\sigma \varphi^{(2\sigma+1)}, \overline{L_1 T^\sigma \varphi^{(2\sigma+1)}} \rangle_\tau \ ; \\ \mu_2 &:= \langle T^\sigma \varphi^{(2\sigma)}, \overline{T^{\sigma+1} \varphi^{(2\sigma+2)}} \rangle_\tau = \frac{-\delta_{1,l}}{2i} \langle T^\sigma \varphi^{(2\sigma)}, \overline{\bar{L}_1 L_1 T^\sigma \varphi^{(2\sigma+2)}} \rangle_\tau \\ &= \frac{\delta_{1,l}}{2i} \langle L_1 T^\sigma \varphi^{(2\sigma+1)}, \overline{L_1 T^\sigma \varphi^{(2\sigma+1)}} \rangle_\tau = \frac{\delta_{1,l}}{2i} \mu_1 .\end{aligned}$$

Then by (4.45), $c^{(0)} = \frac{\delta_{1,l} (-1)^{\sigma-1}}{s^*!} \mu_1$. By Lemma 4.5, we have

$$\langle L_1 T^j \varphi^{(2j+1)}, \overline{L_1 T^{2\sigma-j} \varphi^{(s-2j-1)}} \rangle_\tau = (-1)^{\sigma-j} \mu_1 \text{ for } 1 \leq j \leq 2\sigma - 1.$$

$$\langle T^{j+1} \varphi^{(2j+2)}, \overline{T^{2\sigma-j} \varphi^{(s-2j-2)}} \rangle_\tau = (-1)^{\sigma-j-1} \mu_2 \text{ for } 1 \leq j \leq 2\sigma - 2.$$

By the above equations, (4.47) is reduced to

$$\begin{aligned}(-1)^{\sigma-1} \mu_1 &= \sum_{j=1}^{2\sigma-1} \binom{2\sigma}{j} (-1)^{\sigma-j} \mu_1 + 2i\delta_{1,l} \sum_{j=1}^{2\sigma-2} \binom{2\sigma}{j} (-1)^{\sigma-j-1} \mu_2 \\ &= \left(\sum_{j=1}^{2\sigma-1} \binom{2\sigma}{j} (-1)^{\sigma-j} + \sum_{j=1}^{2\sigma-2} \binom{2\sigma}{j} (-1)^{\sigma-j-1} \right) \mu_1 .\end{aligned}$$

This implies $(-1)^{\sigma-1}\mu_1 = (-1)^{\sigma-1}2\sigma\mu_1$, or $\mu_1 = 2\sigma\mu_1$. Thus we conclude $\mu_1 = c^{(0)} = 0$.

Hence in all cases, it holds that $c^{(0)} = 0$. Once we know $c^{(0)} = 0$, as before, we use an induction argument that all $c^{(2j)}(z) = 0$ for all $0 \le j \le s^*$. Suppose we have proved $c^{(2j)}(z) = 0$ for $0 \le j < k$ for some $1 \le k \le s^*$ and consider the case $j = k$. By assumption, $D_{z^\alpha} D_w^{s^*-k}(g_p^{**})(0) \equiv 0$ for any $|\alpha| = 2k-1$, and any $p \in \mathbb{H}_l^n$ close to 0. As before, by (II.a) we have $L^\alpha T^{s^*-k} f(p) = O_{wt,p}(1)$. Consequently, $D_{z^\alpha} D_w^{s^*-k} g_p(0) = L^\alpha T^{s^*-k} g(p) + O_{wt,p}(2)$. By (4.30) and Proposition 4.2, we have

$$L^\alpha T^{s^*-k} g(p) + Tg(p)\big(-2i\langle D_{z^\alpha} D_w^{s^*-k-1} \tilde{f}_p^*(0), \overline{a(p)}\rangle_{l,l',n}\big) + O_{wt,p}(2) \equiv 0.$$

Using (4.34), we collect the $O_{wt,p}(1)$-term in the above equation to obtain

$$(s^*-k)! D_{z^\alpha} c^{(2k)}(\hat{p}) - 2i\langle L^\alpha T^{s^*-k-1}\varphi^{(s-3)}(p), T\overline{\varphi^{(3)}}\rangle_\tau = 0$$

Furthermore we collect holomorphic terms in $\hat{p}$ to see $D_{z^\alpha} c^{(2k)}(\hat{p}) = 0$. As α is arbitrary, we have $c^{(2k)}(z) \equiv 0$. By induction, $c^{(2j)}(z) \equiv 0$ for all j, and thus $g^{(s)}(z,w) \equiv 0$. This finishes the proof of Lemma 4.10. □

Lemma 4.14. $\langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau \equiv 0$ *for all* $s_1, s_2 \ge 2, s_1 + s_2 = s$.

Proof of Lemma 4.14. When s is odd, it follows from Lemma 4.7 that

$$\left\langle \frac{\partial^{|\alpha_1|+\beta_1}}{\partial z^{\alpha_1}\partial w^{\beta_1}}\varphi^{(s_1)}, \overline{\frac{\partial^{|\alpha_2|+\beta_2}}{\partial z^{\alpha_2}\partial w^{\beta_2}}\varphi^{(s_2)}}\right\rangle_\tau (0) \equiv 0,$$

for any $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. This implies $\langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau(0) \equiv 0$. When s is even and $s = 2s^* = 4\sigma$ for some $\sigma \ge 2$, it follows from Lemma 4.9 and the proof of Lemma 4.10 (see (4.39) and recall we proved $c^{(0)} = 0$) that

$$\left\langle \frac{\partial^{|\alpha_1|+\beta_1}}{\partial z^{\alpha_1}\partial w^{\beta_1}}\varphi^{(s_1)}, \overline{\frac{\partial^{|\alpha_2|+\beta_2}}{\partial z^{\alpha_2}\partial w^{\beta_2}}\varphi^{(s_2)}}\right\rangle_\tau (0) \equiv 0,$$

for any $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. This again implies $\langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}}\rangle_\tau(0) \equiv 0$.

Similarly, if s is even and $s = 2s^* = 4\sigma+2$, then the conclusion follows from Lemma 4.9 and the proof of Lemma 4.10 (see (4.45) and recall we proved $c^{(0)} = 0$). This establishes Lemma 4.14. □

Proposition 4.3 follows from Lemma 4.10 and Lemma 4.14 (except that it remains to prove Lemma 4.6 and Lemma 4.7 for the case $s = 5$; this will be done in §5). □

4.3. Proofs of Theorem 4.1 and Theorem 1.1

Proof of Theorem 4.1. As was mentioned, we prove only the implication from (1) to (2). This easily follows from the following claim.

Claim 4.15. *Let F be as in the assumption in Theorem 4.1. Assume F is CR transversal near 0 and satisfies the normalization in Lemma 3.1. If F is of geometric rank zero near 0, then*

$$f^{(t-1)} \equiv 0, g^{(t)} \equiv 0, \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_\tau \equiv 0, \ \forall s_1, s_2 \geq 2, s_1 + s_2 = t \geq 4. \tag{4.48}$$

We prove the claim by induction on t. First note by Lemma 3.1 and the geometric zero condition, (4.48) holds for $t = 4$. Now assume Claim 4.15 holds for all $4 \leq t < s$ with $s \geq 5$, and consider the case $t = s$. But Claim 4.15 (with $4 \leq t < s$) applies also to F_p^{**}. We thus have (4.11) holds for $4 \leq t < s$. Then by Proposition 4.3, (4.48) holds for $t = s$. Hence by induction, Claim 4.15 holds. □

Proof of Theorem 1.1. Composing F with automorphisms, we assume that $p = [1, 0, \cdots, 0, 1] \in \partial\mathbb{B}_l^n$ and $F(p) = [1, 0, \cdots, 0, 1] \in \partial\mathbb{B}_l^N$. Denote by Ψ_n the Cayley transformation from $\mathbb{S}_l^n$ to $\mathbb{B}_l^n$, and Φ_N the Cayley transformation from $\mathbb{S}_{l,l',n}^N$ to $\mathbb{B}_{l'}^N$ (see §3). Then $G := \Phi_N^{-1} \circ F \circ \Psi_n$ is well-defined in a small neighborhood of $0 \in \mathbb{H}_l^N$. Note F is side-preserving, CR transversal, and of geometric rank zero if and only if G is so. Then Theorem 1.1 follows from Theorem 4.1 and Theorem 2.1. □

5. Completion of the proof

In this section, we complete the proof of Theorem 1.1 by giving a proof of Lemma 4.6 and Lemma 4.7 in the case $s = 5$. More precisely, we prove the following proposition.

Proposition 5.1. *Let $s_1 \geq 2, s_2 \geq 2$ and $s_1 + s_2 = 5$. Assume $(\alpha_1, \beta_1) \in I_{s_1}, (\alpha_2, \beta_2) \in I_{s_2}$. Then*

$$\langle L^{\alpha_1} T^{\beta_1} \varphi^{(s_1)}, \overline{L^{\alpha_2} T^{\beta_2} \varphi^{(s_2)}} \rangle_\tau (p) \equiv 0.$$

Proof of Proposition 5.1. Note $\langle A, \bar{B} \rangle_\tau = \overline{\langle B, \bar{A} \rangle_\tau}$ for two vectors A, B and thus we can always assume $s_1 \leq s_2$. Since $s_1 + s_2 = s = 5$, this implies $s_1 = 2, s_2 = 3$. We will verify the conclusion by a direct computation case by case.

(1). We first consider the case when $L^{\alpha_1} T^{\beta_1} = L_i^2$ for some $1 \leq i \leq n-1$.

(1.a). If $L^{\alpha_2} T^{\beta_2} = L_i T$, then we have

$$\langle L_i^2 \varphi^{(2)}, \overline{L_i T \varphi^{(3)}} \rangle_\tau = \langle L_i^2 \varphi^{(2)}, \overline{T L_i \varphi^{(3)}} \rangle_\tau = \frac{-\delta_{j,l}}{2i} \langle L_i^2 \varphi^{(2)}, \overline{(\bar{L}_j L_j - L_j \bar{L}_j) L_i \varphi^{(3)}} \rangle_\tau.$$

Here we have used identity (3.3) and let $j \neq i$. Furthermore, by Lemma 4.4 it equals

$$\begin{aligned}&\frac{-\delta_{j,l}}{2i}\langle L_i^2\varphi^{(2)},\overline{\bar{L}_jL_jL_i\varphi^{(3)}}\rangle_\tau=\frac{\delta_{j,l}}{2i}\langle L_jL_i^2\varphi^{(3)},\overline{L_jL_i\varphi^{(2)}}\rangle_\tau\\=&\frac{-\delta_{j,l}}{2i}\langle L_iL_j\varphi^{(2)},\overline{L_j\bar{L}_iL_i\varphi^{(3)}}\rangle_\tau=\frac{-\delta_{j,l}}{2i}\langle L_iL_j\varphi^{(2)},\overline{L_j2i\delta_{i,l}T\varphi^{(3)}}\rangle_\tau\\=&\delta_{j,l}\delta_{i,l}\langle L_iL_j\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau.\end{aligned}\tag{5.1}$$

This equals zero by case (2.a).

(1.b). If $L^{\alpha_2}T^{\beta_2}=L_jT$ with $1\le j\ne i\le n-1$, we have by (3.3),

$$\begin{aligned}\langle L_i^2\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau&=\frac{-\delta_{j,l}}{2i}\langle L_i^2\varphi^{(2)},\overline{(\bar{L}_jL_j-L_j\bar{L}_j)L_j\varphi^{(3)}}\rangle_\tau\\&=\frac{-\delta_{j,l}}{2i}\langle L_i^2\varphi^{(2)},\overline{\bar{L}_jL_j^2\varphi^{(3)}}\rangle_\tau+\frac{\delta_{j,l}}{2i}\langle L_i^2\varphi^{(2)},\overline{L_j\bar{L}_jL_j\varphi^{(3)}}\rangle_\tau\end{aligned}\tag{5.2}$$

Furthermore, by Lemma 4.4 and (3.3) respectively, we have

$$\langle L_i^2\varphi^{(2)},\overline{\bar{L}_jL_j^2\varphi^{(3)}}\rangle_\tau=-\langle L_jL_i^2\varphi^{(3)},\overline{L_j^2\varphi^{(2)}}\rangle_\tau=\langle L_jL_i\varphi^{(2)},\overline{\bar{L}_iL_j^2\varphi^{(3)}}\rangle_\tau=0;$$
$$\langle L_i^2\varphi^{(2)},\overline{L_j\bar{L}_jL_j\varphi^{(3)}}\rangle_\tau=\langle L_i^2\varphi^{(2)},\overline{L_j2i\delta_{j,l}T\varphi^{(3)}}\rangle_\tau=-2i\delta_{j,l}\langle L_i^2\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau.$$

Then by (5.2), $\langle L_i^2\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau=-\langle L_i^2\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau$, and thus $\langle L_i^2\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau=0$.

(1.c). If $L^{\alpha_2}T^{\beta_2}=L_i^3$, by Lemma 4.4 and (4.22) we have

$$\begin{aligned}\langle L_i^2\varphi^{(2)},\overline{L_i^3\varphi^{(3)}}\rangle_\tau&=-\langle\bar{L}_iL_i^2\varphi^{(3)},\overline{L_i^2\varphi^{(2)}}\rangle_\tau\\&=-\langle L_i^2\bar{L}_i\varphi^{(3)},\overline{L_i^2\varphi^{(2)}}\rangle_\tau-4i\delta_{i,l}\langle TL_i\varphi^{(3)},\overline{L_i^2\varphi^{(2)}}\rangle_\tau.\end{aligned}$$

Note $\langle L_iT\varphi^{(3)},\overline{L_i^2\varphi^{(2)}}\rangle_\tau=0$ by (1.a). Hence we conclude $\langle L_i^2\varphi^{(2)},\overline{L_i^3\varphi^{(3)}}\rangle_\tau\equiv0$.

(1.d). If $L^{\alpha_2}T^{\beta_2}=L_jL_kL_l$ where $j\ne i$ and k,l may be equal to i, then

$$\langle L_i^2\varphi^{(2)},\overline{L_jL_kL_l\varphi^{(3)}}\rangle_\tau=-\langle\bar{L}_jL_i^2\varphi^{(3)},\overline{L_kL_l\varphi^{(2)}}\rangle_\tau=-\langle L_i^2\bar{L}_j\varphi^{(3)},\overline{L_kL_l\varphi^{(2)}}\rangle_\tau=0.$$

(2). It remains to consider the case where $L^{\alpha_1}T^{\beta_1}=L_iL_j$ for some $1\le i\ne j\le n-1$.

(2.a). If $L^{\alpha_2}T^{\beta_2}=L_jT$, then by (3.3),

$$\begin{aligned}&\langle L_iL_j\varphi^{(2)},\overline{L_jT\varphi^{(3)}}\rangle_\tau=\langle L_iL_j\varphi^{(2)},\overline{TL_j\varphi^{(3)}}\rangle_\tau\\=&\frac{-\delta_{j,l}}{2i}\langle L_iL_j\varphi^{(2)},\overline{(\bar{L}_jL_j-L_j\bar{L}_j)L_j\varphi^{(3)}}\rangle_\tau\\=&\frac{-\delta_{j,l}}{2i}\langle L_iL_j\varphi^{(2)},\overline{\bar{L}_jL_j^2\varphi^{(3)}}\rangle_\tau+\frac{\delta_{i,l}}{2i}\langle L_iL_j\varphi^{(2)},\overline{L_j\bar{L}_jL_j\varphi^{(3)}}\rangle_\tau\end{aligned}$$

Note by Lemma 4.4 and (3.3) respectively,

$$\langle L_iL_j\varphi^{(2)},\overline{\bar{L}_jL_j^2\varphi^{(3)}}\rangle_\tau=-\langle L_iL_j^2\varphi^{(3)},\overline{L_j^2\varphi^{(2)}}\rangle_\tau=-\langle L_j^2\varphi^{(3)},\overline{\bar{L}_iL_j^2\varphi^{(2)}}\rangle_\tau=0;$$

$$\begin{aligned}\langle L_iL_j\varphi^{(2)}, \overline{L_j\bar{L}_jL_j\varphi^{(3)}}\rangle_\tau &= \langle L_iL_j\varphi^{(2)}, \overline{L_j(L_j\bar{L}_j + 2i\delta_{j,l}T)\varphi^{(3)}}\rangle_\tau \\ &= -2i\delta_{j,l}\langle L_iL_j\varphi^{(2)}, \overline{L_jT\varphi^{(3)}}\rangle_\tau.\end{aligned}$$

Then we have $\langle L_iL_j\varphi^{(2)}, \overline{L_jT\varphi^{(3)}}\rangle_\tau = -\langle L_iL_j\varphi^{(2)}, \overline{L_jT\varphi^{(3)}}\rangle_\tau$. Thus

$$\langle L_iL_j\varphi^{(2)}, \overline{L_jT\varphi^{(3)}}\rangle_\tau = 0.$$

(2.b). If $L^{\alpha_2}T^{\beta_2} = L_kT$ with $k \neq i, j$, a similar argument as in (2.a) yields

$$\langle L_iL_j\varphi^{(2)}, \overline{L_kT\varphi^{(3)}}\rangle_\tau = 0.$$

(2.c). If $L^{\alpha_2}T^{\beta_2} = L_j^3$, then by Lemma 4.4 and (3.3),

$$\langle L_iL_j\varphi^{(2)}, \overline{L_j^3\varphi^{(3)}}\rangle_\tau = -\langle \bar{L}_jL_jL_i\varphi^{(3)}, \overline{L_j^2\varphi^{(2)}}\rangle_\tau = -2i\delta_{j,l}\langle TL_i\varphi^{(3)}, \overline{L_j^2\varphi^{(2)}}\rangle_\tau.$$

This equals 0 by (1.b).

(2.d). If $L^{\alpha_2}T^{\beta_2} = L_i^2L_j$, then by Lemma 4.4 and (3.3),

$$\langle L_iL_j\varphi^{(2)}, \overline{L_i^2L_j\varphi^{(3)}}\rangle_\tau = -\langle \bar{L}_iL_iL_j\varphi^{(3)}, \overline{L_iL_j\varphi^{(2)}}\rangle_\tau = -2i\delta_{i,l}\langle TL_j\varphi^{(3)}, \overline{L_iL_j\varphi^{(2)}}\rangle_\tau.$$

This equals 0 by (2.a).

(2.e). If $L^{\alpha_2}T^{\beta_2} = L_kL_\mu L_\nu$ with $k \neq i, j$ and μ, ν may equal i, j, Lemma 4.4 implies

$$\begin{aligned}\langle L_iL_j\varphi^{(2)}, \overline{L_kL_\mu L_\nu\varphi^{(3)}}\rangle_\tau &= -\langle \bar{L}_kL_iL_j\varphi^{(2)}, \overline{L_\mu L_\nu\varphi^{(3)}}\rangle_\tau \\ &= -\langle L_iL_j\bar{L}_k\varphi^{(2)}, \overline{L_\mu L_\nu\varphi^{(3)}}\rangle_\tau = 0.\end{aligned}$$

This proves Proposition 5.1. □

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