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Mapping $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$

Xiaojun Huang[1,⋆], **Shanyu Ji**[2]

[1] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
[2] Department of Mathematics, The University of Houston, Houston, TX 77439, USA

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1. Introduction

In this paper, we are concerned with the classification problem of proper holomorphic maps between balls in complex spaces. Write $\mathbf{B}^n = \{z \in \mathbf{C}^n : |z| < 1\}$ and $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ for the collection of all proper holomorphic maps from $\mathbf{B}^n$ into $\mathbf{B}^N$. We recall that $f, g \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ are said to be equivalent if there are elements $\sigma \in \mathrm{Aut}(\mathbf{B}^n)$ and $\tau \in \mathrm{Aut}\,(\mathbf{B}^N)$ such that $f = \tau \circ g \circ \sigma$. It is a well-known result of Poincaré [Po] and Alexander [Alx] that when $N = n > 1$, then any $f \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^n)$ is equivalent to the identity map. For the case of $N > n > 1$, due to the discovery of inner functions, it is clear that solving the classification problem in $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ is unrealistic. Therefore, one focuses on the important subclass of mappings, $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$, the collection of all rational proper holomorphic mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$. And here, there are already many non-trivial and interesting questions ([DA1]).

A first result along these lines is due to Webster [We], who showed that $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^{n+1})$ has only one equivalence class for $n > 2$. This was proven to hold also for $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ by Faran [Fa2] in the larger codimensional case: $N \le 2n - 2$. For the case $N \ge 2n - 1$, the collection of all equivalence classes, denoted by $\widetilde{R}(n, N)$, of $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ carries a real algebraic structure from the work of [Fo1], [BER]. However, the specific description of $\widetilde{R}(n, N)$ remains to be quite mysterious in general. In [DA2], D'Angelo discovered a continuous family of mutually inequivalent polynomial proper embeddings from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$ (see Example 3), which in particular indicates that the set $\widetilde{R}(n, N)$ contains infinitely many elements when $N \ge 2n$. On the other hand, in a paper of Faran [Fa1] in the early 80's, it was shown that $\widetilde{R}(2, 3)$ has exactly four elements. This then leads to the natural question of

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clarifying the border case of $N = 2n - 1$ with $n > 2$. It is indeed this problem that motivates the present work; and we will show that unlike Faran's result concerning $\widetilde{R}(2, 3)$, there are only two elements in $\widetilde{R}(n, 2n-1)$ for $n \geq 3$. Namely, we provide in this paper the following:

Theorem 1: *When $n \geq 3$, $Rat(n, 2n-1)$ has exactly two equivalence classes. One is generated by the standard linear embedding $L(z_1, \cdots, z_n) := (z_1, \cdots, z_n, 0, \cdots, 0)$, and the other is generated by the Whitney map $W(z_1, \cdots, z_n) := (z_1, \cdots, z_{n-1}, z_n z_1, z_n z_2, \cdots, z_n z_n)$. More precisely, any rational proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$ with $n > 2$ is equivalent either to the standard linear map $L(z)$ or to the Whitney map $W(z)$.*

One of the striking developments of complex analysis in the early $80'$s is the discovery of the fact that $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ is a much larger class than $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ (see [HS], [Low], [Fo2], [Hu2] for related references). This makes it a natural but also a subtle direction in complex analysis to understand the conditions under which an element in $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ becomes linear or rational. (Indeed, for many applications of complex analysis in classical dynamics and geometry, see [Yue] and [LN] and the references therein, a good understanding of such a phenomenon always appears to be quite crucial). Along these studies, there have recently appeared a number of papers (see [Fo2], [Yue], [Hu2], [Hu3] for surveys and related references), in which the interaction of the boundary regularity or the dynamical property of the maps with the rationality and the linearity of the maps was investigated. Here we mention a theorem of Forstneric [Fo1]: If $f : \mathbf{B}^n \to \mathbf{B}^N$, $(1 < n < N)$, is a proper holomorphic mapping which is $(N-n+1)$ continuously differentiable up to the boundary, then f is rational. In a recent work of the first author [Hu1], a new approach was introduced for such a study. It was shown in [Hu1–2] that the linearity and rationality hold in fact for maps which are only assumed to be twice continuously differentiable up to the boundary in case $N < 2n - 1$ or $N \leq 2n - 1$, respectively. While it is still an open question if this is also the case for general codimensions, we mention that these results are good enough for us to reformulate Theorem 1 for proper maps which are only twice differentiable up to the boundary. Notice that a proper holomorphic embedding from $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$ cannot be equivalent to the Whitney map, the previous result of the first author (say, [Corollary 1.2, Hu2]) together with Theorem 1, in particular, provides the following rigidity theorem for holomorphic embeddings between balls. (Notice in case $n = 2$, it is contained in the work of Faran [Fa2]):

Theorem 2: *Let F be a proper holomorphic embedding from $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$ $(n > 1)$, which is twice continuously differentiable up to the boundary. Then F is equivalent to the standard linear embedding $L(z)$.*

Example 3 (D'Angelo [pp87, DA2]): Let $z = (z', z_n)$ and $F_t = (z', \cos(t) z_n, \sin(t) z_n z)$ with $t \in (0, \pi/2)$. Then F_t is a proper holomorphic embedding from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$. Clearly, F_t cannot be linear; for it has degree two. Furthermore, F_t is equivalent to F_s if and only if $t = s$. Hence, $\widetilde{R}(n, 2n)$ has infinitely many elements.

Now, we say a few words about the proof of Theorem 1. Part of the approaches and techniques we need to use are based on the work done in [Hu1]. We first establish the Chern-Moser formal theory for the normalized map, which gives us some sort of the jet relation (namely, a certain kind of the curvature relations). In particular, we will prove the rank of a certain matrix formed by the second derivatives of the map is generically one if the map is not linear. This rank condition holds exclusively for $N = 2n - 1$, but only gives a non-trivial and intrinsic restriction to the map for $n > 2$. Using the large automorphism groups of the balls, we derive from this rank condition a partial differential equation. Unlike the situation considered in [Hu1], it is now very difficult to solve the equations directly. However, by performing certain formal arguments to the equations, we can derive some connection between the third derivatives of the map (Lemma 4.1). With such a relation provided in Lemma 4.1, we will prove the existence of what we call the characteristic direction along which the map is linear and the solutions must take certain specific normal form. We next show the degree of the map as in the theorem is at most two, to further simplify the obtained normal form. Finally, we show that the map is either linear or is equivalent to the Whitney map. Once Theorem 1 is proved, Theorem 2 is an immediate consequence of Theorem 1 and the results in [Hu1–2].

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2. Notations, preliminaries and brief description of the proof of Theorem 1

In this section, we set up some notation and recall some results from [Hu1], which will be used through the paper. We will also discuss briefly the proof of Theorem 1.

First, it is well-known that the ball $\mathbf{B}^n$ is holomorphically equivalent to the Siegel upper-half space $\mathbf{H}_n = \{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$ and the sphere is equivalent to the Heisenberg hypersurface $\partial\mathbf{H}_\mathbf{n}$. Let $L_j = \frac{\partial}{\partial z_j} + 2\sqrt{-1}\overline{z_j}\frac{\partial}{\partial w}$ for $j = 1, \cdots, n-1$. Then $\{L_1, \cdots, L_{n-1}\}$ form a basis of $\mathrm{T}^{(1,0)}\partial\mathbf{H}_n$. We assign the weight of z to be 1 and w to be 2, and use the notation $(\cdot)^{(\sigma)}$ for a holomorphic polynomial of weighted degree σ. For a function h defined over $\partial\mathbf{H}_n$, we say that $h \in o_{wt}(k)$ if $\lim_{t\to 0} \frac{h(tz,t^2w)}{t^k} \to 0$ uniformly on (z, w) over any compact subset of $\partial\mathbf{H}_n$. We also use the notation: $\langle a, b\rangle = \sum_j a_j b_j$ and $|a|^2 = \langle a, \overline{a}\rangle$.

Next, let F be a rational proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$. By a result of Cima-Suffridge [CS2], F extends holomorphically up to the boundary. Hence, it induces naturally a (non-constant) holomorphic map from $\partial \mathbf{H}_n$ into $\partial \mathbf{H}_N$, which will still be denoted by F for simplicity of notation. Since we are only interested in the normalization problem of F up to the action of the automorphisms of the Heisenberg hypersurfaces, we can assume without loss of generality that $F(0) = 0$. Moreover, we have

Lemma 2.1 ([Lemma 5.3, Hu1]): *After composing F with certain automorphisms of the Heisenberg hypersurfaces, the map $F = (f, \phi, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ can be assumed to take the following normal form ($N > n > 1$):*

(2.1)

$$f = z + \frac{i}{2} a^{(1)}(z) w + o_{wt}(3), \ \phi = \phi^{(2)}(z) + o_{wt}(2), \ g = w + o_{wt}(4),$$

with

$$\langle \overline{z}, a^{(1)}(z) \rangle |z|^2 = |\phi^{(2)}|^2. \tag{2.2}$$

As in Lemma 5.3 of [Hu1], the above holds for F which is only assumed to be C^2-smooth over the source hypersurface. Equation (2.2), which comes as a consequence of the normalization as in (2.1), reflects the lowest order CR curvature terms, which may be useful for further understanding of problems for mappings between Heisenberg hypersurfaces. In particular, we mention that the polynomials $a^{(1)}(z)$, $\phi^{(2)}(z)$ contain the information on how far the map is away from a linear one. Equation (2.2) relates the second order jets of F at the point $p = 0$. To make it useful for our analysis, we will produce from the F in Lemma 2.1 a family of maps from $\mathbf{H}_n$ into $\mathbf{H}_N$, denoted by F_p^{**}, with parameter $p \in \partial \mathbf{H}_n$. Notice that if we apply the curvature equation (2.2) to F_p^{**}, we will get a relation of the second order jets of F at any point $p \in \partial \mathbf{H}_n$. Hence, we derive differential equations for F, through which we take the control of the map F.

In case $N = 2n - 1$, we will see in Lemma 3.1 that after composing F with certain automorphisms of the Heisenberg hypersurfaces, we can further simplify $a^{(1)}(z)$ and $\phi^{(2)}(z)$. Moreover, (2.2) implies a rank condition for the second derivatives of the map F at 0 (see Remark 3.1′). It is this rank condition that enables us to find a direction along which the map F is linear. It should be mentioned that such a condition becomes empty when $n = 2$. Hence, our approach here is not usable when $n = 2$. Indeed, by the examples of Alexander [Alx] and Faran [Fa2], when $n = 2$, F may not have any direction at all in which F is linear.

Now, we describe how F_p^{**} is defined. Let F be as in (2.1). F induces a holomorphic map $F_{p_0} : \partial \mathbf{H}_n \to \partial \mathbf{H}_N$ for any $p_0 = (z_0, w_0) \in \partial \mathbf{H}_n$ defined as follows. We write $\sigma_{p_0} \in \mathrm{Aut}(\mathbf{H}_n)$ for the map given by $\sigma_{p_0}(z, w) = (z+z_0, w + w_0 + 2i\langle z, \overline{z_0} \rangle)$ ($\sigma_{p_0}(0) = p_0$), and $\tau_{p_0} \in \mathrm{Aut}(\mathbf{H}_N)$ by $\tau(z^*, w^*) =$

$(z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle)$. Here we write $\widetilde{f} = (f, \phi)$. Then $F_{p_0}(z, w) = (f_{p_0}, \phi_{p_0}, g_{p_0}) = (\widetilde{f_{p_0}}, g_{p_0}) := \tau_{p_0} \circ F \circ \sigma_{p_0}$ is a holomorphic map from $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$ with $F_{p_0}(0) = 0$.

However, the map F_{p_0} constructed in this way does not satisfy (2.1) in general. By further composing F_{p_0} with suitable automorphisms of $\mathbf{H}_N$, we will get $F^{**}_{p_0}$ for which (2.1) and thus (2.2) holds. Such a process and related formulas have been carefully discussed in [§2, §4, Hu1] and are reviewed as follows:

Let $\lambda_{p_0} := \frac{\partial g_{p_0}}{\partial w}|_0 = g'_w(z_0, w_0) - 2i\langle(\widetilde{f})'_w(z_0, w_0), \overline{\widetilde{f}(z_0, w_0)}\rangle$;

$(E_l)_{p_0} := \frac{\partial \widetilde{f}_{p_0}}{\partial z_l}|_0 = L_l(\widetilde{f})(z_0, w_0)$;

$(E_w)_{p_0} := \frac{\partial \widetilde{f}_{p_0}}{\partial w}|_0 = \widetilde{f}'_w(z_0, w_0)$;

and let $(C_l)_{p_0}$ be so chosen that (see [pp 17, Hu1])

$$A_{p_0} = A_{(z_0,w_0)} := \begin{pmatrix} (E_1)_{p_0}/\sqrt{\lambda_{p_0}} \\ \vdots \\ (E_{n-1})_{p_0}/\sqrt{\lambda_{p_0}} \\ (C_1)_{p_0} \\ \vdots \\ (C_{N-n})_{p_0} \end{pmatrix} \text{ is a unitary matrix.}$$

Define $F^*_{p_0} = (\widetilde{f^*}_{p_0}, g^*_{p_0}) = ((f_1)^*_{p_0}, \cdots, (f_{n-1})^*_{p_0}, (\phi_1)^*_{p_0}, \cdots, (\phi_{N-n})^*_{p_0}, g^*_{p_0})$ by

$$\frac{1}{\sqrt{\lambda_{p_0}}} F_{p_0} \cdot \begin{pmatrix} \overline{A^t_{p_0}} & 0 \\ 0 & 1/\sqrt{\lambda_{p_0}} \end{pmatrix}.$$

Then $F^*_{p_0}$ is a proper holomorphic map from $\mathbf{H}_n$ into $\mathbf{H}_N$ with $F^*_{p_0}(0) = 0$ and $\widetilde{f^*}_{p_0} = (z, 0) + o_{wt}(1)$, $g^*_{p_0} = w + o_{wt}(1)$. To further modify $F^*_{p_0}$ so that (2.1) holds, we let

$$\mathbf{a}_{p_0} := \big((a_1)_{p_0}, \cdots, (a_{n-1})_{p_0}, (b_1)_{p_0}, \cdots, (b_{N-n})_{p_0}\big), \text{ with}$$

$(a_l)_{p_0} = \frac{1}{\lambda_{p_0}}(E_w)_{p_0} \cdot \overline{(E_l)_{p_0}}^t$, $(b_l)_{p_0} = \frac{1}{\sqrt{\lambda_{p_0}}}(E_w)_{p_0} \cdot \overline{(C_l)_{p_0}}^t$. It can be seen that $|\mathbf{a}_{p_0}|^2 = \frac{1}{\lambda_{p_0}}|(E_w)_{p_0}|^2$. Also let

$(d_{lj})_{p_0} := \frac{\partial^2 (f^*_l)_{p_0}}{\partial z_j \partial w}|_0 = \frac{1}{\lambda_{p_0}}(\widetilde{f}_{p_0})''_{wz_l}(0) \cdot \overline{(E_j)^t_{p_0}} = \frac{1}{\lambda_{p_0}} L_l(\widetilde{f}'_w)(p_0) \cdot \overline{(E_j)^t_{p_0}}$;

$(c_l)_{p_0} := \frac{\partial^2 g^*_{p_0}}{\partial z_l \partial w}|_0 = \frac{1}{\lambda_{p_0}}(g_{p_0})''_{wz_l}(0) = \frac{1}{\lambda_{p_0}} L_l(g'_w - 2i\widetilde{f}'_w \cdot \overline{\widetilde{f}}^t)(p_0)$;

$r_{p_0} := \frac{1}{2}\frac{\partial^2 (g)^*_{p_0}}{\partial w^2}|_0 = \frac{1}{2\lambda_{p_0}}\mathrm{Re}((g_{p_0})''_{ww}(0)) = \frac{1}{2\lambda_{p_0}}\mathrm{Re}(g''_{ww} - 2i\widetilde{f}''_{ww} \cdot \overline{\widetilde{f}}^t)(p_0)$.

Define

$$G_{p_0}(z^*, w^*) := \left(\frac{z^* - \mathbf{a}_{p_0} w^*}{1 + 2i\langle z^*, \overline{\mathbf{a}_{p_0}}\rangle - (-r + i\langle \mathbf{a}_{p_0}, \overline{\mathbf{a}_{p_0}}\rangle)w^*}, \frac{w^*}{1 + 2i\langle z^*, \overline{\mathbf{a}_{p_0}}\rangle - (-r + i\langle \mathbf{a}_{p_0}, \overline{\mathbf{a}_{p_0}}\rangle)w^*}\right).$$

Then $G_{p_0} \in \mathrm{Aut}(\mathbf{H}_N)$.

Finally, $F^{**}_{p_0}$ is the composition $F^{*}_{p_0}$ with G_{p_0}:

$$F^{**}_{p_0} = \left(f^{**}_{p_0}, \phi^{**}_{p_0}, g^{**}_{p_0}\right) = \left(\widetilde{f}^{**}_{p_0}, g^{**}_{p_0}\right) := G_{p_0} \circ F^{*}_{p_0}.$$

By [§2, Lemma 5.3, Hu1], $F^{**}_{p_0}$ satisfies the normalization conditions (2.1), and thus (2.2) too.

As in [Hu1], it is desirable to know how F^{**}_{p} gives the feedback to the original function F (see Lemma 2.1 of [Hu1], for instance). For our purpose later, we need the following formula, which can been easily obtained by collecting the coefficients of the $z_j w$-terms in the Taylor expansion of the right hand side of [(2.7)′, Hu1]:

(2.3)

$$\frac{\partial^2 (f_l)^{**}_{p_0}}{\partial z_j \partial w}|_0 = (d_{lj})_{p_0} - (a_l)_{p_0}(c_j)_{p_0} - \delta^l_j\left(i|\mathbf{a}_{p_0}|^2 + r_{p_0}\right),\ 1 \le l, j \le n-1.$$

Here $p_0 \in \partial\mathbf{H}_n$ and δ^l_j takes value 1 for $j = l$ and 0 otherwise. We notice that $F^{**}_{p_0}$ depends on p_0 and the choice of $(C_l)_{p_0}$. However, $\left.\frac{\partial^2 (f_l)^{**}_{p_0}}{\partial z_j \partial w}\right|_0$ is independent of the choice of $(C_l)_{p_0}$ by the formula in (2.3).

The key point to the proof of the theorem is to simplify a non-linear map F as in Lemma 2.1 in such a way that we can recognize its equivalence class. Hence, we need a test to distinguish a non-linear map from a linear one. To this aim, we will need the following linearity criterion which will be the starting point for the proof of Theorem 1:

Proposition 2.2 ([Theorem 4.2, Hu1]): *Suppose that for any $p_0 = (z_0, w_0) \in \partial\mathbf{H}_n$ close to the origin $(N > n > 1)$, there is a certain choice of $(C_l)_{p_0}$ so that $\left.\frac{\partial^2 \phi^{**}_{p_0}}{\partial z_l \partial z_k}\right|_0 = 0$ $(1 \le k, l \le n-1)$ for the corresponding $F^{**}_{p_0} = (f^{**}_{p_0}, \phi^{**}_{p_0}, g^{**}_{p_0})$. Then $F(z, w) \equiv (z, 0, w)$.*

The above was actually proved in [Hu1] for F which is only assumed to be C^2-smooth over the source hypersurface. Also we remark in passing that the condition that $\left.\frac{\partial^2 \phi^{**}_{p_0}}{\partial z_l \partial z_k}\right|_0 = 0$ for each l, k is independent of the choice $(C_l)_{p_0}$. (Namely, if it holds for one choice of $(C_l)_{p_0}$, then it holds for any other admissible choices of $(C_l)_{p_0}$.) This can be easily seen from the formula (2.2) and the fact that the formula in (2.3) is independent of $(C_l)_{p_0}$. (It also follows from [Lemma 2.1, Hu1] and the formula for q^j_{kl} there).

For the proof of Theorem 2, we need the following result from [Hu1-2]:

Theorem 2.3 ([Corollary 1.2, Hu2]): *Let M_1 and M_2 be two open connected pieces of the boundaries of the unit balls $\mathbf{B}^n$ and $\mathbf{B}^{2n-1}$, respectively $(n \ge 2)$. Let F be a non-constant twice continuously differentiable CR mapping from M_1 and M_2. Then F extends to a rational proper holomorphic map from $\mathbf{B}^n$ to $\mathbf{B}^{2n-1}$.*

The rest of the paper will be arranged as follows: In what follows, we always assume that $N = 2n-1, n > 2$ and F satisfies (2.1). We also assume that F is not linear. In Sect. 3, we will simplify $a^{(1)}$ and $\phi^{(2)}$ by applying (2.2). In particular, we get a rank condition (Lemma 3.1 (i) or Remark 3.1′) for the second order derivatives of F at 0. For the proof of Theorem 1, we need consider the 5th order formal equation to get relations of the higher order jets of F near 0 (see Lemma 3.2). In Sect. 4, we will seek the direction in which F is linear. Notice that for the g in Lemma 2.1, $g = w + o_{wt}(4)$. Since $\mathrm{Im}(g) \geq 0$, by a generalized version of the Hopf lemma due to Burns-Krantz, to show that $g(0, w) \equiv w$, it suffices to find a further normalization in which $g'''_{www}(0) = 0$. To this aim, we need to get a certain relation between the two third order derivatives of g: $g'''_{z_1 ww}(0)$ and $g'''_{www}(0)$. Since they stay in different weighted-order terms, such a relation can not be obtained just by looking at the Moser formal equation. Our method is to apply the rank condition (Lemma 3.1 (i)) to F_p^{**} and the feedback formula (2.3) to get a system of differential equations for F. Then we perform a formal argument for such equations (instead of the Chern-Moser formal equation) to get the desired formula (Lemma 4.1). With Lemma 4.1 at our disposal, in Sect. 5 we will show that after composing F with suitable automorphisms, we can further assume that the map in Lemma 2.1 has the following partial linearity: $g(0, w) \equiv w$, $\widetilde{f}(0, w) \equiv 0$ and F has degree two when restricted to the Segre variety Q_0. After re-employing F_p^{**}, we can similarly show that the degree of F also has degree two along the Segre variety Q_p for any $p(\approx 0) \in \mathbf{H_n}$. This then allows us to prove that the degree of F is two (Lemmas 5.2–5.3) and takes the special normal form as in (6.1). Once we get Lemma 6.1, the proof of Theorem 1.1 can be achieved by explicitly writing down the maps which intervine F into the Whitney map. We also attach an appendix after Sect. 6, in which we give an explicit map and its Taylor expansion (see (A.I)–(A.III)). The reader may like to compare this example with some of the formulas in Lemmas 3.1, 3.2, and 4.1

3. The formal consideration

We now let $F = (f, \phi, g) = (f_1, ..., f_{n-1}, \phi_1, ..., \phi_{n-1}, g) = (\widetilde{f}, g) : \partial\mathbf{H}_n \to \partial\mathbf{H}_{2n-1}$ be a non-constant rational map, which satisfies the normalization condition (2.1). **Assume in all that follows that F is not equivalent to a linear map and** $n > 2$, $N = 2n-1$, unless otherwise stated specifically. Our purpose is then to show that F is equivalent to the Whitney map.

It is well known that any local non-constant holomorphic map H, which sends $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_n$ and satisfies a normalization condition similar to (2.1), must be the identity map. For a holomorphic map from $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_{2n-1}$, the normalization condition (2.1) is not enough to determine the structure of F. Our first objective is to normalize F to make it take even simpler form. To this aim, we will first look at the formal implication of the functional equation $\mathrm{Im}(g) = |\widetilde{f}|^2$ for $\mathrm{Im}(w) = |z|^2$. It should be mentioned that (2.2) actually comes as a consequence of the normalization condition [(3.1), Hu1] (which is even weaker than (2.1) here), by considering the weighted third and 4th order terms of this functional equation ([Lemma 5.3, Hu1]).

Since F_p^{**} is equivalent to F, by Proposition 2.2, after replacing F by $F_{p_0}^{**}$ for a certain p_0 if necessary, we can assume in what follows that $a^{(1)}(z), \phi^{(2)}(z) \not\equiv 0$. Indeed, by (2.3), we can further assume that the $a^{(1)}(z), \phi^{(2)}(z)$ coming from $F_{p_0}^{**}$ do not vanish identically neither for any p_0 sufficiently close to 0. Write $a^{(1)}(z) = z \cdot A$. Then $\langle \overline{z}, a^{(1)}(z)\rangle = zA\overline{z}^t$ and $\overline{A}^t = A$ by (2.2).

Lemma 3.1: *Assume that F is as above such that $a^{(1)}$, $\phi^{(2)}$ are not identically zero. Then the following holds:*

(i) *Rank* $(A) = 1$*;*
(ii) *F is equivalent to a map which satisfies (2.1) with $a^{(1)}(z) = (z_1, 0, ..., 0)$ and $\phi_j^{(2)} = z_1 z_j$, $1 \le j \le n-1$. Namely, F is equivalent to a map of the following form:*

$$
\begin{cases}
f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \\
f_j = z_j + o_{wt}(3), \quad 2 \le j \le n-1, \\
\phi_j = z_1 z_j + o_{wt}(2), \quad 1 \le j \le n-1, \\
g = w + o_{wt}(4), \quad \textit{near } 0 \in \partial \mathbf{H}_n.
\end{cases} \tag{3.1}
$$

Proof of Lemma 3.1: By (2.2), it is clear that A is a semi-positive Hermitian matrix. Since $A \not\equiv 0$, there is a unitary matrix U such that $U^{-1}AU = diag(\kappa_1, ..., \kappa_k, 0, ..., 0)$ with $\kappa_v > 0$, $1 \le v \le k$. Replacing $F(z,w)$ by $\big(f(zU^{-1}, w) \cdot U, \phi(zU^{-1}, w), g(zU^{-1}, w)\big)$, we can assume without loss of generality that A already takes such a diagonal form.

We claim $k = 1$. In fact, write $\phi_j^{(2)} = \sum_{\alpha \le \beta} b_{\alpha\beta}^{(j)} z_\alpha z_\beta$. Then

$$
\big|\phi_j^{(2)}(z)\big|^2 = \sum_{\alpha \le \beta} \big|b_{\alpha\beta}^{(j)}\big|^2 |z_\alpha|^2 |z_\beta|^2 + \sum_{(\alpha,\beta)\neq(\gamma,l), \alpha \le \beta, \gamma \le l} \left\{ b_{\alpha\beta}^{(j)} \overline{b_{\gamma l}^{(j)}} z_\alpha z_\beta \overline{z_\gamma z_l} \right\}.
$$

Write $B_{\alpha\beta} := (b_{\alpha\beta}^{(1)}, ..., b_{\alpha\beta}^{(n-1)})$. Notice that (2.2) now reads as

$$
\left(\sum_{j=1}^{k} \kappa_j |z_j|^2 \right) |z|^2 \equiv \sum_{j=1}^{n-1} \big|\phi_j^{(2)}(z)\big|^2. \tag{3.2}
$$

Comparing the coefficients of the $|z_1|^2 z_l \overline{z_\beta}$-terms, we get $|B_{1\beta}|^2 > 0$ and $B_{1l} \cdot \overline{B_{1\beta}}^t = 0$, $\forall\, l \neq \beta$. Hence $\{B_{11}, B_{12}, ..., B_{1\, n-1}\}$ forms an orthogonal basis of $\mathbf{C}^{\mathbf{n-1}}$. For B_{sl} with $s \neq 1$, since we can similarly see from (3.2) that $B_{1\beta} \cdot \overline{B_{sl}}^t = 0$ for all β, we get that $B_{sl} = 0$ for $s \neq 1$. Therefore we conclude that $k = 1$, for the right hand side of (3.2) contains z_1 factor.

Combining the above, we conclude that the left hand side of (3.2) becomes $\kappa_1|z_1|^2|z|^2$ and $\phi_j^{(2)} = z_1\sum_\beta b_{1\beta}^{(j)} z_\beta = z_1\widetilde{B_j}\cdot z^t$, where $\widetilde{B_j} := (b_{1\,1}^{(j)}, ..., b_{1\,n-1}^{(j)})$. Write $\begin{pmatrix}\phi_1^{(2)}\\ ...\\ \phi_{n-1}^{(2)}\end{pmatrix} = z_1\widetilde{B}\cdot z^t$, where $\widetilde{B} := \begin{pmatrix}\widetilde{B}_1\\ ...\\ \widetilde{B_{n-1}}\end{pmatrix}$. Applying (3.2), we get $\widetilde{B}\overline{\widetilde{B^t}} = \kappa_1(\mathrm{id})$.

Replacing (f, ϕ, g) by $(f, \phi\frac{\widetilde{B}}{\sqrt{\kappa_1}}, g)$, the new map then takes the form $f_1 = z_1 + \frac{i}{2}\kappa_1 z_1 w + \cdots$, $\phi_j = \sqrt{\kappa_1}z_1 z_j + \cdots$. Next replacing it by

$$(z, w) \mapsto \left(\sqrt{\kappa_1}\widetilde{f}\left(\frac{z}{\sqrt{\kappa_1}}, \frac{w}{\kappa_1}\right), \kappa_1 g\left(\frac{z}{\sqrt{\kappa_1}}, \frac{w}{\kappa_1}\right)\right),$$

then we get a new map, which is equivalent to the original F and is now of the desired form. □

Remark 3.1′: Suppose that F is an arbitrary non-linear holomorphic map from $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$, satisfying the normalization condition (2.1) with $a^{(1)}$, $\phi^{(2)}$ not identically zero. As we mentioned before, (2.2) then holds automatically. Hence Lemma 3.1 (i) shows that for $N = 2n - 1$,

$$\mathrm{Rank}\left.\begin{pmatrix}(f_1)_{z_1 w} & ... & (f_1)_{z_{n-1}w}\\ ... & ... & ...\\ (f_{n-1})_{z_1 w} & ... & (f_{n-1})_{z_{n-1}w}\end{pmatrix}\right|_0 = 1.$$

We remark that when $N < 2n-1$, the rank of the above matrix becomes zero as is shown in [Proposition 3.1, Hu1]. It was such a rank-zero geometric fact that leads to the proof in [Hu1]. For the case considered in this paper, the rank is 1, which appears to be the key geometric fact that leads to the result in Theorem 1. (Notice that when $n = 2$, the rank-one condition simply says that $f''_{zw}(0) \neq 0$, which generically holds for any non-linear map. Namely, the rank condition only gives a crucial restriction for the map when $n > 2$).

In the next section, we will apply the above rank condition to the map $F_{p_0}^{**}$ introduced in Sect. 2 to obtain a certain type of second order non-linear partial differential equations (see (4.5)) of F. Such a system of equations is complicated. For the proof of Theorem 1, we need to carry out a formal argument for this system to derive a formula for the two third order derivatives of g at 0 (Lemma 4.1). To facilitate the formal consideration in Sect. 4, we next study the information contained in the 5th order curvature equation.

In what follows, we assume that F takes the normal form as in (3.1).

Collecting the weighted-5th order terms in the equation $\mathrm{Im}(g) = |\widetilde{f}|^2$ over $\mathrm{Im}(w) = |z|^2$, we obtain

$$(3.3)\quad Im\{g^{(5)} - 2i\overline{z}f^{(4)}\} = 2\mathrm{Re}\sum_{j=1}^{n-1}\phi_j^{(2)}\overline{\phi_j^{(3)}}, \quad \forall (z, w) \in \partial\mathbf{H}_n.$$

Letting $z_1 = 0$ and using (3.1), we get $\mathrm{Re}\sum_{j=1}^{n-1}\phi_j^{(2)}\overline{\phi_j^{(3)}} \equiv 0$. By the Chern-Moser lemma (see [Lemma 2.1, CM] or [Lemma 3.0, Hu1], or by a straightforward computation), we get $g^{(5)}(0,z',w)\equiv 0$ and $f_j^{(4)}(0,z',w)\equiv 0$ for $j>1$, where $z=(z_1,z')$. Therefore, we can write

$$(3.3')\qquad \begin{cases} f_1^{(4)} = a^{(4)}(z)+a^{(2)}(z)w+a_{02}w^2,\\ f_j^{(4)} = z_1(a_j^{(1)}(z)w+a_j^{(3)}(z)),\ \ 2\le j\le n-1,\\ \phi_j^{(3)} = b_j^{(3)}(z)+b_j^{(1)}(z)w,\ \ 1\le j\le n-1,\\ g^{(5)} = z_1\big[c^{(2)}(z)w+c_{12}w^2+c^{(4)}(z)\big]. \end{cases}$$

Lemma 3.2: *With the above notation, the following holds:*

$$a^{(2)}(z)=z_1\widetilde{a^{(1)}(z)},\, a_j^{(1)}(z)=\frac{\mu}{2}z_j,\, b_1^{(1)}(z)=b_1z_1,\ b_j^{(1)}=b_jz_1+\frac{\overline{\mu}}{2}z_j,$$

$$c^{(2)}(z)=0,\, c^{(4)}(z)=0,\, a^{(4)}(z)=0,\, a_j^{(3)}(z)=0$$

where μ, b_1 and $b_j (j>1)$ are complex numbers, $\widetilde{a^{(1)}}(z)$ is a linear function in z. Moreover, we have:

$$(3.4)\qquad \mu = 2c_{12} = 4i\overline{a_{02}}.$$

Proof of Lemma 3.2: Substituting $(3.3)'$ into (3.3), we get

$$\mathrm{Im}\Big\{g^{(5)}-2i\overline{z}\cdot f^{(4)}-2i\sum\phi_j^{(2)}\overline{\phi_j^{(3)}}\Big\}\equiv 0 \text{ or}$$

$$\mathrm{Im}\Big\{z_1\big[c^{(2)}(z)w+c_{12}w^2+c^{(4)}(z)\big]-2i\overline{z_1}\big[a^{(2)}(z)w+a_{02}w^2+a^{(4)}(z)\big]-$$

$$-2i\overline{z'}\cdot\big[z_1(a^{(1)'}(z)w+a^{(3)'}(z))\big]-2iz_1^2\Big(\overline{b_1^{(3)}(z)}+\overline{b_1^{(1)}(z)w}\Big)-$$

$$(3.5)\qquad -2iz_1\sum_{j\ge 2}z_j\Big(\overline{b_j^{(3)}(z)}+\overline{b_j^{(1)}(z)w}\Big)\Big\}\equiv 0,$$

where $a^{(k)'}=(a_2^{(k)},\dots,a_{n-1}^{(k)})$, $(z,w)\in\partial\mathbf{H}_n$.

Since $w=u+i|z|^2$ on $\partial\mathbf{H}_n$, we can replace w by $u+i|z|^2$ in (3.5). By comparing the coefficients of the u^2 terms in (3.5), we get $\mathrm{Im}\{c_{12}z_1-2i\overline{z_1}a_{02}\}=0$ for any z_1, which implies $c_{12}=2i\overline{a_{02}}$. Collecting the harmonic terms in (3.5), we get $c^{(4)}(z)=0$.

Comparing the coefficients of the u terms in (3.5), we get an identity for any $z\in\mathbf{C}^{n-1}$:

$$\operatorname{Im}\Big\{ z_1 c^{(2)}(z) + 2ic_{12} z_1 |z|^2 - 2i\overline{z_1} a^{(2)}(z) - 2i\overline{z_1} a_{02}(2i|z|^2) - \\ -2iz_1 \sum_{j\ge 2} a_j^{(1)}(z)\overline{z_j} - 2iz_1 \sum_{j\ge 1} z_j \overline{b_j^{(1)}(z)} \Big\} \equiv 0.$$

Hence it follows that $c^{(2)}(z) \equiv 0$ and that

$$(3.5') \qquad 2iz_1 \sum_{j\ge 1} z_j \overline{b_j^{(1)}(z)} + 2iz_1 \sum_{j\ge 2} a_j^{(1)}(z)\overline{z_j} + 2ia^{(2)}(z)\overline{z_1} \\ \equiv 2ic_{12} z_1 |z|^2 - 4\overline{a_{02}} z_1 |z|^2.$$

Now collecting in (3.5) terms of the form: $z^\alpha \overline{z}^\beta$ with $|\alpha| = 4,\ |\beta| = 1$, we get $a^{(4)} = 0,\ a^{(3)'} = 0$.

Back to $(3.5)'$, since its left hand side divides z_1, we get $a^{(2)}(z) = z_1 \widetilde{a^{(1)}(z)}$ for a certain $\widetilde{a^{(1)}(z)}$. Write $\mu := 2c_{12} = 4i\overline{a_{02}}$. We have

$$(3.6) \quad z_1 \overline{b_1^{(1)}(z)} + \sum_{j\ge 2} z_j \overline{b_j^{(1)}(z)} + \sum_{j\ge 2} a_j^{(1)}(z)\overline{z_j} + \widetilde{a^{(1)}(z)}\overline{z_1} \equiv \mu |z|^2.$$

Considering the coefficients of $|z_1|^2$, $z_1\overline{z_i}$ $(i > 1)$, $z_j\overline{z_l}$ terms $(l, j > 1)$ in (3.6), respectively, we obtain

$$(3.7) \qquad \mu \equiv \frac{\partial \overline{b_1^{(1)}(z)}}{\partial z_1} + \frac{\partial \widetilde{a^{(1)}(z)}}{\partial z_1},\ \ \frac{\partial \overline{b_1^{(1)}(z)}}{\partial z_i} + \frac{\partial a_i^{(1)}(z)}{\partial z_1} \equiv 0, \\ \frac{\partial \overline{b_j^{(1)}}}{\partial z_l} + \frac{\partial a_l^{(1)}}{\partial z_j} = \begin{cases} 0, & l \ne j, \\ \mu, & l = j. \end{cases}$$

Collecting terms in (3.5) without u-factor and recalling that $c^{(2)}(z)$, $c^{(4)}$, $a^{(4)}$, $a_j^{(3)} \equiv 0$, we get

$$\operatorname{Im}\Big\{ z_1 c_{12}(-|z|^4) - 2i\overline{z_1}\Big(a^{(2)}(z) i|z|^2 - a_{02}|z|^4\Big) - 2i\overline{z'}\cdot\Big[z_1 a^{(1)'}(z)(i|z|^2)\Big] - \\ -2iz_1^2\Big(\overline{b_1^{(3)}(z)} - i|z|^2\overline{b_1^{(1)}(z)}\Big) - 2iz_1\sum_{j\ge 2} z_j\Big(\overline{b_j^{(3)}(z)} - i|z|^2\overline{b_j^{(1)}(z)}\Big)\Big\} \equiv 0.$$

Considering terms with $z^I\overline{z^J}$ factor in the above identity, where $|I| = 3$ and $|J| = 2$, we get

$$(-c_{12}z_1 + 2i\overline{a_{02}}z_1)|z|^4 + 2z_1\overline{z_1}\widetilde{a^{(1)}(z)}|z|^2 + 2z_1\overline{z'}\cdot a^{(1)'}(z)|z|^2 - 2z_1^2\overline{b_1^{(1)}(z)}|z|^2 - \\ -2z_1\Big(\sum_{j\ge 2} z_j\overline{b_j^{(1)}(z)}\Big)|z|^2 - 2i\overline{z_1^2}b_1^{(3)}(z) - 2i\overline{z_1}\sum_{j\ge 2}\overline{z_j}b_j^{(3)}(z) \equiv 0.$$

Since $c_{12} = 2i\overline{a_{02}}$, the above implies

$$z_1|z|^2\Big(\overline{z_1}\widetilde{a^{(1)}(z)} + \overline{z'}\cdot a^{(1)\prime}(z) - z_1\overline{b_1^{(1)}(z)} - \sum_{j\geq 2} z_j\overline{b_j^{(1)}(z)}\Big) \equiv i\overline{z_1}^2 b_1^{(3)}(z) + i\overline{z_1}\sum_{j\geq 2}\overline{z_j}b_j^{(3)}(z). \tag{3.8}$$

Considering the coefficients of the $\overline{z_j z_l}(j \neq 1, l \neq 1)$ terms in (3.8), we find

$$z_l a_j^{(1)}(z) + z_j a_l^{(1)}(z) - z_l z_1 \overline{\frac{\partial b_1^{(1)}}{\partial z_j}} - z_j z_1 \overline{\frac{\partial b_1^{(1)}}{\partial z_l}} - z_j z_l \overline{\frac{\partial b_l^{(1)}}{\partial z_l}} - z_l z_j \overline{\frac{\partial b_j^{(1)}}{\partial z_j}} \equiv 0. \tag{3.9}$$

Further considering the $z_1 z_l$ terms in (3.9) ($l \neq 1$), we get $\frac{\partial a_j^{(1)}(z)}{\partial z_1} - \overline{\frac{\partial b_1^{(1)}}{\partial z_j}} \equiv 0$, $\forall j \neq 1$. Combining it with the second equation in (3.7), we conclude $\frac{\partial b_1^{(1)}(z)}{\partial z_j} \equiv \frac{\partial a_j^{(1)}(z)}{\partial z_1} \equiv 0, \ \ \forall j \neq 1$. Hence

$$\begin{gathered} b_1^{(1)}(z) \equiv b_1 z_1, \text{ for a certain } b_1 \in \mathbf{C}; \\ \text{and } a_j^{(1)}(z) \text{ has no } z_1 \text{ terms for } 2 \leq j \leq n-1. \end{gathered} \tag{3.10}$$

Considering terms with $\overline{z_j}^2$ factor in (3.8), where $j \neq 1$, we get $a_j^{(1)}(z) - z_1\overline{\frac{\partial b_1^{(1)}(z)}{\partial z_j}} - \sum_{k=2}^{n-1} z_k \overline{\frac{\partial b_k^{(1)}(z)}{\partial z_j}} \equiv 0$. Since we have proved $b_1^{(1)}(z) = b_1 z_1$, we have

$$a_j^{(1)}(z) - \sum_{k=2}^{n-1} z_k \overline{\frac{\partial b_k^{(1)}(z)}{\partial z_j}} \equiv 0. \tag{3.11}$$

From this and the third equation in (3.7), it yields that $a_j^{(1)}(z)$ has only z_j term for $2 \leq j \leq n-1$, and $\frac{\partial a_j^{(1)}}{\partial z_j} = \overline{\frac{\partial b_j^{(1)}}{\partial z_j}}, \ \ \forall j \neq 1$. Again by (3.11) and the third equation in (3.7), we get

$$a_j^{(1)} = \frac{\mu}{2} z_j, \quad 2 \leq j \leq n-1. \tag{3.12}$$

From the third equation in (3.7), and (3.12), (3.11), we get $b_j^{(1)} = b_j z_1 + \frac{\overline{\mu}}{2} z_j, \ \ 2 \leq j \leq n-1$, where $b_j \in \mathbf{C}$ are some complex numbers. The proof is complete. □

4. An application of the group structure of the Heisenberg hypersurface

A major step toward the proof of Theorem 1 is to find a direction in which the map is linear (see Lemma 6.1). For this purpose, we need to establish a connection between the weighted 5^{th} order term in g and the weighted 6^{th} order term in g. More specifically, we need to have $c_{03} = |c_{12}|^2$ (Lemma 4.1). (With such a property, we can show in Sect. 5 that after composing F with certain automorphisms, we can make $g(0, w) \equiv w$). However, as explained in Sect. 2, this cannot be done just by the formal consideration as we did in Lemma 3.1 and Lemma 3.2. The way we achieve this is to apply Lemma 3.1 (i) (Remark 3.1′) to F_p^{**} to get a system of differential equations of F (see (4.5)). Then we can obtain the required formula by considering the coefficients of the u and $|z_l|^2$-terms.

Lemma 4.1: *Let F be as in* (3.1). *Write* $c_{03} = \frac{1}{6}\frac{\partial^3 g}{\partial w^3}\Big|_0$ *and* $c_{12} = \frac{1}{2}\frac{\partial^3 g}{\partial z_1 \partial w^2}\Big|_0$. *Then* $c_{03} = |c_{12}|^2$.

Proof of Lemma 4.1: As in Sect. 2, the Heisenberg group structure can be used to produce the map F_p^{**} from F for any $p = (z, w) \in \partial\mathbf{H}_n$. By (2.3) and the formulas preceding it, we have

$$\frac{\partial^2 (f_j^{**})_p}{\partial z_l \partial w}\Big|_0 = \frac{1}{\lambda_p}(\widetilde{f_p})_{wz_l}{}''\Big|_0 \cdot \overline{(E_j)_p}^t - \frac{1}{\lambda_p^2}((E_w)_p \cdot \overline{(E_l)_p})^t (g_p)''_{wz_j}\Big|_0 -$$

$$-\delta_l^j\left(\frac{i|(E_w)_p|^2}{\lambda_p} + \frac{1}{2\lambda_p}\mathrm{Re}\{(g_p)''_{ww}(0)\}\right).$$

Writing $P_l^j := \frac{\partial^2 (f_j)^{**}_{(z,w)}}{\partial z_l \partial w}\Big|_0$ and applying again the formulas in Sect. 2 (or [Sect. 2, 4, Hu1]), we get the following

$$2\lambda P_l^j = 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t - \frac{2}{\lambda}\left(\widetilde{f_w'} \cdot \overline{L_l(\widetilde{f})}^t\right) \cdot L_j\left(g_w' - 2i\widetilde{f_w'} \cdot \overline{\widetilde{f}}^t\right) -$$

$$- 2i\delta_l^j|\widetilde{f_w'}|^2 - \delta_l^j \mathrm{Re}\{g''_{ww}\} + 2\delta_l^j \mathrm{Re}\{i\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t\}, \tag{4.1}$$

$$\text{at any } p = (z, w) \in \partial\mathbf{H}_n,$$

where $\lambda = \lambda(p) := \lambda_p$ is a smooth positive function in p. On the other hand, we can apply T^2 to the basic equation $\mathrm{Im}(g) = |\widetilde{f}|^2$, where $T := \frac{\partial}{\partial u}$ is the real tangent vector field along $\partial\mathbf{H}_n$ and $w = u + i|z|^2$ to get

$$0 = 2i\ \mathrm{Im}\left\{i\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t\right\} + 2i|\widetilde{f_w'}|^2 - i\mathrm{Im}\{g''_{ww}\}, \quad \forall (z, w) \in \partial\mathbf{H}_n.$$

Multiplying it by δ_l^j and then adding it to (4.1), we have:

$$2\lambda P_l^j = 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t - \tfrac{2}{\lambda}\left(\widetilde{f'_w} \cdot \overline{L_l(\widetilde{f})}^t\right) \cdot L_j\left(g'_w - 2i\widetilde{f_w}' \cdot \overline{\widetilde{f}}^t\right)$$
$$-\delta_l^j\left(g''_{ww} - 2i\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t\right),$$

$\forall p = (z, w) \in \partial\mathbf{H}_n$. Applying the operator L_jT to the basic equation $\mathrm{Im}(g) = |\widetilde{f}|^2$, we have

$$L_j\left(g'_w - 2i\widetilde{f_w}' \cdot \overline{\widetilde{f}}^t\right) = 2iL_j(\widetilde{f}) \cdot \overline{\widetilde{f'_w}}^t, \quad \forall(z, w) \in \partial\mathbf{H}_n.$$

Write

$$\lambda^* := 2i\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t - g''_{ww}. \tag{4.1$'$}$$

Notice by Lemma 3.1 and (3.1), $\lambda = 1 + |z_1|^2 + o_{wt}(2)$. We then get

$$\begin{aligned} 2\lambda P_l^j &= 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t \\ &-4i\left(\widetilde{f'_w} \cdot \overline{L_l(\widetilde{f})}^t\right)\left(L_j(\widetilde{f}) \cdot \overline{\widetilde{f'_w}}^t\right) + \delta_l^j\lambda^* + o_{wt}(2), \end{aligned} \tag{4.2}$$

for any $(z, w) \in \partial\mathbf{H}_n$. Writing $q_l^j := 2\lambda P_l^j - \delta_l^j\lambda^*$, one obtains from (4.2)

(4.3)

$$q_l^j = 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t - 4i\left(\widetilde{f'_w} \cdot \overline{L_l(\widetilde{f})}^t\right)\left(L_j(\widetilde{f}) \cdot \overline{\widetilde{f'_w}}^t\right) + o_{wt}(2),$$

and

$$2\lambda P_l^j = q_l^j + \delta_l^j\lambda^*. \tag{4.4}$$

Applying Lemma 3.1 (i) or Remark 3.1$'$ to F_p^{**}, we get $Rank(P_l^j) \equiv 1$. Therefore we obtain $\frac{P_1^1}{P_l^1} = \frac{P_1^l}{P_l^l}$, $2 \le l \le n-1$, i.e., $q_l^l + \lambda^* = \frac{q_l^1}{q_1^1+\lambda^*}q_1^l$, $\quad 2 \le l \le n-1$. (Note that this is the place where the crucial restriction that $n \geq 3$ comes out). Hence, we get the following system of (second order) differential equations:

$$\begin{gathered} (\lambda^*)^2 + \lambda^*\left(q_1^1 + q_l^l\right) + \left(q_1^1 q_l^l - q_l^1 q_1^l\right) \equiv 0, \\ 2 \le l \le n-1, \ \forall(z, w) \in \partial\mathbf{H}_n. \end{gathered} \tag{4.5}$$

(4.5) gives a complicated system of second order partial differential equations for the map. Fortunately, for the proof of Lemma 4.1, it suffices for us to do some formal analysis for (4.5) of weighted order two.

In the following, *we fix an* $l(>1)$ *and only consider the* u *and* $|z_l|^2$ *terms in the Taylor expansion of the left hand side of (4.5).*

We now compute $\lambda^*, q_1^1, q_l^l, q_l^1$ and q_1^l in (4.5) as follows.

By (3.1), (3.3)′ and Lemma 3.2, we have $f_1 = z_1 + \frac{i}{2}z_1 w + a_{02}w^2 + z_1\widetilde{a^{(1)}(z)}w + \widetilde{a_1^{(1)}}(z)w^2 + \widetilde{a_1^{(3)}}(z)w + o_{wt}(5)$, and $f_j = z_j + \frac{\mu}{2}z_1 z_j w + \widetilde{a_j^{(1)}}(z)w^2 + \widetilde{a_j^{(3)}}(z)w + o_{wt}(5)$ for $j > 1$. The fact that f does not contain terms of degree 5 purely in z in its Taylor expansion can be obtained from a simple formal consideration of the weighted-6^{th} order truncation in the equation $\mathrm{Im}(g) = |\widetilde{f}|^2$: $\mathrm{Im}\{g^{(6)} - 2i\overline{z}f^{(5)}\} = 2\,\mathrm{Re}\Big(\sum_j(\phi_j^{(2)}\overline{\phi_j^{(4)}} + \frac{1}{2}\phi_j^{(3)}\overline{\phi_j^{(3)}})\Big)$.

For convenience, we first fix some notation and terminology: For functions $A, B, C_1, \cdots, C_m$, we will write $A = B \bmod(C_1, \cdots, C_m)$ if $A = B + \sum_{j=1}^m k_j C_j$ for some constants $\{k_j\}$. For real analytic functions $H(Z, \overline{Z})$ and and $Q(Z, \overline{Z})$ defined near the origin, we will write $H = Q$ mod(terms other than $Z^{\alpha_1}\overline{Z}^{\beta_1}, \cdots, Z^{\alpha_k}\overline{Z}^{\beta_k}$) if $\frac{\partial^{|\alpha_m|+|\beta_m|}(H-Q)}{\partial Z^{\alpha_m}\partial\overline{Z}^{\beta_m}}\Big|_0 = 0$ for $1 \le m \le k$. Remark that $\partial\mathbf{H}_n$ can be identified with $\mathbf{R}^{2n-1}$ with coordinates (x, y, u) through the map $z = x + iy$ and $w = u + i|z|^2$. Notice that the weight of u is two.

We start with the following expansions which, as explained, come as consequences of (3.1), (3.3)′ and Lemmas 3.1–3.2:

$$f_1 = z_1 + \frac{i}{2}z_1 w + a_{02}w^2 + z_1\widetilde{a^{(1)}(z)}w + \widetilde{a_1^{(1)}}(z)w^2 + \\ + \widetilde{a_1^{(3)}}(z)w + o_{wt}(5) \cap o(|(z,w)|^4), \quad \mathrm{mod}(z_s z_t w^2, z_s w^3, w^3, w^4),$$

$$f_j = z_j + \frac{\mu}{2}z_1 z_j w + \widetilde{a_j^{(1)}}(z)w^2 + \widetilde{a_j^{(3)}}(z)w + o_{wt}(5) \cap o(|(z,w)|^4), \\ \mathrm{mod}(z_s z_t w^2, z_s w^3, w^3, w^4),\ j \ge 2.$$

$$\phi_1 = z_1^2 + b_1 z_1 w + b_1^{(0)}w^2 + b_1^{(3)}(z) + o(|(z,w)|^2) \cap o_{wt}(3),$$

$$\phi_j = z_1 z_j + \left(b_j z_1 + \frac{\overline{\mu}}{2}z_j\right)w + b_j^{(0)}w^2 + b_j^{(2)}(z)w \\ + b_j^{(3)}(z) + o_{wt}(3) \cap o(|(z,w)|^2)\ \ (j > 1),$$

$$g = w + c_{12}z_1 w^2 + c_{03}w^3 + o_{wt}(5) \cap o(|(z,w)|^3).$$

Here, a function of the form $X^{(m)}(z)$ denotes a holomorphic polynomial in z of degree m. Therefore it yields

$$
\begin{cases}
(f_1)'_w = \frac{i}{2}z_1 + 2a_{02}w + z_1\widetilde{a^{(1)}} + 2\widetilde{a_1^{(1)}}w + \widetilde{a_1^{(3)}} + o(|(z,w)|^3), \\
\hfill \mathrm{mod}\big(z_sz_tw,\, z_sw^2,\, w^2,\, w^3\big), \\
(f_1)''_{ww} = 2a_{02} + 2\widetilde{a_1^{(1)}}(z) + o(|(z,w)|^2),\ \mathrm{mod}\big(z_sz_t,\, z_sw,\, w,\, w^2\big), \\
(f_j)'_w = \frac{\mu}{2}z_1z_j + 2\widetilde{a_j^{(1)}(z)}w + \widetilde{a_j^{(3)}}(z) + o(|(z,w)|^3), \\
\hfill \mathrm{mod}\big(z_sz_tw,\, z_sw^2,\, w^2,\, w^3\big), \\
(f_j)''_{ww} = 2\widetilde{a_j^{(1)}}(z) + o(|(z,w)|^2),\ \mathrm{mod}\big(z_sz_t,\, z_sw,\, w,\, w^2\big), \\
(\phi_1)'_w = b_1z_1 + 2b_1^{(0)}w + o(|(z,w)|), \\
(\phi_1)''_{ww} = 2b_1^{(0)} + o(1), \\
(\phi_j)'_w = b_jz_1 + \frac{\overline{\mu}}{2}z_j + 2b_j^{(0)}w + b_j^{(2)}(z) + o_{wt}(2), \\
(\phi_j)''_{ww} = 2b_j^{(0)} + o(1), \\
g''_{ww} = 2c_{12}z_1 + 6c_{03}w + o(|(z,w)|^2), \mathrm{mod}\big(z_sz_j,\, z_sw,\, w^2\big),
\end{cases}
$$

where $2 \le j \le n-1$. Thus we obtain

$$
\lambda^* = 2i\,\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t - g''_{ww} = 4i\frac{\partial \widetilde{a_l^{(1)}}}{\partial z_l}|z_l|^2 - 6c_{03}w, \tag{4.6}
$$
$$
\mathrm{mod}\big(\text{terms other than } 1,\ z_l,\ \overline{z_l},\ |z_l|^2, u\big).
$$

Hence

$$
(\lambda^*)^2 = 0,\ \mathrm{mod}\big(\text{terms other than } |z_l|^2, u\big). \tag{4.7}
$$

Computation for q_1^1: With a direct computation, we have the following:

$$
\begin{cases}
L_1(f_1) = 1 + \frac{i}{2}w \\
L_1\big((f_1)'_w\big) = \frac{i}{2} + \frac{\partial \widetilde{a^{(1)}(z)}}{\partial z_l}z_l + 2\frac{\partial \widetilde{a_1^{(1)}}(z)}{\partial z_1}w \\
L_1(f_j) = 0 \\
L_1\big((f_j)'_w\big) = \frac{\mu}{2}\delta_l^j z_l + 2\frac{\partial \widetilde{a_j^{(1)}}(z)}{\partial z_1}w \\
\hfill \mathrm{mod}\big(\text{terms other than } 1,\ z_l,\ \overline{z_l},\ |z_l|^2,\ u\big) \\
L_1(\phi_1) = b_1w \\
L_1\big((\phi_1)'_w\big) = b_1 + o(1) \\
L_1(\phi_j) = z_j + b_jw \\
L_1\big((\phi_j)'_w\big) = b_j + \frac{\partial b_j^{(2)}(z)}{\partial z_1} + o_{wt}(1),
\end{cases}
$$

where $2 \le j \le n-1$. Hence

$$q_1^1 = 2L_1(\widetilde{f'_w}) \cdot \overline{L_1(\widetilde{f})}^t - 4i\left|\widetilde{f'_w} \cdot \overline{L_1(\widetilde{f})}^t\right|^2 =$$

$$(4.8) \qquad = i + 2\frac{\partial \widetilde{a^{(1)}(z)}}{\partial z_l} z_l + 4\frac{\partial \widetilde{a_1^{(1)}}(z)}{\partial z_1} w + \frac{1}{2}\overline{w} + 2\sum_{j=1}^{n-1} |b_j|^2 \overline{w} + 2b_l\overline{z_l} +$$

$$+ 2\frac{\partial^2 b_l^{(2)}(z)}{\partial z_1 \partial z_l}|z_l|^2, \quad \text{mod}\big(\text{terms other than } 1,\ z_l, \overline{z_l},\ |z_l|^2,\ u\big).$$

Computation for q_l^l: With a straightforward computation, we have the following:

$$\begin{cases} L_l(f_j) = \delta_l^j, \ \ 1 \le j \le n-1 \\ L_l\big((f_1)'_w\big) = 4ia_{02}\overline{z_l} + 4i\frac{\partial \widetilde{a_1^{(1)}}}{\partial z_l}|z_l|^2 + 2\frac{\partial \widetilde{a_1^{(1)}}}{\partial z_l} w \\ L_l\big((f_j)'_w\big) = 4i\frac{\partial \widetilde{a_j^{(1)}}}{\partial z_l}|z_l|^2 + 2\frac{\partial \widetilde{a_j^{(1)}}}{\partial z_l} w, \ j \ne 1 \\ L_l(\phi_1) = o_{wt}(1), \qquad\qquad \text{mod}\big(\text{terms other than } 1, z_l, \overline{z_l}, |z_l|^2, u\big) \\ L_l(\phi_k) = \delta_k^l i\overline{\mu}|z_l|^2 + \frac{\overline{\mu}}{2}\delta_k^l w, \\ L_l\big((\phi_1)'_w\big) = o(1), \\ L_l\big((\phi_k)'_w\big) = \frac{\overline{\mu}}{2}\delta_k^l + 4ib_k^{(0)}\overline{z_l} + \frac{\partial b_k^{(2)}(z)}{\partial z_l} + o_{wt}(1), \end{cases}$$

where $2 \le k \le n-1$. Hence

$$q_l^l = 2L_l(\widetilde{f'_w}) \cdot \overline{L_l(\widetilde{f})}^t - 4i\left|\widetilde{f'_w} \cdot \overline{L_l(\widetilde{f})}^t\right|^2 = 2\left(4i\frac{\partial \widetilde{a_l^{(1)}(z)}}{\partial z_l}|z_l|^2 + 2\frac{\partial \widetilde{a_l^{(1)}}}{\partial z_l} w\right) +$$

$$(4.9) \quad + 2\left(\frac{\overline{\mu}}{2} + 4ib_l^{(0)}\overline{z_l} + \frac{\partial b_l^{(2)}(z)}{\partial z_l}\right)\overline{\left(i\overline{\mu}|z_l|^2 + \frac{\overline{\mu}}{2} w\right)} + o_{wt}(2) =$$

$$= 8i\frac{\partial \widetilde{a_l^{(1)}(z)}}{\partial z_l}|z_l|^2 + 4\frac{\partial \widetilde{a_l^{(1)}(z)}}{\partial z_l} w - i|\mu|^2|z_l|^2 + \frac{|\mu|^2}{2}\overline{w},$$

mod(terms other than $1,\ z_l,\ \overline{z_l},\ |z_l|^2,\ u$). Therefore from (4.8), we get

$$(4.10) \quad q_1^1 q_l^l = -8\frac{\partial \widetilde{a_l^{(1)}(z)}}{\partial z_l}|z_l|^2 + 4i\frac{\partial \widetilde{a_l^{(1)}(z)}}{\partial z_l} w + |\mu|^2|z_l|^2 + \frac{i}{2}|\mu|^2\overline{w},$$

mod(terms other than $|z_l|^2,\ u$), $\forall (z, w) \in \partial \mathbf{H}_n$.

Computation for q_1^l and q_l^1: Similarly, we can get

$$(4.11) \qquad \begin{cases} q_1^l = \mu z_l, \ \text{mod(terms other than } 1, z_l, \overline{z_l}), \\ q_l^1 = 8ia_{02}\overline{z_l} + \overline{\mu z_l}, \ \text{mod(terms other than } 1, \overline{z_l}). \end{cases}$$

Since by Lemma 3.2, $a_{02} = \frac{i\overline{\mu}}{4}$, we thus obtain from (4.11):

$$(4.13) \quad q_1^l q_l^1 = -|\mu|^2|z_l|^2, \ \text{mod}\big(\text{terms other than } |z_l|^2, u\big), \forall (z, w) \in \partial \mathbf{H}_n.$$

Now from (4.6) (4.8) and (4.9), we get

(4.14)
$$\lambda^*\big(q_1^1 + q_l^l\big) = -i6c_{03}w - 4\frac{\widetilde{\partial a_l^{(1)}(z)}}{\partial z_l}|z_l|^2, \quad \mathrm{mod}\big(\text{terms other than } |z_l|^2, u\big).$$

Therefore by (4.7), (4.10), (4.11), (4.13), and (4.14) and by writing up the u and $z_i\overline{z_j}$-terms in (4.5), we get over $\partial\mathbf{H}_n$ the following, which holds modifying terms other than u and $|z_l|^2$. Namely, we have

$$-i6c_{03}w - 4\frac{\widetilde{\partial a_l^{(1)}(z)}}{\partial z_l}|z_l|^2 - 8\frac{\widetilde{\partial a_l^{(1)}(z)}}{\partial z_l}|z_l|^2 + 4i\frac{\widetilde{\partial a_l^{(1)}(z)}}{\partial z_l}w + |\mu|^2|z_l|^2 +$$

$$+\frac{i}{2}|\mu|^2\overline{w} - \big(-|\mu|^2|z_l|^2\big) \equiv 0, \ \mathrm{mod}\big(\text{terms other than } |z_l|^2, u\big). \tag{4.15}$$

Collecting the u terms in (4.15), we get the identity $-6ic_{03} + 4i\gamma + \frac{i}{2}|\mu|^2 = 0$, where $\gamma := \frac{\widetilde{\partial a_l^{(1)}(z)}}{\partial z_l}$, i.e.,

$$12c_{03} = 8\gamma + |\mu|^2. \tag{4.16}$$

Collecting the $|z_l|^2$ terms in (4.15), we get

$$-i6c_{03}(i) - 4\gamma - 8\gamma + 4i\gamma(i) + |\mu|^2 + \frac{i}{2}|\mu|^2(-i) + |\mu|^2 = 0. \tag{4.17}$$

From (4.16), (4.17) and the fact $\mu = 2c_{12}$, we get $c_{03} = \frac{1}{4}|\mu|^2 = |c_{12}|^2$. The proof of Lemma 4.1 is complete. □

5. Linear direction and degree estimate

With the normalization (3.1), we get $g = w + o_{wt}(4)$. It was mentioned that if we have in addition $c_{03} = 0$, then we get immediately $g(0, w) \equiv w$ by applying a generalized Hopf lemma due to Burns-Krantz (see [BK] or the argument following the proof of Lemma 5.1). This will certainly make the map more accessible. c_{03} may not be zero in general. However, we will show in this section, by making use of Lemma 4.1 that if we replace F by a suitable $\hat{F}_c$ in the same equivalence class, c_{03} will vanish while (3.1) still holds. With the simple expression of $\hat{F}_c$, we will show that it has degree 2 along the Segre variety Q_0. By considering F_p^{**} instead of F, we will further show that F has degree two when restricted to Q_p. This will force $\deg(F) = 2$. In Lemma 6.1, with the analysis in this section, we will further show that F is linear when restricted to some subspace (see Lemma 6.1), which then plays an essential role in proving Theorem 1.

Let us define $\hat{F}_c$ as follows. Write $p_1 = -\overline{c_{12}}$, $\mathbf{p} = (p_1, 0') \in \mathbf{C}^{n-1}$ and $(\mathbf{p}, 0') \in \mathbf{C}^{\mathbf{2n-2}}$. Let $\hat{F}_c = \tau_c \circ F \circ \sigma_c$, where the two automorphisms $\sigma_c \in \mathrm{Aut}(\partial\mathbf{H}_n)$ and $\tau_c \in \mathrm{Aut}(\partial\mathbf{H}_{2n-1})$ are given by

$$(5.1)\qquad \sigma_c(z,w)=\left(\frac{z+w\mathbf{p}}{1-2i\overline{p_1}z_1-i|p_1|^2w},\frac{w}{1-2i\overline{p_1}z_1-i|p_1|^2w}\right),$$

$$(5.2)\qquad \tau_c(z^*,w^*)=\left(\frac{z^*-w^*(\mathbf{p},0')}{1+2i\overline{p_1}z_1^*-i|p_1|^2w^*},\frac{w^*}{1+2i\overline{p_1}z_1^*-i|p_1|^2w^*}\right).$$

Lemma 5.1: *Let F be as in (3.1) and $\hat{F}_c=\tau_c\circ F\circ\sigma_c=(\hat{f}_c,\hat{\phi}_c,\hat{g}_c)=(\widetilde{\hat{f}_c},\hat{g}_c)=(\hat{f}_{1,c},\cdots,\hat{\phi}_{n-1,c},\hat{g}_c)$ be defined as above ($n>2$). Then $\hat{F}_c$ satisfies the normalization condition* (3.1). *Moreover, we have $\frac{\partial^3\hat{g}_c}{\partial w^3}\Big|_0=0$.*

Proof of Lemma 5.1: First, by a direct computation, it is clear that $\hat{g}_c=w+o(|(z,w|)$, $\hat{f}_c=z+o(|(z,w)|)$ and $\hat{\phi}_c=o(|(z,w)|)$. By [Lemma 5.3, Hu1], to verify that $\hat{F}_c$ satisfies (2.1), it suffices to show that $\mathrm{Re}(\frac{\partial^2\hat{g}_c}{\partial w^2})\Big|_0=0$, for $\frac{\partial^2\hat{g}_c}{\partial z_k\partial z_l}\Big|_0=0$ holds automatically (see Sect. 2 of [Hu1]).

We next calculate the coefficients of w^2 and w^3 in the expansion of $\hat{g}_c$. For this, let $z=0$ and consider

$$\hat{g}_c(0,w)=\frac{g\left(\frac{p_1w}{1-i|p_1|^2w},0,\frac{w}{1-i|p_1|^2w}\right)}{1+2i\overline{p_1}f_1\left(\frac{p_1w}{1-i|p_1|^2w},0,\frac{w}{1-i|p_1|^2w}\right)-i|p_1|^2g\left(\frac{p_1w}{1-i|p_1|^2w},0,\frac{w}{1-i|p_1|^2w}\right)}.$$

Recall $f_1(z_1,0,w)=z_1+\frac{i}{2}z_1w+a_{02}w^2+z_1\widetilde{a^{(1)}(z_1},0)w+o_{wt}(5)$, and $g(z_1,0,w)=w+c_{12}z_1w^2+c_{03}w^3+o(|(z_1,0,w)|^3)$. Then, by a Taylor expansion consideration we get

$$\begin{aligned}\hat{g}_c(0,w)=&\left(\frac{w}{1-i|p_1|^2w}+c_{12}p_1w^3+c_{03}w^3\right)\times\\&\times\Bigg[1-2i\overline{p_1}\left(\frac{p_1w}{1-i|p_1|^2w}+\frac{i}{2}\frac{p_1w^2}{(1-i|p_1|^2w)^2}+a_{02}w^2\right)+\\&+i|p_1|^2\frac{w}{1-i|p_1|^2w}+\left(-2i\overline{p_1}\frac{p_1w}{1-i|p_1|^2w}+i|p_1|^2w\right)^2\Bigg]+\\&+o(w^3)\\=&\left(w+i|p_1|^2w^2+(-|p_1|^4+c_{12}p_1+c_{03})w^3\right)\times\\&\times\Bigg\{1-2i\overline{p_1}\Big[p_1w\big(1+i|p_1|^2w\big)+\frac{i}{2}p_1w^2++a_{02}w^2\Big]+\\&+i|p_1|^2\Big[w\big(1+i|p_1|^2w\big)\Big]+\\&+\Big[-2i|p_1|^2w(1+i|p_1|^2w)+i|p_1|^2w)\Big]^2\Bigg\}+o(w^3)\end{aligned}$$

$$= w + \Big(i|p_1|^2 - 2i|p_1|^2 + i|p_1|^2\Big)w^2 + \Big(- |p_1|^4 + c_{12}p_1 + c_{03} + \\ + 2|p_1|^4 - |p_1|^4 + 2|p_1|^4 + |p_1|^2 - 2ia_{02}\overline{p_1} - |p_1|^4 - |p_1|^4\Big) \times \\ \times w^3 + o(w^3) = w + o(w^3).$$

Here we used *the key fact* that $p_1 = -\overline{c_{12}} = 2ia_{02}$ and $c_{03} = |c_{12}|^2 = |p_1|^2$, as provided by Lemma 3.2 and Lemma 4.1.

Hence, (2.1) (2.2) hold for $\hat{F}_c$. Notice that

$$\hat{\phi}_{c,j}(z,0) = \frac{\phi_j\Big(\frac{z}{1-2i\overline{p_1}z_1}, 0\Big)}{1 + 2i\overline{p_1}f_1\Big(\frac{z}{1-2i\overline{p_1}z_1}, 0\Big) - i|p_1|^2 g\Big(\frac{z}{1-2i\overline{p_1}z_1}, 0\Big)}.$$

Then from the normalization in (3.1) for F, it follows easily that $(\hat{\phi}_j)_c = z_1 z_j + o_{wt}(2)$. In a similar manner, we can verify that $\hat{f}_{1,c} = z_1 + \frac{i}{2}z_1 w + o_{wt}(3)$ and $\hat{f}_{1,j} = z_j + o_{wt}(3)$ for $j \geq 2$. Therefore, $\hat{F}_c$ satisfies the normalization in (3.1) and $(\hat{g}_c)'''_{www}(0) = 0$. This completes the proof of Lemma 5.1. □

Now, since $\mathrm{Im}(\hat{g}_c(0, w)) \geq 0$ for $\mathrm{Im}\, w \geq 0$ and $\hat{g}_c(0, w) = w + o(|w|^3)$, by a generalized version of the Hopf Lemma due to Burns-Krantz [BK], it follows that $\hat{g}_c(0, w) \equiv w$. Indeed, consider the harmonic function $h(w) = \mathrm{Im}\left(\frac{1}{w} - \frac{1}{\hat{g}_c(0', w)}\right)$ over the upper half plane $H^+ = \{w \in \mathbf{C}^1 : \mathrm{Im} w > 0\}$. Then it is easy to verify that $h(w) = o(|w|)$ as $w \to 0$ and $\underline{\lim}_{w(\in H^+)\to x\in(\mathcal{R}\cup\infty)} h(w) \geq 0$. Hence 0 is the minimum value of $h(w)$. By the Hopf lemma, it follows that $h(w) \equiv 0$. Namely, $\hat{g}_c(0', w) \equiv w$. Since $\mathrm{Im}(\hat{g}_c(0, w)) = |\widetilde{\hat{f}_c}(0, w)|^2$ for $\mathrm{Im}\, w = 0$, it follows that $\widetilde{\hat{f}_c}(0, w) \equiv 0$. Hence for the $\hat{F}_c$ defined above, we have the following weighted expansion:

$$\begin{cases} \hat{F}_c(0, w) = (0, w), \\ \hat{f}_{1,c} = z_1 + \frac{i}{2}z_1 w + z_1\widetilde{a^{(1)}}(z)w + o_{wt}(4), \\ \hat{f}_{l,c} = z_l + o_{wt}(4),\ 2 \leq l \leq n-1, \\ \hat{\phi}_{j,c} = z_1 z_j + b_j z_1 w + b_j^{(3)}(z) + o_{wt}(3),\ 1 \leq j \leq n-1, \\ \hat{g}_c = w + o(|(z, w)|^3). \end{cases} \tag{5.3}$$

Here we used the fact that for the $c_{03} = \frac{1}{6}(\hat{g}_c)'''_{www}(0)$, $c_{12} = (\hat{g}_c)'''_{wwz_1}(0)$, and $a_{02} = \frac{1}{2}(\hat{f}_{1,c})''_{ww}(0)$ corresponding to $\hat{F}_c$, we have $|c_{12}|^2 = c_{03} = 0$ (by Lemma 4.1) and $2i\overline{a_{02}} = c_{12} = 0$ (by Lemma 3.2).

We will show in the rest of this section that F has degree two, by applying some basic Segre family theory.

We write $\mathcal{L}_j$ for the complexification of L_j, namely, $\mathcal{L}_j = 2i\overline{\zeta_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$. Then $\{\mathcal{L}_1, ..., \mathcal{L}_{n-1}\}$ form a basis of the holomorphic tangent space of the Segre variety $Q_{(\zeta,\eta)} = \left\{(z, w) : \frac{w-\overline{\eta}}{2i} = \sum z_j\overline{\zeta_j}\right\}$ for any (ζ, η).

Lemma 5.2: *When restricted to $Q_0 = \{w = 0\}$, the complex tangent space of $\partial\mathbf{H}_n$ at 0, $\hat{F}_c = (f_{j,c}, \phi_{j,c}, \hat{g}_c)$ has degree 2. Namely, $\hat{F}_c(z, 0) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials in z with $deg(P), deg(Q) \le 2$.*

Proof of Lemma 5.2: Along any $Q_{(\zeta,\eta)}$, we have

(5.4)
$$\frac{\hat{g}_c(z, w) - \overline{\hat{g}_c(\zeta, \eta)}}{2i} = \sum_{l=1}^{n-1} \hat{f}_{l,c}(z, w)\overline{\hat{f}_{l,c}(\zeta, \eta)} + \sum_{l=1}^{n-1} \hat{\phi}_{l,c}(z, w)\overline{\hat{\phi}_{l,c}(\zeta, \eta)}.$$

Applying $\mathcal{L}_j$ and $\mathcal{L}_1\mathcal{L}_j$ to the above equation, using (5.3) and letting $(z, w) = 0,\ \eta = 0$, we get

$$\begin{pmatrix} \overline{\zeta_1} \\ \cdots \\ \overline{\zeta_{n-1}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ A_{(n-1)\times(n-1)} & B_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} \overline{\hat{f}_c(\zeta, 0)} \\ \overline{\hat{\phi}_c(\zeta, 0)} \end{pmatrix}.$$

Here $I_{(n-1)\times(n-1)}$ is the identical $(n-1) \times (n-1)$ matrix,

$$A_{(n-1)\times(n-1)} = A = \begin{pmatrix} -2\overline{\zeta_1} & 0 & \cdots & 0 \\ -\overline{\zeta_2} & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 0 \\ -\overline{\zeta_{n-1}} & 0 & \cdots & 0 \end{pmatrix} \ and$$

$$B_{(n-1)\times(n-1)} = B = \begin{pmatrix} 2 + 4ib_1\overline{\zeta_1} & 4ib_2\overline{\zeta_1} & \dots & 4ib_{n-1}\overline{\zeta_1} \\ 2ib_1\overline{\zeta_2} & 1 + 2ib_2\overline{\zeta_2} & \dots & 2ib_{n-1}\overline{\zeta_2} \\ \dots & \dots & \dots & \dots \\ 2ib_1\overline{\zeta_{n-1}} & 2ib_2\overline{\zeta_{n-1}} & \dots & 1 + 2ib_{n-1}\overline{\zeta_{n-1}} \end{pmatrix}.$$

Briefly, one has

$$\overline{\tilde{\hat{f}}_c(\zeta, 0)}^t = C^{-1}\begin{pmatrix} \overline{\zeta}^t \\ 0 \end{pmatrix} \ where\ C = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix},\ \overline{\zeta} = (\overline{\zeta_1}, \dots, \overline{\zeta_{n-1}}).$$

Next we notice that $C^{-1} = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix}$. Hence

$$\overline{\tilde{\hat{f}}_c(\zeta, 0)}^t = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix}\begin{pmatrix} \overline{\zeta}^t \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\zeta}^t \\ -B^{-1}A\overline{\zeta}^t \end{pmatrix}.$$

It remains to study $B^{-1}A\overline{\zeta}^t$. We write $B = D + \widetilde{B}$ with

$$D = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \ and\ \widetilde{B} = \begin{pmatrix} 4i\overline{\zeta_1} \\ 2i\overline{\zeta_2} \\ \dots \\ 2i\overline{\zeta_{n-1}} \end{pmatrix}\left(b_1, b_2, \dots, b_{n-1}\right).$$

Write $B^{-1} = (D + \widetilde{B})^{-1} = (I + B^*)^{-1}D^{-1}$, where $B^* = 2i\overline{\zeta}^t \cdot \mathbf{b}$ and $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$. Note that $B^{*2} = (2i)l(\zeta)B^*$ with $l(\zeta) = \sum_{j\geq 1} b_j\overline{\zeta_j}$, $B^{*3} = (2i)^2l^2(\zeta)B^*, \cdots$. We have

$$B^{-1} = \left(\sum_{j=0}^{\infty}(-1)^j B^{*j}\right)D^{-1} = \left(I - \sum_{j=0}^{\infty}(-1)^j(2i)^j l(\zeta)^j B^*\right)D^{-1}$$

$$= \left(I - \tfrac{1}{1+2il(\zeta)}B^*\right)D^{-1},$$

$$-B^{-1}A\overline{\zeta}^t = \left(I - \tfrac{2i\overline{\zeta}^t\cdot\mathbf{b}}{1+2il(\zeta)}\right)\begin{pmatrix}\overline{\zeta_1}^2\\ \overline{\zeta_1\zeta_2}\\ \overline{\cdots}\\ \overline{\zeta_1\zeta_{n-1}}\end{pmatrix}, \ \textit{and}$$

$$\overline{\widetilde{\hat{f}_c}}^t(\zeta, 0) = \begin{pmatrix}\overline{\zeta}^t\\ \left(I - \tfrac{2i\overline{\zeta}^t\cdot\mathbf{b}}{1+2il(\zeta)}\right)\overline{\zeta_1\zeta}^t\end{pmatrix} = \begin{pmatrix}\overline{\zeta}^t\\ \tfrac{\overline{\zeta_1\zeta}^t}{1+2il(\zeta)}\end{pmatrix}.$$

Therefore

$$\widetilde{\hat{f}_c}(z, 0) = \left(z,\ \frac{z_1 z}{1 - 2i\sum_{j\geq 1}\overline{b_j}z_j}\right). \tag{5.5}$$

Finally, by putting $z = w = \eta = 0$ in (5.4), we get $\overline{\hat{g}_c(\zeta, 0)} = 0$ by (5.5). Hence, it is clear that $\hat{F}_c(z, 0)$ can be written as the quotient of a vector-valued quadratic polynomial with a linear function. □

For any rational map $H \not\equiv 0$, write $H = \frac{(P_1, \cdots, P_m)}{R}$, where P_j, R are holomorphic polynomials and $(P_1, \cdots, P_m, R) = 1$. We then define

$$\deg(H) = \max(\deg(P_j)_{j=1,\cdots,m}, \deg(R)).$$

(When $H \equiv 0$, we set $\deg(H) = -\infty$.) For any $q = (q_1, \cdots, q_n) \approx 0$, we define the degree of $H|_{Q_q}$ to be the degree of the rational mapping $H(z_1, \cdots, z_{n-1}, q_n + 2i\sum_{j=1}^{n-1}\overline{q_j}z_j)$ in z. Recall that H is said to be linear if $\deg(H) = 1$. Notice that any automorphism of $\mathbf{H}_m$ is linear and has degree 1.

Lemma 5.3: *Assume that F takes the normal form as in (3.1). Then $\deg F = 2$.*

Proof of Lemma 5.3: For each $p_0 = (z_0, w_0) \in \partial\mathbf{H}_n$ close to the origin, starting from $F^{**}_{p_0}$, we can similarly construct the map $(\hat{F^{**}_{p_0}})_c$ such that Lemma 5.2 is applicable to $(\hat{F^{**}_{p_0}})_c$. Notice that there is a $\tau \in \mathrm{Aut}(\mathbf{H}_{2n-1})$ and $\sigma \in \mathrm{Aut}(\mathbf{H}_n)$ such that $\tau \circ (F^{**}_{p_0})_c \circ \sigma = F \circ \sigma_{p_0}$ by the way $F^{**}_{p_0}$ and

$\widehat{(F^{**}_{p_0})}_c$ were constructed, where $\sigma(0) = 0$ and $\sigma_{p_0}(z, w) = (z+z_0, w+w_0+2i < z, \overline{z_0} >)$ as defined in Sect. 2. Also notice that $\sigma_{p_0}(Q_0) = Q_{p_0}$ and $\sigma_{p_0}|_{Q_0} = (z+z_0, w_0+2i\langle z, \overline{z_0}\rangle)$. Since $\deg(\widehat{F^{**}_{p_0}})_c|_{Q_0} = 2$ by Lemma 5.2 and since $\sigma(Q_0) = Q_0$, it is clear that $\deg((\widehat{F^{**}_{p_0}})_c \circ \sigma)|_{Q_0} = 2$. Since τ is a linear map, it follows easily that $\deg\Big((\tau\circ(\widehat{F^{**}_{p_0}})_c\circ \sigma)|_{Q_0}\Big) = \deg((\widehat{F^{**}_{p_0}})_c \circ \sigma)|_{Q_0} = 2$. Namely, we proved that

$$\forall p(\approx 0) \in \partial\mathbf{H}_n, \quad \deg(F)\big|_{Q_p} = 2.$$

Now the rest of the proof follows from the following lemma. □

Lemma 5.4: *Let $H = \frac{(P_1,\cdots,P_m)}{R}$ be a rational map from $\mathbf{C}^n$ into $\mathbf{C}^m$, where P_j, R are holomorphic polynomials with $(P_1, \cdots, P_m, R) = 1$ $(m > n > 1)$. Assume for each $p \in \partial\mathbf{H}_n$ close to the origin, $deg(H|_{Q_p}) \leq k$ with $k > 0$ a fixed integer. Then $deg(H) \leq k$.*

Proof of Lemma 5.4: Seeking a contradiction, suppose $deg(H) \geq k+1$. Consider the irreducible decomposition of the affine algebraic variety: $Zero(P, R) = \{P_1 = \cdots = P_m = R = 0\} := Z_1 \cup Z_2 \cup \ldots \cup Z_l$. Since $(P_1, \cdots, P_m, R) = 1$, $Z'_j s$ are of at least codimension two in $\mathbf{C^n}$.

Notice that for any polynomial $h(z, w) = \sum_{|\alpha|+j\leq s} a_{\alpha j} z^\alpha w^j$ of degree $s > 0$, there is a proper real analytic subvariety S of $\partial\mathbf{H}^n$ such that for any $p \in \partial\mathbf{H}^n \setminus S$, $\deg(h|_{Q_p}) = s$. Indeed, note that $h|_{Q_p} = \sum_{|\alpha|+j=s} a_{\alpha j} z^\alpha (2i \sum_{j=1}^{n-1} \overline{p_j} z_j)^j +$lower order terms. Hence, there is a proper complex analytic subvariety A of $\mathbf{C}^{n-1}$ such that

$$\sum_{|\alpha|+j=s} a_{\alpha j} z^\alpha (2i \sum_{j=1}^{n-1} \overline{p_j} z_j)^j \not\equiv 0$$

for any $(p_1, \cdots, p_{n-1}) \notin A$. Now, S can be taken as $(A \times \mathbf{C}) \cap \partial\mathbf{H}_n$.

Back to our H, we thus can find a real analytic subvariety S_0 inside $\partial\mathbf{H}_n$ such that $\deg(P_j) = \deg(P_j|_{Q_p})$ and $\deg(R) = \deg(R|_{Q_p})$ for any $p \in \partial\mathbf{H}^n \setminus S_0$. For such a p, from the hypotheses, we conclude that $\{P_1|_{Q_p}, \cdots, P_m|_{Q_p}, R|_{Q_p}\}$ must have a (non-constant) common polynomial factor H^*_p, whose zero induces a complex analytic subvariety, denoted by $Z(p)$, in Q_p. Since $Z(p) \subset Zero(P|_{Q_p}, R|_{Q_p})$ and $Z(p)$ is of codimension one in Q_p, there must be some irreducible component Z_{j_0} with $Z_{j_0} \cap Q_p$ containing an irreducible component of maximum dimension of $Z(p)$. (Here j_0 may depend on p). Since the irreducible variety Z_{j_0} is of at least codimension 2 in $\mathbf{C^n}$ and since $Z_{j_0} \cap Q_p$ has to be of codimension 1 in Q_p, it yields easily that $Z_{j_0} \subset Q_p$.

Since there are only finitely many choices of such Z_{j_0}'s and since for any Z_j the set $\{p : Z_j \subset Q_p\}$ is a closed subset in $\partial\mathbf{H}_n \setminus S$, we see that there is a non-empty open subset $U \subset \partial\mathbf{H}_n$ such that for any $p \in U$,

Q_p contains a certain fixed Z_j, which has codimension two in $\mathbf{C}^n$. This is clearly a contradiction. Indeed, if not, pick a point $z^* \in Z_j$. Since $z \in Q_{z^*}$ if and only if $z^* \in Q_z$, we would have Q_{z^*} containing an open piece of $\partial \mathbf{H}_n$. That is impossible. □

6. Completion of the proof of Theorem 1

We now give the proof of the main theorem. First we let F be normalized to $\hat{F}_c$ as in Lemma 5.1. Still write $F = (f, \phi, g)$. Recall that $deg(F) = 2$, $F(0, w) = (0, w)$, $f(z, 0) = z$, $\phi(z, 0) = \frac{z_1 z}{1-2iR(z)}$ with $R(z) = \sum_{j=1}^{n-1} \overline{b_j} z_j$, $g = w + O(|(z, w)|^4)$ from (5.5) and (5.3). We first show that F is partially linear:

Lemma 6.1: *It holds that $f_l = z_l$ for $l \geq 2$, $g = w$, $\phi_j = z_1 \widetilde{\phi_j}$, $f_1 = z_1 \widetilde{f_1}$, where $\Phi = (\widetilde{f_1}, \widetilde{\phi_1}, \ldots, \widetilde{\phi_{n-1}})$ extends as a biholomorphic map from $\mathbf{H}_n$ to $\mathbf{B}^n$.*

Proof of Lemma 6.1: Write $\widetilde{g} = g - w = \frac{P}{Q}$ with $deg(P), deg(Q) \leq 3$. Then $\widetilde{g} Q = P$. Assume that $P \not\equiv 0$. Since $deg(P) \leq 3$, there is an index α with $|\alpha| \leq 3$ such that $D^\alpha P|_0 \neq 0$. However, $D^\beta(\widetilde{g})|_0 \equiv 0$ for any $|\beta| \leq 3$ because $g = w + O(|(z, w)|^4)$. This is a contradiction. Hence $g \equiv w$.

Next by (5.5), (5.3), Lemma 5.3 and the fact that $F(0, w) = (0, w)$, ϕ_l takes the form:

$$\phi_l = \frac{z_1 z_l + b_l z_1 w}{1 - 2iR(z) + B^{(0)} w + \widetilde{B^{(1)}(z)} w + \widetilde{B^{(0)}} w^2}.$$

In particular, we get $\phi_l(0, z', w) \equiv 0$, where $z' = (z_2, \ldots, z_{n-1})$.

Now, returning to the equation $\mathrm{Im}\, g = |f|^2 + |\phi|^2$ and letting $z_1 = 0$, we get

$$\mathrm{Im}(w) = |f(0, z', w)|^2, \quad \text{when } \mathrm{Im}(w) = |z'|^2. \tag{6.0}$$

We claim that

$$f(0, z', w) \equiv (0, z'). \tag{6.1}$$

In fact, when $n \geq 4$, by (6.0) and [We], the map $(f(0, z', w), w)$ is a linear map. Since $f_1(0, z', w) = o(|(z', w)|)$, it follows that $f_1(0, z', w) \equiv 0$. Since $f_l(0, z', w) = z_l + o_{wt}(4)$ by (5.3), it follows easily that $f_l(0, z', w) \equiv z_l$ for $l > 1$.

When $n = 3$, by Lemma 5.3, the map $(f(0, z_2, w), w) : \mathbf{H}_2 \to \mathbf{H}_3$ can be written as $(\frac{P_1}{Q}, \frac{P_2}{Q})$ where P_1, P_2 and Q are polynomials with degree ≤ 2. Since $f(0, w) = 0$ and $f(z, 0) = z$ by (5.5), this map can be written as

$$\begin{cases} f_1(0, z_2, w) = \frac{A_1 z_2 w}{1+B_1 z_2+B_2 w+B_3 z_2^2+B_4 z_2 w+B_5 w^2}, \\ f_2(0, z_2, w) = \frac{z_2+A_2 z_2 w+A_3 z_2^2}{1+B_1 z_2+B_2 w+B_3 z_2^2+B_4 z_2 w+B_5 w^2}. \end{cases}$$

Since $f_1(0, z_2, w) = o_{wt}(4)$, it follows that $A_1 = 0$, namely $f_1(0, z_2, w) \equiv 0$. Since $|z_2|^2 = |f_2(0, z_2, w)|^2$ holds for $\mathrm{Im}(w) = |z_2|^2$ by (6.0), we can easily get $f_2(0, z_2, w) \equiv z_2$. Hence, (6.1) also holds when $n = 3$. This proves (6.1) for any $n > 2$.

Next, by Lemma 5.3, we can write

$$f_l = \frac{z_l + A_l^{(1)}(z)w + A_l^{(2)}(z) + A_l^{(0)}w + \widetilde{A_l^{(0)}}w^2}{1 + \mathcal{A}^{(1)}(z)w + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(2)}(z) + \mathcal{A}^{(0)}w + \widetilde{\mathcal{A}^{(0)}}w^2}, \quad \forall l \geq 2.$$

By (5.5) and letting $w = 0$, we have $f_l(z, 0) = \frac{z_l + A_l^{(2)}(z)}{1 + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(2)}(z)} = z_l$. It follows that $A_l^{(2)} = z_l\widetilde{\mathcal{A}^{(1)}(z)}$ and $\mathcal{A}^{(2)}(z) \equiv 0$. Since $f_l(0, w) = \frac{A_l^{(0)}w + \widetilde{A_l^{(0)}}w^2}{1 + \mathcal{A}^{(0)}(z)w + \widetilde{\mathcal{A}^{(0)}}w^2} \equiv 0$, we get $A_l^{(0)} = \widetilde{A_l^{(0)}} = 0$. Hence

$$f_l = \frac{z_l + A_l^{(1)}(z)w + z_l\widetilde{\mathcal{A}^{(1)}}(z)}{1 + \mathcal{A}^{(1)}(z)w + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(0)}w + \widetilde{\mathcal{A}^{(0)}}w^2}, \quad \forall l \geq 2.$$

By (6.1), we have

$$z_l = \frac{z_l + A_l^{(1)}(0, z')w + z_l\widetilde{\mathcal{A}^{(1)}}(0, z')}{1 + \mathcal{A}^{(1)}(0, z')w + \widetilde{\mathcal{A}^{(1)}(0, z')} + \mathcal{A}^{(0)}w + \widetilde{\mathcal{A}^{(0)}}w^2}.$$

This implies $\widetilde{\mathcal{A}^{(0)}} = 0$, $\mathcal{A}^{(1)}(z) = az_1$ and $A_l^{(1)}(z) = a_l^* z_1 + z_l\mathcal{A}^{(0)}$. So

$$f_l = \frac{z_l + \left(a_l^* z_1 + \mathcal{A}^{(0)}z_l\right)w + z_l\widetilde{\mathcal{A}^{(1)}}(z)}{1 + az_1w + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(0)}w}, \quad \forall l \geq 2.$$

Since $f_l = z_l + o_{wt}(4)$, by comparing the coefficients of the z_1w-terms, we get $a_l^* = 0$. Hence,

$$f_l = \frac{z_l + \mathcal{A}^{(0)}z_lw + z_l\widetilde{\mathcal{A}^{(1)}}(z)}{1 + az_1w + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(0)}w} = z_l - az_1z_lw + o_{wt}(4). \quad \forall l \geq 2.$$

We conclude also that $a = 0$ and thus

$$f_l = \frac{z_l + \mathcal{A}^{(0)}z_lw + z_l\widetilde{\mathcal{A}^{(1)}}(z)}{1 + \widetilde{\mathcal{A}^{(1)}(z)} + \mathcal{A}^{(0)}w} = z_l, \quad \forall l \geq 2.$$

Finally, in terms of (6.1) and the expression obtained for ϕ, we know that $f_1 = z_1\widetilde{f}_1, \phi = z_1\widetilde{\phi}$. Moreover, it is easy to conclude that $|\widetilde{f}_1|^2 + |\widetilde{\phi}|^2 \equiv 1$ over $\partial\mathbf{H}_n$. Notice that $\widetilde{\phi}_l = z_l + b_lw + o(|(z, w)|^2)$, $\widetilde{f}_1 = 1 + \frac{i}{2}w + o(|(z, w)|^3)$. $\Phi = (\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{n-1}, \widetilde{f}_1)$ is biholomorphic from $\mathbf{H}_n$ to $\mathbf{B}^n$ with $\Phi(0) = (0, \cdots, 0, 1)$, by the classical Poincaré-Tanaka-Alexander Theorem [Po], [Ta]. □

Next, return to the map $H : \mathbf{B}^n \to \mathbf{B}^{2n-1}$ by letting $H = \Psi_{2n-1} \circ F \circ \Psi_n^{-1}$, where

$$\Psi_k(z, w)(z, w) = \left(\frac{2z}{i+w}, \frac{i-w}{i+w}\right)$$

is the Cayley transformation from $\mathbf{H}_k$ into $\mathbf{B}^k$. Then it can be easily verified that

$$H = \left(z_1\widetilde{h}_1, z_2, \ldots, z_{n-1}, z_1\widetilde{h}_2, \ldots, z_1\overline{h_{n-1}}, w\right).$$

Here $(\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h_{n-1}}) \in \mathrm{Aut}(\mathbf{B}^n)$.

Summarizing what we did so far, we get

Proposition 6.2: *Any non-linear rational proper holomorphic map from $\mathbf{B}_n$ into $\mathbf{B}_{2n-1}$ with $n > 2$ is equivalent to a map of the form: $H(z) := (z_1, \cdots, z_{n-1}, z_n h(z))$ with $h \in Aut(\mathbf{B}_n)$.*

Next, we show the following:

Lemma 6.3: *Let $H = (z_1, z_2, \ldots, z_{n-1}, z_n h_1, z_n h_2, \ldots, z_n h_n)$ be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$ with $(h_1, h_2, \ldots, h_n) \in Aut(\mathbf{B}^n)$. Then F is equivalent to the Whitney map.*

Proof of Lemma 6.3: We first remark that in Lemma 6.3, we need only to assume that $n \geq 2$.

Let $h = (h_1, \ldots, h_n)$ be such that $h(0) = p_0$. Then there is a unitary matrix U such that $h(0)U = (0, 0, \ldots, 0, b)$ with $1 > b \geq 0$. Replacing H by $(z', z_n hU)$ where $z' = (z_1, \ldots, z_{n-1})$, we can assume that $h(0) = (0, 0, \ldots, , b)$ *with* $b \in [0, 1)$. Hence $h = \varphi_b \cdot \widetilde{U}$, where

$$\varphi_b(z', z_n) = \left(\frac{s_b z'}{1 - bz_n}, \frac{b - z_n}{1 - bz_n}\right) \tag{6.2}$$

$s_b = \sqrt{1 - b^2}$ and $\widetilde{U}$ is a certain unitary matrix. Therefore we can assume, without loss of generality, that

$$H = (z', z_n\varphi_b) \ \ with\ b \in [0, 1). \tag{6.3}$$

On the other hand, starting from the Whitney map $W(z', z_n) = (z', z'z_n, z_n^2)$ and for any $a \in [0, 1)$,

$$W \circ \varphi_a = \left(\frac{s_a z'}{1 - az_n}, \frac{s_a z'(a - z_n)}{(1 - az_n)^2}, \frac{(a - z_n)^2}{(1 - az_n)^2}\right).$$

We take $\widetilde{\varphi_{a^2}} \in \mathrm{Aut}(\mathbf{B}^{2n-1})$ as follows:

$$\widetilde{\varphi_{a^2}}\left(z^{*\prime}, z^*_{2n-1}\right) = \left(\frac{s_{a^2} z^{*\prime}}{1 - a^2 z^*_{2n-1}}, \frac{a^2 - z^*_{2n-1}}{1 - a^2 z^*_{2n-1}}\right),$$

where $z^{*\prime} = (z_1^*, \ldots, z_{2n-2}^*)$. Then

$$\widetilde{\varphi_{a^2}} \circ W \circ \varphi_a = \left(\frac{s_{a^2} z'(1 - az_n)}{s_a(1+a^2-2az_n)}, \frac{s_{a^2} z'(a - z_n)}{s_a(1+a^2-2az_n)}, \frac{2az_n - (1+a^2)z_n^2}{1 + a^2 - 2az_n} \right).$$

Let U_a be a $(2n-1) \times (2n-1)$ unitary matrix given by

$$U_a := \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $A = \frac{1}{\sqrt{1+a^2}} I$, $B = -\frac{a}{\sqrt{1+a^2}} I$, $C = \frac{a}{\sqrt{1+a^2}} I$, $D = \frac{1}{\sqrt{1+a^2}} I$ and I is the $(n-1) \times (n-1)$ identity matrix. Define

$$\Psi := U_a \circ \widetilde{\varphi_{a^2}} \circ W \circ \varphi_a = (I, II, III),$$

where

$$\begin{cases} I = \frac{z'(1-a^2)}{1+a^2-2az_n}, \\ II = \frac{z'[2a-(a^2+1)z_n]}{1+a^2-2az_n}, \\ III = \frac{z_n[2a-(a^2+1)z_n]}{1+a^2-2az_n}. \end{cases}$$

Put $\beta := \frac{2a}{1+a^2}$. Then Ψ can be written as

$$\Psi = \left(\frac{s_\beta z'}{1 - \beta z_n}, \frac{z'(\beta - z_n)}{1 - \beta z_n}, \frac{z_n(\beta - z_n)}{1 - \beta z_n} \right).$$

Define $(z'^*, z_n^*) := \varphi_\beta(z', z_n)$. Then it follows that

$$\Psi \circ \varphi_\beta^{-1} = \left(z'^*, z_n^* \varphi_\beta\left(z'^*, z_n^*\right)\right). \tag{6.4}$$

Here we need to use the fact that $\varphi_\beta^{-1} = \varphi_\beta$. By choosing a suitable a such that $b = \beta$, we see that the Whitney map is equivalent to (6.5) which is the same as the map in (6.4). □

Proof of Theorem 1 and Theorem 2: The proof of Theorem 1 follows from Proposition 6.2 and Lemma 6.3, while Theorem 2 follows from Theorem 1 and Theorem 2.3. (We remark that since Lemma 3.1 holds for maps which are only assumed to be C^2-smooth up to the boundary, the proof of Theorem 2.3 can also be obtained by applying Proposition 2.2 and by an argument similar to that for the proof of Lemma 5.2). □

Appendix: The explicit calculation of an example

In this appendix, for convenience of the reader, we give an explicit example to demonstrate some of the complicated calculations performed in the paper. The computation here was verified in Maple V Release 4.

Consider the Whitney map from $\mathbf{B}^3$ to $\mathbf{B}^5$: $H(z_1, z_2, w) = (z_1^2, z_1z_2, z_1w, z_2, w)$.

Let Ψ_n be the Cayley transformation from $\mathbf{H}_n$ into $\mathbf{B}_n$ as defined in Sect. 6 right before Lemma 6.2. Considering the map $\Psi_5 \circ H \circ \Psi_3^{-1}$ and then multiplying it on the right by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we get a map F from $\mathbf{H}_3$ into $\mathbf{H}_5$:

$$F = \left(\frac{z_1(1+iw)}{1-iw}, z_2, \frac{2z_1^2}{1-iw}, \frac{2z_1z_2}{1-iw}, w \right).$$

Take a point $p_0 = (z_0, w_0) := (1, 0, 2+i) \in \partial\mathbf{H}_3$, and consider the map $F_{p_0} := F_{(z_0,w_0)} = \tau_{p_0} \circ F \circ \sigma_{p_0}$ as defined in Sect. 2. Then, it is given by:

$$\begin{cases} (f_1)_{p_0}(z, w) = \frac{-4z_1^2+2iz_1w+(-2+2i)z_1+(-1+i)w}{2[2z_1-iw+(2-2i)]}, \\ (f_2)_{p_0}(z, w) = z_2, \\ (\phi_1)_{p_0}(z, w) = \frac{4z_1^2+(6-2i)z_1+(-1+i)w}{2[2z_1-iw+(2-2i)]}, \\ (\phi_2)_{p_0}(z, w) = \frac{2(z_1+1)z_2}{2z_1-iw+(2-2i)}, \\ g_{p_0}(z, w) = \frac{-iw^2+(3-i)z_1w+3(1-i)w}{2z_1-iw+(2-2i)}. \end{cases}$$

Computing its first derivatives and choosing $(C_j)p_0$ as required in Sect. 2, we have:

$$\begin{cases} (E_1)_{p_0} = \left(-\frac{1}{2}, 0, \frac{2+i}{2}, 0\right), \\ (E_2)_{p_0} = \left(0, 1, 0, \frac{1+i}{2}\right), \\ \lambda_{p_0} = (E_1)_{p_0} \cdot \overline{(E_1)_{p_0}} = (E_2)p_0 \cdot \overline{(E_2)_{p_0}} = \frac{3}{2}, \\ (C_1)_{p_0} = \frac{1}{\sqrt{6}}(2-i, 0, 1, 0), \\ (C_2)_{p_0} = \sqrt{\frac{3}{2}}\left(0, \frac{-1+i}{3}, 0, \frac{2}{3}\right), \\ (E_{p_0})_w = \left(-\frac{1}{4}, 0, -\frac{1}{4}, 0\right). \end{cases}$$

Then by the definition of $F^*_{p_0}$ in Sect. 2, one obtains

$$
\begin{aligned}
f^*_{p_0} &= \left(\left(f^*_1\right)_{p_0}, \left(f^*_2\right)_{p_0}\right) = \frac{1}{\lambda_{p_0}} \widetilde{f} \cdot \overline{E}^t_{p_0} \\
&= \left(-\frac{1}{3}(f_1)_{p_0} + \frac{2-i}{3}(\phi_1)p_0,\ \frac{2}{3}(f_2)_{p_0} + \frac{1-i}{3}(\phi_2)p_0\right), \\
\phi^*_{p_0} &= \left(\left(\phi^*_1\right)_{p_0}, \left(\phi^*_1\right)_{p_0}\right) = \frac{1}{\sqrt{\lambda_{p_0}}} \widetilde{f} \cdot \overline{C}^t_{p_0} \\
&= \left(\frac{2+i}{3}(f_1)_{p_0} + \frac{1}{3}(\phi_1)_{p_0},\ \frac{-1-i}{3}(f_2)_{p_0} + \frac{2}{3}(\phi_2)_{p_0}\right), \\
g^*_{p_0} &= \frac{g_{p_0}}{\lambda_{p_0}} = \frac{2}{3} g_{p_0}.
\end{aligned}
$$

Moreover, $(f^*_{p_0})'_w|_0 = \left(\frac{-1+i}{12}, 0\right)$, $(\phi^*_{p_0})'_w|_0 = \left(\frac{-3-i}{12}, 0\right)$, $\mathbf{a} = \left(\frac{-1+i}{12}, 0, \frac{-3-i}{12}, 0\right)$, *and* $\|\mathbf{a}\|^2 = \frac{1}{12}$, and $r := \frac{1}{2} Re\left\{ \frac{\partial^2 g^*_{p_0}}{\partial w^2}\Big|_0 \right\} = -\frac{1}{12}$. Now, $F^{**}_{p_0}$ is given as follows:

$$
\left(f^{**}_1\right)_{p_0}(z, w) = \frac{12(3-i)z_1^2 + (-1-i)w^2 + (2-10i)z_1 w + 36(1-i)z_1}{(-1+i)w^2 + 2(1-i)z_1 w + 12(3-i)z_1 - 12iw + 36(1-i)},
$$

$$
\left(f^{**}_2\right)_{p_0}(z, w) = \frac{12z_2[(3-i)z_1 - iw + 3(1-i)]}{(-1+i)w^2 + 2(1-i)z_1 w + 12(3-i)z_1 - 12iw + 36(1-i)},
$$

$$
\left(\phi^{**}_1\right)_{p_0}(z, w) = \frac{-12(1+i)z_1^2 + (1-3i)w^2 + (4+12i)z_1 w}{(-1+i)w^2 + 2(1-i)z_1 w + 12(3-i)z_1 - 12iw + 36(1-i)},
$$

$$
\left(\phi^{**}_2\right)_{p_0}(z, w) = \frac{6z_2[(2-2i)z_1 + (-1+i)w]}{(-1+i)w^2 + 2(1-i)z_1 w + 12(3-i)z_1 - 12iw + 36(1-i)},
$$

$$
g^{**}_{p_0}(z, w) = \frac{12[-iw^2 + (3-i)z_1 w + 3(1-i)w]}{(-1+i)w^2 + 2(1-i)z_1 w + 12(3-i)z_1 - 12iw + 36(1-i)}.
$$

Remark that $F^{**}_{p_0}$ satisfies (2.1). With this map, we can define a new map, which satisfies the normalization in (3.1):

$$(z, w) \mapsto \left(\kappa\left(f_1^{**}\right)_{p_0}\left(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}, \frac{w}{|\kappa|^2}\right), \kappa\left(f_2^{**}\right)_{p_0}\left(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}, \frac{w}{|\kappa|^2}\right),\right.$$
$$\left.\kappa\left(\phi_1^{**}\right)_{p_0}\left(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}, \frac{w}{|\kappa|^2}\right), \kappa\left(\phi_2^{**}\right)_{p_0}\left(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}, \frac{w}{|\kappa|^2}\right), |\kappa|^2 g_{p_0}^{**}\left(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}, \frac{w}{|\kappa|^2}\right)\right),$$

where $\kappa = -\frac{i}{3}$. This new map, still denoted by $F = (f_1, f_2, \phi_1, \phi_2, g)$ for simplicity, is explicitly given by:

$$\text{(A.I)} \quad \begin{aligned} f_1(z, w) &= \frac{(-1+2i)z_1^2 + \frac{3}{4}w^2 + \frac{3-2i}{2}z_1 w + z_1}{-\frac{9}{4}w^2 + \frac{3i}{2}z_1 w + (-1+2i)z_1 + \frac{3(1-i)}{2}w + 1}, \\ f_2(z, w) &= \frac{(-1+2i)z_1 z_2 + \frac{3(1-i)}{2}z_2 w + z_2}{-\frac{9}{4}w^2 + \frac{3i}{2}z_1 w + (-1+2i)z_1 + \frac{3(1-i)}{2}w + 1}, \\ \phi_1(z, w) &= \frac{z_1^2 + \frac{3(-1-2i)}{4}w^2 + (-1+2i)z_1 w}{-\frac{9}{4}w^2 + \frac{3i}{2}z_1 w + (-1+2i)z_1 + \frac{3(1-i)}{2}w + 1}, \\ \phi_2(z, w) &= \frac{z_1 z_2 + \frac{3i}{2}z_2 w}{-\frac{9}{4}w^2 + \frac{3i}{2}z_1 w + (-1+2i)z_1 + \frac{3(1-i)}{2}w + 1}, \\ g(z, w) &= \frac{\frac{3(1-i)}{2}w^2 + (-1+2i)z_1 w + w}{-\frac{9}{4}w^2 + \frac{3i}{2}z_1 w + (-1+2i)z_1 + \frac{3(1-i)}{2}w + 1}. \end{aligned}$$

For it, we have the following Taylor expansion:

$$\text{(A.II)} \quad \begin{aligned} f_1(z, w) &= z_1 + \frac{i}{2}z_1 w - \frac{3}{4}w^2 + (1-i)wz_1^2 + \frac{9}{4}z_2 w^2 - \\ &\quad - \left(3 + \frac{3}{2}i\right)z_1^2 z_2 w + o_{wt}(5), \\ f_2(z, w) &= z_2 - \frac{3}{2}iz_1 z_2 w + \frac{9}{4}z_2 w^2 - 3wz_2 z_1^2 - \frac{3}{2}iz_1^2 z_2 w + \\ &\quad + o_{wt}(5), \\ \phi_1(z, w) &= z_1^2 + (-1+2i)z_1 w + (1-2i)z_1^3 + o_{wt}(3), \\ \phi_2(z, w) &= z_1 z_2 + \frac{3}{2}iz_2 w + (1-2i)z_1^2 z_2 + o_{wt}(3), \\ g(z, w) &= w - \frac{3}{2}iz_1 w^2 + \frac{9}{4}w^3 - \left(3 + \frac{3}{2}i\right)z_1^2 w^2 + o_{wt}(6). \end{aligned}$$

For this map, the data defined in Lemma 3.2 and Lemma 4.1 are as follows:

$$\mu = -3i, c_{12} := \frac{1}{2}g'''_{z_1ww}(0) = -\frac{3i}{2}, a_{02} := \frac{1}{2}(f_1)''_{ww}(0) = -\frac{3}{4},$$
$$c_{03} := \frac{1}{6}g'''_{www}(0) = \frac{9}{4}, \text{ and } \gamma := \frac{1}{2}(f_2)'''_{z_2ww}(0) = \frac{9}{4}.$$

In such an example, one easily sees the following: (The reader may compare (A.III) with the results (for the general maps) in Lemma 3.2, Lemma 4.1 and (4.16))

$$\text{(A.III)} \qquad \mu = 2c_{12} = 4i\overline{a_{02}},\ c_{03} = |c_{12}|^2 \text{ and } 12c_{03} = 8\gamma + |\mu|^2.$$

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