An Overview of Lie Superalgebras

Graduate Algebraic Representation Theory Seminar

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Structure

- Background
- Definitions & Examples
- Structure & Representation Theory
- Supersymmetry
- My recent work
Background

- In physics: Supersymmetry regarding particles of different statistics (Bosons & Fermions)
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- In physics: Supersymmetry regarding particles of different statistics (Bosons & Fermions)
- In mathematics: Graded Lie algebras in deformation theory
Background

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Background

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 Still, it finds applications to various other fields beyond particle physics.

Super Lie theory still has a lot of unsolved mysteries. It lives on as some pure mathematical pursuit.
Lie Superalgebras

General Principle of Superization

A (good) $\mathbb{Z}_2$-grading for everything!

$$\mathbb{Z}_2 = \{0, 1\} = \{\text{even}, \text{odd}\}$$
Lie Superalgebras

General Principle of Superization

A (good) $\mathbb{Z}_2$-grading for everything!

$$\mathbb{Z}_2 = \{0, 1\} = \{\text{even, odd}\}$$

Let’s review Lie algebras!
Lie Algebras

Definition ([Hum78])
A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear map $[-,-]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew symmetric and satisfies the Jacobi identity, that is

1. $[X,Y] = -[Y,X]$  
2. $[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$

Classic example: $\text{End} V$ equipped with the usual commutator $([A,B] = AB - BA)$. As a Lie algebra, we denote it as $\mathfrak{gl}(V)$. 
Definition

Let $\mathfrak{g}$ be a Lie algebra. A representation is a pair $(\pi, V)$ such that $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a linear map preserving the Lie bracket, i.e. $\pi([X, Y]) = [\pi(X), \pi(Y)]$. We say $V$ is a $\mathfrak{g}$-module.
A bit of rep theory

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Weyl’s Theorem: If \(\mathfrak{g}\) is complex semisimple, then any finite dimensional \(\mathfrak{g}\)-module is completely reducible.

Know simples, know all.

“Redefine” \(\mathfrak{g}\)

Jacobian id. \(\iff\) \(\mathfrak{g}\) is a \(\mathfrak{g}\)-module.
Linear Superalgebra

Definition

A vector superspace \( V \) is a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \). A vector \( v \in V_0 \) (resp. \( V_1 \)) is said to be even (resp. odd) and write \( |v| = 0 \) (resp. 1). Denote the vector superspace over \( k \) with even subspace \( k^m \) and odd subspace \( k^n \) as \( k^{m|n} \). Its dimension is denoted as \( (m|n) \) while the superdimension is defined as \( m - n \).
A vector superspace $V$ is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. A vector $v \in V_0$ (resp. $V_1$) is said to be even (resp. odd) and write $|v| = 0$ (resp. 1). Denote the vector superspace over $k$ with even subspace $k^m$ and odd subspace $k^n$ as $k^m|n$. Its dimension is denoted as $(m|n)$ while the superdimension is defined as $m - n$.

A linear map $f : V \to W$ is even if it preserves parity, that is, $|f(v)| = |v|$ is even. If $|f(v)| = -|v|$, it is defined to be odd. An even or odd map is said to be homogeneous. We have

$$\begin{align*}
\text{Hom}(V, W)_0 & := \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_1) \\
\text{Hom}(V, W)_1 & := \text{Hom}(V_0, W_1) \oplus \text{Hom}(V_1, W_0)
\end{align*}$$
SVect

Just a quick comment:

SVect

The category of all vector superspaces, denoted as $\text{SVect}$, is a rigid symmetric monoidal category.
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**SVect**

The category of all vector superspaces, denoted as $\text{SVect}$, is a rigid symmetric monoidal category.

It means $\otimes$ is well-defined as follows:

$$(V \otimes W)_i := \bigoplus_{j+k=i} V_j \otimes W_k$$

with a natural isomorphism from $V \otimes W$ to $W \otimes V$:

$$s_{V,W} : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$
Linear Superalgebra

What do the matrices look like?
What do the matrices look like?
\[
\begin{pmatrix}
A_{m\times m} & 0_{m\times n} \\
0_{n\times m} & D_{n\times n}
\end{pmatrix}
\]
correspond to even linear maps, and
\[
\begin{pmatrix}
0_{m\times m} & B_{m\times n} \\
C_{n\times m} & 0_{n\times n}
\end{pmatrix}
\]
to odd linear maps.
An Overview of Lie Superalgebras

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Lie Superalgebras

Definition ([Kac77])

A Lie superalgebra is a vector superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew supersymmetric and satisfies the super Jacobi identity:

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

Note:

1. Everything looks similar!
2. The sign can be explained by SVect.
3. $\mathfrak{g}_0$ is just a Lie algebra, $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-mod.
4. The bracket is symmetric on $\mathfrak{g}_1$. 
Super $\mathfrak{gl}$

We write $\text{End}(\mathbb{C}^m|n)$ as $\mathfrak{gl}(m|n)$. As matrices:

\[
\begin{pmatrix}
A_{m \times m} & 0_{m \times n} \\
0_{n \times m} & D_{n \times n}
\end{pmatrix}
\]

$\mathfrak{gl}_0$:

\[
\begin{pmatrix}
A_{m \times m} & 0_{m \times n} \\
0_{n \times m} & D_{n \times n}
\end{pmatrix}
\]

$\mathfrak{gl}_1$:

\[
\begin{pmatrix}
0_{m \times m} & B_{m \times n} \\
C_{n \times m} & 0_{n \times n}
\end{pmatrix}
\]

The superbracket is the supercommutator $[X, Y] := XY - (-1)^{|X||Y|} YX$. 
We write $\mathrm{End}(\mathbb{C}^m|n)$ as $\mathfrak{gl}(m|n)$. As matrices: $
abla \begin{pmatrix} A_{m \times m} & B_{m \times n} \\ C_{n \times m} & D_{n \times n} \end{pmatrix}$

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The superbracket is the supercommutator $[X, Y] := XY - (-1)^{|X||Y|} YX$. 
We aim to give an overview of the classification of complex, simple, and finite dimensional Lie superalgebras.

“Simple $\iff$ no non-trivial ideals” as usual.
An Overview of Lie Superalgebras

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Lie Algebras

7 types:

1. $A_n, B_n, C_n, D_n$: classical, (countably!) infinite families;
2. $E_6, 7, 8, F_4, G_2$: exceptional. Dimensions: 78, 133, 248, 52, 14
Lie Algebras

\( \mathfrak{sl}(n+1) \) = the Lie algebra of \((n+1) \times (n+1)\) traceless matrices; special linear Lie algebra; \( A_n \)

\( \mathfrak{so}(n) := \{ X \in \text{End}(k^n) : f(Xv, w) = -f(v, Xw), \forall v, w \} \), where \( f \) is a non-degenerate symmetric bilinear form; orthogonal Lie algebra; \( B_n \) for \( 2n + 1 \) and \( D_n \) for \( 2n \). VERY DIFFERENT DESPITE THEIR SIMILAR APPEARANCES

\( \mathfrak{sp}(2n) := \{ X \in \text{End}(k^{2n}) : f(Xv, w) = -f(v, Xw), \forall v, w \} \), where \( f \) is a non-degenerate symplectic bilinear form. Non-degeneracy \( \Rightarrow \) dim must be even; symplectic Lie algebra; \( C_n \)
Lie Algebras

The condition $f(Xv, w) = -f(v, Xw)$ comes from the corresponding Lie group condition

$$f(gv, gw) = f(v, w)$$

with $g = e^{tX}$ and differentiating at $t = 0$. 
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$$f(gv, gw) = f(v, w)$$

with $g = e^{tX}$ and differentiating at $t = 0$.

Matrix form of $f$:

1. Orthogonal: $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$

2. Symplectic: $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

The condition: $fX = -X^T f$
Classification Scheme

So what do with the LSAs?

\[^1\text{also includes } \mathfrak{gl}!\]
Classification Scheme

So what do with the LSAs?
Remember, simple Lie algebras are simple Lie superalgebras already. That’s already a quarter of the zoo of LSA!

\[ \begin{align*}
\text{Simple} & \\
\{ & \text{Classical} \\
& \begin{cases}
\{ & \text{even non-deg. inv. form} \\
\end{cases}
\end{align*} \]

\begin{align*}
\text{Strange} & \\
\text{The Cartan Series} & \\
\text{weird ones} & \\
\end{align*}

Good references: [Mus12, CW12]

not so good reference: [Kac77]

\[ ^1 \text{also includes } gl! \]
Classification Scheme

So what do with the LSAs?

Remember, simple Lie algebras are simple Lie superalgebras already. That’s already a quarter of the zoo of LSA!

Simple
\[
\begin{cases} 
\text{Classical} & \text{if } g_T \text{ is a completely reducible } g_0 \text{-mod} \\
\text{The Cartan Series} & \text{weird ones}
\end{cases}
\]

Basic \(^1\) even non-deg. inv. form

Strange

Good references: [Mus12, CW12]

\(^1\) also includes \(gl\)!
So what do with the LSAs?
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Simple \[ \begin{cases} \mathfrak{g}_T \text{ is a completely reducible } \mathfrak{g}_0\text{-mod} \\ \text{The Cartan Series} \\ \text{weird ones} \end{cases} \]

Classical \[ \begin{cases} \text{even non-deg. inv. form} \\ \text{Strangwe} \end{cases} \]

Good references: [Mus12, CW12]

not so good reference: [Kac77]

\[^{1}\text{also includes } \mathfrak{gl}!\]
Define the supertrace of $X = \begin{pmatrix} A_{m \times m} & B_{m \times n} \\ C_{n \times m} & D_{n \times n} \end{pmatrix}$, denoted $\text{str}(X)$, as $\text{tr}(A) - \text{tr}(D)$.\(^2\)

Define $\mathfrak{sl}(m|n) := \{X \in \mathfrak{gl}(m|n) : \text{str}(X) = 0\}$. Guaranteed to be simple, except...
Define the *supertrace* of \( X = \begin{pmatrix} A_{m \times m} & B_{m \times n} \\ C_{n \times m} & D_{n \times n} \end{pmatrix} \), denoted \( \text{str}(X) \), as 
\[
\text{tr}(A) - \text{tr}(D).^2
\]

Define \( \mathfrak{sl}(m|n) := \{ X \in \mathfrak{gl}(m|n) : \text{str}(X) = 0 \} \). Guaranteed to be simple, except...
when \( m = n \), then \( I_{n|n} = \text{diag}(I_n, I_n) \in \mathfrak{sl}(n|n) \) which is central. We take the quotient to get \( \mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\mathbb{C}I_{n|n} \).

\[\text{\footnotesize 2 Makes sense! Think sdim.}\]
Define the *supertrace* of \( X = \left( \begin{array}{c|c} A_{m \times m} & B_{m \times n} \\ \hline C_{n \times m} & D_{n \times n} \end{array} \right) \), denoted \( \text{str}(X) \), as 
\[ \text{tr}(A) - \text{tr}(D). \]

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This is the Type A analog.

\[ A(m, n) := \mathfrak{sl}(m + 1|n + 1), m > n \geq 0 \text{ and } A(n, n) := \mathfrak{psl}(n + 1|n + 1). \]

\(^2\)Makes sense! Think sdim.
Orthosymplectic LSA

Let’s look at Type $BCD$ analogs all at once.

**Definition**

Let $V_0 \oplus V_\uparrow$ be a vector superspace. A bilinear form $f : V \times V \to \mathbb{C}$ is said to be **even** if $f(V_i, V_{1-i}) = 0$.

In terms of matrices, $f$ has top-right and bottom-left blocks equal to 0 matrices.

**Definition**

A bilinear form $f$ is said to be **supersymmetric** if $f(v \otimes w) = f(s_{V,V}(v \otimes w))$ for any $v, w \in V$.

If $f$ is even, then $f |_{V_0 \times V_0}$ is symmetric and $f |_{V_\uparrow \times V_\uparrow}$ is skew-symmetric.

---

3 slight abuse of notation
Orthosymplectic LSA

Skew-symmetric + non-degeneracy = symplectic

\( \mathfrak{osp} \)

For \( i \in \mathbb{Z}_2 \), \( \mathfrak{osp}(V)_i := \{ X \in \text{End}(V)_i : f(Xv, w) = -(-1)^{|v|} f(v, Xw), \forall v, w \} \)
Orthosymplectic LSA

Skew-symmetric + non-degeneracy = symplectic

**osp**

For $i \in \mathbb{Z}_2$, $\text{osp}(V)_i := \{ X \in \text{End}(V)_i : f(Xv, w) = -(-1)^{|v|} f(v, Xw), \forall v, w \}$

Matrix form of $f$:

1. $V = \mathbb{C}^{2m+1|2n}$:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & I_m \\
0 & I_m & 0 \\
0 & I_n & 0 \\
-I_n & 0 & 0
\end{pmatrix}
$$

2. $V = \mathbb{C}^{2m|2n}$: delete the first row and column.

The condition is now $fX = -X^{s^\top} f$ where $(-)^{s^\top} : (A \; B) \mapsto (A^\top \; C^\top) (-B^\top \; D^\top)$ is called the supertranspose.
Orthosymplectic LSA

**BCD**

\[ B(m, n) := \mathfrak{osp}(2m + 1|2n), \quad m \geq 0, n \geq 1; \]
\[ D(m, n) := \mathfrak{osp}(2m|2n), \quad m \geq 2, n \geq 1; \]
\[ C(n) := \mathfrak{osp}(2|2n - 2), \quad n \geq 2. \]
Exceptional LSAs (still basic!)

1. $D(2,1,\alpha)$, a (continuum) one-parameter family, dim = (6|8);
2. $F(4)$, aka $AB(1|3)$, dim = (24|16);
3. $G(3)$, aka $AG(1|2)$. dim = (17|14).
Denote $g = D(2, 1, \alpha)$. Let $g^{(i)} = \mathfrak{sl}(2)$ and $V^{(i)}$ be the standard representation of $\mathfrak{sl}(2)$ for $i = 1, 2, 3$. Then $g_{\bar{0}} := g^{(1)} \oplus g^{(2)} \oplus g^{(3)}$ and $g_{\bar{T}} := V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ (an irreducible $g_{\bar{0}}$-module).
Denote \( g = D(2, 1, \alpha) \). Let \( g^{(i)} = \mathfrak{sl}(2) \) and \( V^{(i)} \) be the standard representation of \( \mathfrak{sl}(2) \) for \( i = 1, 2, 3 \). Then \( g_0 := g^{(1)} \oplus g^{(2)} \oplus g^{(3)} \) and \( g_T := V^{(1)} \otimes V^{(2)} \otimes V^{(3)} \) (an irreducible \( g_0 \)-module).

Not enough to define a Lie superalgebra. Need to define the symmetric bracket from \( g_T \times g_T \) to \( g_0 \). Do that by using three parameters \( \alpha_1, \alpha_2, \alpha_3 \). Let \( g = g(\alpha_1, \alpha_2, \alpha_3) \). Must have \( \sum \alpha_i = 0 \).
Denote $\mathfrak{g} = D(2, 1, \alpha)$. Let $\mathfrak{g}^{(i)} = \mathfrak{sl}(2)$ and $V^{(i)}$ be the standard representation of $\mathfrak{sl}(2)$ for $i = 1, 2, 3$. Then $\mathfrak{g}_0 := \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}$ and $\mathfrak{g}_\top := V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ (an irreducible $\mathfrak{g}_0$-module).

Not enough to define a Lie superalgebra. Need to define the symmetric bracket from $\mathfrak{g}_\top \times \mathfrak{g}_\top$ to $\mathfrak{g}_0$. Do that by using three parameters $\alpha_1, \alpha_2, \alpha_3$. Let $\mathfrak{g} = \mathfrak{g}(\alpha_1, \alpha_2, \alpha_3)$. Must have $\sum \alpha_i = 0$.

**Redundancy:** $\mathfrak{g}(\alpha_1, \alpha_2, \alpha_3) = \mathfrak{g}(c\alpha_1, c\alpha_2, c\alpha_3) = \mathfrak{g}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)})$ for any non-zero $c \in \mathbb{C}$ and any $\sigma \in S_3$. 
D(2, 1, α)

Denote \( \mathfrak{g} = D(2, 1, \alpha) \). Let \( \mathfrak{g}^{(i)} = \mathfrak{sl}(2) \) and \( V^{(i)} \) be the standard representation of \( \mathfrak{sl}(2) \) for \( i = 1, 2, 3 \). Then \( \mathfrak{g}_0 := \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \) and \( \mathfrak{g}_\Gamma := V^{(1)} \otimes V^{(2)} \otimes V^{(3)} \) (an irreducible \( \mathfrak{g}_0 \)-module).

Not enough to define a Lie superalgebra. Need to define the \textit{symmetric} bracket from \( \mathfrak{g}_\Gamma \times \mathfrak{g}_\Gamma \) to \( \mathfrak{g}_0 \). Do that by using three parameters \( \alpha_1, \alpha_2, \alpha_3 \). Let \( \mathfrak{g} = \mathfrak{g}(\alpha_1, \alpha_2, \alpha_3) \). Must have \( \sum \alpha_i = 0 \).

\textbf{Redundancy:} \( \mathfrak{g}(\alpha_1, \alpha_2, \alpha_3) = \mathfrak{g}(c\alpha_1, c\alpha_2, c\alpha_3) = \mathfrak{g}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}) \) for any non-zero \( c \in \mathbb{C} \) and any \( \sigma \in S_3 \).

It turns out that

\[ D(2, 1, \alpha) := \mathfrak{g}(\alpha, 1, -1 - \alpha) \]

is simple when \( \alpha \neq -1, 0 \). Regarding the name, notice \( D(2, 1, 1) = D(2, 1) = \mathfrak{osp}(4|2) \).
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\[ F(4) \text{ aka } AB(1|3), \text{ and } G(3) \text{ aka } AG(1|2) \]

\[ F(4)_0 := \mathfrak{sl}(2) \oplus \mathfrak{so}(7). \quad F(4)_\uparrow := \text{Natural Rep. of } \mathfrak{sl}(2) \otimes \text{Spin Rep. of } \mathfrak{so}(7). \]
$F(4)$ aka $AB(1|3)$, and $G(3)$ aka $AG(1|2)$

$$F(4)_0 := \mathfrak{sl}(2) \oplus \mathfrak{so}(7).\ F(4)_{\uparrow} := \text{Natural Rep. of } \mathfrak{sl}(2) \otimes \text{Spin Rep. of } \mathfrak{so}(7).\ G(3)_0 := \mathfrak{sl}(2) \oplus G_2,\ G(3)_{\uparrow} := \text{Natural Rep. of } \mathfrak{sl}(2) \otimes \text{The Fund. Rep. of } G_2.$$
$F(4)$ aka $AB(1|3)$, and $G(3)$ aka $AG(1|2)$

\[ F(4)_0 := \mathfrak{sl}(2) \oplus \mathfrak{so}(7). \]
\[ F(4)_1 := \text{Natural Rep. of } \mathfrak{sl}(2) \otimes \text{Spin Rep. of } \mathfrak{so}(7). \]
\[ G(3)_0 := \mathfrak{sl}(2) \oplus G_2, \quad G(3)_1 := \text{Natural Rep. of } \mathfrak{sl}(2) \otimes \text{The Fund. Rep. of } G_2. \]

Details see [Mus12]
Strange Ones

The previous ones all have even non-deg. invariant bilinear forms (e.g. Killing forms). They are called basic.
The strange ones do NOT!
Let $n \geq 2$. We let

$$p(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) : \text{tr} A = 0, B^\top = B, C^\top = -C \right\}.$$ 

$p(n)_0 \cong \mathfrak{sl}(n+1)$, and $p(n)_\uparrow = p(n)_{-1} \oplus p(n)_1$ as a $p(n)_0$-module. Here $p(n)_0$ is the diagonal even part, while $p(n)_{\pm 1}$ are the $B$ and $C$ parts respectively.
Periplectic LSA $\mathfrak{p}(n)$

Let $n \geq 2$. We let

$$\mathfrak{p}(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in \mathfrak{gl}(n + 1|n + 1) : \text{tr} A = 0, B^\top = B, C^\top = -C \right\}.$$ 

$\mathfrak{p}(n)_0 \cong \mathfrak{sl}(n + 1)$, and $\mathfrak{p}(n)_{-1} = \mathfrak{p}(n)_{-1} \oplus \mathfrak{p}(n)_1$ as a $\mathfrak{p}(n)_0$-module. Here $\mathfrak{p}(n)_0$ is the diagonal even part, while $\mathfrak{p}(n)_{\pm 1}$ are the $B$ and $C$ parts respectively. Can be regarded as the subalgebra of $\mathfrak{sl}(n + 1|n + 1)$ preserving certain odd symmetric form.
Let $n \geq 2$. We let $\hat{q}(n) := \{ (\begin{array}{cc} A & B \\ B & A \end{array}) \in \mathfrak{gl}(n+1|n+1) : \text{tr} B = 0 \}$, and define

$$q(n) := \frac{[\hat{q}(n), \hat{q}(n)]}{\mathbb{C}I_{n+1|n+1}}$$
Let $n \geq 2$. We let $\hat{q}(n) := \{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) : \text{tr} B = 0 \}$, and define

$$q(n) := [\hat{q}(n), \hat{q}(n)]/\mathbb{C}I_{n+1|n+1}$$

When people study $q(n)$, the $\hat{q}(n)$ version is often used for computations.
Those I just include for the sake of the completion of the discussion...
The Cartan Series

Those I just include for the sake of the completion of the discussion...

\( W(n), S(n), \tilde{S}(2n), H(n) \)

Let \( \Lambda(n) \) be the exterior algebra on \( n \) letters \( \xi_i, i = 1, \ldots, n \). \( \Lambda(n) \) has a natural parity grading induced by \( \deg \xi_i = \bar{1} \). Then we define

\[
W(n) := \text{Der} \, \Lambda(n)
\]

with \( W(n)_i = \{ D \in \text{End}_i(\Lambda(n)) : D(ab) = D(a)b + (-1)^{|a|}aD(b) \} \)
The Cartan Series

Those I just include for the sake of the completion of the discussion...

$W(n), S(n), \tilde{S}(2n), H(n)$

Let $\bigwedge(n)$ be the exterior algebra on $n$ letters $\xi_i, i = 1, \ldots, n$. $\bigwedge(n)$ has a natural parity grading induced by $\text{deg} \, \xi_i = \bar{1}$. Then we define

$$W(n) := \text{Der} \, \bigwedge(n)$$

with $W(n)_i = \{D \in \text{End}_i(\bigwedge(n)) : D(ab) = D(a)b + (-1)^{|a|}aD(b)\}$

In particular, any homogeneous derivation can be expressed in the form of

$$\sum_{i=1}^{n} p_i \frac{\partial}{\partial \xi_i}$$

with $p_i \in \bigwedge(n)$. The other three are subalgebras of $W(n)$. 
Classification Theorem

The following is a complete list of finite dimensional simple Lie superalgebras over $\mathbb{C}$, up to some low rank isomorphisms:

1. A finite dimensional simple Lie algebra;
2. $A(m, n)$, $m > n \geq 0$; $A(n, n)$, $n \geq 1$; $B(m, n)$, $m \geq 0, n \geq 1$; $C(n)$, $n \geq 2$; $D(m, n)$, $m \geq 2, n \geq 1$ (basic);
3. $D(2, 1, \alpha)$ for $\alpha \neq -1, 0$, $F(4)$, $G(3)$ (exceptional, basic);
4. $\mathfrak{p}(n)$, $\mathfrak{q}(n)$ for $n \geq 2$ (strange);
5. $W(n)$, $S(n)$, $\tilde{S}(2n)$, $H(n)$ (Cartan).
Classification Theorem

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Kac used non-degeneracy/degeneracy of Killing forms, rep. theory of $\mathfrak{g}_0$, grading/filtration, etc. Pretty lengthy. Real forms and Kac–Moody superalgebras are studied by Vera Serganova [Ser83, Ser11].
Not so good news

1. Recall we may construct a semisimple Lie algebra using Cartan matrix + Serre’s relations. The same can be said for the basic LSAs. But it fails for other LSAs.
2. A result by Djokovic and Hochschild says that the only not purely even LSAs with Weyl’s complete reducibility is $\mathfrak{osp}(1|2n)$.
3. Unlike the Lie algebra case, the Borel subalgebras are not conjugate to each other. The choice of positivity matters.
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Yes, I didn’t talk about root systems but they exist.
Root Systems

We will look at Type A OF COURSE!
Root Systems

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One can “diagonalize” the adjoint action of the diagonal matrices in $\mathfrak{gl}(m|n)$ as usual. It’s the same as $\mathfrak{gl}(n)$ in the non-super setting. The plot twist is the form/inner product on roots is different now. Remember we defined supertrace. Instead of $(\epsilon_i, \epsilon_j) = \delta_{ij}$, we now have $(\epsilon_i, \epsilon_j) = - (\delta_i, \delta_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$, where $\epsilon_i, \delta_j$ are standard coordinates of the diagonal matrices. Note $\epsilon_m - \delta_1$ has ODD root space and its length is 0. This means it’s isotropic.
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Remember, $\epsilon_m - \delta_1$ is a simple root with length 0. The reflection is not well-defined.

Remark: For other simple roots, they generate $S_m \times S_n$ which just permutes $\epsilon$'s with $\epsilon$'s and $\delta$'s with $\delta$'s.
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   Schur polynomials are symmetric in the usual sense as the Weyl group is $S_n$.

Important question: how do we do these for $\mathfrak{gl}(m|n)$?
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\[ \lambda' = \lambda - \alpha \text{ for } (\lambda, \alpha) \neq 0 \text{ else } \lambda' = \lambda. \]

So $(\cdots, \times, \cdot, \cdots)$ becomes

- $(\cdots, y, x, \cdots)$ if $x = -y$, or
- $(\cdots, y + 1, x - 1, \cdots)$ if $x \neq -y$.

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   Yes, super Schur polynomials exist!
Supersymmetric Polynomials

Let $V$ be an $m + n$ dimensional vector space with the standard basis $\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n$ and coordinates $x_1, \ldots, x_m, y_1, \ldots, y_n$. Let $W_0$ be $S_m \times S_n$ which acts on $x_i$ and $y_j$ separately. Let $f \in \mathcal{P}(V)$ be a polynomial on $V$. In $V$, we set $\Pi_{\epsilon_i - \delta_j} := \{ v \in V : x_i(v) + y_j(v) = 0 \}$.

We say $f$ is supersymmetric if
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We say $f$ is supersymmetric if

1. $f \in \mathfrak{P}(V)^{W_0}$;
2. $f(X + \epsilon_i - \delta_j) = f(X)$ if $x_i + y_j = 0$, i.e. $f(X + \alpha) = f(X)$ for $X \in \Pi_{\alpha = \epsilon_i - \delta_j}$.

The first condition is the usual symmetry, while the second one captures some “odd” condition.
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The first condition is the usual symmetry, while the second one captures some "odd" condition. Super Schur polynomials appear as characters of certain simple f.d. modules. They are supersymmetric and basis for the ring of supersymmetric polynomials.
Weyl Groupoids

Group = Small category of one object with invertible morphisms.
Groupoid = Multi-object version of a group!
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Group action: a group homomorphism from $W$ to $GL(V)$, equiv. to a functor from $W$ to $GL(V)$.

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The object is sent to $V$, while a morphism is sent to a linear isomorphism of $V$.

Then how does a groupoid $\mathcal{W}$ act on a vector space $V$?
Let $\mathcal{AF}(V)$ be the category in which

1. Objects: all affine subspaces of $V$;
2. Morphisms: $\text{Hom}(U, W) := \{\text{affine linear } f : U \to W\}$
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1. **Objects**: all affine subspaces of $V$;
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Let $\mathcal{W}$ be a groupoid, then we say

**(Groupoid Action)**

$\mathcal{W}$ acts on $V$ if there is a functor $\mathcal{C}$ from $\mathcal{W}$ to $\mathcal{AF}(V)$.

This degenerates to the usual group action if there is only one object $\ast$ and $\mathcal{C}(\ast) = V$. 
Weyl Groupoids

Denote the set of isotropic roots as $\Sigma$. Let $W_0$ be the Weyl group which is generated by the reflections of anisotropic roots.
Denote the set of isotropic roots as $0\Sigma$. Let $W_0$ be the Weyl group which is generated by the reflections of anisotropically roots. The \textit{isotropic roots groupoid} $0\mathcal{S}$ is a groupoid such that Obj($0\mathcal{S}$) = $0\Sigma$, with non-trivial morphisms $\tau_\alpha : \alpha \to -\alpha$. Thus \[
\text{Hom}_{0\mathcal{S}}(\alpha, \beta) = \begin{cases} 
\emptyset & \text{if } \beta \neq \pm \alpha \\
\{ \tau_\alpha \} & \text{if } \beta = -\alpha \\
\{ \text{id}_\alpha \} & \text{if } \beta = \alpha
\end{cases}\]

One can define the semidirect product of $W_0$ and $0\mathcal{S}$ via the action of $W_0$ on $0\Sigma$. Let us define the Weyl groupoid as follows

$$\mathcal{W} \coloneqq W_0 \sqcup W_0 \ltimes 0\mathcal{S}$$
Weyl Groupoids

An action ([SV11]) of \( \mathfrak{W} \) on \( \mathfrak{h}^* \) is given by (loosely speaking),

1. sending \( * \in \text{Obj}(W_0) \) to the entire \( V \), and \( W_0 \) acts on \( V \) as usual;
2. sending \( \alpha \in \text{Obj}(0S) = 0\Sigma \) to \( \Pi_\alpha := \{ \mu \in \mathfrak{h}^*: (\mu, \alpha) = 0 \} \), and \( \bar{\tau} \) to \( \tau: \mu \mapsto \mu + \alpha \) in \( \Pi_\alpha \);
3. making sure that \( W_0 \)'s action and \( 0S \)'s action are compatible.
Weyl Groupoids

A function \( f \) on \( V \) is \( W \)-invariant if \( f(wx) = f(x) \) for any \( w \in W \). Similarly, we can define groupoid invariance:

**Invariance**

Let \( \mathcal{M} \) act on \( V \) via \( \mathcal{C} \). Then a function \( F \) defined on \( V \) is said to be \( \mathcal{M} \)-invariant if \( F|_{\mathcal{C}(x)} = F|_{\mathcal{C}(y)} \circ \mathcal{C}(f) \) for any \( f : x \to y \) in \( \mathcal{M} \). Thus, \( F(\mathcal{C}(f)x) = F(x) \).

**Punchline**

Supersymmetric polynomials on \( \mathfrak{h}^* \) are \( \mathcal{M} \)-invariant w.r.t. the action above.
My Work

I started with $\mathfrak{gl}$, but ended up with Type $BC$ supersymmetry (even supersymmetry) as I used restricted root systems.
Set-up

Let’s consider $g = gl(2m|2n), \mathfrak{k} = gl(m|n) \oplus gl(m|n)$. Such a pair comes from certain symmetric superspace.
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where $\mathfrak{p}^\pm$ are abelian.

Turns out that as a $\mathfrak{k}$-module, $\mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{p}^+)$ is completely reducible and multiplicity free. The components $W^*_\lambda \otimes W_\lambda$ are nicely parametrized by certain partitions/Young diagrams ($\lambda$).
One may choose " $1 \in \text{End}_k(W_\lambda)\text{Id}_\lambda$" canonically.

\[
W_\lambda^* \otimes W_\lambda \xrightarrow{\mathfrak{U}(p^-) \otimes \mathfrak{U}(p^+)} \mathfrak{U}(\mathfrak{g})^\xi \xrightarrow{\Gamma} \mathfrak{S}(\mathfrak{a})^{\mathfrak{m}_0}
\]

\[1 \xrightarrow{\text{def.}} D_\lambda \xrightarrow{\Gamma(D_\lambda)} \]

Here $\Gamma$ is the restricted root system version of Harish-Chandra isomorphism.
Results

Proposition ([Zhu22])

The algebra $\text{Im} \Gamma$ consists precisely of the symmetric polynomials on $a^*$ with Type BC supersymmetry property.

Can be reformulated as

$$\text{Im} \Gamma = \mathfrak{S}(a)^\mathfrak{W} \cong \mathfrak{P}(a^*)^\mathfrak{W}$$

Theorem ([Zhu22])

Assuming a conjecture, the Harish-Chandra image of the super Shimura operator associate with $\mu$, $\Gamma(D_\mu)$, is equal to some non-zero multiple of a Type BC supersymmetric interpolation polynomial.
Thank you!
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