MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 1

(1) (Problem 7)
   (a) In how many ways can 3 boys and 3 girls sit in a row?
   (b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
   (c) In how many ways if only the boys must sit together?
   (d) In how many ways if no two people of the same sex are allowed to sit together?

(2) (Problem 9) A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

(3) (Problem 12) How many digit numbers xyz with x, y, z all ranging from 0 to 9 have at least two of their digits equal? How many have exactly two equal digits?

(4) (Problem 15) Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

(5) (Problem 17) A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

(6) (Problem 22) A person has 8 friends, of whom 5 will be invited to a party.
   (a) How many choices are there if 2 of the friends are feuding and will not attend together?
   (b) How many choices if 2 of the friends will only attend together?

(7) (Theoretical Exercises 8&9) Prove that

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k},$$

and

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

(8) You prepare decorations for a party. How many ways are there to arrange 12 blue balloons, 9 green lanterns and 6 red ribbons in a row, such that no two ribbons are next to each other?

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 2

(1) (Problem 1) A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

(2) (Problem 8) Suppose that A and B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$. What is the probability that
  (a) either A or B occurs?
  (b) A occurs but B does not?
  (c) both A and B occur?

(3) (Problem 11) A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.
  (a) What percentage of males smokes neither cigars nor cigarettes?
  (b) What percentage smokes cigars but not cigarettes?

(4) Let E be the set

$$E = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Let A, B, C be the subsets

$$A = \{1, 3, 5\}, \quad B = \{5, 7, 8\}, \quad C = \{1, 7, 8\}.$$

Compute the following sets

$$A \cap B \cap C, \quad A^{\mathsf{c}} \cap B \cap C, \quad A \cap B^{\mathsf{c}} \cap C, \quad A \cap B \cap C^{\mathsf{c}}, \quad A^{\mathsf{c}} \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}, \quad A \cup B \cup C.$$

(5) A survey of 100 students taken over the last year revealed the following:
  (a) 30 of them smoke,
  (b) 50 of them drink,
  (c) 20 of them smoke and drink,
  (d) 25 go to the gym regularly, and of these, none smokes nor drinks.

Calculate how many of the 100 students neither smoke, nor drink, nor go to the gym.

(6) What is the probability that if you toss a fair coin n times, you would get exactly k tails?

(7) What is the probability that among 30 people, at least two of them have the same birthday? Assume that no one is born on February 29th, and that the remaining 365 days are equally likely.

(8) There is a box consisting of 20 red balls, 30 green balls, and 40 yellow balls, and you draw 12 balls without replacing them. What is the probability that you would pick 3 red balls, 4 green balls, and 5 yellow balls?

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 3

(1) (Problem 1) Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

(2) (Problem 5) An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 black?

(3) (Problem 11) Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces, let A_S be the event that the ace of spades is chosen, and let A be the event that at least one ace is chosen. Find $P(B|A_S)$ and $P(B|A)$.

(4) (Problem 18) In a certain community, 36 percent of the families own a dog and 22 percent of the families that own a dog also own a cat. In addition, 30 percent of the families own a cat. What is
   (a) the probability that a randomly selected family owns both a dog and a cat?
   (b) the conditional probability that a randomly selected family owns a dog given that it owns a cat?

(5) (Problem 24) Urn I contains 2 white and 4 red balls, whereas urn II contains 1 white and 1 red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is
   (a) the probability that the ball selected from urn II is white?
   (b) the conditional probability that the transferred ball was white given that a white ball is selected from urn II?

(6) (Problem 28) Suppose that 5 percent of men and 0.25 percent of women are color blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?

(7) (Problem 31) There are 15 tennis balls in a box, of which 9 have not previously been used. Three of the balls are randomly chosen, played with, and then returned to the box. Later, another 3 balls are randomly chosen from the box. Find the probability that none of these balls has ever been used.

(8) (Problem 34) A family has children with probability p_j, where $p_1 = 0.1$, $p_2 = 0.25$, $p_3 = 0.35$, $p_4 = 0.3$. A child from this family is randomly chosen. Given that this child is the eldest child in the family, find the conditional probability that the family has 4 children.

(9) (Problem 35) On rainy days, Joe is late to work with probability 0.3; on nonrainy days, he is late with probability 0.1. With probability 0.7, it will rain tomorrow.
   (a) Find the probability that Joe is early tomorrow.
   (b) Given that Joe was early, what is the conditional probability that it rained?

(10) (Problem 38) Stores A, B and C have 50, 75, and 100 employees, respectively, and 50, 60, and 70 percent of them respectively are women. Resignations are equally likely among all employees, regardless of sex. One woman employee resigns. What is the probability that she works in store C?

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 4

(1) (Problem 2) Two fair dice are rolled. Let X equal the product of the 2 dice. Compute $E[X]$.

(2) (Problem 11) The random variable X is said to follow the distribution of Benford's Law if

$$P(X = i) = \log_{10}\left(\frac{i+1}{i}\right), \quad i = 1, 2, \ldots, 9.$$

Find the CDF(cumulative distribution function) of X.

(3) (Problem 39) If $E[X] = 1$, $\mathrm{Var}[X] = 5$, find $E[(2 + X)^2]$, $\mathrm{Var}[4 + 3X]$.

(4) A sample of 3 items is chosen at random from a box containing 20 items of which 4 are defective. Let X be the number of defective items in the sample. Find $E[X]$, $\mathrm{Var}[X]$, $\mathrm{SD}[X]$, $E[20 - X]$, $\mathrm{Var}[20 - X]$ and $\mathrm{SD}[20 - X]$.

(5) The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}, \quad n \geqslant 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

(6) Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. Find $E[X]$ and $\mathrm{Var}[X]$.

(7) On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student will get 4 or more correct answers just by guessing?

(8) (Hard)Let N be a random number of fair coins, where N has the Poisson distribution with parameter 2. You toss each coin once. Let X be the total number of heads. Compute the pmf p_X to show that $X \sim \mathrm{Pois}(1)$, that is, that X has the Poisson distribution with parameter 1.

(9) A certain loaded die has

$$P(X = 1) = 0.001, \quad P(X = 2) = 0.009, \quad P(X = 3) = 0.09,$$
$$P(X = 4) = 0.2, \quad P(X = 5) = 0.3, \quad P(X = 6) = 0.4$$

If you roll it 1000 times, let X be the number of times that it lands on 1. Find $P(X = 2)$ directly and using Poisson distribution to approximate X.

(10) A sample of 3 items is selected at random from a box containing 20 items, of which 4 are defective. Find the expected value and variance of defective items in the sample.

(11) Consider a roulette wheel consisting of 38 numbers 1 through 36, 0, and 00. Smith always bets that the outcome will be any one of the numbers 1 through 12.

  (a) What is the probability that Smith's first win will occur on his fourth round?
  (b) If we learn that Smith losses in his first four rounds, what is the probability that Smith's first win will occur on his seventh round?
  (c) What is the probability that Smith's third win will occur on his ninth round?

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 5

(1) (Problem 1) Let X be the random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the value of c and the cumulative distribution function of X.

(2) (Problem 7) The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leqslant x \leqslant 1 \\ & 0\text{otherwise} \end{cases}.$$

If $E[X] = \frac{3}{5}$, find a and b.

(3) (Problem 11) A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

(4) (Problem 15) $X \sim N(10, 36)$, find $P(X > 5)$, $P(4 < X < 16)$, $P(X < 8)$, $P(X < 20)$ and $P(X > 16)$.

(5) (Problem 16) The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches? What assumptions are you making?

(6) (Problem 18) Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.2$, find $\text{Var}[X]$.

(7) (Problem 19) Let X be a normal random variable with mean 12 and variance 4. Find the value fo c such that $P(X > c) = 0.1$.

(8) (Problem 23) One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.

(9) (Problem 32) The time in hours required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is the probability that a repair time exceeds 2 hours? What is the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTERS 6 AND 7

(1) (Chapter 5 Problem 33) If $X \sim U(0,1)$ is a uniformly distributed r.v., find the distribution of $Y = -\log X$.

(2) (Chapter 5 Problem 34) If X is an exponential r.v. with parameter λ and $c > 0$, find the distribution of $Y = cX$.

(3) (Chapter 5 Problem 41) If X is uniformly distributed over (a, b), find a and b such that $E[X] = 10$, $\mathrm{Var}[X] = 48$.

(4) (Chapter 6 Problem 10) The joint pdf of X and Y is given by

$$f_{X,Y}(x, y) = e^{-(x+y)}, \quad x, y > 0.$$

Find $P(X < Y)$. Identity the distributions of X and Y. Are they independent?

(5) (Chapter 6 Problem 15) The random vector (X, Y) is said to be uniformly distributed over a region R if, for some constant c, its joint density is

$$f_{X,Y}(x, y) = \begin{cases} c, & (x, y) \in R \\ 0, & \text{otherwise} \end{cases}.$$

(a) Show that $1/c =$ area of region R.
(b) Suppose (X, Y) is uniformly distributed over the square centered at $(0, 0)$ and with sides of length 2. Show that X and Y are independent, with each being distributed uniformly over $(1, 1)$.
(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P(X^2 + Y^2 \leqslant 1)$.

(6) (Chapter 6 Problem 19, Chapter 7 Problem 4) Let $f_{X,Y}(x, y) = c/x$, $0 < y < x < 1$ be the joint pdf of X and Y.
(a) Find c.
(b) Find the marginal densities of X and Y. Are X and Y independent?
(c) Find $E[X]$ and $E[Y]$.
(d) Find $E[XY]$.

(7) (Chapter 6 Problem 27) If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2. Find the distribution of $Z = X_1/X_2$, and compute $P(X_1 < X_2)$.

(8) (Chapter 6 Problem 28) Let X and Y be independently and identically exponentially distributed with parameter 1.
(a) For $t > 0$, find $P(Y + t < X)$.
(b) Find $P(X + Y < 2)$.

(9) (Chapter 6 Problem 30) Let $X \sim N(170, 20^2)$, $Y \sim N(160, 15^2)$ be independent normal random variables.
(a) Find $P(X < Y)$.
(b) Find $P(X + Y > 350)$.

(10) (Chapter 6 Problem 42) Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number Y at random from the set $\{1, \ldots, X\}$.
(a) Find the joint pmf of X and Y.
(b) Find the conditional probability mass function of X given that $Y = j$, for $j = 1, 2, 3, 4, 5$. That is, find the pmf of $X|_{Y=j}$.

(c) Are X and Y independent?

(11) (Chapter 6 Problem 43) Two dice are rolled. Let X and Y denote, resp., the largest and the smallest values obtained. Find the joint pmf and the conditional density of X given Y. Are X and Y independent?

(12) Let X, Y be random variables with joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{ke^{-x}}{\sqrt{x}}, & -\sqrt{x} < y < \sqrt{x}, x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find c.

(b) Find the marginal density $f_X(x)$ of X and identify the distribution of X.

(c) Find the conditional density $f_{Y|X}(y|x)$ of Y given $X = x$ and identify the distribution of $Y|_{X=x}$.

(d) Find the marginal density $f_Y(y)$ of Y and express in terms the $\Phi(t)$, the cdf of the standard normal distribution.

(13) (Chapter 7 Problem 6) A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

(14) (Chapter 7 Problem 42) The joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{y}e^{-(y+x/y)}, \quad x, y > 0.$$

Find $E[X]$, $E[Y]$, and show that $\text{Cov}(X, Y) = 1$.

(15) If $\text{Var}[X] = 5$, $\text{Var}[Y] = 7$ and $\text{Var}[X + Y] = 14$. Find $\text{Var}[2X + 3Y]$ and $\text{Var}[X - Y]$.

(16) The number of earthquakes, each year, of moment magnitude 5 or larger in the US is modeled by $N = 30 + Y$, where $Y \sim \text{Bin}(100, \frac{1}{2})$. The moment magnitude of each such earthquake is modeled by $M = 5 + X$, where X is an exponentially distributed r.v. with mean $\log 10$. You can assume the moment magnitude of each earthquake is independent of N and of other earthquakes. Find the MGF $M_Y(t)$ and calculate the probability that there will be no earthquake of moment magnitude 6.5 or larger in some year.

(17) Let $M_X(t) = (1 - 3t)^{-3}$, find $E[X]$ and $\text{Var}[X]$.

(18) Let X, Y be random variables, with $X \geqslant 0$, $E[X] = \mu$, $\text{Var}[X] = \sigma^2$. Suppose that, given $X = x$, the conditional distribution of Y is the uniform distribution over $(x, 2x)$, i.e., $Y|_X \sim U(X, 2X)$.

(a) Find $E[Y|X]$ and $E[Y]$.

(b) Find $\text{Var}[Y]$.

(c) Find $\text{Cov}(X, Y)$.

MATH 477, NON-HAND-IN PROBLEMS FOR CHAPTER 8

(1) Let $X \sim U(0, 100)$. What is $P(10 < X < 90)$? What does Chebyshev's inequality say about this probability?

(2) If 30 fair dice are rolled, find the approximate probability that the average number of dots is between 3 and 4.

(3) Let $X_1, X_2, \ldots, X_{20}$ be independent and identically distributed(i.i.d.) random variables, each with pdf $f_{X_i}(x) = 3x^2$, $0 < x < 1$. Approximate $P(S < 16)$, where $S = \sum_{i=1}^{20} X_i$.

(4) (Problem 14) A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least 0.95?