

# Every Real Ellipsoid in $\mathbf{C}^2$ Admits CR Umbilical Points

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To the memory of Professor S. S. Chern

## 1 Introduction

By the Cartan-Chern-Moser theory [CM], the germ of a strongly pseudoconvex real analytic hypersurface  $M \subset \mathbf{C}^n$  is determined, up to a local biholomorphic map, by a set of complete invariants which can be expressed by the curvatures of a connection or the coefficients in a normal form.

When  $n \geq 3$ , the fourth-order pseudoconformal curvature tensor  $\mathcal{S}$  of Chern-Moser [CM] is of fundamental importance because it generates other invariants by differentiation. It is known that  $\mathcal{S} \equiv 0$  if and only if  $M$  is locally biholomorphic to the sphere  $\partial\mathbf{B}^n$ . When  $n = 2$ , the fourth-order curvature tensor vanishes identically and its role is played by the Cartan six-order invariant curvature tensor  $\mathcal{P}$  [Car]. Similarly,  $\mathcal{P} \equiv 0$  if and only if  $M$  is locally biholomorphic to the 3-sphere  $\partial\mathbf{B}^2$ . In both cases, a point on  $M$ , at which the Chern-Moser tensor  $\mathcal{S}$  (or the Cartan curvature tensor  $\mathcal{P}$  for the case of  $n = 2$ ) vanishes, is called a CR umbilical point, or briefly, an umbilical point ([CM]). CR umbilical points are biholomorphic differential invariants of  $M$ .

The study of CR umbilical points on a compact strongly pseudoconvex hypersurface  $M$  provides useful information for the holomorphic structure of its enclosed domain, as well as the intrinsic CR structure of  $M$  itself. However, different from the situation in the classical Differential Geometry, except in the trivial spherical case, where  $\mathcal{S}$  or  $\mathcal{P} \equiv 0$ , computing umbilical points seems to be a very difficult problem. This is because the explicit formula for the fundamental Cartan-Chern-Moser curvature tensors is too complicated. Indeed, the situation is already non-trivial even in the simplest non-spherical case— where  $M$  is a real

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ellipsoid. Recently, based on his previous work on the complex dynamics property of real ellipsoids, Webster proved the following: (See §3 for the definitions)

**Theorem 1.1**(Webster [We2]): *A generic real ellipsoid in  $\mathbf{C}^n$  with  $n \geq 3$  does not admit any umbilical point.*

Umbilical points on a certain class of real hypersurfaces of revolution were also studied by Webster [We3].

A natural question arising from [We2] is then to ask whether a generic real ellipsoid in  $\mathbf{C}^2$  shares the same property as its analogy in higher dimensions. It is indeed this problem that motivated our present work, and we provide, in this paper, the following:

**Theorem 1.1** *Every real ellipsoid  $M \subset \mathbf{C}^2$  admits at least four umbilical points.*

Theorem 1.2 resembles the classical result for the umbilical points on the ellipsoids in  $\mathbf{R}^3$  [pp222, Spv]. A famous theorem of Hamburger [Ham] states that every compact real analytic convex surface in  $\mathbf{R}^3$  admits at least two umbilical points. We do not know if there is a CR version of the Hamburger theorem. More precisely, it is an open question to us if every compact strongly convex hypersurface in  $\mathbf{C}^2$  admits at least two CR umbilical points. Notice that only for  $n = 2$ , the fundamental curvature tension reduces to a function. It may not be surprising that it is more likely to find umbilical points on a hypersurface in  $\mathbf{C}^2$  than to find umbilical points for a hypersurface in  $\mathbf{C}^n$  ( $n \geq 3$ ).

The proof of Theorem 1.2 uses Chern's inhomogeneous coordinates for the projective  $G$ -structure bundle of the Segre family of a real analytic strongly pseudoconvex hypersurface [Ch] [CJ], and a formula derived in Huang-Ji-Yau [Theorem 3.1, HJY] for the complexified Cartan fundamental curvature tension represented under these coordinates. The formula of [HJY] seems to fit particularly well with the computation here.

In the classical Differential Geometry [Spv], surfaces in  $\mathbf{R}^3$  without umbilical points must be diffeomorphic to a torus. The boundary of a small thickening of the unit circle in  $\mathbf{R}^2$  provides examples of closed surfaces without any umbilical point. However, this type of examples does not give compact CR manifolds without CR umbilical points. The following theorem gives a precise description for the set of umbilical points for the thickening of a closed real curve. It is not clear to us if there is any embeddable three dimensional compact CR manifold which has no CR umbilical points.

**Theorem 1.2** *Let  $M_\epsilon \subset \mathbf{C}^2$  be the boundary of the  $\epsilon$ -thickening of the unit circle  $\{|z| = 1, w = 0\}$  in  $\mathbf{C}^2$ , defined by the equation  $(1 - |z|)^2 + |w|^2 = \epsilon^2$ , where  $\epsilon$  is a sufficiently small positive number. Then the set of all umbilical points of  $M_\epsilon$  forms a disjoint union of a closed real analytic curve and two two-dimensional totally real analytic tori.*

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## 2 Umbilical Points of Real Hypersurfaces In $\mathbf{C}^2$

In this section, we briefly review the Cartan-Chern-Moser theory (cf. [C][HJ] [Hu]). We restrict ourselves to the case of  $n = 2$ . Let

$$M = \{(z, w) \in \mathbf{C}^2 : r(z, w, \bar{z}, \bar{w}) = 0\} \quad (1)$$

be a Levi non-degenerate smooth real analytic hypersurface with  $(z_0, w_0) \in M$ . Its complexification, called the *Segre family* of  $M$ , is then the complex three-fold

$$\mathcal{M} = \{(z, w, \zeta, \eta) \mid r(z, w, \zeta, \eta) = 0\} \subset \mathbf{C}^4.$$

Clearly  $(z_0, w_0, \bar{z}_0, \bar{w}_0) \in \mathcal{M}$ . Assume that

$$r_w(z_0, w_0, \bar{z}_0, \bar{w}_0) := \frac{\partial r}{\partial w}(z_0, w_0, \bar{z}_0, \bar{w}_0) \neq 0. \quad (2)$$

Define

$$S : \mathcal{M} \rightarrow \widetilde{\mathcal{M}} := S(\mathcal{M}) \subset \mathbf{C}^2 \times \mathbf{P}^1, \quad (z, w, \zeta, \eta) \mapsto \left( z, w, \left[ \frac{\partial r}{\partial z} : \frac{\partial r}{\partial w} \right] (z, w, \zeta, \eta) \right). \quad (3)$$

$S$  is locally biholomorphic by the Levi non-degeneracy condition. (See Proposition 4.1 of [CJ]). With the assumption in (2), we can regard  $(z, w, \zeta)$  as a local non-homogeneous coordinates system for  $\mathcal{M}$ , and we can write  $S(z, w, \zeta) = (z, w, -\frac{r_z}{r_w})$ . Then we use  $(z, w, p)$  as a local coordinates system, called the Chern coordinates system, for  $\widetilde{\mathcal{M}}$ , where

$$p = -\frac{r_z}{r_w}. \quad (4)$$

Making use of the implicit function theorem, we can find a unique holomorphic function (in its argument)  $h(z, \bar{z}, \bar{w})$  near  $(z_0, \bar{z}_0, \bar{w}_0)$  with  $h(z_0, \bar{z}_0, \bar{w}_0) = w_0$  such that  $w = h(z, \bar{z}, \bar{w})$  solves the equation:  $r(z, w, \bar{z}, \bar{w}) = 0$ . Then for  $(z, w, \zeta, \eta) \in \mathcal{M}$ , we have

$$p(z, w, \zeta) = \frac{\partial h(z, \zeta, \eta)}{\partial z}. \quad (5)$$

We have  $dp = p_{11}dz + \hat{p}_1^1 d\zeta + \hat{p}_1^2 d\eta$ , where  $p_{11} = \frac{\partial^2 h}{\partial z^2}$ ,  $\hat{p}_1^1 = \frac{\partial^2 h}{\partial \zeta \partial z}$  and  $\hat{p}_1^2 = \frac{\partial^2 h}{\partial \eta \partial z}$ ; and we have the following identity:

$$-pdz + dw - \hat{p}^1 d\zeta - \hat{p}^2 d\eta = 0 \quad (6)$$

where  $p = p_1 = \frac{\partial h}{\partial z}$ ,  $\hat{p}^1 = \frac{\partial h}{\partial \zeta}$  and  $\hat{p}^2 = \frac{\partial h}{\partial \eta}$ . Therefore, we obtain

$$dp|_{\mathcal{M}} = \left( p_{11} - \frac{\hat{p}_1^2 \hat{p}_1^1}{\hat{p}^2} \right) dz + \frac{\hat{p}_1^2}{\hat{p}^2} dw + \left( p_1^1 - \frac{p_1^2 p^1}{p^2} \right) d\zeta. \quad (7)$$

Hence, we have the following holomorphic coframe on  $\widetilde{\mathcal{M}}$ :

$$\begin{aligned} \theta &= dw - pdz = dw - p_1 dz, \\ \theta^1 &= dz, \\ \theta_1 &= \frac{\hat{p}_1^2}{\hat{p}^2} \theta + \left( \hat{p}_1^1 - \frac{\hat{p}_1^2 \hat{p}^1}{\hat{p}^2} \right) d\zeta = dp - p_{11} dz. \end{aligned}$$

We emphasize again that  $p_{11}$  is a holomorphic function in  $(z, w, p)$  near  $(z_0, w_0, p_0)$  with  $(z_0, w_0) \in M$  and  $p_0 = p(z_0, w_0, \bar{z}_0)$ ; and  $p_{11}$  is given by the following formula:

$$p_{11} = \frac{\partial^2 h}{\partial z^2} \quad (8)$$

Define the holomorphic coframes over  $\mathcal{M}$ :

$$\omega = u\theta, \quad \omega^1 = u^1\theta + u_1^1\theta^1, \quad \omega_1 = v_1\theta + v_1^1\theta_1 \quad (9)$$

where  $u, u_1^1, u^1, v_1$  are holomorphic functions with  $u = iu_1^1 v_1^1 \neq 0$ .

Now, the fundamental Cartan-Chern-Moser theory [CM] gives the following:

Let  $M = \{r = 0\} \subset \mathbf{C}^2$ ,  $(z_0, w_0) \in M$  such that (2) is satisfied and let  $\widetilde{\mathcal{M}}$  be as in (3). Let  $\widetilde{\pi} : \widetilde{\mathcal{Y}} \rightarrow \widetilde{\mathcal{M}}$  be the corresponding holomorphic projective structure bundle. Then besides the 3 holomorphic 1-forms in (9), there exist 5 more holomorphic 1-forms  $\phi, \phi_1^1, \phi^1, \phi_1, \psi$ , defined over  $\widetilde{\mathcal{Y}}$ , with holomorphic coordinates  $z, w, p, u, u_1^1, u^1, v_1, t$ , with  $u, u_1^1 \neq 0$ . These holomorphic 1-forms are  $\mathbf{C}$ -linearly independent, and satisfy the following structure equations

$$\begin{aligned} d\omega &= i\omega^1 \wedge \omega_1 + \omega \wedge \phi, \\ d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\ d\omega_1 &= \phi_1^1 \wedge \omega_1 + \omega_1 \wedge \phi + \omega \wedge \phi_1, \\ d\phi &= i\omega^1 \wedge \phi_1 + i\phi^1 \wedge \omega_1 + \omega \wedge \psi, \\ d\phi_1^1 &= i\omega_1 \wedge \phi^1 - 2i\phi_1 \wedge \omega^1 - \frac{1}{2}\psi \wedge \omega, \\ d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \frac{1}{2}\psi \wedge \omega^1 + L^{11}\omega \wedge \omega_1, \\ d\phi_1 &= \phi_1^1 \wedge \phi_1 - \frac{1}{2}\psi \wedge \omega_1 + P_{11}\omega \wedge \omega^1, \\ d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \phi_1 + H_1\omega \wedge \omega^1 + K^1\omega \wedge \omega_1. \end{aligned} \quad (10)$$

All of these forms  $\omega, \omega^1, \omega_1, \phi, \phi_1^1, \phi^1, \phi_1$  and  $\psi$ , as well as all of the curvature functions  $L^{11}, P_{11}, H_1$  and  $K^1$ , have been calculated explicitly in [Theorem 3.1, HJY]. In particular, we have

$$\begin{aligned}
L^{11} &= -\frac{i(u_1^1)^2}{6w^3} \frac{\partial^4 p_{11}}{\partial p^4}, \\
P_{11} &= \frac{i}{u(u_1^1)^2} \left[ \frac{\partial^2 p_{11}}{\partial w^2} - \frac{1}{2} \frac{\partial p_{11}}{\partial w} \frac{\partial^2 p_{11}}{\partial p^2} + \frac{2}{3} \frac{\partial p_{11}}{\partial p} \frac{\partial^2 p_{11}}{\partial p \partial w} + \frac{p_{11}}{6} \frac{\partial^3 p_{11}}{\partial p^2 \partial w} \right. \\
&\quad - \frac{1}{6} \frac{\partial p_{11}}{\partial p} \left( \frac{\partial^3 p_{11}}{\partial p^2 \partial z} + p \frac{\partial^3 p_{11}}{\partial p^2 \partial w} \right) - \frac{2}{3} \left( \frac{\partial^3 p_{11}}{\partial z \partial w \partial p} + p_{11} \frac{\partial^3 p_{11}}{\partial p^2 \partial w} + p \frac{\partial^3 p_{11}}{\partial p \partial w^2} \right) \\
&\quad + \frac{1}{6} \left( \frac{\partial^4 p_{11}}{\partial p^2 \partial z^2} + p_{11} \frac{\partial^4 p_{11}}{\partial p^3 \partial z} + p \frac{\partial^4 p_{11}}{\partial p^2 \partial z \partial w} \right) + \frac{1}{6} \frac{\partial^3 p_{11}}{\partial p^3} \left( \frac{\partial p_{11}}{\partial z} + p \frac{\partial p_{11}}{\partial w} \right) \\
&\quad \left. + \frac{p_{11}}{6} \left( \frac{\partial^4 p_{11}}{\partial z \partial p^3} + p_{11} \frac{\partial^4 p_{11}}{\partial p^4} + p \frac{\partial^4 p_{11}}{\partial p^3 \partial w} \right) + \frac{p}{6} \left( \frac{\partial^4 p_{11}}{\partial z \partial p^2 \partial w} + p_{11} \frac{\partial^4 p_{11}}{\partial p^3 \partial w} + p \frac{\partial^4 p_{11}}{\partial p^2 \partial w^2} \right) \right]. \tag{11}
\end{aligned}$$

On the CR structure bundle  $\hat{Y}$  over  $\hat{M} = S(\{(z, w, \bar{z}, \bar{w}) : (z, w) \in M\})$ , there are  $\mathbf{R}$ -linearly independent 1-forms  $\omega, \omega^1, \bar{\omega}^1, \phi_1^1, \phi = \phi_1^1 + \bar{\phi}_1^1, \phi^1, \bar{\phi}^1, \psi$  satisfying the structure equations

$$\begin{aligned}
d\omega &= i\omega^1 \wedge \bar{\omega}^1 + \omega \wedge \phi, \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\
d\phi_1^1 &= i\bar{\omega}^1 \wedge \phi^1 - 2i\bar{\phi}^1 \wedge \omega^1 - \frac{1}{2}\psi \wedge \omega, \\
d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \frac{1}{2}\psi \wedge \omega^1 + \hat{L}^{11}\omega \wedge \bar{\omega}^1, \\
d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \bar{\phi}_1^1 + (-\hat{H}_1\omega^1 - \overline{\hat{H}_1\omega^1}) \wedge \omega. \tag{12}
\end{aligned}$$

It is known that the projective connection underlines the CR connection [C] [F]. Hence the structure equations (10), when restricted on  $\hat{Y}$ , reduce to (12). Consequently,  $\hat{L}^{11} = L^{11}|_{\hat{Y}} = \overline{P_{11}}|_{\hat{Y}}$ .  $\hat{L}^{11}$ , when pulled back to  $(Y, \pi, M)$ , is the Cartan fundamental curvature function. Hence,  $(z_0, w_0) \in M$  is an *umbilical point* if and only if  $L^{11}|_{\hat{Y}} = 0$  along the fiber  $\hat{\pi}^{-1}(z_0, w_0)$ , where  $\hat{\pi} : \hat{Y} \rightarrow \hat{M}$  is the natural projection. Notice that  $(z_0, w_0)$  is an umbilical point of  $M$  if and only if there is a biholomorphic change of coordinates under which  $(z_0, w_0)$  is mapped to the origin and  $\hat{M}$  is defined by an equation of the form:  $\text{Im}(w) = |z|^2 + o(|z|^6)$ . (See [CM]).

From (11), we notice that  $L^{11}$  vanishes at a point in the fiber  $\hat{\pi}^{-1}(S(z_0, w_0, \bar{z}_0, \bar{w}_0))$  if and only if  $L^{11}$  vanishes along the whole fiber  $\hat{\pi}^{-1}(S(z_0, w_0, \bar{z}_0, \bar{w}_0))$ . Since  $u \neq 0, u_1^1 \neq 0$  in (11), we obtain

**Theorem 2.1** *Let  $M = \{r = 0\} \subset \mathbf{C}^2$ . Let  $r$  and  $(z_0, w_0) \in M$  be as in (1). Assume that*

(2) is satisfied. Then  $(z_0, w_0) \in M$  is an umbilical point if and only if

$$\frac{\partial^4 p_{11}}{\partial p^4}(z_0, w_0, p_0) = 0$$

where  $p_0 = -\frac{r_z}{r_w}(z_0, w_0, \bar{z}_0, \bar{w}_0)$ .

### 3 Umbilical points of ellipsoids in $\mathbf{C}^2$

Recall that a real ellipsoid  $M \subset \mathbf{C}^n$  is the image of the unit sphere  $\partial\mathbf{B}^n$  under a real-affine transformation of  $\mathbf{R}^{2n} := \mathbf{C}^n$ . It is known [We1] that after a holomorphic affine transformation, any real ellipsoid is given by an equation of the form:  $\sum_{j=1}^n (A_j x_j^2 + B_j y_j^2) = 1$  where  $A_j \geq B_j > 0$  and  $z_j = x_j + iy_j$ . The complex structure of ellipsoids was first studied by Webster in his famous paper [We1]. He showed that when  $n \geq 2$ , two ellipsoids are biholomorphically equivalent if and only if the set of ratios  $(A_j - B_j)/(A_j + B_j)$  is the same for the two. Hence any ellipsoid  $M$  can be made into the form:

$$\sum_{j=1}^n \left( (1 + a_j)x_j^2 + y_j^2 \right) = 1 \quad (13)$$

where  $a_j \geq 0$ . Notice that  $M$  is spherical if and only if  $a_j = 0$  for all  $j$ . In particular, after a holomorphically linear change of coordinates, any ellipsoid  $M$  in  $\mathbf{C}^2$  can be given by

$$(1 + a_1)x_1^2 + y_1^2 + (1 + a_2)x_2^2 + y_2^2 = 1, \quad a_1, a_2 \geq 0; \quad (14)$$

or equivalently,

$$a_1 z^2 + a_1 \bar{z}^2 + 2(2 + a_1)z\bar{z} + a_2 w^2 + a_2 \bar{w}^2 + 2(2 + a_2)w\bar{w} = 4. \quad (15)$$

We notice from (14) that  $M$  can be parameterized by three real parameters  $\alpha, \beta \in [0, 2\pi]$ ,  $c \in [0, 1]$  through the following equation:

$$z = \frac{c}{\sqrt{1+a_1}} \cos \alpha + i c \sin \alpha, \quad w = \frac{\sqrt{1-c^2}}{\sqrt{1+a_2}} \cos \beta + i \sqrt{1-c^2} \sin \beta \quad (16)$$

In fact, for any  $c \in [0, 1]$ , consider  $w = x_2 + iy_2$  with  $(1 + a_2)x_2^2 + y_2^2 = 1 - c^2$ . Then  $w = \frac{\sqrt{1-c^2}}{\sqrt{1+a_2}} \cos \beta + i \sqrt{1-c^2} \sin \beta$  for  $\beta \in [0, 2\pi]$ . Since  $(1 + a_1)x_1^2 + y_1^2 = c^2$ , the formula for  $z = x_1 + iy_1 = \frac{c}{\sqrt{1+a_1}} \cos \alpha + i c \sin \alpha$  follows.

Complexifying (15), we obtain the Segre family  $\mathcal{M} \subset \mathbf{C}^2 \times \mathbf{C}^2$  of  $M$ , defined by the equation:

$$a_1 z^2 + a_1 \zeta^2 + 2(2 + a_1)z\zeta + a_2 w^2 + a_2 \eta^2 + 2(2 + a_2)w\eta = 4. \quad (17)$$

Choose the defining function of  $M$  to be  $r := a_1 z^2 + a_1 \bar{z}^2 + 2(2 + a_1)z\bar{z} + a_2 w^2 + a_2 \bar{w}^2 + 2(2 + a_2)w\bar{w} - 4$ . Then a point  $(z, w)$  satisfies (2) if and only if  $a_2 w + (2 + a_2)\bar{w} \neq 0$ . By (16), this is equivalent to the condition that  $c \neq 1$ , or equivalently,  $w \neq 0$ . We assume

$$c \neq 1, \quad \text{i.e., } w \neq 0. \quad (18)$$

Then making use of the implicit function theorem, we have a unique function  $w = h(z, \bar{z}, \bar{w})$ , which solves the equation  $r = 0$  near the point under study. Applying  $\frac{\partial}{\partial z}$  and  $\frac{\partial^2}{\partial z^2}$  to (17), we get  $a_1 z + (2 + a_1)\zeta + a_2 w \frac{\partial h}{\partial z} + (2 + a_2)\eta \frac{\partial h}{\partial z} = 0$  and  $a_1 + a_2 \left( \frac{\partial h}{\partial z} \right)^2 + a_2 w \frac{\partial^2 h}{\partial z^2} + (2 + a_2)\eta \frac{\partial^2 h}{\partial z^2} = 0$ . Since  $p = \frac{\partial h}{\partial z}$  and  $p_{11} = \frac{\partial^2 h}{\partial z^2}$  on  $\mathcal{M}$ , we obtain

$$a_1 z + (2 + a_1)\zeta + a_2 w p + (2 + a_2)\eta p = 0 \quad \text{and} \quad (19)$$

$$a_1 + a_2 p^2 + a_2 w p_{11} + (2 + a_2)\eta p_{11} = 0. \quad (20)$$

At the point  $(z, w, \bar{z}, \bar{w}) \in \mathcal{M}$ , we then have

$$p = -\frac{a_1 z + (2 + a_1)\bar{z}}{a_2 w + (2 + a_2)\bar{w}}. \quad (21)$$

Now, we can use (17-20) to cancel out  $\xi, \eta$  as follows:

Multiplying  $(2 + a_1)^2$  to the equation (17) and making use of the equality:  $(2 + a_1)\zeta = -a_1 z - a_2 w p - (2 + a_2)\eta p$  from (19), we have

$$\begin{aligned} & (2 + a_1)^2 a_1 z^2 + a_1 \left[ a_1 z + a_2 w p + (2 + a_2)\eta p \right]^2 \\ & + 2(2 + a_1)^2 z \left( -a_1 z - a_2 w p - (2 + a_2)\eta p \right) \\ & + a_2 (2 + a_1)^2 w^2 + a_2 (2 + a_1)^2 \eta^2 \\ & + 2(2 + a_1)^2 (2 + a_2) w \eta = 4(2 + a_1)^2. \end{aligned} \quad (22)$$

Multiplying (22) by  $(2 + a_2)^2 p_{11}^2$  and making use of (20):  $(2 + a_2)\eta p_{11} = -a_1 - a_2 p^2 - a_2 w p_{11}$ ,

we obtain the following

$$\begin{aligned}
& a_1(2+a_1)^2(2+a_2)^2z^2p_{11}^2 + a_1(2+a_2)^2\left(a_1zp_{11} - a_1p - a_2p^3\right)^2 \\
& - 2(2+a_1)^2(2+a_2)^2p_{11}z(a_1zp_{11} - a_1p - a_2p^3) \\
& + a_2(2+a_1)^2(2+a_2)^2p_{11}^2w^2 + a_2(2+a_1)^2\left(a_1 + a_2p^2 + a_2wp_{11}\right)^2 \\
& - 2(2+a_1)^2(2+a_2)^2wp_{11}(a_1 + a_2p^2 + a_2wp_{11}) = 4(2+a_1)^2(2+a_2)^2p_{11}^2.
\end{aligned} \tag{23}$$

Write (23) as

$$\tilde{A}p_{11}^2 + 2\tilde{B}p_{11} + \tilde{C} = 0, \text{ where} \tag{24}$$

$$\tilde{A} = -4a_1(1+a_1)(2+a_2)^2z^2 - 4a_2(1+a_2)(2+a_1)^2w^2 - 4(2+a_1)^2(2+a_2)^2, \tag{25}$$

$$\tilde{B} = 4(a_1 + a_2p^2) \left[ (1+a_1)(2+a_2)^2zp - (2+a_1)^2(1+a_2)w \right], \tag{26}$$

$$\tilde{C} = (a_1 + a_2p^2)^2 \left[ a_1(2+a_2)^2p^2 + a_2(2+a_1)^2 \right]. \tag{27}$$

Assume that  $\tilde{A} \neq 0$  at the point  $(z, w) \in M$  with  $w \neq 0$ . We can then solve  $p_{11}$  from (24):

$$p_{11} = \frac{-\tilde{B} \pm \tilde{H}}{\tilde{A}} \tag{28}$$

where

$$\begin{aligned}
\tilde{H}^2 = \tilde{B}^2 - \tilde{A}\tilde{C} = & 4(a_1 + a_2p^2)^2 \left\{ 4 \left[ (1+a_1)(2+a_2)^2zp - (1+a_2)(2+a_1)^2w \right]^2 \right. \\
& + \left[ a_1(2+a_2)^2p^2 + a_2(2+a_1)^2 \right] \cdot \left[ a_1(1+a_1)(2+a_2)^2z^2 \right. \\
& \left. \left. + a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2 \right] \right\}.
\end{aligned} \tag{29}$$

After taking out the common factor  $2(a_1 + a_2p^2)$ , (28) can be simplified as

$$p_{11} = \frac{-\hat{B} \pm \hat{H}}{\tilde{A}} \cdot 2(a_1 + a_2p^2) \tag{30}$$

where  $2(a_1 + a_2 p^2)\hat{B} = \tilde{B}$ , and

$$\begin{aligned} \hat{H}^2 = & 4 \left[ (1 + a_1)(2 + a_2)^2 z p - (1 + a_2)(2 + a_1)^2 w \right]^2 + \left[ a_1(2 + a_2)^2 p^2 + a_2(2 + a_1)^2 \right] \\ & \cdot \left[ a_1(1 + a_1)(2 + a_2)^2 z^2 + a_2(1 + a_2)(2 + a_1)^2 w^2 + (2 + a_1)^2(2 + a_2)^2 \right]. \end{aligned} \quad (31)$$

Write

$$\hat{H}^2 = Ap^2 + Bp + C, \text{ where} \quad (32)$$

$$\begin{aligned} A = & 4(1 + a_1)^2(2 + a_2)^4 z^2 \\ & + a_1(2 + a_2)^2 \left[ a_1(1 + a_1)(2 + a_2)^2 z^2 + a_2(1 + a_2)(2 + a_1)^2 w^2 + (2 + a_1)^2(2 + a_2)^2 \right], \end{aligned} \quad (33)$$

$$B = -8(1 + a_1)(1 + a_2)(2 + a_1)^2(2 + a_2)^2 zw, \quad (34)$$

$$\begin{aligned} C = & 4(1 + a_2)^2(2 + a_1)^4 w^2 \\ & + a_2(2 + a_1)^2 \left[ a_1(1 + a_1)(2 + a_2)^2 z^2 + a_2(1 + a_2)(2 + a_1)^2 w^2 + (2 + a_1)^2(2 + a_2)^2 \right]. \end{aligned} \quad (35)$$

Assume that  $\hat{H}^2 = Ap^2 + Bp + C \neq 0$  at the point  $(z, w) \in M$  with  $p$  being given as before. Notice that  $\tilde{A}$  is independent of  $p$  and that the degree of  $\hat{B}$  in  $p$  is 1. From the formula of  $p_{11}$  in (30), it follows that at  $(z, w, \bar{z}, \bar{w})$ ,

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \Leftrightarrow \frac{\partial^4}{\partial p^4} \left( (a_1 + a_2 p^2) \hat{H} \right) = 0. \quad (36)$$

Assume that  $\hat{H}(z^*, w^*, p^*) = 0$  with  $(z^*, w^*) \in M$  and  $p^* = p(z^*, w^*, \bar{z}^*)$ , where

$$w^*, A(z^*, w^*), \tilde{A}(z^*, w^*) \neq 0.$$

Since  $p_{11}(z, w, p)$  is a holomorphic function for  $(z, w, p) \approx (z^*, w^*, p^*)$ , we easily see from (30) that  $J(z, w, p) := \hat{H} \cdot (a_1 + a_2 p^2)$  is also holomorphic for  $(z, w, p) \approx (z^*, w^*, p^*)$ . In particular,  $J(z^*, w^*, p)$  is holomorphic in  $p$  for  $p \approx p^*$ . Now, suppose that  $2A(z^*, w^*)p^* + B(z^*, w^*) \neq 0$ . Then  $\hat{H} = \pm(p - p^*)^{1/2} h^*$  with  $h^* \neq 0$  holomorphic for  $p \approx p^*$ , by (32). This clearly

contradicts the fact that  $J(z^*, w^*, p)$  is holomorphic in  $p$  for  $p \approx p^*$ . Hence, we conclude that  $\hat{H}(z^*, w^*, p^*) = 0$  can only occur at the point where

$$2A(z^*, w^*)p^* + B(z^*, w^*) = 0. \quad (37)$$

Next, we have

$$\frac{\partial^4}{\partial p^4} \left( (a_1 + a_2 p^2) \hat{H} \right) = 12a_2 \frac{\partial^2 \hat{H}}{\partial p^2} + 8a_2 p \frac{\partial^3 \hat{H}}{\partial p^3} + (a_1 + a_2 p^2) \frac{\partial^4 \hat{H}}{\partial p^4}. \quad (38)$$

Since  $\hat{H}^2 = Ap^2 + Bp + C$ , we get  $2\hat{H} \frac{\partial \hat{H}}{\partial p} = 2Ap + B$ . We continue to differentiate it to get  $\left( \frac{\partial \hat{H}}{\partial p} \right)^2 + \hat{H} \frac{\partial^2 \hat{H}}{\partial p^2} = A$ . Hence

$$\frac{\partial^2 \hat{H}}{\partial p^2} = \frac{A - \left( \frac{\partial \hat{H}}{\partial p} \right)^2}{\hat{H}} = \frac{4A\hat{H}^2 - (2Ap+B)^2}{4\hat{H}^3} = \frac{4A(Ap^2+Bp+C) - (4A^2p^2 + 4ABp + B^2)}{4\hat{H}^3} = \frac{4AC - B^2}{4\hat{H}^3}. \quad (39)$$

Continuing differentiation on  $\left( \frac{\partial \hat{H}}{\partial p} \right)^2 + \hat{H} \frac{\partial^2 \hat{H}}{\partial p^2} = A$ , we obtain  $3 \frac{\partial \hat{H}}{\partial p} \frac{\partial^2 \hat{H}}{\partial p^2} + \hat{H} \frac{\partial^3 \hat{H}}{\partial p^3} = 0$  and thus

$$\frac{\partial^3 \hat{H}}{\partial p^3} = -\frac{3}{\hat{H}} \cdot \frac{\partial \hat{H}}{\partial p} \cdot \frac{\partial^2 \hat{H}}{\partial p^2} = -\frac{3}{\hat{H}} \cdot \frac{2Ap+B}{2\hat{H}} \cdot \frac{4AC-B^2}{4\hat{H}^3} = -\frac{3}{8\hat{H}^5} (2Ap+B)(4AC-B^2). \quad (40)$$

Again from the equation  $3 \frac{\partial \hat{H}}{\partial p} \frac{\partial^2 \hat{H}}{\partial p^2} + \hat{H} \frac{\partial^3 \hat{H}}{\partial p^3} = 0$ , we get by differentiation

$$3 \left( \frac{\partial^2 \hat{H}}{\partial p^2} \right)^2 + 4 \frac{\partial \hat{H}}{\partial p} \frac{\partial^3 \hat{H}}{\partial p^3} + \hat{H} \frac{\partial^4 \hat{H}}{\partial p^4} = 0, \text{ and thus}$$

$$\frac{\partial^4 \hat{H}}{\partial p^4} = \frac{1}{\hat{H}} \left[ -3 \left( \frac{\partial^2 \hat{H}}{\partial p^2} \right)^2 - 4 \frac{\partial \hat{H}}{\partial p} \frac{\partial^3 \hat{H}}{\partial p^3} \right] = \frac{3(4AC - B^2)}{16\hat{H}^7} \left( B^2 - 4AC + 4(2Ap + B)^2 \right). \quad (41)$$

By Theorem 2.1, (36), (38), (39), (40) and (41),  $(z, w) \in M$  is an umbilical point if and only if

$$\begin{aligned} & \frac{a_2(4AC - B^2)}{\hat{H}^3} - \frac{a_2 p(2Ap + B)(4AC - B^2)}{\hat{H}^5} \\ & + \left( a_1 + a_2 p^2 \right) \frac{(4AC - B^2)[B^2 - 4AC + 4(2Ap + B)^2]}{16\hat{H}^7} = 0, \end{aligned}$$

which amounts to say that either  $4AC - B^2 = 0$  or

$$a_2\hat{H}^4 - a_2p(2Ap + B)\hat{H}^2 + \frac{1}{16}[a_1 + a_2p^2][B^2 - 4AC + 4(2Ap + B)^2] = 0. \quad (42)$$

Since  $\hat{H}^2 = Ap^2 + Bp + C$ , it follows from (42) that

$$4a_2(Bp + 2C)^2 + 4a_1(2Ap + B)^2 + (a_1 + a_2p)(B^2 - 4AC) = 0.$$

Hence, we have proved the following criterion on umbilical points.

**Theorem 3.1** *Let  $M \subset \mathbf{C}^2$  be as in (14). Let  $(z, w) \in M$  be such that  $w \neq 0$ ,  $\tilde{A}(z, w) \neq 0$  and  $\hat{H}(z, w, p(z, w, \bar{z})) = Ap^2 + Bp + C \neq 0$ . Then  $(z, w)$  is an umbilical point if and only if either  $4AC - B^2 = 0$  or*

$$4a_2(Bp + 2C)^2 + 4a_1(2Ap + B)^2 + (a_1 + a_2p)(B^2 - 4AC) = 0 \quad (43)$$

at  $(z, w, p)$ . Here  $p$  is as in (21);  $A, B$  and  $C$  are as in (33), (34) and (35).

## 4 Proof of Theorem 1.2

**Lemma 4.1** *Let  $M$  be as in (14). Assume that  $a_1 > 0$ . If  $16a_1 + 16a_1a_2 + 3a_1a_2^2 - 4a_2^2 > 0$ , then  $M$  is umbilical at  $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2}) \in M$  for a certain  $c \in (0, 1)$ .*

*Proof:* Consider the curve  $\Gamma \subset M$  with the parameter  $c \in [0, 1]$ , defined by:

$$z(c) = \frac{c}{\sqrt{1+a_1}}, \quad (44)$$

$$w(c) = i\sqrt{1-c^2}, \quad 0 \leq c < 1. \quad (45)$$

Then along  $\Gamma$ , from (21), we have

$$p(c) = -\frac{a_1z + (2+a_1)\bar{z}}{aw + (2+a)\bar{w}} = -\frac{i(\sqrt{1+a_1})c}{\sqrt{1-c^2}}. \quad (46)$$

By (33), (34) and (35), we have

$$\begin{aligned} A(c) &= 4(1+a_1)(2+a_2)^4c^2 \\ &+ a_1(2+a_2)^2 \left[ a_1(2+a_2)^2c^2 - a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2 \right], \end{aligned} \quad (47)$$

$$B(c) = -8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2 ic \frac{\sqrt{1-c^2}}{\sqrt{1+a_1}}, \quad (48)$$

$$C(c) = -4(1+a_2)^2(2+a_1)^4(1-c^2) + a_2(2+a_1)^2 \left[ a_1(2+a_2)^2 c^2 - a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2 \right]. \quad (49)$$

By Theorem 3.1, it is enough to show that there is a certain  $c \in (0, 1)$  such that at the point  $(z(c), w(c), p(c)) \in \widetilde{\mathcal{M}}$

$$\widetilde{A} \neq 0, \quad (50)$$

$$Ap^2 + Bp + C \neq 0, \quad \text{and} \quad (51)$$

$$4a_2(Bp + 2C)^2 + 4a_1(2Ap + B)^2 + (a_1 + a_2p^2)(B^2 - 4AC) = 0. \quad (52)$$

We first prove that (50) holds for any point in  $\Gamma$ . By (25),  $\widetilde{A} = 0$  at  $(z(c), w(c)) \in \Gamma$  if and only if

$$-4a_1(2+a_2)^2c^2 + 4a_2(1+a_2)(2+a_1)^2(1-c^2) - 4(2+a_1)^2(2+a_2)^2 = 0,$$

namely,

$$-4a_1(2+a_2)^2c^2 - 4(2+a_1)^2[4 + 3a_2 + c^2a_2 + c^2a_2^2] = 0.$$

But this is a contradiction because the left hand side is strictly negative for any  $c \in [0, 1]$ .

We also notice that  $A > 0$  along  $\Gamma$ , too.

Next, after restricted to  $\Gamma$ , (52) can be written as

$$\begin{aligned} & \left[ 4a_2B^2 + 16a_1A^2 + a_2(B^2 - 4AC) \right] p^2 + (16a_2BC + 16a_1AB)p \\ & + \left[ 16a_2C^2 + 4a_1B^2 + a_1(B^2 - 4AC) \right] = 0. \end{aligned} \quad (53)$$

In order to solve the equation (52), by (46) and (53), it is enough to show that there exists a point  $c \in (0, 1)$  such that  $K(c) = 0$ , where

$$\begin{aligned} K(c) : &= \left[ 4a_2B^2 + 16a_1A^2 + a_2(B^2 - 4AC) \right] \left( a_1z + (2+a_1)\bar{z} \right)^2 \\ & - (16a_2BC + 16a_1AB) \left( a_1z + (2+a_1)\bar{z} \right) \left( a_2w + (2+a_2)\bar{w} \right) \\ & + \left[ 16a_2C^2 + 4a_1B^2 + a_1(B^2 - 4AC) \right] \left( a_2w + (2+a_2)\bar{w} \right)^2. \end{aligned} \quad (54)$$

By (54) (47) (48) and (49),  $K(c)$  is a real-valued function defined on  $[0, 1]$ . When  $c = 0$ , we have  $z = 0$ ,  $w = i$  and

$$\begin{aligned} A &= a_1(2 + a_2)^2 \left[ -a_2(1 + a_2)(2 + a_1)^2 + (2 + a_1)^2(2 + a_2)^2 \right] \\ &= a_1(2 + a_1)^2(2 + a_2)^2(4 + 3a_2), \quad B = 0, \end{aligned}$$

$$\begin{aligned} C &= -4(1 + a_2)^2(2 + a_1)^4 + a_2(2 + a_1)^2 \left[ -a_2(1 + a_2)(2 + a_1)^2 + (2 + a_1)^2(2 + a_2)^2 \right] \\ &= -(2 + a_1)^4(2 + a_2)^2. \end{aligned}$$

Hence

$$K(0) = -16C(4a_2C - a_1A) < 0, \quad (55)$$

by noticing that  $C < 0$  and  $A > 0$ .

When  $c = 1$ , we have  $z = \frac{1}{\sqrt{1+a_1}}$ ,  $w = 0$  and

$$\begin{aligned} A &= 4(1 + a_1)(2 + a_2)^4 + a_1(2 + a_2)^2 \left[ a_1(2 + a_2)^2 + (2 + a_1)^2(2 + a_2)^2 \right] \\ &= (2 + a_2)^4(1 + a_1)(2 + a_1)^2, \quad B = 0, \end{aligned}$$

$$\begin{aligned} C &= a_2(2 + a_1)^2 \left[ a_1(2 + a_2)^2 + (2 + a_1)^2(2 + a_2)^2 \right] \\ &= a_2(2 + a_1)^2(2 + a_2)^2(1 + a_1)(4 + a_1) \end{aligned}$$

Hence

$$\begin{aligned} K(1) &= 4A(4a_1A - a_2C)4(1 + a_1) \\ &= d^*[4a_1(2 + a_2)^2 - a_2^2(4 + a_1)]. \end{aligned} \quad (56)$$

Here  $d^* > 0$ . Hence, when

$$4a_1(2 + a_2)^2 - a_2^2(4 + a_1) = 16a_1 + 16a_1a_2 + 3a_1a_2^2 - 4a_2^2 > 0,$$

$K(0) < 0$  and  $K(1) > 0$ . Thus,  $K(c) = 0$  for a certain  $c \in (0, 1)$ . Namely, we showed that (52) holds for a certain  $c$ .

It remains to prove that (51) cannot hold for the above  $c \in (0, 1)$ . Suppose that  $\hat{H}(c)^2 = 0$ . Since  $\tilde{A}(c) > 0$  and  $A(c) > 0$ , we conclude by (37), that  $2Ap + B = 0$ . Making use of (46), (47) and (48), we thus have

$$-8(1 + a_1)(1 + a_2)(2 + a_1)^2(2 + a_2)^2 \frac{ic\sqrt{1-c^2}}{\sqrt{1+a_1}} = \frac{2i(\sqrt{1+a_1})c}{\sqrt{1-c^2}} \cdot A(c) \quad (57)$$

This is a contradiction, for after dividing the fact  $i$ , the left hand side of (57) is negative, while its right hand side is strictly positive. The proof of Lemma 4.1 is complete.  $\blacksquare$

*Proof of Theorem 1.2:* If  $M$  is spherical, then every point is umbilical point. We assume that  $M$  is not spherical. Then  $a_1 + a_2 > 0$ . We notice that  $(1 + a_1)x_1^2 + y_1^2 + (1 + a_2)x_2^2 + y_2^2 = 1$  is holomorphically equivalent to the ellipsoid defined by  $(1 + a_2)x_1^2 + y_1^2 + (1 + a_1)x_2^2 + y_2^2 = 1$  through the map  $(z, w) \rightarrow (w, z)$ . Hence, we need only to prove Theorem 1.2 for the case when  $a_1 \geq a_2$ . Then the assumption in Lemma 4.1 holds automatically and thus we have an umbilical point of the form  $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2})$  ( $c \in (0, 1)$ ). Notice that  $M$  has automorphisms sending  $(z, w)$  to  $(\pm z, \pm w)$ . We easily conclude that  $M$  possesses at least four umbilical points.  $\blacksquare$

## 5 Proof of Theorem 1.2

The  $\epsilon$ -thickening  $\Omega_\epsilon$  of the unit circle  $\{|z| = 1, w = 0\}$  is defined to be the set of points whose distance to the circle is less than  $\epsilon$ . It is straightforward to verify that the boundary  $M_\epsilon$  of  $\Omega_\epsilon$  is defined by the following equation, which is strictly plurisubharmonic when  $0 < \epsilon < 1/4$ :

$$|z|^2 - 2|z| + 1 + |w|^2 = \epsilon^2. \quad (58)$$

Here and in what follows, we assume  $0 < \epsilon \ll 1$ . Also, since  $\Omega_\epsilon$  is a Reinhardt domain, we need only to study the points  $(z, w) \in M_\epsilon$  with  $z = x_1 \geq 0$  and  $w = x_2 \geq 0$ . Also, we assume that  $x_2 > 0$ . Notice that when  $\epsilon \ll 1$ ,  $x_2 \approx 1$ .

The complexification of (58) is given by

$$r := z\zeta - 2(z\zeta)^{1/2} + 1 + w\eta - \epsilon^2 = 0. \quad (59)$$

As we did in §3, we have

$$r_z = \zeta - (z\zeta)^{-1/2}\zeta + p\eta = 0, \text{ and} \quad (60)$$

$$r_{zz} = \frac{1}{2}(z\zeta)^{-3/2}\zeta^2 + p_{11}\eta = 0. \quad (61)$$

From (60), we have

$$z\zeta - (z\zeta)^{1/2} + pz\eta = 0. \quad (62)$$

Subtracting (59) from (62), we obtain

$$(z\zeta)^{1/2} = 1 - \epsilon^2 + (w - pz)\eta. \quad (63)$$

Returning to (61) and making use of (63), we get

$$1 - \epsilon^2 + (w - pz)\eta + 2\eta z^2 p_{11} = 0. \quad (64)$$

Here, we remark that near the point under study,  $\eta \approx x_2 \neq 0$ . Hence  $\frac{1-\epsilon^2}{\eta} + (w-pz) + 2z^2 p_{11} = 0$  and

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4}{\partial p^4} \left( \frac{1}{\eta} \right) = 0. \quad (65)$$

Set  $X = \frac{1}{\eta}$ . Substituting (63) into (59), we get

$$\begin{aligned} & \left[ (1 - \epsilon)^2 + (w - pz)\eta \right]^2 - 2 \left[ (1 - \epsilon^2) + (w - pz)\eta \right] + 1 + w\eta - \epsilon^2 = 0, \text{ or} \\ & (w - pz)^2 \eta^2 + \left[ 2(1 - \epsilon^2)(w - pz) - 2(w - pz) + w \right] \eta + (1 - \epsilon^2)^2 - 2(1 - \epsilon^2) + (1 - \epsilon^2) = 0, \\ & -\epsilon^2(1 - \epsilon^2)X^2 + \left[ -2\epsilon^2(w - pz) + w \right] X + (w - pz)^2 = 0. \end{aligned}$$

Hence

$$X = \frac{-(-2\epsilon^2(w - pz) + w) \pm H}{-2\epsilon^2(1 - \epsilon^2)} \quad (66)$$

where

$$H^2 := (2\epsilon^2(w - pz) - w)^2 + 4\epsilon^2(1 - \epsilon^2)(w - pz)^2.$$

Hence

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4 H}{\partial p^4} = 0. \quad (67)$$

Write  $H^2 = Ap^2 + Bp + C$  where

$$\begin{aligned} A &= 4\epsilon^2 z^2 + 4\epsilon^2(1 - \epsilon^2)z^2 = 4\epsilon^2 z^2, \\ B &= -4\epsilon^2 z(2\epsilon^2 w) - 8\epsilon^2(1 - \epsilon^2)wz = -4\epsilon^2 wz, \\ C &= \epsilon^2 w^2. \end{aligned} \quad (68)$$

By (41), we conclude that  $\frac{\partial^4 p_{11}}{\partial p^4} = 0$  if and only if

$$\text{either } 4AC - B^2 = 0 \text{ or } B^2 - 4A + 4(2Ap + B)^2 = 0. \quad (69)$$

Since  $4AC - B^2 = 4\epsilon^2(zw)^2(1 - 4\epsilon^2) \neq 0$ , the first equality in (69) never occurs. The second equality in (69) is equivalent to  $4AC - B^2 = 4(2Ap + B)^2$ , namely,

$$2\epsilon zw \sqrt{1 - 4\epsilon^2} = \pm 2(2Ap + B). \quad (70)$$

At the point in  $M$  with  $z = x_1 > 0$  and  $w = x_2 > 0$ , by (60), we find  $x_1 - 1 + px_2 = 0$ , or

$$p = \frac{1 - x_1}{x_2}. \quad (71)$$

Hence we get from (68)

$$A = 4\epsilon^2 x_1^2, \quad B = -4\epsilon^2 x_1 x_2 \text{ and } C = \epsilon^2 x_2^2. \quad (72)$$

Then (70) is equivalent to

$$2\epsilon x_1 x_2 \sqrt{1 - 4\epsilon^2} = \pm 2 \left( 8\epsilon^2 x_1^2 \cdot \frac{1 - x_1}{x_2} - 4\epsilon^2 x_1 x_2 \right). \quad (73)$$

Since  $x_1 \approx 1$ , we get from (73):

$$x_2^2 \sqrt{1 - 4\epsilon^2} = \pm \left( 8\epsilon(x_1 - x_1^2) - 4\epsilon x_2^2 \right). \quad (74)$$

Recall  $x_2^2 = \epsilon^2 - (1 - x_1)^2$ . Let  $T = 1 - x_1$ . Then  $x_1 - x_1^2 = T - T^2$  and  $x_2^2 = \epsilon^2 - T^2$ . Hence (74) is equivalent to

$$(\epsilon^2 - T^2) \sqrt{1 - 4\epsilon^2} = \pm 4\epsilon(2T - T^2 - \epsilon^2), \quad (75)$$

or

$$f(T) := (\sqrt{1 - 4\epsilon^2} \mp 4\epsilon)T^2 \pm 8\epsilon T + (-\epsilon^2 \sqrt{1 - 4\epsilon^2} \mp \epsilon^2) = 0.$$

Notice that  $-\epsilon < T < \epsilon$ . From the fact that

$$f'(T) = 2(\sqrt{1 - 4\epsilon^2} \mp 4\epsilon)T \pm 8\epsilon = 0 \iff |T| \approx 4\epsilon$$

for  $\epsilon \ll 1$ , we conclude that the real-valued function  $f(T)$  is monotonic for  $T \in (-\epsilon, \epsilon)$ . We further compute

$$f(-\epsilon) = (\sqrt{1 - 4\epsilon^2} \mp 4\epsilon)\epsilon^2 \mp 8\epsilon^2 + (-\epsilon^2 \sqrt{1 - 4\epsilon^2} \mp 4\epsilon^3) \approx \mp 8\epsilon^2$$

and

$$f(\epsilon) = (\sqrt{1 - 4\epsilon^2} \mp 4\epsilon)\epsilon^2 \pm 8\epsilon^2 + (-\epsilon^2 \sqrt{1 - 4\epsilon^2} \mp 4\epsilon^3) \approx \pm 8\epsilon^2$$

for  $\epsilon \ll 1$ . Then we see that (69) has two solutions in  $(-\epsilon, \epsilon)$ . A little more effort actually shows that these two solutions are different. Therefore, by Theorem 2.1, we conclude that  $M$  admits two distinct umbilical points with  $z = x_1 > 0, w = x_2 > 0$ . One can similarly verify that points in  $M$  with  $w = 0$  are umbilical points. The statement Theorem 1.3 thus follows from the Reinhardt property of  $\Omega_\epsilon$ .  $\blacksquare$

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