A Second Course in Differential Geometry

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Contents

Chapter 1

Differential Forms and Hodge Theory

"Thoroughly conscious ignorance is the prelude to every real advance in science."

—James Clerk Maxwell

1.1 Differential forms and de Rham theory

We recall basic notions about differential forms and de Rham cohomology. Details can be found in any introductory differential geometry textbook.

1.1.1 Linear algebra

Let V be a real vector space. Its dual space V^* is defined as the space of linear functionals $f : V \to \mathbb{R}$. For $k \geq 0$, let

$$\Lambda^k V^*$$

be the space of alternating k-linear functionals, i.e., functions $f(v_1, \ldots, v_k)$ which are linear in each variable and which satisfy

$$f(\cdots, v_i, \cdots, v_j, \cdots) = -f(\cdots, v_j, \cdots, v_i, \cdots).$$

Then $\Lambda^1 V^* = V^*$ and we define

$$\Lambda^0 V^* = \mathbb{R},$$

i.e., constant functions. If $(e_1, \dots, e_n)$ is a basis of V with $(e^1, \dots, e^n)$ dual basis of V^*, then

$$e^I := e^{i_1} \wedge \cdots \wedge e^{i_k}, \text{ where } I = (1 \leq i_1 < \cdots < i_k \leq n)$$

is a basis of $\Lambda^k V^*$. As V is the dual space of V^*, $\Lambda^k V$ can be understood similarly. Define

$$\Lambda^\bullet V = \bigoplus_{i \geq 0} \Lambda^i V.$$

There is also a "wedge product"

$$\Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V$$

which is bilinear, associative, and "supercommutative"

$$\alpha \wedge \beta = (-1)^{\deg\alpha \cdot \deg\beta} \beta \wedge \alpha.$$

1.1.2 Differential forms

Let M be a smooth n-dimensional manifold. For each $x \in M$, the tangent space $T_x M$ and cotangent space $T_x^* M$ are dual to each other. If $(x^1, \dots, x^n)$ are local coordinates near x, then the natural basis of $T_x M$ is

$$\Big(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\Big)$$

and the dual basis of $T_x^* M$ is

$$\Big(dx^1, \dots, dx^n\Big).$$

One can consider the exterior power

$$\Lambda^k T_x^* M.$$

A **differential k-form** is locally written as

$$\sum_I f_I(x^1, \ldots, x^n) dx^I$$

which "transforms" in the correct way: when $y^j = y^j(x^1, \ldots, x^n)$ is another local coordinate system, then

$$dy^{j_1} \wedge \cdots \wedge dy^{j_k} = \sum_{i_1, \ldots, i_k} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_k}}{\partial x^{i_k}} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Let $\Omega^k(M)$ be the space of differential k-forms on the manifold M. There is the wedge product of differential forms

$$\Omega^k(M) \otimes \Omega^l(M) \to \Omega^{k+l}(M).$$

If we understand $x \mapsto y$ is a smooth map $\phi : M \to N$ between two different manifolds written in terms of local coordinates, the above formula gives the "pullback" of differential forms, written as

$$\phi^* \alpha \in \Omega^k(N).$$

This is contravariant in the sense that if $\psi : N \to P$ is another map, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

There is the natural **exterior differential** of any differential form, determined as follows. For a smooth function f (a 0-form), define

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

Then extend this operator to all forms by the **Leibniz Rule**

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

The exterior differentiation respects the pullback:

$$d(\phi^* \alpha) = \phi^* d\alpha.$$

In other words, for a smooth map $\phi : M \to N$ the diagram commutes.

$$\begin{tikzcd} \Omega^k(N) \arrow[r, "d"] \arrow[d, "\phi^*"'] & \Omega^{k+1}(N) \arrow[d, "\phi^*"] \\ \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \end{tikzcd}$$

1.1.3 de Rham cohomology

The *Poincaré lemma* says that $d^2 = 0$. Then consider the chain complex of (infinite-dimensional) vector spaces.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

Poincaré lemma implies that $\mathrm{Im}(d_{k-1}) \subset \ker(d_k)$.

Definition 1.1.1. The k-th **de Rham cohomology** of M is

$$H^k_{\mathrm{dR}}(M) := \ker(d_k)/\mathrm{Im}(d_{k-1}).$$

It is a slighly nontrivial fact that for a compact smooth manifold M, $H^k_{\mathrm{dR}}(M)$ is finite-dimensional. It is more nontrivial to have

Theorem 1.1.2. *(de Rham Theorem) For a smooth manifold M,*

$$H^k_{\mathrm{dR}}(M) \cong H^k(M; \mathbb{R}).$$

For a proof of the de Rham theorem, see the book of Bott–Tu [BT13].

1.1.4 Pullback and homotopy invariance

Let $f : M \to N$ be a smooth map. Because pullback commutes with d, it follows that f pulls back closed forms to closed forms and pulls back exact forms to exact forms. As a consequence, f induces a linear map

$$f^* : H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M).$$

A crucial feature is the invariance of this pullback map under homotopy.

Definition 1.1.3. Two smooth maps $f_0, f_1 : M \to N$ are called **smoothly homotopic** if there is a smooth map $F : M \times [0,1] \to N$ such that

$$F|_{M\times\{i\}} = f_i,\ \ i = 0, 1.$$

Proposition 1.1.4. *Suppose $f_0, f_1 : M \to N$ are smoothly homotopic, then $f_0^* = f_1^*$ as maps between de Rham cohomologies.*

Proof. Let $\alpha \in \Omega^k(N)$ be any closed k-form. It suffices to show that $f_1^*\alpha - f_0^*\alpha$ is an exact form on M. Choose a smooth homotopy $F : M \times [0,1] \to N$. Consider the k-form $F^*\alpha \in \Omega^k(M \times [0,1])$ which can be written as

$$F^*\alpha = \beta + dt \wedge \gamma$$

where $\beta(t)$ is a smooth family of k-forms on M and $\gamma(t)$ is a smooth family of $k-1$ forms on M. As $F^*\alpha$ is closed on $M \times [0,1]$, one has

$$0 = dF^*\alpha = d^M F^*\alpha + dt \wedge \frac{\partial}{\partial t} F^*\alpha = d^M\beta - dt \wedge d^M\gamma + dt \wedge \frac{\partial \beta}{\partial t}.$$

Compare the terms with dt, one has

$$\frac{\partial \beta}{\partial t} = d^M\gamma.$$

Notice that $\beta(0) = f_0^*\alpha$ and $\beta(1) = f_1^*\alpha$. Then

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 dt \wedge \frac{\partial \beta}{\partial t} = \int_0^1 dt \wedge d^M\gamma = d^M\left(\int_0^1 dt \wedge \gamma\right)$$

which is exact. □

1.2 Harmonic Forms

1.2.1 Linear algebra

An inner product $\langle \cdot, \cdot \rangle$ on a real vector space induces inner products on all tensor spaces, including the dual space V^* as well as exterior power $\Lambda^k V$. If $e_1, \ldots, e_k$ is an orthonormal basis of V, then the elements

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k},\ \ I = (i_1 < \cdots < i_k)$$

form an orthonormal basis of $\Lambda^k V$.

An **orientation** on a real vector space V is an equivalence class of basis: two bases are equivalent if the transition matrix between them has positive determinant. Each vector space has precisely two orientations. Equivalently, an orientation is given by a nonzero top form $v_1 \wedge \cdots \wedge v_n$ where two top forms represent the same orientation if they differ by a positive factor.

If V is equipped with both an inner product and orientation, then there is a unique "positive" volume form

$$\mathrm{dvol} := e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$$

if $(e_1, \ldots, e_n)$ is a positively oriented orthonormal basis.

1.2.2 Riemannian metric

Recall that a Riemannian metric is a symmetric 2-tensor

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j \in \Gamma(T^*M \otimes T^*M)$$

which is also positive definite. Equivalently, it defines an inner product on tangent spaces. Therefore, it induces an inner product on cotangent spaces and hence on differential forms.

On a Riemannian manifold (M, g) one can do integrations. If M is oriented, then there is an associated **volume form**

$$\mathrm{dvol}_g \in \Omega^n(M)$$

which is locally of the form

$$\sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

Then for any function $f : M \to \mathbb{R}$, its integral is defined as the integration of the top form

$$\int_M f \mathrm{dvol}_g.$$

Since we can define inner products on $\Lambda^k T_x^* M$ for all $x \in M$, one can also define a pairing on differential forms. Define a bilinear form on $\Omega^k(M)$ by

$$\langle\langle \alpha, \beta \rangle\rangle := \int_M \langle \alpha, \beta \rangle \mathrm{dvol}_g$$

which is usually called the L^2-pairing. It is an inner product on $\Omega^k(M)$ and induces the L^2-norm. However, the norm is not complete.

1.2.3 Variantional definition of harmonic forms

For a differential form $\alpha \in \Omega^k(M)$, define its **energy** by

$$E(\alpha) := \frac{1}{2}\|\alpha\|_{L^2}^2.$$

We restrict to forms representing cohomology classes, i.e., closed forms.

Definition 1.2.1. Given a de Rham cohomology class, a closed k-form α representing this class is called a **harmonic form** if

$$E(\alpha) \le E(\alpha + d\beta) \ \forall \beta \in \Omega^{k-1}(M).$$

Like the case of geodesics, the shortest one may not exist (if the metric is not complete). Nonetheless, the principle is to see the differential equation required on minimizers via the "variation."

Suppose α is a harmonic k-form. Consider a path

$$\alpha(t) = \alpha + td\beta \in \Omega^k(M)$$

and

$$F(t) := \frac{1}{2}\|\alpha(t)\|_{L^2}^2 = \frac{1}{2}\int_M \langle \alpha(t), \alpha(t) \rangle \mathrm{dvol}_g.$$

Then one has

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} F(t) \\
&= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \int_M \langle \alpha + td\beta, \alpha + td\beta \rangle \mathrm{dvol}_g \\
&= \int_M \langle \alpha, d\beta \rangle \mathrm{dvol}_g.
\end{aligned}$$

Hence a necessary condition for α being a harmonic form is that

$$\int_M \langle \alpha, d\beta \rangle \mathrm{dvol}_g = 0 \ \forall \beta \in \Omega^{k-1}(M).$$

1.2.4 Hodge star operator

We would like to define the **formal adjoint** of d, which is an operator

$$d^* : \Omega^{k+1}(M) \to \Omega^k(M)$$

such that

$$\langle\langle \alpha, d\beta \rangle\rangle = \langle\langle d^*\alpha, \beta \rangle\rangle.$$

There is a convenient way to find a formal ajoint, via the **Hodge star operator**, when the manifold is oriented.

Definition 1.2.2. Let V be an n-dimensional oriented inner product space. Let $(e_1, \ldots, e_n)$ be an oriented orthonormal basis. Define a linear operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ by

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$$

if $(e_{i_1}, \cdots, e_{i_k}, e_{j_1}, \cdots, e_{j_{n-k}})$ is an even permuation of $(e_1, \ldots, e_n)$.

In particular, one has

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n, \quad *(e_{k+1} \wedge \cdots \wedge e_n) = (-1)^{k(n-k)} e_1 \wedge \cdots \wedge e_k.$$

Therefore,

$$*^2 = (-1)^{k(n-k)} \mathrm{Id}_{\Omega^k(M)}.$$

Lemma 1.2.3. *The Hodge star operator preserves the inner product.*

Proof. Exercise. $\square$

Lemma 1.2.4. *For $\alpha, \beta \in \Lambda^k V$, one has*

$$\langle \alpha, \beta \rangle \mathrm{dvol}_g = \alpha \wedge *\beta.$$

Proof. Exercise. □

Definition 1.2.5. Define $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$d^*\alpha = (-1)^{nk+n+1} * d * \alpha.$$

Lemma 1.2.6. *d^* is a formal adjoint of d.*

Proof. Suppose $\deg\alpha = k-1$ and $\deg\beta = k$. One has

$$\begin{aligned}
\langle\langle d\alpha, \beta\rangle\rangle &= \int_M d\alpha \wedge *\beta \\
&= \int_M d(\alpha \wedge *\beta) - (-1)^{\deg\alpha} \alpha \wedge d(*\beta) \\
&= -(-1)^{\deg\alpha}(-1)^{\deg(d*\beta)(n-\deg(d*\beta))} \int_M \alpha \wedge \Big(**(d*\beta)\Big) \\
&= (-1)^k (-1)^{(n+k+1)(k+1)} \langle \alpha, *d*\beta\rangle \\
&= \langle\langle \alpha, d^*\beta\rangle\rangle. \qquad \square
\end{aligned}$$

Therefore, if α is a harmonic form, then

$$d\alpha = d^*\alpha = 0.$$

1.2.5 The Hodge Laplacian

Definition 1.2.7. The Hodge Laplacian is the operator

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M).$$

As $d^2 = 0$, it follows $(d^*)^2 = 0$ and hence

$$\Delta = (d + d^*)^2.$$

The operator $d + d^*$ is a type of **Dirac operator**.

ocal_Laplacian **Exercise 1.2.8.** *Prove that with respect to any local basis $\alpha^1, \dots, \alpha^{m_k}$ of $\Lambda^k T^*M$ (with respect to which a k-form is written as a vector-valued function $\mathbf{f}$ in any local coordinates $x^1, \dots, x^n$), the Hodge Laplacian has the form*

$$\Delta\mathbf{f} = -\sum_{i=1}^n \frac{\partial^2 \mathbf{f}}{(\partial x^i)^2} + \text{lower order terms.}$$

Proposition 1.2.9. *The following conditions are equivalent.*

1. α *is a harmonic form.*
2. $d\alpha = 0$ *and* $d^*\alpha = 0$.
3. $\Delta\alpha = 0$.
4. $(d + d^*)\alpha = 0$.
5. $dd^*\alpha = d^*d\alpha = 0$.

Proof. We only prove the equivalence between (2) and (5). The rest is left as exercise. Obviously (2) implies (5). Suppose $dd^*\alpha = 0$, then

$$\langle\langle dd^*\alpha, \alpha\rangle\rangle = 0.$$

As d^* is the formal adjoint of d, the above is equal to

$$\langle\langle d^*\alpha, d^*\alpha\rangle\rangle = \|d^*\alpha\|^2 = 0.$$

Therefore $d^*\alpha = 0$. The implication $d^*d\alpha = 0 \Longrightarrow d\alpha = 0$ is similar. $\square$

Exercise 1.2.10. *Prove that if* M *is compact and connected, then the only harmonic functions are constants and the only harmonic top forms are constant multiples of the volume form.*

Notice that if we use the second order viewpoint, then the above statement follows from the maximal principle.

1.3 Hodge theorem

a priori we do not know if a de Rham cohomology class admits a harmonic representative, or an energy minimizer. The main statement of Hodge theorem is that it is the case. If the de Rham complex were finite-dimensional, this is a simple algebraic fact. The hard part is that the de Rham complex is infinite-dimensional and the differential operators are unbounded. The most crucial feature is the ellipticity of the Hodge Laplacian.

Theorem 1.3.1 (Hodge theorem)**.** *Let (M, g) be a compact oriented Riemannian manifold.*

1. *$\mathcal{H}^k(M)$ is finite-dimensional.*

2. *There exists an orthogonal decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) \oplus \mathrm{Im}(d^* : \Omega^{k+1}(M) \to \Omega^k(M)).$$

The orthogonality of the three components are obvious. However the finite-dimensionality, as well as that the three summands generate all differential forms are nontrivial.

Corollary 1.3.2. *The natural map*

$$\mathcal{H}^k(M) \to H^k_{\mathrm{dR}}(M)$$

is an isomorphism. Namely, each de Rham cohomology class has a unique harmonic representative.

Proof. If α_1, α_2 are both harmonic, then the Hodge decomposition theorem implies that their difference $\alpha_1 - \alpha_2$ is orthogonal to the image of d, hence cannot be an exact form. So the map $\mathcal{H}^k(M) \to H^k_{\mathrm{dR}}(M)$ is injective. On the other hand, if α is a closed k-form, then the Hodge decomposition induces a unique decomposition

$$\alpha = \alpha_0 + d\beta + d^*\gamma$$

where α_0 is harmonic. As $d\alpha = 0$, it follows that $dd^*\gamma = 0$ which implies $d^*\gamma = 0$. Hence $\alpha = \alpha_0 + d\beta$, i.e., the cohomology class of α is also represented by the harmonic form α_0. $\square$

One can get some topological facts easily from the Hodge theorem (although it is not really the point). For example, there is a bilinear pairing

$$H^k_{\mathrm{dR}}(M) \otimes H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}, \ [\alpha] \cap [\beta] := \int_M \alpha \wedge \beta.$$

(It is independent of the representing closed forms α and β.) Let's call it the **intersection form**.

Exercise 1.3.3 (Poincaré duality)**.** *Prove that the intersection pairing is nondegenerate. Namely, if* $[\alpha] \in H^k_{\mathrm{dR}}(M)$ *and*

$$[\alpha] \cap [\beta] = 0 \ \forall [\beta] \in H^{n-k}_{\mathrm{dR}}(M),$$

then $[\alpha] = 0$*. In particular, the pairing defines a linear isomorphism*

$$H^k_{\mathrm{dR}}(M) \cong (H^{n-k}_{\mathrm{dR}}(M))^*.$$

Remark 1.3.4. When $\dim M = 4k$, the intersection form on $H^{2k}_{\mathrm{dR}}(M)$ is a quadratic form. The **signature** of M is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form.

1.3.1 Proof of Hodge decomposition

The technical theorems

The crucial fact one should remember is that the infinite-dimensional space $\Omega^k(M)$ is not complete and the Hodge Laplacian is unbounded with respect to the L^2-norm. When dealing with variational problems, it is fundamental to have a completion of $\Omega^k(M)$ and consider objects in the completion.

Definition 1.3.5. Let S be a normed vector space. A linear functional $T : S \to \mathbb{R}$ is said to be **bounded** if there exists $C > 0$ such that

$$\|Ts\| \leq C\|s\| \ \forall s \in S.$$

The set of bounded linear functionals S^* is a normed vector space with norm defined by

$$\|T\| := \sup\Big\{\|Ts\| \mid \|s\| = 1\Big\}.$$

The dual space S^* is always complete, i.e., Cauchy sequences always converge.

When the norm on S is induced from an inner product, then each $a \in S$ induces a linear functional $T_a : S \to \mathbb{R}$ by

$$T_a(s) = \langle\langle s, a \rangle\rangle$$

and $\|T_a\| = \|a\|$. Moreover, the map $a \mapsto T_a$ embeds S into S^*. If S is complete, i.e., a Hilbert space, then $S \cong S^*$.

In our case, $S = \Omega^k(M)$ equipped with the L^2 inner product. It is possible that a Cauchy sequence does not converge in S. However, it always converges in the larger dual space.

Definition 1.3.6. Given $\beta \in \Omega^k(M)$. A **weak solution** to the equation $\Delta\alpha = \beta$ is a bounded linear functional

$$\varphi : \Omega^k(M) \to \mathbb{R}$$

such that

$$\varphi(\Delta\gamma) = \langle\langle \beta, \gamma \rangle\rangle \;\forall \gamma \in \Omega^k(M).$$

Theorem 1.3.7 (Regularity Theorem). *If φ is a weak solution to $\Delta\alpha = \beta$, then there exists $\alpha \in \Omega^k(M)$ such that $\Delta\alpha = \beta$ and such that*

$$\varphi(\gamma) = \langle\langle \alpha, \gamma \rangle\rangle.$$

The next theorem is about when a convergence (even in the complete space) could happen. It tells another feature of elliptic equations.

Theorem 1.3.8 (Compactness Theorem). *Let $\alpha_n \in \Omega^k(M)$ be a sequence of differential forms such that*

$$\|\alpha_n\|_{L^2} \leq c, \|\Delta\alpha_n\|_{L^2} \leq c.$$

Then a subsequence α_{n_k} is a Cauchy sequence, i.e.,

$$\|\alpha_{n_k} - \alpha_{n_l}\|_{L^2} \to 0.$$

Proof assuming the regularity and compactness theorems

We first prove that $\mathcal{H}^k(M)$ is finite-dimensional. If not the case, then one can choose an infinite orthonormal sequence $\alpha_n \in \mathcal{H}^k(M)$. Then by the compactness theorem, there is a subsequence α_{n_k} which is a Cauchy sequence. However, an orthonormal sequence cannot be a Cauchy sequence.

For the decomposition, from basic linear algebra, one can see that the splitting follows from an orthogonal splitting

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{Im}(\Delta).$$

Now consider the orthogonal complement

$$(\mathcal{H}^k(M))^\perp \subset \Omega^k(M).$$

It remains to prove that

$$\mathrm{Im}(\Delta) = (\mathcal{H}^k(M))^\perp.$$

As Δ is formally self-adjoint, it is easy to see that $\Delta(\mathcal{H}^k(M)) \subset (\mathcal{H}^k(M))^\perp$. Hence it remains to prove that $(\mathcal{H}^k(M))^\perp \subset \mathrm{Im}(\Delta)$. In other words, for each $\beta \in (\mathcal{H}^k(M))^\perp$, solve the equation

$$\Delta\alpha = \beta.$$

We first find a weak solution φ, which should satisfy

$$\varphi(\Delta\gamma) = \langle\langle \gamma, \beta \rangle\rangle.$$

Notice that this determines the value of φ on $\mathrm{Im}(\Delta)$. Moreover, if $\Delta\gamma = \Delta\gamma'$, then $\varphi(\Delta\gamma) - \varphi(\Delta\gamma') = \langle\langle \gamma - \gamma', \beta \rangle\rangle$ which is zero as β is orthogonal to $\mathcal{H}^k(M)$. Hence one obtains a linear functional

$$\varphi : \mathrm{Im}(\Delta) \to \mathbb{R}.$$

In order to find a weak solution, one needs that φ extends to a bounded linear functional on $\Omega^k(M)$.

Claim. There exists $C > 0$ such taht

$$\|\beta\| \leq C\|\Delta\beta\| \ \forall \beta \in (\mathcal{H}^k(M))^\perp. \tag{1.3.1}$$ estimate

Proof of the claim. If it is not the case, then there exist sequence $\beta_i \in (\mathcal{H}^k(M))^\perp$ with $\|\beta_i\| = 1$ and $\|\Delta\beta_i\| \to 0$. Then by the compactness theorem, there is a subsequence (still indexed by i for simplicity) which is a Cauchy sequence. Hence β_i converges in the L^2 space and one can define a linear functional

$$\varphi(\alpha) := \lim_{i\to\infty} \langle\langle \beta_i, \alpha\rangle\rangle.$$

One can check that φ is a weak solution: for any test form γ,

$$|\varphi(\Delta\gamma)| = \left|\lim_{i\to\infty} \langle\langle \beta_i, \Delta\gamma\rangle\rangle\right| = \left|\lim_{i\to\infty} \langle\langle \Delta\beta_i, \gamma\rangle\rangle\right| \le \lim_{i\to\infty} \|\Delta\beta_i\|_{L^2}\|\gamma\|_{L^2} = 0.$$

Then by the regularity theorem, there is $\beta_\infty \in \mathcal{H}^k(M)$ representing φ. Moreover, $\beta_i \to \beta_\infty$ in L^2 norm. However, it is impossible as β_i are orthogonal to $\mathcal{H}^k(M)$ and $\|\beta_i\| = 1$. Therefore the claim is proved. □

Then because $\Delta\gamma \in \Delta(\Omega^k(M)) \subset (\mathcal{H}^k(M))^\perp$, by (1.3.1), one has $\|\gamma\| \le C\|\Delta\gamma\|$. Hence

$$|L_\beta(\Delta\gamma)| \le \|\beta\|\|\gamma\| \le C\|\beta\|\|\Delta\gamma\|.$$

Then by Hahn–Banach theorem, one can extend L_β to a bounded linear functional on the whole $\Omega^k(M)$. Then the definition of L_β implies that it is a weak solution to $\Delta\alpha = \beta$. By the regularity theorem, $\beta \in \Delta(\Omega^k(M))$.

1.4 Proof of the technical results

The regularity and compactness results are basic features of elliptic operators. There are many ways to establish them. The approach used here (from Warner's book) is quite elementary after setting up the notations (mostly from Fourier analysis).

1.4.1 Sobolev spaces via Fourier calculus

Consider complex-valued functions $f : \mathbb{R}^n \to \mathbb{C}$ which are periodic, i.e.

$$f(\cdots, x^i + 2\pi, \cdots) = f(\cdots, x^i, \cdots).$$

For example, the Fourier monomials for $\xi_1, \ldots, \xi_n \in \mathbb{Z}$:

$$e^{i\xi\cdot x} := e^{\mathbf{i}(\xi_1 x^1 + \cdots + \xi_n x^n)}.$$

The values of f is determined by its restriction on the cube

$$Q := \Big\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid -\pi \le x^i \le \pi\Big\}.$$

We also view such f as functions on the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

For each multiindex (usually representing the order of a higher order partial derivative or the exponents of a multivariable monomial)

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$$

denote

$$[\alpha] = \alpha_1 + \cdots + \alpha_n, \ \|\alpha\| = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}.$$

In particular, we can regard x or α as an n-dimensional complex vector. For such vectors $\xi = (\xi_1, \ldots, x_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$, let

$$\xi \cdot \eta := \sum_i \xi_i \bar{\eta}_i$$

be the standard Hermitian pairing. Denote

$$D_i = -\sqrt{-1}\frac{\partial}{\partial x^i}$$

and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \left(-\sqrt{-1}\right)^{[\alpha]} \frac{\partial^{[\alpha]}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}.$$

Fourier transform

The most crucial notation is the notion of Fourier series. Given an integrable periodic function u, the associated Fourier series is a function on $\xi \in \mathbb{Z}^n$ with

$$\hat{u}(\xi) := \frac{1}{(2\pi)^n} \int_Q u(x) e^{-\mathbf{i}\xi\cdot x} dx.$$

A very basic fact, coming from integration by parts and the periodicity of the function, is that

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi)$$

for all multiindex α. Formally, one should have

$$u(x) = \sum_\xi \hat{u}(\xi) e^{\mathbf{i}\xi\cdot x}.$$

However, the right hand side may not converge or converge nicely.

_fourier_decay **Lemma 1.4.1.** *Suppose $u \in \mathcal{P}$. Then for all $t \in \mathbb{R}$, one has*

$$\sum_\xi \left(1 + |\xi|^2\right)^t |\hat{u}(\xi)|^2 < \infty. \tag{1.4.1}$$ decay

Proof. By integration by parts (using the periodicity), for any $k \in \mathbb{N}$, for some $C_k > 0$ (independent of ξ), one has

$$\|\hat{u}(\xi)\| \le \frac{C_k}{(1 + |\xi|^2)^{\frac{k}{2}}} \sup_{|\alpha| \le k} \sup_Q |D^\alpha u| \,. \qquad \square$$

lemma_Sobolev **Lemma 1.4.2** (Sobolev Lemma)**.** *If $t > \frac{n}{2}$ and $\hat{u} : \mathbb{Z}^n \to \mathbb{C}$ is a function satisfying (1.4.1), then the Fourier series*

$$\sum_\xi \hat{u}(\xi) e^{\mathbf{i}\xi\cdot x}$$

converges uniformly to a continuous periodic function. Moreover, if $\hat{u}$ is the Fourier series of a smooth periodic function u, then the limit of the Fourier series is u.

Proof. For the uniform convergence, it suffices to show the absolute convergence

$$\sum_\xi |\hat{u}(\xi)| < \infty.$$

Indeed, for any $N > 0$, one has

$$\begin{aligned}\sum_{|\xi|\le N} |\hat{u}(\xi)| &= \sum_{|\xi|\le N} (1 + |\xi|^2)^{-\frac{t}{2}} (1 + |\xi|^2)^{\frac{t}{2}} |\hat{u}(\xi)| \\ &\le \sum_{|\xi|\le N} (1 + |\xi|^2)^{-t} \sum_{|\xi|\le N} (1 + |\xi|^2)^t |\hat{u}(\xi)|^2\end{aligned}$$

which is uniformly bounded. Therefore, the Fourier series converges uniformly to a periodic continuous function ϕ. We prove that $u = \phi$.

Indeed, by the fact that the trignometric nomonials $e^{\mathbf{i}\xi\cdot x}$ are L^2-orthogonal, one can see that $\hat{\phi} = \hat{u}$. In particular, $\phi - u$ is orthogonal to all finite Fourier series. However, by Stone–Weierstrass theorem, the trigonometric nomonials form a complete basis. Hence $\phi - u = 0$. □

Corollary 1.4.3. *If $t > \frac{n}{2} + m + 1$ and $\hat{u} : \mathbb{Z}^n \to \mathbb{C}$ satisfies (1.4.1), then the Fourier series $\sum_\xi u_\xi e^{ix\cdot\xi}$ converges (together with all derivatives up to order m) uniformly to a C^m periodic function.*

Proof. Exercise. □

Sobolev spaces

Definition 1.4.4. Let $\mathcal{S}$ (the Schwarz space) be the space of all functions $\hat{u} : \mathbb{Z}^n \to \mathbb{R}$ (or just Fourier series). For any $s \in \mathbb{R}$, define the **Sobolev space**

$$H_s := \Big\{\varphi \in \mathcal{S} \mid \sum_\xi (1 + |\xi|^2)^s |\varphi_\xi|^2 < \infty\Big\}.$$

Define a Hermitian inner product on H_s by

$$\langle\langle u, v\rangle\rangle_s := \sum_\xi (1 + |\xi|^2)^s u_\xi \cdot v_\xi$$

where the dot is the Hermitian inner product of complex vectors.

Then H_s is the L^2-space of a certain discrete measure space. Hence H_s is complete, i.e., a Hilbert space.

Remark 1.4.5. When s is a nonnegative integer, we can interprete the Sobolev norm as follows. Because

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi)$$

one has

$$\begin{aligned}\sum_{[\alpha]\leq s}\|D^\alpha u\|_{L^2}^2 &= \sum_{[\alpha]\leq s}\sum_{\xi}|\widehat{D^\alpha u}(\xi)|^2\\ &= \sum_{[\alpha]\leq s}\sum_{\xi}|\xi^\alpha|^2|\hat{u}(\xi)|^2\\ &\leq C\sum_{\xi}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\\ &= C\|u\|_s^2.\end{aligned}$$

On the other hand, one can also control

$$(1+|\xi|^2)^s \leq C'\sum_{[\alpha]\leq s}|\xi^\alpha|^2.$$

Hence when s is an integer, the Sobolev norm is roughly the sum of L^2-norms of all derivatives up to order s. A function lies in H_s if all its partial derivatives up to order s are L^2-integrable.

Let $\mathcal{P}$ denote the space of smooth periodic functions. The Fourier transform defines a linear inclusion

$$\mathcal{P}\hookrightarrow\mathcal{S}.$$

In some sense, H_s is a family of complete spaces sitting between the space of smooth functions and the space of Fourier series. The L^2-space is exactly H_0. Lemma 1.4.1 and Lemma 1.4.2 imply that

$$\mathcal{P}=\bigcap_{s\in\mathbb{R}}H_s=\lim_{s\to+\infty}H_s.$$

As Fourier transform changes derivatives to multiplications, we can extend the differentiation to the Sobolev spaces by

$$(D^\alpha)\hat{u}(\xi)=\xi^\alpha\hat{u}(\xi).$$

emma_bounded_1 **Lemma 1.4.6.** *D^α defines a bounded linear operator from $H_{s+[\alpha]}$ to H_s.*

Proof. Exercise. □

lemma_bounded_2 **Lemma 1.4.7.** *Multiplying by a smooth periodic function is a bounded linear operator from H_s to H_s.*

Proof. Exercise. We only need the case when s is an integer. In this case, one can use integration by parts. □

Rellich lemma

Lemma 1.4.8. *Let u^i be a sequnece of elements in H_t with $\|u^i\|_t$ uniformly bounded. Then for any $s < t$, a subsequence of u^i converges in H_s.*

Proof. We inductively construct a subsequence. For each ξ, one can choose a subsequence of u^i_ξ which converges. THen by the diagonal argument, one can find a subsequence such that u^i_ξ converges for all ξ. We claim that it is a Cauchy sequence in H_s. Indeed, for any N one has

$$\begin{aligned}
\|u^i - u^j\|_s^2 &= \sum_\xi \left(1 + \|\xi\|^2\right)^s |u^i_\xi - u^j_\xi|^2 \\
&= \sum_\xi \left(1 + \|\xi\|^2\right)^{s-t} \left(1 + \|\xi\|^2\right)^t |u^i_\xi - u^j_\xi|^2 \\
&= \sum_{\|\xi\| \le N} \left(1 + \|\xi\|^2\right)^{s-t} \left(1 + \|\xi\|^2\right)^t |u^i_\xi - u^j_\xi|^2 \\
&\quad + \sum_{\|\xi\| > N} \left(1 + \|\xi\|^2\right)^{s-t} \left(1 + \|\xi\|^2\right)^t |u^i_\xi - u^j_\xi|^2.
\end{aligned}$$

Then given ϵ, by the uniformly boundedness of $\|u^i\|_t$, one can choose N such that the second term on the right is less than ϵ, for all i and j; then by the convergence of u^i_ξ, one can guarantee that when i and j are large, the first term is at most ϵ. □

1.4.2 Elliptic operators

Differential operators

Consider a differential operator with smooth coefficients varying with $x \in \mathbb{R}^n$

$$L = \sum_{\alpha} A_\alpha(x) D^\alpha$$

where $A_\alpha(x)$ is an $N \times N$ complex matrix. Such an operator acts on smooth $\mathbb{C}^N$-valued functions on $\mathbb{R}^n$. Its order is the maximal $[\alpha]$ such that $A_\alpha \neq 0$. When the coefficients $A_\alpha(x)$ are periodic, it acts on periodic functions.

Associated to L, consider the **principal symbol**

$$\sigma_L(x)(\xi) := \sum_{[\alpha]=k} \xi^\alpha A_\alpha(x) \in \mathbb{C}^{N\times N}$$

which is a homogeneous polynomial of degree k.

Definition 1.4.9. L is called **elliptic** at x if for all $\xi \neq 0$, $\sigma_L(x)(\xi)$ is an invertible matrix.

Now we assume that the coefficients $A_\alpha(x)$ are all periodic in x. Then as $\sigma_L(\xi)$ is homogeneous, one can see that if L is elliptic, then for some constant $C > 0$ and any $v \in \mathbb{C}^N$, one has

$$|\sigma_L(\xi)(v)| \geq \frac{1}{C}|\xi|^k|v| \geq \frac{1}{C}(1+|\xi|^2)^{\frac{k}{2}}|v|, \forall \xi \neq 0.$$

For example, the standard Laplacian is

$$\Delta_0 = \sum_{i=1}^n D_i^2.$$

Its symbol is

$$\sigma_{\Delta_0}(\xi) := \sum_{i=1}^n \xi_i^2 = |\xi|^2.$$

Hence the Laplacian is elliptic.

The importance of ellipticity is the associated estimate. In the simplest case when the coefficients A_α are all constants, on Fourier series, L acts like

$$\widehat{Lu}(\xi) = \sum_\alpha \xi^\alpha A_\alpha \hat{u}(\xi).$$

lemma_bounded **Lemma 1.4.10.** *Let L be a differential operator of order k. Then for all $s \in \mathbb{R}$, L induces a bounded linear operator from H_{s+k} to H_s.*

Proof. It is a consequence of Lemma 1.4.6 and Lemma 1.4.7. □

The following is the crucial **elliptic estimate**. Its importance is analogous to the fundamental ϵ-δ language in mathematical analysis.

Proposition 1.4.11. *Suppose L is an ellptic operator of degree k with periodic coefficients. Then there exists a constant $C > 0$ such that for all $u \in H_{s+k}$*

$$\|u\|_{s+k} \leq C\left(\|Lu\|_s + \|u\|_s\right).$$

Proof. We only prove for the case when L has only the maximal degree part and has constant coefficients. The proof for the general case can be found in [War83]. One has

$$\widehat{Lu}(\xi) = \sum_\xi \sum_\alpha A_\alpha \xi^\alpha \hat{u}(\xi).$$

The ellipticity implies that

$$\begin{aligned}
\|u\|_{s+k}^2 &= \sum_\xi (1+|\xi|^2)^{s+k} |\hat{u}(\xi)|^2 \\
&= \sum_\xi (1+|\xi|^2)^s (1+|\xi|^2)^k |\hat{u}(\xi)|^2 \\
&= |\hat{u}(0)|^2 + \sum_{\xi \neq 0} (1+|\xi|^2)^s (1+|\xi|^2)^k |\hat{u}(\xi)|^2 \\
&\leq \|u\|_s^2 + C_1 \sum_{\xi \neq 0} (1+|\xi|^2)^s |\sigma_L(\xi)(\hat{u}(\xi))|^2 \\
&\leq \|u\|_s^2 + C_2 \sum_{\xi \neq 0} (1+|\xi|^2)^s |\widehat{Lu}(\xi)|^2 \\
&\leq C_3\left(\|u\|_s^2 + \|Lu\|_s^2\right).
\end{aligned}$$

□

1.4.3 Proofs of the technical results for periodic functions

The regularity theorem

On the flat torus T^n (with periodic coordinates $x^1, \ldots, x^n$), consider a differential operator (on $\mathbb{C}^N$-valued functions) of the form

$$\Delta = \Delta_0 + \rho$$

where

$$\Delta_0 = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}$$

is the standard Laplacian and ρ is an operator of order at most 1 with periodic and smooth coefficients.

rop_regularity **Proposition 1.4.12.** *Suppose $\alpha \in H_{-\infty} = \mathcal{S}$ and $\beta \in H_s$ such that $\Delta_0 \alpha = \beta$, then $\alpha \in H_{s+2}$.*

Proof. In terms of Fourier coefficients, one has

$$|\xi|^2 \hat{\alpha}(\xi) = \hat{\beta}(\xi).$$

As $\hat{\beta} \in H_s$, then we see

$$\begin{aligned}
\|\hat{\alpha}\|_{s+2}^2 &= \sum_{\xi} (1+|\xi|^2)^s |\hat{\alpha}(\xi)|^2 \\
&= |\hat{\alpha}(0)|^2 + \sum_{\xi \neq 0} (1+|\xi|^2)^{s+2} |\hat{\alpha}(\xi)|^2 \\
&\leq |\hat{\alpha}(0)|^2 + C \sum_{\xi} |\xi|^2 (1+|\xi|^2)^s |\hat{\alpha}(\xi)|^2 \\
&= |\hat{\alpha}(0)|^2 + C \sum_{\xi} (1+|\xi|^2)^s |\hat{\beta}(\xi)|^2 \\
&= |\hat{\alpha}(0)|^2 + C \|\hat{\beta}\|_s^2
\end{aligned}$$

which converges. Hence $\hat{\alpha} \in H_{s+2}$. $\square$

Now consider the differential equation

$$\Delta u = \beta \in \mathcal{P}.$$

Suppose α is a weak solution. Then α is L^2-integrable, hence $\alpha \in H_0$. Then by Lemma 1.4.10, $\rho\alpha \in H_{-1}$. The equation can be rewritten as

$$\Delta_0 \alpha = \beta - \rho\alpha.$$

Then as $\beta \in H_\infty \subset H_{-1}$ and $\rho\alpha \in H_{-1}$, it follows from Proposition 1.4.12 that $\alpha \in H_1$.

Suppose we have proved that $\alpha \in H_s$ for some s. Then by Lemma 1.4.10, $\rho\alpha \in H_{s-1}$. Since $\beta \in H_{s-1}$, it follows from Proposition 1.4.12 that $\alpha \in H_{s+1}$. This is the so-called **elliptic bootstrapping**. Then inductively

$$\alpha \in \bigcap_s H_s = H_\infty = \mathcal{P}.$$

Namely α is smooth.

The compactness theorem

Suppose β_i is a sequence of $\mathbb{C}^N$-valued periodic functions satisfying

$$\|\beta_i\|_0 \le C,\ \ \|\Delta\beta_i\|_0 \le C.$$

Then by the elliptic estimate for the operator Δ_0, one has

$$\begin{aligned}
\|\beta_i\|_1 &\le C\Big(\|\beta_i\|_{-1} + \|\Delta_0\beta_i\|_{-1}\Big)\\
&\le C\Big(\|\beta_i\|_{-1} + \|\Delta\beta_i\|_{-1} + \|\rho\beta_i\|_{-1}\Big)\\
&\le C\Big(\|\beta_i\|_{0} + \|\Delta\beta_i\|_{0} + \|\beta_i\|_{0}\Big).
\end{aligned}$$

Notice that one obtained the elliptic estimate for Δ which is periodic but possibly has nonconstant coefficients. The right hand side is uniformly bounded by the assumption. Then by Rellich's lemma for the inclusion $H_1 \hookrightarrow H_0$, a subsequence converges in H_0.

1.4.4 The general case

Compactness theorem

We first need to reduce the problem to a local one where we can use previous results about periodic operators.

Lemma 1.4.13. *There exist a finite cover of M by coordinate charts $\phi_\nu : U_\nu \to \mathbb{R}^n$, a subordinate partition of unity $\rho_\nu : U_\nu \to [0,1]$, a local orthonormal basis $(u^1, \ldots, u^N)$ of $\Lambda^k T^* M$, and a constant $C > 0$ satisfying the following conditions: on each U_ν, with respect to the basis $(u^1, \ldots, u^N)$, the Hodge Laplacian is written as*

$$\Delta = \sum_{i=1}^n D_i^2 + \sum_{j=1}^n V^j(x) D_i + W(x)$$

such that

$$\sup_{U_\nu} \left(|V^j(x)| + |W(x)| + |\nabla \rho_v(x)| + |\nabla^2 \rho_v(x)| \right) \leq C.$$

Now, regarding the compactness theorem, suppose $\alpha \in \Omega^k(M)$. Then over U_ν we can regard α as a $\mathbb{C}^N$-valued function. As we are using an orthonormal basis, the L^2-norm is preserved. Hence one has

$$\|\Delta \alpha\|_{L^2(U_\nu)} + \|\alpha\|_{L^2(U_\nu)} \leq C.$$

Then consider $\rho_\nu \alpha$. As we can regard U_ν to be included in a cube and ρ_ν is supported in U_ν, $\rho_\nu \alpha$ is zero near the boundary of the cube, hence can also be regarded as a periodic function on $\mathbb{R}^n$. We then need to translate the norm bound for $\rho_\nu \alpha$. Indeed, one has

$$\|\rho_\nu \alpha\|_{L^2} \leq \|\alpha\|_{L^2(U_\nu)}.$$

On the other hand, the coefficients of Δ may not be periodic. However, we can modify the coefficients at where $\rho_\nu \alpha$ vanishes to have a periodic operator Δ_ν such that

$$\Delta_\nu(\rho_\nu \alpha) = \Delta(\rho_\nu \alpha).$$

On the other hand,

$$\Delta_\nu(\rho_\nu\alpha) = \rho_\nu\Delta_\nu(\alpha) + [\Delta_\nu, \rho_\nu](\alpha)$$

where $[\Delta_\nu, \rho_\nu]$ is a first-order operator whose coefficients are functions coming from ρ_ν and its derivative, as well as coefficients of Δ_ν. To be slightly more rigorous, find another function ρ'_ν supported in the cube which is identically 1 where $\rho_\nu \neq 0$. Then $\rho_\nu = \rho_\nu\rho'_\nu$. Then we can rewrite it as

$$\Delta_\nu(\rho_\nu\alpha) = \rho_\nu\Delta_\nu(\rho'_\nu\alpha) + [\Delta_\nu, \rho_\nu](\rho'_\nu\alpha).$$

Then all terms are periodic. Then by the elliptic estimate, one has

$$\begin{aligned}
\|\rho_\nu\alpha\|_1 &\leq C\left(\|\rho_\nu\alpha\|_{-1} + \|\Delta_\nu(\rho_\nu\alpha)\|_{-1}\right) \\
&\leq C\left(\|\alpha\|_{L^2} + \|\rho_\nu(\Delta_\nu\alpha)\|_{-1} + \|[\Delta_\nu, \rho_\nu](\rho'_\nu\alpha)\|_{-1}\right) \\
&\leq C\left(\|\alpha\|_{L^2} + \|\Delta\alpha\|_{L^2} + \|\alpha\|_{L^2}\right)
\end{aligned}$$

which is uniformly bounded. Then by the Rellich lemma, a subsequence of $\rho_\nu\alpha_n$ will converge on the cube. As finitely many cubes are involved, one has a well-defined limit on the whole manifold.

Regularity theorem

Let φ be a weak solution to $\Delta\alpha = \beta$. Its restriction to each U_ν is still an L^2-integrable function. So is $\rho_\nu\varphi$. We can treat it as peroidic, hence an element $\varphi_\nu \in H_0$. Similar as before, one can choose another cut-off function ρ'_ν supported in the cube which is identically 1 on the support of ρ_ν. Then

$$\Delta_\nu(\rho_\nu\varphi) = \rho_\nu\Delta_\nu(\rho'_\nu\varphi) + [\Delta_\nu, \rho_\nu](\rho'_\nu\varphi) = \rho_\nu\beta + [\Delta_\nu, \rho_\nu](\rho'_\nu\varphi).$$

The first term on the right hand side is smooth. Moreover, as $\rho'\varphi$ is in H_0 and $[\Delta_\nu, \rho_\nu]$ is of first order, the second term is in H_{-1}. Hence by the regularity theorem, $\rho_\nu\varphi$ is in H_1.

Viewing this idea, one can find a nested sequence of open subsets and cut-off functions to show that near any point, there is a neighborhood on which φ is smooth.

1.5 References and Remarks

The first instance where people essentially have the idea of harmonic forms is the Maxwell equation in electromagnetism, where electromagnetic fields are described as differential forms in space time and the Maxwell equation says that the electromagnetic field is harmonic (in vacuum).

A classical reference to the theory of differential forms is the book [BT13], where you also get an exposure of homological algebra, homotopy theory, spectral sequences, and characteristic classes.

The theory of harmonic forms can be phrased in more physical way as a model for "supersymmetric quantum mechanics." Witten wrote a very influential paper [Wit82] which relates de Rham theory with Morse theory and which popularized this idea to the mathematical community.

It is possible to extend the Hodge decomposition to the case of manifolds with boundary. In this case, there are two relevant cohomology groups $H^*(M)$ and $H^*(M, \partial M)$, corresponding to two different boundary conditions of the Laplace equation (Dirichlet and Neumann).

One can also extend the theory of harmonic forms to complex or Kähler geometry. For the Kähler case, you can look at [GH14] or [WGP80].

The Hodge-de Rham theory is surely very elegant. However, students should not be misguided as if this is the only way to understand (co)homology. In fact de Rham theory is a model for cohomology theory which has both its advantage and disadvantage: for example, torsion classes in integral (co)homology cannot be detected by differential forms. There are also fascinating stories of other models of (co)homology theory as well as other generalized (co)homology theories.

A classical reference for second order elliptic operators is [GT77].

Chapter 2

Vector Bundles and Connections

"It is by logic that we prove, but by intuition that we discover."

— Henri Poincaré

During this section we will shift from real situation to complex situations. There are many reasons to prefer complex numbers to real numbers. For example, the set of invertible complex numbers is connected.

2.1 Vector bundles

Definition 2.1.1. A **real vector bundle** over a topological space M consists of a topological space E, a continuous surjective map $\pi : E \to M$, and structures of real vector spaces on the **fibre** $E_x = \pi^{-1}(x)$, which satisfy the **local triviality** condition

- For each $x \in M$, there exists a pair (U_x, φ_x) where $U_x \subset M$ is an open neighborhood of x and

$$\varphi_x : \pi^{-1}(U_x) \cong U_x \times \mathbb{R}^k \tag{2.1.1}$$

  is a homeomorphism such that for each $y \in U_x$, $\varphi_x|_{E_y}$ is mapped to $\{y\} \times \mathbb{R}^k$ and is a linear isomorphism.

A **local trivialization** of a vector bundle $\pi : E \to M$ is a pair (U, φ) such that $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a homeomorphism and maps E_x isomorphically onto $\{x\} \times \mathbb{R}^k$ for all $x \in U$. Given two local trivializations $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β), the **transition function** is

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

which must be of the form

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \ni (x, v) \mapsto (x, g_{\alpha\beta}(x)v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

where

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k; \mathbb{R})$$

is a continuous map.

Exercise 2.1.2. *Following the pattern of defining smooth manifolds (after defining topological manifolds), define the notion of* **smooth vector bundles** *over smooth manifolds. One must choose an "atlas" among all local trivializations which are "compatible" in a certain sense.*

From now on we only consider smooth vector bundles over smooth manifolds. Immediate examples include the tangent bundle, cotangent bundle, tensor bundles, etc. In the smooth case, the total space E is itself a smooth manifold and the projection $\pi_E : E \to M$ is a smooth submersion.

We can easily consider the complex case, where fibres are complex vector spaces and transition functions take values in complex matrices.

2.1.1 Sections

Definition 2.1.3. A **cross section** (or **section**) of a vector bundle $E \to M$ is a smooth map $s : M \to E$ such that

$$\pi_E \circ s = \mathrm{Id}_M.$$

Let $\Gamma(E)$ denote the set of all smooth sections of E.

$\Gamma(E)$ is (typically) an infinite-dimensional vector space and is a module over $C^\infty(M)$. A section of the tangent bundle TM is just a vector field.

Exercise 2.1.4. *Given a smooth vector bundle $E \to M$ and any $v \in E_x$, there exists a section $s : M \to E$ such that $s(x) = v$.*

The zero locus of a section $s : M \to E$ is defined to be

$$s^{-1}(0) = \{x \in M \mid s(x) = 0 \in E_x\}.$$

Theorem 2.1.5 (Hairy Ball Theorem)**.** *Any section of $TS^2 \to S^2$ has nonempty zero locus.*

2.1.2 Bundle maps

Definition 2.1.6. A map of (smooth) vector bundles over M from E_1 to E_2 is a smooth map $f : E_1 \to E_2$ which satisfies $\pi_{E_2} \circ f_1 = \pi_{E_1}$ and which maps fibres of E_1 linearly to the corresponding fibres of E_2. f is said to be an **isomorphism** if its fibrewise restrictions are linear isomorphisms.

Subbundle and quotient bundles

Definition 2.1.7. Let $E \to M$ be a vector bundle of rank k. A subbundle of rank l is a bundle map $F \to E$ which is fibrewise injective. We may regard F as a union of fibrewise subspaces.

Exercise 2.1.8. *Suppose E has rank k and a subbundle $F \subset E$ has rank l. Prove that there exist a bundle atlas whose transition functions are block-upper triangular. (Hint: any basis of a subspace can be extended to a basis of the total space).*

Given a subbundle $F \subset E$, one can form the quotient bundle

$$E/F \to M$$

whose fibre at $x \in M$ is the quotient space E_x/F_x. There is then a bundle map $E \to E/F$. In fact there is an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow E/F \longrightarrow 0\,.$$

Trivial bundles

Definition 2.1.9. A vector bundle $E \to M$ is **trivial** if it is isomorphic to **the** trivial bundle $E_0 = M \times \mathbb{R}^k$.

Exercise 2.1.10. *Prove that a rank k vector bundle $E \to M$ is trivial if and only if there are k sections $s_1, \dots, s_k \in \Gamma(E)$ such that for all $x \in M$, $s_1(x), \dots, s_k(x)$ form a basis of E_x.*

2.1.3 Operations

Pullback

Definition 2.1.11. Let $f : N \to M$ be a smooth map and $\pi : E \to M$ be a smooth vector bundle. The **pullback** of E by f is

$$f^*E := \Big\{(y, v) \subset N \times E \mid y \in N, v \in E,\ f(y) = \pi(v) \in M\Big\}.$$

Exercise 2.1.12. *Prove that the natural map $f^*E \to N$ has the structure of a smooth vector bundle.*

Exercise 2.1.13. *Two C^∞ maps $f_0, f_1 : N \to M$ are* **smoothly homotopic** *if there is a smooth map $\tilde{f} : [0,1] \times N \to M$ such that $\tilde{f}(i, \cdot) = f_i(\cdot)$ for $i = 0, 1$. Prove that if f_0 and f_1 are smoothly homotopic, then*

$$f_0^*E \cong f_1^*E.$$

(First try to prove that under the condition that N is compact. The proof will become much easier once we use connections.)

Direct sums, tensor products, etc.

All operations on vector spaces can be generalized to vector bundles. The direct sum of two vector bundles $E_1, E_2 \to M$ is a vector bundle $E_1 \oplus E_2 \to M$ whose fibre at $x \in M$ is the direct sum $E_{1,x} \oplus E_{2,x}$. Then tensor product

$E_1 \otimes E_2$ is defined similarly. The dual vector bundle E^* is fibrewise the dual space of E. When $E \to M$ is a complex vector bundle, one has its conjugate

$$\overline{E} \to M.$$

Moreover, there are the exterior powers $\Lambda^k E$. When $E = T^*M$, $\Lambda^k T^*M$ is the bundle hosting differential k-forms.

Definition 2.1.14. An **inner product** on a vector bundle $E \to M$ is a section of $E^* \otimes E^*$ which is fibrewise a symmetric, nondegenerate bilinear form (i.e., an inner product) on E_x.

A Riemannian metric on M is just an inner product on TM. All smooth vector bundles admits an inner product. When E is a complex vector bundle, one can also consider Hermitian inner product, which is a section of $E^* \otimes \bar{E}^*$.

Lemma 2.1.15. *A real vector bundle $E \to M$ is isomorphic to its dual E^*. A complex vector bundle $E \to M$ is isomorphic to $\overline{E}^*$.*

Proof. Choose a (Euclidean or Hermitian) inner product on E. □

Let $E \to M$ be a vector bundle equipped with an inner product. Let $F \subset E$ be a subbundle and $F^\perp$ be the orthogonal complement. Then $F^\perp$ is also a subbundle. Moreover, one has the obvious isomorphisms

$$F \oplus F^\perp \cong E \text{ and } F^\perp \cong E/F.$$

2.1.4 An analogue of Whitney embedding theorem

One has the Whitney embedding theorem saying that all smooth manifolds embeds into $\mathbb{R}^n$. Similarly, all vector bundles embeds into a trivial bundle. We present the proof for the compact case.

Theorem 2.1.16. *Let M be a compact smooth manifold and $E \to M$ be a smooth vector bundle. Then there exist $N > 0$ and an injective bundle map*

$$\tilde{f} : E \to \underline{\mathbb{R}}^N.$$

Proof. By compactness, one can find a finite bundle atlas $\varphi_i : \pi_E^{-1}(U_i) \to U_i \times \mathbb{R}^k$ for $i = 1, \ldots, m$. For $j = 1, \ldots, k$, denote

$$e_{i,j} : U_i \to E|_{U_i}$$

be the section defined by

$$e_{i,j}(x) = \varphi_i^{-1}(x, e_j)$$

where $e_j \in \mathbb{R}^k$ is the j-th standard basis vector. Choose a subbordinate partition of unity ρ_i. Then $\rho_i e_{i,j}$ is a global section. Choose an inner product on E. Denote $N = mk$. Define

$$\tilde{f}(x, v) = (x, f_{i,j}(x, v))$$

where $f_{i,j} : E \to \mathbb{R}$ is the function defined by

$$f_{i,j}(x, v) = \langle v, \rho_i(x) e_{i,j}(x) \rangle.$$

Then $\tilde{f}$ is clearly a bundle map. We verify that it is injective. For each $(x, v) \in E$. There exists i such that $\rho_i(x) \neq 0$. Then there exists j such that $\langle v, \rho_i(x) e_{i,j}(x) \rangle \neq 0$. Hence $\tilde{f}$ is injective. □

Corollary 2.1.17. *For any smooth vector bundle $E \to M$, there exists another vector bundle $F \to M$ such that $E \oplus F$ is trivial.*

Proof. By the above theorem, choose a bundle injection $E \to \underline{\mathbb{R}}^N$. □

2.2 Connections on vector bundles

Definition 2.2.1. A **connection** (or called **covariant differentiation**) on a vector bundle $E \to M$ is a linear map

$$\nabla^E : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

which satisfies the Leibniz rule:

$$\nabla^E(fs) = df \otimes s + f \nabla^E s.$$

Equivalently, one can consider the covariant derivative in a direction

$$\nabla_X^E s := \langle X, \nabla^E s\rangle$$

where X is a tangent vector.

Exercise 2.2.2. *Connections are not tensors. In particular, they do not have "pointwise" values (contained in any finite-dimensional space). On the other hand prove that if $E, F \to M$ are vector bundles and if a linear operator*

$$T : \Gamma(E) \to \Gamma(F)$$

satisfies

$$T(fs) = fT(s)$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$, then T has pointwise values. Namely, there exists a section $T' \in \Gamma(\mathrm{Hom}(E, F))$ such that $T(s)(x) = T'(x)(s(x))$ $\forall x \in M$.

We would like to see that on any smooth vector bundles there always exist (a lot of) connections.

Proposition 2.2.3. *There exists a connection on a smooth vector bundle $E \to M$.*

Proof. First we know that on the trivial bundle $\underline{\mathbb{R}}^N \to M$ there is the trivial connection (just the ordinary differentiation). Choose a locally finite covering $\{U_\alpha\}$ of M such that $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ is a local trivialization with a subordinate partition of unity $\rho_\alpha : M \to [0, 1]$. Define

$$\nabla_\alpha := \varphi_\alpha^{-1} \circ d \circ \varphi_\alpha : \Gamma(E|_{U_\alpha}) \to \Gamma(T^*U_\alpha \otimes E|_{U_\alpha})$$

to be the connection on $E|_{U_\alpha}$ corresponding to the trivial differentiation. Then for any $s \in \Gamma(E)$, define

$$\nabla^E s = \sum_\alpha \rho_\alpha \nabla_\alpha s|_{U_\alpha}.$$

This is clearly a linear map. Moreover, for any smooth function f, one has

$$\begin{aligned}\nabla^E(fs) &= \sum_\alpha \rho_\alpha \nabla_\alpha (fs|_{U_\alpha}) = \sum_\alpha \rho_\alpha (df \otimes s|_{U_\alpha} + f\nabla_\alpha s|_{U_\alpha}) \\ &= df \otimes \sum_\alpha \rho_\alpha s|_{U_\alpha} + f\sum_\alpha \rho_\alpha \nabla_\alpha (s|_{U_\alpha}) = df \otimes s + f\nabla^E s.\end{aligned}$$

Hence ∇^E is a connection. □

Connections form an affine space.

Proposition 2.2.4. *The set of all connections on E is an affine space modelled on $\Gamma(T^*M \otimes \mathrm{End}E)$.*

Proof. Given two connections, by the Lebniz rule, their difference is a $T : \Gamma(E) \to \Gamma(T^*M \otimes E)$ which commutes with multiplying with a function. By the previous exercise, T is a section of $\Gamma(\mathrm{Hom}(E, T^*M \otimes E)) = \Gamma(T^*M \otimes \mathrm{End}E)$.

On the other hand, if ∇^E is a connection and $T \in \Gamma(T^*M \otimes \mathrm{End}E)$, one has

$$(\nabla^E + T)(fs) = df \otimes s + f(\nabla^E + T)(s).$$

Hence $\nabla^E + T$ is also a connection. □

2.2.1 Connection matrices

Given a local trivialization

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k,$$

which is equivalent to k sections $e_1, \ldots, e_k : U \to E|_U$ which are everywhere linearly independent over U, we can write the connection in the local form. We have

$$\nabla^E e_\nu = \sum_\mu \omega_{\mu\nu} e_\mu$$

where $\omega_{\mu\nu}$ is a 1-form on the open set U. The matrix $\boldsymbol{\omega} = (\omega_{\mu\nu})$ (whose entries are 1-forms) is called the connection matrix of ∇^E with respect to this local trivialization/frame. Then locally, if a section is written as

$$s = y^1 e_1 + \cdots + y^k e_k = \begin{bmatrix} e_1 & \dots & e_k \end{bmatrix} \begin{bmatrix} y^1 \\ \cdots \\ y^k \end{bmatrix}$$

where $y^1, \dots, y^k$ are functions on U, one has

$$\nabla^E s = \nabla^E \left(\sum_{\mu=1}^{k} s^\mu e_\mu \right) = \begin{bmatrix} e_1 & \dots & e_k \end{bmatrix} \left(\begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} + \boldsymbol{\omega} \right) \begin{bmatrix} y^1 \\ \cdots \\ y^k \end{bmatrix}.$$

Lemma 2.2.5. *Let $\boldsymbol{\omega}_\alpha$, $\boldsymbol{\omega}_\beta$ be the connection matrices of ∇^E with respect to local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, $\varphi_\beta : U_\beta \times \mathbb{R}^k$. Then on the overlap $U_\alpha \cap U_\beta$ there holds*

$$\boldsymbol{\omega}_\beta = g_{\alpha\beta}^{-1} \boldsymbol{\omega}_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}.$$

Proof. Let $s : U_\alpha \cap U_\beta \to E|_{U_\alpha \cap U_\beta}$ be any section, which can be written as

$$s = \sum_{\mu=1}^{k} y_\alpha^\mu e_{\alpha,\mu} = \sum_{\mu=1}^{k} y_\beta^\mu e_{\beta,\mu}.$$

Then by the definition of transition function, one has

$$\varphi_\alpha(s) = \begin{bmatrix} y_\alpha^1 \\ \vdots \\ y_\alpha^k \end{bmatrix} = g_{\alpha\beta} \varphi_\beta(s) = g_{\alpha\beta} \begin{bmatrix} y_\beta^1 \\ \vdots \\ y_\beta^k \end{bmatrix}.$$

Then

$$
\begin{aligned}
\nabla^E s &= \begin{bmatrix} e_{\alpha,1} & \dots & e_{\alpha,k} \end{bmatrix} \left(\begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} + \boldsymbol{\omega}_\alpha \right) \begin{bmatrix} y_\alpha^1 \\ \cdots \\ y_\alpha^k \end{bmatrix} \\
&= \begin{bmatrix} e_{\beta,1} & \dots & e_{\beta,k} \end{bmatrix} g_{\alpha\beta}^{-1} \left(\begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} + \boldsymbol{\omega}_\alpha \right) g_{\alpha\beta} \begin{bmatrix} y_\beta^1 \\ \cdots \\ y_\beta^k \end{bmatrix} \\
&= \begin{bmatrix} e_{\beta,1} & \dots & e_{\beta,k} \end{bmatrix} \left(\begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} + g_{\alpha\beta}^{-1} d g_{\alpha\beta} + g_{\alpha\beta}^{-1} \boldsymbol{\omega}_\alpha g_{\alpha\beta} \right) \begin{bmatrix} y_\beta^1 \\ \cdots \\ y_\beta^k \end{bmatrix}.
\end{aligned}
$$

The claim then follows. $\square$

Connections do not transform like tensors. In fact one has the following trick to turn connection matrix to zero at any point.

Lemma 2.2.6. *Given $x_0 \in M$, there exists a local trivialization φ near x such that the associated connection matrix vanishes at x_0.*

Proof. Choose a local trivialization φ with associated connection matrix $\boldsymbol{\omega}$. We only need to solve the equation

$$0 = g^{-1}\boldsymbol{\omega} g + g^{-1} dg \Longleftrightarrow \boldsymbol{\omega} + dg g^{-1} = 0$$

at x_0. Choose a local coordinate system $(x^1, \dots, x^n)$ such that x_0 is sent to the origin. In these coordinates, we can write

$$\boldsymbol{\omega} = \sum_{i=1}^{n} dx^i \boldsymbol{\omega}_i$$

where $\boldsymbol{\omega}_i(x^1, \dots, x^n)$ is a $k \times k$ matrix, then consider

$$g(x^1, \dots, x^n) = \exp(-x^1 \boldsymbol{\omega}_1(x_0)) \cdots \exp(-x^n \boldsymbol{\omega}_n(x_0)).$$

We have $g(x_0) = \mathrm{Id}$ and

$$dg(x_0) = -\sum_{i=1}^{n} dx^i \boldsymbol{\omega}_i(x_0). \qquad \square$$

2.2.2 Parallel transport

A connection induces a notion of parallel transport. Let $\gamma : [0,1] \to M$ be a smooth curve. For any section $s \in \Gamma(\gamma^* E)$, i.e., a section along the curve, one has the covariant derivative.

$$\nabla_t^E s \in \Gamma(\gamma^* E).$$

If γ is short, then one can find a local coordinate system and trivialize E around the image of γ by a local frame $(e_1, \ldots, e_k)$ with corresponding connection matrix $\boldsymbol{\omega} = (\omega_{ij})$. We write $\gamma(t) = (x^1(t), \ldots, x^n(t))$ and $s(t) = y^1(t)e_1 + \cdots + y^k(t)e_k$. Then

$$\begin{aligned} &\nabla_t^E s \\ &= \begin{bmatrix} e_1 & \ldots & e_k \end{bmatrix} \left(\begin{bmatrix} \frac{d}{dt} & & \\ & \ddots & \\ & & \frac{d}{dt} \end{bmatrix} + \boldsymbol{\omega}(\dot{\gamma}(t)) \right) \begin{bmatrix} y^1 \\ \vdots \\ y^k \end{bmatrix}. \end{aligned}$$

This is a linear differential operator on the vector value function $[y^1, \ldots, y^k]^T$. Hence given an initial value $s(0) \in E_{\gamma(0)}$, there is a unique smooth solution to

$$\nabla_t^E s(t) \equiv 0$$

subject to the initial condition; moreover, the solution is linear in the initial value. Hence there is an induced family of linear maps

$$\Phi_{t,s} : E_{\gamma(t)} \to E_{\gamma(s)}, \ t, s \in [0,1]$$

such that $\Phi_{t,t} = \mathrm{Id}_{E_{\gamma(t)}}$. This definition clearly extends to arbitrarily long piecewise smooth paths, with the obvious property

$$\Phi_{s,r} \circ \Phi_{t,s} = \Phi_{t,r}.$$

Now let's prove the homotopy invariance of vector bundle pullbacks.

Proposition 2.2.7. *Let $f_0, f_1 : M \to N$ be two smoothly homotopic maps. Let $E \to N$ be a vector bundle. Then f_0^*E is isomorphic to f_1^*E.*

Proof. Consider a smooth homotopy $F : M \times [0,1] \to N$ and denote $F(x,t)$ by $f_t(x)$. Consider the pullback bundle $F^*E \to M \times [0,1]$. The restriction of F^*E to $M \times \{i\}$ is exactly f_i^*E for $i = 0, 1$. Now choose a connection ∇ on F^*E. For each $x \in M$ and $t \in [0,1]$, let

$$\phi_x : F^*E|_{(x,0)} \to F^*E|_{(x,1)}$$

be the parallel transport along the path (x,s) for $0 \le s \le 1$ with respect to ∇. This induces a map

$$\Phi : f_0^*E = \tilde{F}|_{M\times\{0\}} \to \tilde{F}|_{M\times\{1\}} = f_1^*E$$

of bundles over M. As parallel transports are linear isomorphisms, Φ is a bundle isomorphism. □

Corollary 2.2.8. *If M is smoothly contractible, then any vector bundle over M is trivial.*

Parallel transport can provide particular kinds of local trivializations.

2.2.3 Connections in the sense of Ehresmann

The total space of a vector bundle $\pi : E \to M$ is a smooth manifold. The tangent bundle TE contains a special subbundle called the **vertical tangent bundle** $T^{\mathrm{vert}}E$ consisting of vectors tangent to fibres. Then one has

$$T^{\mathrm{vert}}E \cong \pi^*E.$$

Then one has the exact sequence

$$0 \longrightarrow \pi^*E \xrightarrow{\iota} TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0\,. \tag{2.2.1}$$

horizontal_ver

A **splitting** of this exact sequence is one of the following equivalent items.

1. A left inverse of ι, i.e., a bundle map $\phi : TE \to \pi^*E$ such that $\phi \circ \iota = \mathrm{Id}_{\pi^*E}$. This bundle map can be regarded as a 1-form
$$\theta \in \Omega^1(E, \pi^*E)$$
such that $\theta \circ \iota = \mathrm{Id}_{\pi^*E}$.

2. A right inverse of $d\pi$, i.e., a bundle map $\psi : \pi^*TM \to TE$ such that $d\pi \circ \psi = \mathrm{Id}_{\pi^*TM}$.

3. A subbundle $T^{\mathrm{hor}}M \subset TM$, called the **horizontal distribution** such that $TE = T^{\mathrm{vert}}E \oplus T^{\mathrm{hor}}E$.

Ehresmann connections are certain splittings.

Definition 2.2.9. A linear **Ehresmann connection** on E is a splitting of the exact sequence (2.2.1), given by a 1-form $\theta \in \Omega^1(E, \pi^*E)$ such that for each $\lambda \in \mathbb{R}$, if $m_\lambda : E \to E$ is the fibrewise scalar multiplication by λ, then
$$m_\lambda^*\theta = \lambda\theta.$$

Theorem 2.2.10. *There is a one-to-one correspondence between connections on E and linear Ehresmann connections.*

Proof. Suppose E is equipped with a connection ∇^E. Given a local trivialization $\varphi : E|_U \to U \times \mathbb{R}^k$, let $\boldsymbol{\omega} = (\omega_{\mu\nu})$ be the connection matrix. The local trivialization induces bundle coordinates
$$(x^1, \dots, x^n, y^1, \dots, y^k).$$
Consider the 1-forms
$$\theta = \theta^\mu \otimes e_\mu = (dy^\mu + \omega_{\mu\nu}y^\nu) \otimes e_\mu \in \Omega^1(E|_U, \pi^*E).$$
We prove that this is independent of local trivialization. Indeed, if $\varphi' : E|_{U'} \to U' \times \mathbb{R}^k$ is another local trivialization with local frames $e'_1, \dots, e'_k$ and induced

bundle coordinates $z^1, \dots, z^k$, and $g = \varphi \circ (\varphi')^{-1} : U \cap U' \to GL(k; \mathbb{R})$, then one has

$$y^\mu = g_{\mu\nu} z^\nu$$

and

$$\boldsymbol{\omega}' = g^{-1} dg + g^{-1} \boldsymbol{\omega} g.$$

Then one has

$$\begin{aligned}
(dy^\mu + \omega_{\mu\nu} y^\nu) \otimes e_\mu &= \Big(d(g_{\mu\nu} z^\nu) + \omega_{\mu\nu} (g_{\nu\kappa} z^\kappa) \Big) \otimes e_\mu \\
&= \Big(g_{\mu\nu} dz^\nu + dg_{\mu\nu} z^\nu + \omega_{\mu\nu} g_{\nu\kappa} z^\kappa \Big) \otimes e_\mu \\
&= g_{\mu\nu} \Big(dz^\nu + (g^{-1} dg)_{\nu\kappa} z^\kappa + (g^{-1} \boldsymbol{\omega} g)_{\nu\kappa} z^\kappa \Big) \otimes e_\mu \\
&= g_{\mu\nu} \Big(dz^\nu + \omega'_{\nu\kappa} z^\kappa \Big) \otimes e_\mu .
\end{aligned}$$

Hence the horizontal distribution is well-defined. It is easy to see that it is linear, hence a linear Ehresmann connection.

On the other hand, if one has a linear Ehresmann connection, then in terms of a local coordinates, one can write it as

$$\theta = \theta^\mu \otimes e_\mu = (dy^\mu + A_i^\mu dx^i) \otimes e_\mu .$$

The linear condition implies that

$$A_i^\mu = -\omega_{i,\mu\nu}(x) y^\nu .$$

In order for θ to be well-defined, when changing local frames, $\omega_{i,\mu\nu}$ undergoes the same transformation as the way of connection matrices. □

Using the notion of Ehresmann connection one can describe the covariant derivative in a different way. Let $s : M \to E$ be a section. Then viewing E as a manifold, the smooth map s has a derivative

$$ds : T_x M \to T_{s(x)} E.$$

By using the decomposition $TE \cong T^{\text{vert}} E \oplus T^{\text{hor}} E$, its projection to $T^{\text{vert}} E \cong \pi^* E$ is a map

$$\text{proj}_{T^{\text{vert}} E} \circ ds : T_x M \to T^{\text{vert}}_{s(x)} E \cong E_x .$$

One can check that this map is linear and satisfies the Leibniz rule.

2.2.4 Induced connections

Let $E, F \to M$ be vector bundles and ∇^E resp. ∇^F be connections. The direct sum connection is the connection

$$\nabla^E \oplus \nabla^F$$

on the bundle $E \oplus F$. The tensor product is the connection

$$\nabla^E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \nabla^F.$$

∇^E induces a connection on the dual bundle E^* by the corresponding Leibniz rule:

$$d\langle s, s^* \rangle := \langle \nabla^E s, s^* \rangle + \langle s, \nabla^{E^*} s^* \rangle.$$

Proposition 2.2.11. *The induced connection on* $\mathrm{End}E \cong E^* \otimes E$ *is given by*

$$\nabla^{\mathrm{End}E} \phi = [\nabla^E, \phi].$$

Proof. Given $s \in \Gamma(E)$ and $\phi \in \Gamma(\mathrm{End}E)$, one has

$$\nabla^E(\phi(s)) = (\nabla^{\mathrm{End}E}\phi)(s) + \phi(\nabla^E s).$$

So

$$(\nabla^{\mathrm{End}E}\phi)(s) = [\nabla^E, \phi](s).$$

As s is arbitrary, the proposition follows. □

2.2.5 Curvature

Definition 2.2.12. Let ∇^E be a connection on $E \to M$. Its curvature is the tensor

$$R^E \in \Gamma(T^*M \otimes T^*M \otimes \mathrm{End}(E))$$

defined by

$$R^E(X, Y) := \nabla^E_X \nabla^E_Y s - \nabla^E_Y \nabla^E_X s - \nabla^E_{[X,Y]} s.$$

Exercise 2.2.13. *Prove that R^E defined above is a tensor (although the expression requires extending X and Y locally to vector fields). Prove that $R^E(X,Y) = -R^E(Y,X)$.*

As its alternating in X and Y, it can be regarded as a differential form taking values in $\mathrm{End}(E)$, or a homomorphism with coefficients in 2-forms

$$R^E \in \Omega^2(M, \mathrm{End}(E)).$$

Another way to, perhaps more elegant, is to first extend the connection. Consider instead of $\Gamma(E)$ the space $\Omega^k(M;E)$, i.e., sections of E with differential form coefficients. Then define

$$\nabla^E : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$$

by applying the Leibniz rule:

$$\nabla^E(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\mathrm{deg}\alpha} \alpha \wedge ds.$$

Then we can define simply

$$R^E := \nabla^E \circ \nabla^E : \Omega^k(M;E) \to \Omega^{k+2}(M;E).$$

Indeed, as ∇^E increases the degree by 1, this is also equal to

$$R^E = \frac{1}{2}[\nabla^E, \nabla^E]$$

Exercise 2.2.14. *Prove that R^E defined in the above way is local, hence an element in $\Omega^2(M; \mathrm{End}(E))$.*

Exercise 2.2.15. *Prove that the curvature of a direct sum connection is $R^E \oplus R^F$; the curvature of a tensor product connection is $R^E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes R^F$.*

Curvature matrices

Take a local trivialization $\varphi : E|_U \to U \times \mathbb{R}^k$, with the associated connection matrix $\boldsymbol{\omega}$, we can write the local form of the connection as $d + \boldsymbol{\omega}$. Then the curvature is the operator

$$\Omega = (d + \boldsymbol{\omega}) \circ (d + \boldsymbol{\omega}) = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}.$$

Notice that although $\boldsymbol{\omega}$ is a 1-form, $\boldsymbol{\omega} \wedge \boldsymbol{\omega}$ is not zero because matrix multiplication is not commutative.

A different way to formulate the fact that curvature is a tensor is the following lemma.

Lemma 2.2.16. *Let Ω_α and Ω_β be the curvature matrix associated to local trivializations $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ and $\varphi_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{R}^k$. Then one has*

$$\Omega_\beta = g_{\alpha\beta}^{-1} \Omega_\alpha g_{\alpha\beta}.$$

Proof. Notice that since $gg^{-1} \equiv 1$, one has

$$0 = dgg^{-1} + gd(g^{-1}) \Longrightarrow d(g^{-1}) = -g^{-1}dgg^{-1}.$$

Then we have

$$\begin{aligned}
\Omega_\beta &= d\boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\beta \wedge \boldsymbol{\omega}_\beta \\
&= d(g^{-1}\boldsymbol{\omega}_\alpha g + g^{-1}dg) + (g^{-1}\boldsymbol{\omega}_\alpha g + g^{-1}dg) \wedge (g^{-1}\boldsymbol{\omega}_\alpha g + g^{-1}dg) \\
&= -g^{-1}dgg^{-1} \wedge \boldsymbol{\omega}_\alpha g + g^{-1}d\boldsymbol{\omega}_\alpha g - g^{-1}\boldsymbol{\omega}_\alpha \wedge dg - g^{-1}dg \wedge g^{-1}dg \\
&\quad + g^{-1}\boldsymbol{\omega}_\alpha \wedge \boldsymbol{\omega}_\alpha g + g^{-1}dgg^{-1} \wedge \boldsymbol{\omega}_\alpha g + g^{-1}\boldsymbol{\omega}_\alpha \wedge dg + g^{-1}dg \wedge g^{-1}dg \\
&= g^{-1}(d\boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_\alpha \wedge \boldsymbol{\omega}_\alpha)g \\
&= g^{-1}\Omega_\alpha g
\end{aligned}$$

□

Bianchi identity

The Bianchi identity says that the curvature is covariantly constant. Recall that ∇^E induces a connection on the bundle $\mathrm{End}(E)$. If we view sections of $\mathrm{End}(E)$ as linear maps from $\Gamma(E)$ to $\Gamma(E)$, then for $t \in \Gamma(\mathrm{End}(E))$, this connection is

$$\nabla^{\mathrm{End}(E)}(t) = [\nabla^E, t] = \nabla^E \circ t - t \circ \nabla^E.$$

Then this connection extends to $\Omega^k(\mathrm{End}(E))$, which contains the curvature $R^E \in \Omega^2(\mathrm{End}(E))$. As $R^E = \nabla^E \circ \nabla^E$, it follows that

Theorem 2.2.17 (Bianchi identity)**.** $[\nabla^E, R^E] = 0$.

2.2.6 Structural groups

In general the transition functions $g_{\alpha\beta}$ take value in the Lie group $GL(k;\mathbb{R})$ (in the real case) or $GL(l;\mathbb{C})$) (in the complex case). Additional structures on the vector bundles can be viewed as certain "reduction" to smaller groups.

We only consider the following important classes of subgroups.

1. The **orthogonal group** $O(k) \subset GL(k;\mathbb{R})$ which is the group of matrices which preserves the standard Euclidean inner product.

2. The **special orthogonal group** $SO(k) \subset O(k)$ of orthogonal matrices whose determinant is 1.

3. The **unitary group** $U(l) \subset GL(l;\mathbb{C})$ consists of unitary matrices.

4. The **special unitary group** $SU(l) \subset U(l)$ of unitary matrices whose determinant is 1.

5. The group of block-upper triangular, block-lower triangular, or block-diagonal matrices.

In general G is a subgroup of GL and a closed submanifold. Without introducing the abstract notion of Lie algebras, these classical groups have their Lie algebras. The Lie algebra of $GL(k;\mathbb{R})$ resp. $GL(l;\mathbb{C})$ is $\mathfrak{gl}(k;\mathbb{R})$ resp. $\mathfrak{gl}(l;\mathbb{C})$ which is the space of all square matrices of the fixed size. Moreover,

1. $\mathfrak{o}(k) = \mathfrak{so}(k)$ is the space of all skew-symmetric $k \times k$ matrices.

2. $\mathfrak{u}(l)$ is the space of all skew-Hermitian matrices.

3. $\mathfrak{su}(l)$ is the space of all traceless skew-Hermitian matrices.

The Lie bracket of two matrices A and B is

$$[A, B] = AB - BA.$$

The above spaces of matrices are all closed under Lie bracket. Moreover, there is an exponential map

$$\exp : \mathfrak{g} \to G$$

given by

$$\exp(A) = \sum_{m=0}^{\infty} \frac{A^m}{m!}.$$

Definition 2.2.18. Let $E \to M$ be a smooth vector bundle of rank k. Let $G \subset GL(k)$ be a subgroup. Two bundle charts $\phi_i : E|_{U_i} \to U_i \times \mathbb{R}^k$ are G-compatible if their transition functions take value in G. A **structural group reduction** of E to G consists of a bundle atlas consisting of local trivializations

$$\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

such that the transition functions $g_{\alpha\beta}$ take value in the subgroup G.

Examples

Proposition 2.2.19. *A Euclidean resp. Hermitian inner product on a real resp. complex vector bundle $E \to M$ is equivalent to a structural group reduction to $O(k)$ resp. $U(l)$.*

Proof. Suppose E is equipped with an inner product. We consider the subset of local trivializations ϕ_α such that $\phi_\alpha^{-1}(e_1), \ldots, \phi_\alpha^{-1}(e_k)$ form a local orthonormal basis of each fibre. Such local trivalizations exist because of Gram–Schmidt. This is a structural group reduction to $O(k)$ or $U(l)$. □

Proposition 2.2.20. *Let G be the group of block upper triangular matrices with diagonal blocks of sizes $k_1 \times k_1$ and $k_2 \times k_2$. Then a structural group reduction of E to G is equivalent to a subbundle of rank k_1.*

Proof. Exercise. □

Connections respecting the reduction

Suppose $E \to M$ is a vector bundle, then there are associated bundles such as $\mathrm{End}(E)$. When the structural group is reduced to $G \subset GL(k)$, we can consider a subbundle $\mathfrak{g}(E) \subset \mathrm{End}(E)$ as follows. A transformation $T \in \mathrm{End}(E)$ is said to be in $\mathfrak{g}(E)$ if for any local trivialization

$$\phi : E|_U \to U \times \mathbb{F}^k$$

belonging to the G-atlas, the induced linear map

$$\phi \circ T \circ \phi^{-1} \in \mathfrak{gl}(k)$$

belongs to the Lie algebra $\mathfrak{g}$.

Now we consider connections. A connection ∇^E is said to be a G-connection if, when using a local G-frame, the connection matrix Ω takes value in the Lie algebra $\mathfrak{g}$. When $G = O(k)$ or $U(l)$, i.e., E is equipped with a Euclidean or Hermitian inner product, such a connection is said to **respect the inner product**.

Lemma 2.2.21. *Let E be a Euclidean vector bundle over M. Then a connection ∇^E respects the inner product if and only if*

$$d\langle s_1, s_2\rangle = \langle \nabla^E s_1, s_2\rangle + \langle s_1, \nabla^E s_2\rangle.$$

Proof. If ∇^E respects the inner product, then given a local orthonormal frame $e_1, \ldots, e_k$, let $\boldsymbol{\omega}$ be the connection matrix. Let s_1, s_2 be two local sections of E which are written as

$$s_1 = y_1^\mu e_\mu,\ \ s_2 = y_2^\mu e_\mu.$$

Then

$$\langle s_1, s_2 \rangle = y_1^\mu \overline{y_2^\mu}.$$

Then

$$\begin{aligned} d\langle s_1, s_2 \rangle &= d(y_1^\mu \overline{y_2^\mu}) \\ &= dy_1^\mu \overline{y_2^\mu} + y_1^\mu d\overline{y_2^\mu} \\ &= dy_1^\mu \overline{y_2^\mu} + y_1^\mu d\overline{y_2^\mu} + y_1^\nu \boldsymbol{\omega}_{\mu\nu} \overline{y_2^\mu} + y_1^\nu \overline{\boldsymbol{\omega}_{\nu\mu} y_2^\mu} \\ &= (dy_1^\mu + \omega_{\mu\nu} y_1^\nu)\overline{y_2^\mu} + y_1^\mu(\overline{dy_2^\mu} + \overline{\omega_{\mu\nu} y_2^\nu}) \\ &= \langle \nabla^E s_1, s_2 \rangle + \langle s_1, \nabla^E s_2 \rangle. \square \end{aligned}$$

Lemma 2.2.22. *Let E be a Euclidean vector bundle on M. Then there exists a connection ∇^E which respects the inner product.*

Proof. Choose an arbitrary connection $(\nabla^E)'$ on E. Define another connection $(\nabla^E)''$ by the formula

$$d\langle s_1, s_2 \rangle = \langle (\nabla^E)' s_1, s_2 \rangle + \langle s_1, (\nabla^E)'' s_2 \rangle.$$

Then the average $\nabla^E := \frac{1}{2}(\nabla^E)' + \frac{1}{2}(\nabla^E)''$ is such a connection. $\square$

Proposition 2.2.23. *If ∇^E is a G-connection on E, then its curvature form R^E is in $\Omega^2(\mathfrak{g}(E))$.*

2.2.7 Gauge transformations

Bundle isomorphisms can transfer connections between different vector bundles.

Lemma 2.2.24. *Let $E, F \to M$ be two smooth vector bundles and $g : E \to F$ is a bundle isomorphism. Suppose ∇^E is a connection on E, then the operator*

$$g \circ \nabla^E \circ g^{-1}$$

is a connection on F.

Proof. Exercise. $\square$

Consider the space of automorphisms

$$g : E \to E,\ g(x) : E_x \to E_x$$

We can view

$$\mathcal{G}(E)$$

be an infinite-dimensional group.

2.3 Characteristic classes

2.3.1 First Chern class

Let $L \to M$ be a complex line bundle equipped with a Hermitian inner product. This is equivalent to a structural group reduction to $U(1)$. The Lie algebra is $\mathbf{i}\mathbb{R}$. Consider a Hermitian connectino ∇, whose local connection matrix is a purely imaginary 1-form α. Then the curvature is

$$R^{\nabla} = d\alpha \in \mathbf{i}\Omega^2(M).$$

This 2-form is closed, either by the Bianchi identity or by its expression (locally exact). Therefore, it represents a de Rham cohomology class.

Now suppose ∇_0, ∇_1 are two connections. Then $\nabla_0 = \nabla_1 + \beta$ where $\beta \in \Omega^1(M, \mathfrak{g}(L)) = \mathbf{i}\Omega^1(M)$. Then

$$R^{\nabla_0} = R^{\nabla_1} + d\beta$$

Therefore the two curvatures differ by an exact 2-form, hence represent the same de Rham cohomology class. Hence there is an invariant of the bundle L, which is an element of $H^2_{\mathrm{dR}}(M)$.

We normalize the curvature in the following way: define the "Chern form" of the Hermitian connection to be the differential form

$$-\frac{1}{2\pi\sqrt{-1}} R^{\nabla} \in \Omega^2(M).$$

Its de Rham class is called the **first Chern class** of L, denoted by

$$c_1(L) \in H^2_{\mathrm{dR}}(M).$$

2.3.2 Chern–Weil construction

A function $f : \mathfrak{g} \to \mathbb{C}$ is called **invariant** if

$$f(g^{-1}\xi g) = f(\xi)\ \forall g \in G,\ \xi \in \mathfrak{g}.$$

For example, trace and determinant are invariant functions on $\mathfrak{gl}(k)$.

Consider an invariant polynomial function

$$f : \mathfrak{g} \to \mathbb{C}$$

homogeneous of degree m. Given a connection ∇^E, consider its curvature matrices Ω (which depends on a local trivialization). Applying f, one obtains an even form

$$f(R^E) \in \Omega^{2m}(M)$$

which is well-defined since different local trivializations give conjugate curvature matrices.

Theorem 2.3.1 (Chern–Weil)**.** *$f(R^E)$ is closed. Its de Rham cohomology class is independent of the choice of ∇^E.*

Proof. f can be regarded as a m-linear function on $\mathfrak{g}$, written as $f(a_1, \ldots, a_m)$. The G-invariance implies for all $\xi \in \mathfrak{g}$,

$$\sum_{i=1}^{m} f(a_1, \ldots, [\xi, a_i], \ldots, a_m) = 0.$$

Then by Leibniz rule

$$df(\Omega, \cdots, \Omega) = \sum_{i=1}^{m} f(\Omega, \cdots, d\Omega, \cdots, \Omega) = \sum_{i=1}^{m} f(\Omega, \cdots, d\Omega + [\boldsymbol{\omega}, \Omega], \cdots, \Omega) = 0$$

where the last equality is the Bianchi identity.

To show that $[f(R^E)]$ is independent of ∇^E, consider a family ∇^E_t. On the product $[0, 1] \times M$, the curvature of the connection $\nabla^{\tilde{E}} = \nabla^E_t + dt \wedge \frac{\partial}{\partial t}$ is

$$R^E_t + dt \wedge \frac{d\nabla^E_t}{dt}.$$

Then we have

$$
\begin{aligned}
0 = d^{\tilde{M}} f(R^{\tilde{E}}) &= d^{\tilde{M}} f(R_t^E) + d^{\tilde{M}} \sum_{i=1}^m f\left(R_t^E, \cdots, dt \wedge \frac{d\nabla_t^E}{dt}, \cdots, R_t^E\right) \\
&= dt \wedge \frac{\partial}{\partial t} f(R_t^E) + d^M \sum_{i=1}^m f\left(R_t^E, \cdots, dt \wedge \frac{d\nabla_t^E}{dt}, \cdots, R_t^E\right).
\end{aligned}
$$

We see

$$
\begin{aligned}
f(R_1^E) - f(R_0^E) = & \int_0^1 dt \wedge \frac{\partial}{\partial t} f(R_t^E) \\
= & - d^M \int_0^1 \sum_{i=1}^m f(R_t^E, \cdots, dt \wedge \frac{d\nabla_t^E}{dt}, \cdots, R_t^E)
\end{aligned}
$$

which is exact. □

Therefore, the class

$$[f(R^E)] \in H^{2m}_{\mathrm{dR}}(M)$$

is an invariant of the bundle E. We call it the **characteristic class** associated to the invariant function f.

2.3.3 Chern classes

Consider the unitary group $G = U(n)$ with Lie algebra $\mathfrak{u}(n)$ the space of anti-Hermitian matrices. For $T \in \mathfrak{u}(n)$, consider

$$f(t) = \det\left(tI_n - \frac{T}{2\pi\mathbf{i}}\right) = t^n + f_1(T)t^{n-1} + \cdots + f_n(T)$$

where

$$f_1(T) = \operatorname{tr}\left(-\frac{T}{2\pi\mathbf{i}}\right), \ f_n(T) = \det\left(-\frac{T}{2\pi\mathbf{i}}\right)$$

Notice that if T is diagonalized, then $f_i(T)$ is just the i-th (up to a constant) elementary symmetric function on the eigenvalues.

Then the associated characteristic classes are the 1st, 2nd, $\cdots$, top Chern classes.

2.3.4 Properties

Lemma 2.3.2. *Let $\overline{E}$ be the conjugate of E, i.e., the same bundle with conjugate complex structure. Then*

$$\sum_{i\geq 0} c_i(\overline{E})t^i = \sum_{i\geq 0} c_i(E)(-t)^i.$$

Proof. If $\phi : E|_U \to U \times \mathbb{C}^k$ is a local trivialization of E, then

$$\overline{\phi} := \tau \circ \phi : E|_U \to U \times \mathbb{C}^k$$

where τ is the complex conjugation on $\mathbb{C}^k$, is a local trivialization of $\overline{E}$. In this way, the connection ∇^E induces a connection on $\overline{E}$. If $\boldsymbol{\omega}$ is the connection matrix of ∇^E, then the corresponding connection matrix of $\nabla^{\overline{E}}$ is $\overline{\boldsymbol{\omega}}$. Then $\overline{\Omega}$ is the corresponding curvature matrix. Then

$$\sum_{i\geq 0} c_i(\overline{E})t^i = \det\left(tI_k - \frac{\overline{\Omega}}{2\pi\mathbf{i}}\right).$$

Because Ω is skew-Hermitian, the eigenvalues of Ω are exactly opposite to those of $\overline{\Omega}$. Hence

$$f_m(\overline{\Omega}) = (-1)^m f_m(\Omega).$$

Therefore one has the stated property. □

Corollary 2.3.3. *Let $E \to M$ be a real vector bundle. Then $c_{2k-1}(E\otimes\mathbb{C}) = 0$.*

Exercise 2.3.4. *Prove the Whitney sum formula of Chern classes. Let $E_1, E_2 \to M$ be two complex vector bundles. Prove that*

$$c_{\mathrm{total}}(E_1 \oplus E_2) = c_{\mathrm{total}}(E_1) \wedge c_{\mathrm{total}}(E_2).$$

Exercise 2.3.5. *Prove the* **functoriality** *of Chern classes: suppose $f : N \to M$ is a smooth map and $E \to M$ is a complex vector bundle. Then*

$$c_k(f^*E) = f^*c_k(E) \in H^{2k}_{\mathrm{dR}}(N).$$

Exercise 2.3.6. *Consider $M = S^2$ and $L \to M$ being a complex line bundle. L is characterized by two local trivializations on the north and south hemispheres, and a transiation function, which is determined by a map $g : S^1 \to U(1)$.*

Write down a connection on L in the two local trivializations (related by the transiation function), compute the curvature and the integral

$$\int_M -\frac{F_A}{2\pi \mathbf{i}}$$

and show that it is equal to the winding number of $g : S^1 \to U(1) \cong S^1$.

In particular, this shows that the first Chern class is integral.

Now we consider an important example. Consider the complex projective space $\mathbb{CP}^n$ and the **tautological line bundle** $\mathcal{O}(-1) \to \mathbb{CP}^n$. Recall that the cohomology ring of $\mathbb{CP}^n$ is generated by the **hyperplane class** $H \in H^2(\mathbb{CP}^n; \mathbb{Z})$.

Proposition 2.3.7.

1. *When $n = 1$, $\mathbb{CP}^n \cong S^2$. Prove that $c_1(\mathcal{O}(-1)) = -H$.*
2. *Prove that for general n, $c_1(\mathcal{O}(-1)) = -H$.*

Proof. Consider $\mathbb{CP}^1$ first. Consider the homogeneous coordinates $[z_0, z_1]$. There are two standard coordinate charts

$$U_0 = \{z_0 \neq 0\} \ni [z_0, z_1] \mapsto \frac{z_1}{z_0} \in \mathbb{C}$$

$$U_1 = \{z_1 \neq 0\} \ni [z_0, z_1] \mapsto \frac{z_0}{z_1} \in \mathbb{C}.$$

We view U_0 as the piece near the south pole and U_1 the piece near the north pole. Now consider the tautological line bundle. Over U_0, the natural trivialization is

$$\phi_0(z_0, z_1) \mapsto ([z_0, z_1], z_1)$$

and over U_1 the natural trivialization is

$$\phi_1(z_0, z_1) \mapsto ([z_0, z_1], z_0).$$

Then the transition function is

$$\phi_{01} = \phi_0\phi_1^{-1}([z_0, z_1], \xi) = \phi_0(\xi, \frac{z_1}{z_0}\xi) = ([z_0, z_1], \frac{z_1}{z_0}\xi).$$

So it is the map.....

Now consider $\mathbb{CP}^n$. Notice that the cohomology class c_1 is determined by its evaluation on the homology class given by a line $\mathbb{CP}^1$. $\square$

Exercise 2.3.8. *Let $L \to M$ be a complex line bundle.*

1. *Prove that there exist a natural number n and $n+1$ smooth sections $s_0, \dots, s_n \in \Gamma(L)$ such that for all $x \in M$, for some i, $s_i(x) \neq 0$.*

2. *Let $s_0, \dots, s_n$ be sections satisfying the above condition. Consider the map*

   $$f : M \to \mathbb{CP}^n,\ f(x) = [s_0(x), \dots, s_n(x)].$$

   Prove that there exists an isomorphism

   $$f^*\mathcal{O}(1) \cong L.$$

Exercise 2.3.9. *Let $E \to M$ be a complex vector bundle.*

2.3.5 Pontrajin classes

Consider the group $O(k)$ whose Lie algebra $\mathfrak{so}(k)$ is the space of $k \times k$ skew-symmetric matrices. We know that their canonical forms are those block-diagonal matrices

$$\begin{bmatrix} 0 & -\lambda_1 & \\ \lambda_1 & 0 & \\ & & \ddots \end{bmatrix}.$$

Depending on the parity of k, there could be a 1×1 block at the end which is zero. Notice that up to conjugation (by orthogonal matrices), the order of the 2×2 blocks can permute; λ and $-\lambda$ can also switch. In this way, the invariant functions are indeed symmetric functions of λ_i^2.

There is a cheap way to define Pontrajin classes.

Definition 2.3.10. Let $E \to M$ be a real vector bundle. The j-th Pontrajin class of E is the class

$$p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}) \in H^{4j}_{\mathrm{dR}}(M).$$

Notice that for odd k, $c_k(E \otimes \mathbb{C}) = 0$.

2.3.6 Euler class

Euler classes are defined for *oriented* real vector bundles, namely, vector bundles whose structural groups are reduced to $GL_+(k; \mathbb{R})$. By choosing an inner product, it is essentially equivalent to $SO(k)$-bundles.

There is a special invariant polynomial, the Pfaffian. For any skew-symmetric $2k \times 2k$ matrices, one can "diagonalize" to the canonical form

$$\begin{bmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_k \\ & & & \lambda_k & 0 \end{bmatrix}.$$

Notice that changing λ_i with $-\lambda_i$ changes the orientation. Then the Pfaffian of such matrices is defined to be

$$\mathrm{Pf}(A) := \lambda_1 \cdots \lambda_k$$

which is sort of the "positive" square root of the determinant. For $(2k+1) \times (2k+1)$ skew symmetric matrices, simply define the Pfaffian to be zero.

Remark 2.3.11. Notice that this is a speical feature of the group $SO(2k)$. The symmetric functions on λ_i^2 are invariants. But the Pfaffian is an additional invariant.

Definition 2.3.12. Let $E \to M$ be an oriented real vector bundle of rank k equipped with an inner product and a compatible connection ∇^E. The Euler class of E is the de Rham cohomology class

$$\mathrm{Euler}(E) := \left[\mathrm{Pf}\left(-\frac{R^E}{2\pi}\right)\right] \in H^k_{\mathrm{dR}}(M).$$

Theorem 2.3.13 (Gauss–Bonnet–Chern)**.** *Let (M, g) be a compact $2n$-dimensional oriented Riemannian manifold. Then*

$$\chi(M) = -\frac{1}{(2\pi)^n}\int_M \mathrm{Pf}(F^{TM}).$$

Notice that for odd-dimensional oriented manifolds, the Euler characteristic is always zero; this can be seen from Poincaré duality.

Exercise 2.3.14. *Let $E \to M$ be a complex vector bundle of rank k which is automatically an oriented real vector bundle of rank $2k$ ($U(k) \to SO(2k)$). Prove that*

$$c_k(E) = \mathrm{Euler}(E).$$

2.4 Flat connection

A connection ∇^E is called **flat** if $R^E \equiv 0$. By Chern–Weil, if E admits a flat connection, then all characteristic classes vanish. So only special vector bundles admit flat connections. However there are other interesting invariants of such bundles and connections.

Lemma 2.4.1. *Let $p, q \in M$ and $\gamma : [a, b] \to M$ be a smooth path connecting p and q. Then the parallel transport with respect to a flat connection only depends on the smooth homotopy class of γ.*

Proof. Let $u : [a, b] \times [0, 1] \to M$ be a smooth homotopy between $\gamma(\cdot, 0)$ and $\gamma(\cdot, 1)$. Consider the pullback bundle $u^*E \to [a, b] \times [0, 1]$ and the pullback connection ∇^{u^*E}. Denote the covariant derivative in the s and t directions by

∇_s and ∇_t respectively. Given any $\xi \in E_p = E_{u(a,t)}$, let $\xi(s,t)$ be the parallel transport along the s-curve, namely,

$$\nabla_s \xi(s,t) \equiv 0,\ \xi(a,t) \equiv \xi.$$

Then the flatness implies that

$$\nabla_s \nabla_t \xi = \nabla_t \nabla_s \xi = 0.$$

Therefore, the vector field $\nabla_t \xi$ is parallel along the s-curves. Notice that

$$\nabla_t \xi(a,t) \equiv 0.$$

As parallel transport is a linear isomorphism, $\nabla_t \xi(b,t) \equiv 0$. This implies that ξ is independent. $\square$

One way to see flat connections is to consider special atlas.

Lemma 2.4.2. *A vector bundle $E \to M$ admits a flat connection if and only if it admits a bundle atlas whose transition functions are all locally constants.*

Proof. Suppose E admits an atlas $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{F}^k$ with locally constant transition functions. Then take a collection of connection matrices $\boldsymbol{\omega}_\alpha \in \Omega^1(U_\alpha, \mathbb{F}^{k\times k})$ which are all zero. Because $g_{\alpha\beta}$ are constants, we see

$$g_{\alpha\beta}^{-1} \boldsymbol{\omega}_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta} = 0 = \boldsymbol{\omega}_\beta = 0.$$

So these collection of local connection matrices indeed come from a global connection. Obviously it is flat.

On the other hand, suppose ∇^E is flat. Then cover the manifold M by contractible coordinate charts. One can use the parallel transports in each chart to define local trivializations. In this way, the connection matrices are all zero. Then for the transition functions, one has

$$g_{\alpha\beta}^{-1} d g_{\alpha\beta} \equiv 0,$$

which means $g_{\alpha\beta}$ are locally constants. $\square$

2.4.1 In terms of Ehresmann connection

Using the viewpoint of Ehresman connection, the horizontal distribution is

$$T^{\mathrm{hor}}E \subset TE.$$

Proposition 2.4.3. ∇^E *is flat if and only if the corresponding horizontal distribution is integrable.*

Proof. Recall that using a local frame $e_1, \ldots, e_m$ of E and local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$, there are induced local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ on $E|_U$. If the connection matrix is $\omega_{\mu\nu}$, then the horizontal distribution is the kernel of

$$\theta = \sum_\mu e_\mu \otimes (dy^\mu + \omega_{\mu\nu} y^\nu),$$

or, equivalently, the distribution defined by the 1-forms

$$\theta^\mu = dy^\mu + \omega_{\mu\nu} y^\nu, \ \mu = 1, \ldots, k.$$

Recall that a distribution locally defined by k linearly independent 1-forms θ^μ is integrable if and only if

$$d\theta^\mu \equiv 0 \bmod \theta^\mu.$$

For the current collection of 1-forms, one can see

$$d\theta^\mu = d\omega_{\mu\nu} y^\nu - \omega_{\mu\nu} dy^\nu \equiv d\omega_{\mu\nu} y^\nu + \omega_{\mu\nu}\omega_{\nu\kappa} y^\kappa = (d\omega_{\mu\nu} + \omega_{\mu\kappa}\omega_{\kappa\nu})\, y^\nu \bmod \theta^\mu$$

Notice that modulo the ideal generated by θ^μ, the above is a form which does not have dy^μ term. Hence it vanishes if and only if the curvature is zero. $\square$

2.4.2 Representation of fundamental group

Now suppose $E \to M$ is a vector bundle and ∇^E is a flat connection. Then given $p \in M$, the parallel transport along loops induces a representation

$$\tilde{\rho} : \pi_1(M, p) \to \mathrm{Aut}(E_p).$$

If we choose a trivialization $\phi : E_p \to \mathbb{F}^k$ at p, then one obtains

$$\phi \circ \rho \circ \phi^{-1} \in GL(k).$$

If we use a different trivialization ϕ', one get

$$\phi' \circ \rho \circ (\phi')^{-1} = (\phi' \circ \phi^{-1}) \circ (\phi \circ \rho \circ \phi^{-1}) \circ (\phi' \circ \phi^{-1})^{-1}$$

which is a conjugate representation.

On the other hand, if we choose a different point q, then there exists (not canonically) an isomorphism $\pi_1(M, p) \cong \pi_1(M, q)$, induced by a homotopy class of paths from p to q. Then one essentially get the same representation (up to conjugation) of $\pi_1(M)$.

More generally, if the structural group of E is reduced to $G \subset GL(k)$ and the connection ∇^E is a G-connection, then the representation lands in G.

On the other hand, if there is a representation $\rho : \pi_1(M, p) \to G \subset GL(k)$, then one can cook up a flat connection. Indeed, take the universal covering

$$\tilde{M}_p := \Big\{\gamma : [0, 1] \to M \mid \gamma(0) = p\Big\}/\text{homotopy}.$$

There is the covering map $\tilde{M}_p \to M$ by $\gamma \mapsto \gamma(1)$. The group of deck transformations is $\pi_1(M, p)$, which acts on $\tilde{M}_p$ (from the left by concatenating with a based loop at p. Then cook up a vector bundle by the quotient

$$(\tilde{M}_p \times \mathbb{F}^k)/\pi_1(M, p)$$

where a homotopy class $\alpha \in \pi_1(M, p)$ acts on (γ, v) by $(\alpha \cdot \gamma, \rho(\alpha)(v))$. We see the action is free and hence the quotient is a manifold E_ρ. Moreover, this is a vector bundle $E_\rho \to M$.

Proposition 2.4.4. *E_ρ admits a flat connection.*

Proof. The trivial product $\tilde{M}_p \times \mathbb{F}^k$ admits a trivial connection which is obviously flat. Then on the total space there is the integrable horizontal distribution $T^{\text{hor}}\tilde{E}_p$. As $\pi_1(M, p)$ preserves this distribution, it descends to a integrable distribution on E_ρ. $\square$

Proposition 2.4.5. *If $\rho_1, \rho_2 : \pi_1(M,p) \to G$ are conjugate by an element $g_0 \in G$, then g_0 induces an isomorphism $E_{\rho_1} \cong E_{\rho_2}$ which carries the flat connection on E_{ρ_1} to that on E_{ρ_2}.*

Proof. Consider the isomorphism $\tilde{M}_p \times \mathbb{F}^k \to \tilde{M}_p \times \mathbb{F}^k$ defined by

$$\tilde{g}_0(\gamma, v) = (\gamma, g_0 v).$$

Then we have

$$\tilde{g}_0\Big(\alpha(\gamma, v)\Big) = \tilde{g}_0\Big(\alpha \cdot \gamma, \rho_1 v\Big) = \Big(\alpha \cdot \gamma, g_0\rho_1(\alpha)(v)\Big) = \Big(\alpha \cdot \gamma, \rho_2(\alpha)(g_0 v)\Big).$$

This implies that $\tilde{g}_0$ descends to a map

$$\tilde{g}_0 : E_{\rho_1} \to E_{\rho_2}.$$

This is the bundle isomorphism. □

Let's consider the case of surfaces and restrict to the case that $G = U(n)$. For $M = S^2$, since π_1 is trivial, the only representation is the trivial one, hence gives the trivial bundle with trivial connection. For $M = T^2$, the representation is two commuting matrices $A, B \in U(n)$ up to conjugation:

$$\Big\{(A, B) \in U(n) \times U(n) \mid AB = BA\Big\}/\sim$$

Basically we know that two matrices commute if they share the same eigenspaces and the conjugation will change eigenbasis. However, eigenvalues of A and B can change for different representations. We will look at flat bundles more carefully when discussing Yang–Mills connections.

2.5 Referneces and Remarks

The most classical textbook on vector bundles and characteristic classes is Milnor–Stasheff [MS74]. The so-called Stiefel–Whitney classes cannot be defined using differential forms as they are indeed cohomology class of $\mathbb{Z}_2$-coefficients.

For the origin of Chern classes and Chern–Weil theory, you are strongly recommended to read Chern's original papers [Che44] and [Che46].

For connection on principal bundles, Kobayashi–Nomizu [KN96].

Chapter 3

Complex Manifolds and Holomorphic Vector Bundles

3.1 Complex manifolds

3.1.1 Linear algebra

Let V be a real vector space (such as the tangent space of a manifold). A **complex structure** on V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{Id}$. The standard complex space $\mathbb{C}^n$, when regarded as a real $2n$-dimensional space, has the standard complex structure J as multiplying vectors by $\boldsymbol{i}$.

Let V be a real vector space. Its **complexification** is the space

$$V \otimes_{\mathbb{R}} \mathbb{C}$$

which is a complex vector space.

Given a vector space V with a complex structure J, consider the (real) dual space $V^* = \Lambda^1 V^*$, i.e., the space of real linear functionals on V. Its complexification

$$\Lambda^1 V^* \otimes \mathbb{C}$$

can be regarded as the space of real-linear, complex-valued functionals. There is a splitting (of complex vector space)

$$\Lambda^1 V^* \otimes \mathbb{C} \cong \Lambda^{1,0} V^* \oplus \Lambda^{0,1} V^*$$

where $\Lambda^{1,0}V^*$ resp. $\Lambda^{0,1}V^*$ is the space of complex linear resp. conjugate linear functionals, i.e. real linear functionals $f : V \to \mathbb{C}$ satisfying

$$f(Jv) = \sqrt{-1} f(v) \text{ resp. } f(Jv) = -\sqrt{-1} f(v).$$

The Grassmann algebra construction also works for complex vector spaces. Notice that if $W = W' \oplus W''$, then one has

$$\Lambda^k W \cong \bigoplus_{p+q=k} \Lambda^p W' \otimes \Lambda^q W''.$$

Then

$$\Lambda^k (V^* \otimes \mathbb{C}) = \Lambda^k (\Lambda^{1,0} V^* \oplus \Lambda^{0,1} V^*) \cong \bigoplus_{p+q=k} \Lambda^p (\Lambda^{1,0} V^*) \oplus \Lambda^q (\Lambda^{0,1} V^*).$$

We denote a summand on the right by

$$\Lambda^{p,q} V^* \subset \Lambda^k (V^* \otimes \mathbb{C}).$$

3.1.2 Complex manifolds

Recall that a complex-valued function $f(z^1, \ldots, z^k)$ in complex variables are called **holomorphic** or **complex analytic** if

$$\frac{\partial f}{\partial \bar{z}^i} = \frac{1}{2} \left(\frac{\partial f}{\partial x^i} + \sqrt{-1} \frac{\partial f}{\partial y^i} \right) = 0.$$

Definition 3.1.1. A **complex analytic atlas** of a smooth manifold M is a C^∞ atlas consisting of charts

$$\phi_\alpha : U_\alpha \to \mathbb{C}^k \cong \mathbb{R}^{2k}$$

such that for each pair α, β, the coordinate change

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a holomorphic map. A **complex manifold** is a smooth manifold M equipped with an equivalence class of complex analytic atlases.

Exercise 3.1.2. *Show that complex manifolds are always orientable.*

On a complex manifold, tangent spaces have complex structures. Namely, for each $p \in M$, a complex analytic chart containing p maps T_pM to $\mathbb{C}^n$, equipping T_pM a complex structure $J_p : T_pM \to T_pM$ (which is independent of the chart). Then the complexified space $T_p^*M \otimes \mathbb{C}$ can be decomposed as

$$\Lambda^{1,0}T_p^*M \oplus \Lambda^{0,1}T_p^*M.$$

If $z^1, \ldots, z^n$ are local holomorphic coordinates with $z^j = x^j + \sqrt{-1}y^j$, then $\Lambda^{1,0}T_p^*M$ is spanned by

$$dz^j = dx^i + \sqrt{-1}dy^i.$$

More generally, for each k, one has the complex-linear decomposition

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{p,q}T^*M.$$

Using the previously discussed Grassman algebra notations, the space of complex-valued k-forms can be decomposed as

$$\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The exterior differentiation can then be decomposed as

$$d : \bigoplus_{p+q=k} \Omega^{p,q}(M) \to \bigoplus_{p+q=k+1} \Omega^{p,q}(M).$$

It is easy to see that d has only two nonzero components, the one which increases the p grading by 1 and the one which increases the q grading by 1. Write them as

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),\ \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

In local coordinates,

$$\partial f = \sum_j \frac{\partial f}{\partial z^j} dz^j := \frac{1}{2} \sum_j \left(\frac{\partial f}{\partial x^j} - \mathbf{i} \frac{\partial f}{\partial y^j} \right) dz^j$$

$$\overline{\partial} f = \sum_j \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j := \frac{1}{2} \sum_j \left(\frac{\partial f}{\partial x^j} + \mathbf{i} \frac{\partial f}{\partial y^j} \right) d\bar{z}^j$$

Riemann surfaces

Riemann surfaces are just 1-dimensional complex manifolds. In particular, they are oriented surfaces. We are particularly interested in compact Riemann surfaces. By the classification theorem of compact oriented surfaces, we know that up to homeomorphism or diffeomorphism, compact Riemann surfaces are classified by their genus g (or Euler characteristic $\chi = 2 - 2g$). However, up to biholomorphism, they are different.

Maximal principle

Recall that a holomorphic function $f(z)$ defined on a domain cannot achieve its maximal modulus in the interior unless it is a constant. This implies the following fact: on a compact complex manifold, the only holomorphic functions are constants.

Almost complex structure

A smooth almost complex structure on a smooth manifold M is a smooth family of complex structures on TM. It does not necessarily come from a complex analytic atlas. The Newlander–Nirenberg theorem says that the almost complex structure is integrable if and only if there exists a complex analytic structure on M.

3.2 Holomorphic vector bundles

On complex manifolds one can talk about holomorphic functions or more general holomorphic objects. In particular, one can consider holomorphic vector bundles. Notice that the Lie group $GL(k)$ is an open subset of $\mathbb{C}^{k\times k}$.

Definition 3.2.1. A holomorphic vector bundle over a complex manifold M is a smooth complex vector bundle $E \to M$ equipped with a bundle atlas whose transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k)$$

are holomorphic functions.

Exercise 3.2.2. *Show that the total space E is a complex manifold.*

Definition 3.2.3. A holomorphic section of E is a section $S : M \to E$ which is holomorphic.

Remark 3.2.4. There is a clear distinction between complex vector bundles and holomorhpic vector bundles. For example, if $F \subset E$ is a complex subbundle, then there is always a complement F'; for example one can use a Hermitian metric on E and choose $F' = F^\perp$. However, if F and E are holomorphic vector bundles, there may not exist a holomorphic subbundle $F' \subset E$ such that $F \oplus F' \cong E$ as holomorhpic vector bundles.

On the other hand, complex vector bundles are purely determined by the topological type. However, holomorphic vector bundles have a **moduli**, i.e, their holomorphic structures can vary continuously.

The tangent bundle

Notice that TM is now understand as a real vector bundle equipped with a complex structure $J \in \Gamma(\mathrm{End}TM)$. After tensoring with $\mathbb{C}$, there is a decomposition

$$TM \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$$

where

$$T^{1,0}M = \left\{ X - \sqrt{-1}J(X) \mid X \in TM \right\}$$

and

$$T^{0,1}M = \left\{ X + \sqrt{-1}J(X) \mid X \in TM \right\}.$$

Notice that there is a canonical map $TM \to T^{1,0}M$ sending X to $\frac{1}{2}\left(X - \sqrt{-1}JX\right)$ which is J-linear.

We would like to make $T^{1,0}M$ a holomorphic vector bundle. Indeed, given local holomorphic coordinates $(z^1, \dots, z^n)$, one has the induced real coordinates $(x^1, y^1, \dots, x^n, y^n)$. Then $T^{1,0}M$ has local frame

$$\left(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\right) \text{ where } \frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

This local frame induces a local trivialization of $T^{1,0}M$. We show that on overlapping regions, the transition functions are holomorphic. Indeed, if

$$(w^1, \dots, w^n)$$

is another local coordinate system which overlaps with the previous one, then we have

$$\frac{\partial}{\partial z^i} = \frac{\partial w^j}{\partial z^i}\frac{\partial}{\partial w^j}.$$

Because the coordinates are holomorphically compatible, the complex matrices $(\frac{\partial w^j}{\partial z^i}(z^1, \dots, z^n))$ is a holomorphic function. Hence one obtains a holomorphic atlas. So $T^{1,0}M$ becomes a holomorphic vector bundle. We will usually regard it as the holomorphic tangent bundle.

Tensors of holomorphic vector bundles are holomorphic. For example, $T^*M := \mathrm{Hom}(TM, \mathbb{C})$ is the holomorphic cotangent bundle, and

$$\Lambda^{k,0}T^*M := \Lambda^k T^*M$$

is the bundle hosting all $(k, 0)$-forms. In particular, the bundle

$$K_M := \Lambda^{n,0}T^*M = \det T^*M$$

is called the **canonical line bundle** of M.

Later will will focus on vector bundles on Riemann surfaces. Topologically, they are classified by their first Chern classes. For any smooth complex line bundle $L \to M$, define

$$\mathrm{deg} L := \int_M c_1(L) \in \mathbb{Z}.$$

The Cauchy–Riemann operator

Let $E \to M$ be any complex vector bundle. Consider the bundle

$$\Lambda^{p,q}T^*M \otimes E$$

whose space of sections is denoted by $\Omega^{p,q}(E)$. When E is a holomorphic vector bundle, there is a well-defined operator

$$\overline{\partial}^E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$$

defined as follows. For any local holomorphic frame $(e_1, \ldots, e_k)$ of E, an element $\alpha \in \Omega^{p,q}(E)$ can be written as

$$\alpha = \sum_{i=1}^{k} \alpha_i \otimes e_i$$

where α_i is a (p,q)-form. Then define

$$\overline{\partial}\alpha = \sum_{i=1}^{k} \overline{\partial}\alpha_i \otimes e_i.$$

We prove that this is well-defined. Indeed, if $e'_1, \ldots, e'_k$ is another local holomorphic frame, then there exists transition matrix g_{ij} between the two local trivializations. Then

$$\sum_{i=1}^{k} \overline{\partial}\alpha_i \otimes e_i = \sum_{i,j} \overline{\partial}\alpha_i \otimes g_{ij}e'_j.$$

Because g_{ij} is a holomorphic function, $\overline{\partial}g_{ij} = 0$. Hence the above is equal to

$$\sum_{i,j} \overline{\partial}(\alpha_i g_{ij}) \otimes e'_j$$

which agrees the definition using the frame $e'_1, \ldots, e'_k$. Therefore, the operator $\overline{\partial}^E$ is well-defined.

Lemma 3.2.5. *A smooth section s of E is holomorphic if and only if $\overline{\partial}^E s = 0$.*

Triviality over contractible regions

For smooth or continuous vector bundles, we know that if the base spaces are contractible, then the bundles are trivial. This is no longer true for holomorphic vector bundles. However, over domains such as $\mathbb{C}^n$, noncompact Riemann surfaces, (or more generally Stein manifolds), the holomorphic classification agrees with the smooth classification. This is the so-called Oka–Grauert principle (due to Oka [Oka39] and Grauert [Gra58]).

3.2.1 Dolbeault cohomology

Let M be a compact complex manifold and $E \to M$ be a holomorphic vector bundle. Then remember there is the Cauchy–Riemann operator

$$\overline{\partial}^E : \Gamma(E) \to \Omega^{0,1}(E).$$

It extends to a complex

$$0 \longrightarrow \Omega^{0,0}(E) \xrightarrow{\overline{\partial}^E} \Omega^{0,1}(E) \xrightarrow{\overline{\partial}^E} \cdots \xrightarrow{\overline{\partial}^E} \Omega^{0,n}(E) \longrightarrow 0\ .$$

As $\overline{\partial}^E \circ \overline{\partial}^E = 0$, one can consider the cohomology $H^k(M, E)$.

Theorem 3.2.6. *$H^k(M, E)$ is finite-dimensional.*

Proof. One way to prove this is similar to the proof of de Rham theorem, using the theory of sheaves. The other way is to use an analogue of Hodge theory. □

Theorem 3.2.7 (Hirzebruch–Riemann–Roch)**.** *The holomorphic Euler characteristic of E*

$$\chi(E) := \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(M; E)$$

is a characteristic number which is

$$\int_M \mathrm{Td}(TM) \wedge \mathrm{ch}(E).$$

When M is 1-dimensional, this gives the classical Riemann–Roch theorem

$$\dim H^0(M,E) - \dim H^1(M,E) = \mathrm{rank} E(1-g) + \deg(E)$$

When $E = L$ is a line bundle, the formula is

$$\dim H^0(M,L) - \dim H^1(M,L) = 1 - g + \deg L.$$

Remark 3.2.8. The classical Riemann–Roch theorem was firstly observed by Riemann for line bundles. For Hirzebruch's generalization, see the book [Hir66]. This is also a special case of the Atiyah–Singer index theorem.

Theorem 3.2.9 (Serre duality). *There is a canonical isomorphism*

$$H^k(M,E) \cong H^{n-k}(M, E^* \otimes K_M)^*.$$

Proof. Roughly, if $\alpha \in H^k(M,E)$ is represented by a form $\alpha \in \Omega^{0,k}(E)$ and $\beta \in H^{n-k}(M, E^* \otimes K_M)$ is represented by a form $\beta \in \Omega^{0,n-k}(E^* \otimes K_M) \cong \Omega^{n,n-k}(E^*)$, then α and β can be paired to be a top form. Then the integration provides a perfect pairing. □

3.2.2 Splitting and Extensions

There is an important distinction between smooth vector bundles and holomorphic vector bundles. Namely, holomorphic subbundles maynot admit splitting. Suppose $F \subset E$ is a subbubndle, then there is an exact sequence

$$0 \longrightarrow F \xrightarrow{\iota} E \xrightarrow{\pi} Q \longrightarrow 0$$

where Q is the quotient bundle. It splits if there exists a bundle homomorphism from $f : Q \to E$ such that $\pi \circ f = \mathrm{Id}_Q$.

On the other hand, given such an exact sequence, one obtains a short exact sequnence of the Dolbeault complex, i.e.,

$$0 \longrightarrow \Omega^{0,q}(F) \longrightarrow \Omega^{0,q}(E) \longrightarrow \Omega^{0,q}(Q) \longrightarrow 0$$

This induces the long exact sequence

$$0 \longrightarrow H^0(F) \longrightarrow H^0(E) \longrightarrow H^0(Q) \longrightarrow H^1(F) \longrightarrow \cdots .$$

Now given an exact sequence, there is an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}(Q,F) \longrightarrow \mathrm{Hom}(Q,E) \longrightarrow \mathrm{Hom}(Q,Q) \longrightarrow 0 .$$

The long exact sequence becomes

$$\cdots \longrightarrow H^0(\mathrm{Hom}(Q,E)) \longrightarrow H^0(\mathrm{Hom}(Q,Q)) \longrightarrow H^1(\mathrm{Hom}(Q,F)) \longrightarrow \cdots$$

Notice that $H^0(\mathrm{Hom}(Q,Q))$ has a canonical identity element 1_Q. If there exists a splitting $f : Q \to E$, then the image of 1_Q in $H^1(\mathrm{Hom}(Q,F)) := \mathrm{Ext}^1(Q,F)$ is zero. This element, called the **extension class** of this exact sequence, is responsible for whether the exact sequence split or not.

3.3 Vector bundles over Riemann surfaces

We fix a compact Riemann surface Σ. We try to classify vector bundles over Σ, especially when the genus of Σ is 0 and 1.

3.3.1 Meromorphic functions

Recall that a meromorphic function f on a Riemann surface Σ is a holomorphic function defined on the complement of finitely many points (poles), such that at around each pole with local coordinate z, the function is equal to

$$f(z) = \frac{g(z)}{z^m}$$

for some locally defined holomorphic function g and an integer $m \geq 1$. The minimal m is called the **order** of the pole. On the other hand, we can look at zeroes of f near which the function can be written as $f(z) = z^m g(z)$ where $g(z)$ is a no-vanishing holomorphic function and $m \geq 1$.

Lemma 3.3.1. *A meromorphic function f on Σ is equivalent to a holomorphic map $f : \Sigma \to \mathbb{CP}^1$.*

Proof. Define $f(z) = [f(z), 1]$ where at a pole, define its value to be $[1, 0]$. It remains to prove it is holomorphic, which is easy. $\square$

Lemma 3.3.2. *Let f be a nonzero meromorphic function on Σ, then*

$$\sum_{p \in \mathrm{Zero}(f)} m_p + \sum_{q \in \mathrm{Pole}(f)} m_q = 0.$$

Proof. The total order of zeroes and the total order of poles are both equal to the degree of the map $f : \Sigma \to \mathbb{CP}^1$. $\square$

3.3.2 Divisors and line bundles

More generally, for a holomorphic line bundle $L \to \Sigma$, one can consider **meromorphic sections**, which are holomorphic sections defined on the complement of finitely many poles such that around each pole, with respect to a local trivialization, the section corresponds to a meromorphic function. In particular, a global meromorphic function on Σ is a meromorphic section of the trivial bundle.

Remark 3.3.3. It is still nontrivial that given L there exist nonzero meromorphic sections. It can be proved using the Riemann–Roch theorem after some preparation. In a different way, for any $p \in \Sigma$, on the open Riemann surface $\Sigma \setminus \{p\}$ the bundle L is trivial (by Oka–Grauert principle). Then the transition function near p provides a possible pole of the trivial section.

orphic_section **Lemma 3.3.4.** *If s_1, s_2 are both meromorphic sections of $L \to \Sigma$, then there exists a unique meromorphic function f on Σ such that $s_1 = f s_2$.*

Each nonzero meromorphic section s of E has finitely many zeroes (with positive orders) and finitely many poles (with negative orders). Define

$$(s) := \sum_{\text{zeroes}} m_p p + \sum_{\text{poles}} m_q q$$

which is a formal linear combination of points.

Definition 3.3.5. A **divisor** on a Riemann surface Σ is a formal linear combination of points with integral coefficients, written as

$$D = \sum_{i=1}^{k} m_i p_i, \ m_i \in \mathbb{Z} \setminus \{0\}.$$

One can see divisors form an abelian group.

Definition 3.3.6. Two divisors D_1 and D_2 are called **linearly equivalent** if there exists a meromorphic function f such that $D_1 - D_2 = (f)$.

From the above lemma, one can see that given a holomorphic line bundle $L \to \Sigma$, the divisors arising from meromorphic sections of L are all linearly equivalent to each other.

From divisors to line bundles

Now given a divisor $D = \sum_{i=1}^{k} m_i p_i$ one can construct a holomorphic line bundle. Indeed, consider an open cover of Σ by

$$\Sigma := \cup_{i=0}^{k} U_i$$

where $U_0 := \Sigma \setminus \{p_1, \dots, p_k\}$ and U_i, $i \geq 1$ is a small open neighborhood of p_i which is biholomorphic to a small disk. Then define

$$[D] := \left(\bigsqcup_{i=0}^{k} U_i \times \mathbb{C} \right) / \sim$$

where the equivalence relation $\sim$ is given by the transition function

$$g_i(z) = z^{m_i}.$$

Then there is an obvious meromorphic section whose induced divisor is D.

Lemma 3.3.7. *One has* $[D_1 + D_2] \cong [D_1] \otimes [D_2]$.

Proof. First, by allowing zero as coefficients in a divisor, we may write

$$D_1 = \sum_{i=1}^{k} m_i p_i, \ D_2 = \sum_{i=1}^{k} n_i p_i.$$

Using the same open cover of Σ, the transition functions of $[D_1] \otimes [D_2]$ are the product of the transition functions of the two line bundles, which are

$$z^{m_i+n_i}.$$

This is exactly the transition functions for the line bundle $[D_1 + D_2]$. $\square$

Lemma 3.3.8. *For a meromorphic function f, the line bundle $[(f)]$ is trivial.*

Proof. Suppose $(f) = \sum_{i=1}^{k} m_i p_i$. Now on the open subset $\Sigma \setminus \{p_1, \ldots, p_k\}$, the function $1/f$ defines a holomorphic section of $[(f)]$ over this open subset. One can check that this section extends to a nonwhere vanishing section on Σ. Hence $[(f)]$ is the trivial bundle. $\square$

Lemma 3.3.9. *$[D_1]$ and $[D_2]$ are isomorphic if and only if D_1 and D_2 are linearly equivalent.*

Proof. Let $s_1 \in H^0([D_1])$ and $s_2 \in H^0([D_2])$ be the meromorphic sections giving the divisors D_1 and D_2. If $[D_1] \cong [D_2]$, then s_1 and s_2 are meromorphic sections of the same line bundle. Hence by Lemma 3.3.4, D_1 is linearly equivalent to D_2.

On the other hand, if D_1 and D_2 are linearly equivalent, then there exists a meromorphic function f such that $D_1 = D_2 + (f)$. Then $[D_1] = [D_2 + (f)] = [D_2] \otimes [(f)] = [D_2]$, as $[(f)]$ is the trivial bundle. $\square$

Therefore, one has established a one-to-one correspondence

$$\{\text{divisors}\}/\text{linear equivalence} \to \{\text{holomorphic line bundle}\}/\text{isomorphism}.$$

Lastly, we look at the degree of the line bundle determined by the divisor.

Lemma 3.3.10. *Given any divisor $D = \sum_{i=1}^{n} m_i p_i$, one has*

$$\deg[D] = \sum_{i=1}^{n} m_i.$$

3.3.3 Line bundles on Riemann surfaces

Rational curve

Since we understand meromorphic functions on $\mathbb{C}$ very well, one can easily prove the following.

Lemma 3.3.11. *If D is a degree zero divisor on $\mathbb{CP}^1$, then $D = (f)$ for some meromophic function f.*

Proof. Suppose $D = \sum_{i=1}^k m_i p_i + m_\infty p_\infty$ where p_∞ is the point at infinity and m_∞ is possibly zero. Consider the function

$$f(z) = \prod_{i=1}^k (z - p_i)^{m_i}$$

It is a meromorphic function on $\mathbb{CP}^1$. □

Corollary 3.3.12. *For each $d \in \mathbb{Z}$, there exists a unique holomorphic line bundle $\mathcal{O}(d) \to \mathbb{CP}^1$ of degree d up to isomorphism.*

Proof. Let L be a holomorphic line bundle of degree d. Choose a meromorphic section s whose divisor is $(s) = \sum_{i=1}^k m_i p_i$. The previous lemma shows that there exists a meromorphic section whose divisor is dp_0. Hence L is isomorphic to the line bundle generated by this "canonical" divisor. Therefore, there is only one such line bundle. □

We first show that line bundles over $\mathbb{CP}^1$ are classified by the degree.

Definition 3.3.13. The **tautological line bundle** is the bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$ whose fibre over $l \in \mathbb{CP}^1$ is the complex line $l \subset \mathbb{C}^2$ represented by l.

Denote $\mathcal{O}(1) = \mathcal{O}(-1)^*$ and for $k > 0$,

$$\mathcal{O}(\pm k) = \underbrace{\mathcal{O}(\pm 1) \otimes \cdots \otimes \mathcal{O}(\pm 1)}_{k}.$$

We know smooth complex line bundles, up to isomorphism, are classified by their degrees, i.e., the integer l. It is in fact also the classification of holomorphic line bundles.

Lemma 3.3.14. *The tangent bundle $T\mathbb{CP}^1$ is isomorphic to $\mathcal{O}(2)$.*

Proof. Exercise. □

On the other hand, one has the following fact.

Proposition 3.3.15. *The space of holomorphic sections of $\mathcal{O}(k)$ is isomorphic to the space of complex polynomials of degree at most k. In particular, when $k \leq -1$, $H^0(\mathcal{O}(k)) = 0$ and*

$$\dim H^0(\mathcal{O}(k)) = k + 1.$$

Proof. A holomorphic section shall generate an effective divisor, which cannot have negative degree. On the other hand, by Serre duality,

$$H^1(\mathcal{O}(k)) \cong (H^0(\mathcal{O}(-k-2)))^*$$

which vanishes when $k \geq 0$. By Riemann–Roch, we know

$$\dim H^0(\mathcal{O}(k)) - \dim H^1(\mathcal{O}(k)) = 1 + k.$$

Hence the assertion follows. □

Higher genus curves

Lemma 3.3.16. *Let Σ_1, Σ_2 be connected compact Riemann surfaces. If $f : \Sigma_1 \to \Sigma_2$ is a holomorphic map of degree 1, then f is an isomorphism.*

Proof. We know that f is one-to-one away from finitely many critical points. At a critical point $p \in \Sigma_1$, using local coordinates f, we can write f as a holomorphic function in z near 0 such that $f(z) = 0$, $df(0) = 0$. Then f has at least an order two zero near 0. This means for any point near 0, it has at least two preimages, which contradicts the condition that f has degree 1. Hence f has no critical point and hence globally one-to-one. □

Corollary 3.3.17. *There is no meromorphic function f on any Riemann surface of genus $g \geq 1$ whose divisor has the form $(f) = p - q$ with $p \neq q$.*

Proof. If such f exists, then f defines a degree 1 map from Σ to $\mathbb{CP}^1$. This contradicts the condition that $g(\Sigma) \geq 1$. $\square$

Corollary 3.3.18. *For any Riemann surface Σ of genus $g \geq 1$, the map*

$$\Sigma \to \{\text{isomorphism classes of holomorphic line bundles}\}$$

sending p to the isomorphism class of $[p]$ is injective.

Proof. If $[p] \cong [q]$, then there exists a meromorphic function f such that $(f) = p - q$, which contradicts the last corollary. $\square$

This shows that holomorphic line bundles do not only contain the topological information of the degree.

Remark 3.3.19. In fact the set (or moduli space) of isomorphism classes of line bundles of a fixed degree is a g-dimensional complex torus. This coincides with the identification of the Yang–Mills moduli space.

3.3.4 Grothendieck's theorem

The purpose of this subsection is to classify holomorphic vector bundles over the genus zero curve $\mathbb{CP}^1$.

Theorem 3.3.20 (Grothendieck [Gro57]). *Any holomorphic vector bundle $E \to \mathbb{CP}^1$ is isomorphic to the direct sum of line bundles $L_1 \oplus \cdots \oplus L_k$ and the summands are unique up to reordering.*

Remark 3.3.21. If we only consider smooth decomposition, then this is always true on any surfaces and the decomposition is not unique: only the first Chern class is the invariant. However, holomorphic decomposition is more rigid and not always possible.

Sections and line bundles

We first prove a useful lemma.

Lemma 3.3.22. *Let $E \to \Sigma$ be a holomorphic vector bundle. If $s : \Sigma \to E$ is a nonzero holomorphic section, then there exists a unique holomorphic line bundle $L \subset E$ satisfying 1) the image of s is contained in L and 2), if $L' \subset E$ is another subbundle containing the image of s, then there exists a holomorphic bundle map from L to L'.*

Proof. If s is nowhere vanishing, the s spanns a subbundle L. Now suppose s vanishes at finitely many points $p_1, \ldots, p_k$. Around each p_i, choose a local coordinate z and local trivializations of E so s becomes

$$s(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_k(z) \end{bmatrix}$$

where f_i is a holomorphic function in z with $f_i(0) = 0$. Hence $f_i(z) = z^{m_i} h_i(z)$ with $h_i(0) \neq 0$ and $m_i \geq 1$. Let $m_p = \min_i m_i \geq 1$. Then consider the divisor

$$D = -\sum_{p \in \mathrm{Zero}(s)} m_p p.$$

Then there is a meromorphic function f on Σ with $(f) = D$. Then consider the meromorphic section fs of $E \otimes [D]$. One can check that fs is in fact holomorphic and nowhere vanishing. Hence there is a subbundle $L \subset E \otimes [D]$. Then $[-D] \otimes L \subset E$ is a subbundle. □

Lemma 3.3.23. *For any holomorphic vector bundle $E \to \Sigma$, for* $\mathrm{deg} E$ *sufficiently large, $H^0(E) \neq 0$.*

Proof. By Riemann–Roch, one has

$$\mathrm{dim} H^0(E) \geq \mathrm{rank}(E)(1 - g) + \deg(E).$$

Then when the degree is sufficiently large, there exists holomorphic sections.

□

Corollary 3.3.24. *Any holomorphic vector bundle $E \to \Sigma$ has a line subbundle $L \subset E$.*

Proof. Given any E one can choose a line bundle L of a sufficiently large degree so that

$$\deg(E \otimes L) = \deg E + \mathrm{rank} E \cdot \deg L$$

is sufficiently large. By the previous lemmas, $E \otimes L$ has a nonzero holomorphic section, hence has a line subbunle. It implies E also has a line subbundle. $\square$

Another ingredient is that line subbundles have an upper bound on the degree.

lemma3525 **Lemma 3.3.25.** *Let $E \to \Sigma$ be a holomorphic vector bundle. Then there exists $K > 0$ such that for all holomorphic subbundles $F \subset E$, $\deg F \leq K$.*

Proof. Since holomorphic sections of F are also holomorphic sections of E, one has

$$H^0(F) \subset H^0(E).$$

Then by Riemann–Roch, one has

$$\dim H^0(F) - \dim H^1(F) = \mathrm{rank}(F)(1-g) + \deg F.$$

Then

$$\begin{aligned} &\deg F = \dim H^0(F) - \dim H^1(F) - \mathrm{rank}(F)(1-g) \\ &\leq \dim H^0(E) - \mathrm{rank}(F)(1-g) - \dim H^1(F) \leq \dim H^0(E) - \mathrm{rank}(F)(1-g). \end{aligned}$$

The right hand side clearly has an upper bound. $\square$

Splitting over $\mathbb{CP}^1$

We can have a lemma showing on $\mathbb{CP}^1$ when an exact sequence split.

Lemma 3.3.26. *Given an exact sequence*

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0$$

of holomorphic vector bundles on $\mathbb{CP}^1$ such that F and Q are line bundles. If

$$\deg(Q^* \otimes F) = \deg F - \deg Q \geq -1$$

then it splits.

Proof. Suppose $Q^* \otimes F \cong \mathcal{O}(k)$. Then we know

$$\mathrm{Ext}^1(Q, F) \cong H^1(\mathcal{O}(k)) \cong (H^0(\mathcal{O}(-k) \otimes K_{\mathbb{CP}^1}))^* \cong H^0(\mathcal{O}(-k-2))^*$$

which vanishes when $k \geq -1$. □

Corollary 3.3.27. *Any rank 2 vector bundle over $\mathbb{CP}^1$ splits as direct sum of line bundles.*

Proof. Let E have rank 2. We may assume that $\dim E = 0, -1$. Now, by Riemann–Roch, one has

$$\dim H^0(E) - \dim H^1(E) = 2(1-g) + \deg(E) \geq 1.$$

Hence $H^0(E) \neq 0$. Hence there exists a holomorphic section s, generating a subbundle $F \subset E$ which has nonnegative degree. Then we see

$$\begin{aligned} \deg Q^* \otimes F &= \deg(Q^*) + \deg(F) = \deg F - \deg Q \\ &= \deg F - (\deg E - \deg F) = 2\deg F - \deg E \geq -\deg E \geq 0. \end{aligned}$$

Hence the exact sequence split. □

Now we prove Grothendieck's theorem by induction. Suppose $\mathrm{rank} E = k$ and we have proved Grothendieck's theorem for all bundles of smaller rank.

By Corollary 3.5.24, E has a line subbundle L. Then there is an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0\,.$$

By the induction hypothesis, Q splits as direct sums

$$Q = \bigoplus_{j=1}^{k-1} L_j.$$

By the extension long exact sequence, this exact sequence splits if

$$H^1(Q^* \otimes L) = H^1\left(\bigoplus_{j=1}^{k-1} L_j^* \otimes L\right) \cong \bigoplus_{j=1}^{k-1} H^1(L_j^* \otimes L) = 0.$$

It suffices to show that $\deg L \geq \deg L_j$. Then we shall take L whose degree is the greatest among all line subbundles (which has an upper bound by Lemma 3.3.25). In this case, suppose on the contrary that $L_j \subset Q$ is a subbundle with $\deg L_j > \dim L$. Then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\mathrm{Id}_L"'] & F \arrow[r] \arrow[d] & L_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & E \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

where F is a rank 2-bundle. Hence $F \cong L \oplus L_j$ by the previous Corollary. Hence L_j is another line subbundle, contradicting the condition that L has the maximal degree.

3.4 References and Remarks

A classical reference to complex manifold theory was Chern's book [Che67]. A good reference for theories of holomorphic vector bundles is the one by Kobayashi [Kob14].

The Hitchin–Kobayashi correspondence is a universal principle. It basically provides a bridge between complex algebraic geometry and differential geometry, or a bridge between complex and symplectic world.

Chapter 4

Yang–Mills Equations

4.1 Yang–Mills functional

We focus on the case of complex vector bundles.

Before we start, we want to change the notations which are more frequently used in gauge theory contexts. Let M be a compact Riemannian manifold. Let $E \to M$ be a complex vector bundle, typically equipped with a Hermitian inner product. Let $\mathcal{A}(E)$ be the space of connections on E; when E has a Hermitian inner product, this notation means the space of Hermitian connections. A connection is denoted by A; however, the associated operator on $\Gamma(E)$ is denoted by

$$D_A : \Gamma(E) \to \Omega^1(E).$$

For any connection $A \in \mathcal{A}(E)$, there is the curvature

$$F_A \in \Omega^2(M, \mathrm{End}(E)).$$

We need an inner product on $\mathrm{End}(E)$; the Hermitian inner product on E induces an inner product on $\mathrm{End}(E)$. More explicitly, one has:

Lemma 4.1.1. *Let V be a Hermitian vector space. Then the pairing*

$$(A, B) \mapsto \mathrm{Re}\left(\mathrm{Tr}(AB^\dagger)\right)$$

is an inner product on the space of skew-Hermitian linear transformations.

Proof. It suffices to check for the standard Hermitian vector space $\mathbb{C}^k$. We have

$$\operatorname{Re}\left(\operatorname{Tr}(AB^\dagger)\right) = \operatorname{Re}\left(\sum_{i,j} A_{ij}\overline{B_{ij}}\right) = \sum_{i,j}\left(A'_{ij}B'_{ij} + A''_{ij}B''_{ij}\right)$$

which is obviously a real inner product. $\square$

4.1.1 The Yang–Mills functional

Define the **Yang–Mills functional** of A to be

$$\mathcal{YM}(A) := \frac{1}{2}\int_M \|F_A\|^2 \mathrm{dvol}_g.$$

We would like to do the same variational approach to find the Euler–Lagrange equation. A deformation of A is $A+\alpha$ for $\alpha \in \Omega^1(\operatorname{End}(E))$. We want to calculate the variation of the curvature. With respect to a local trivialization, write

$$A = d + \boldsymbol{\omega}$$

where the connection matrix $\boldsymbol{\omega}$ takes value in $\mathfrak{g}$; meanwhile, α also takes value in $\mathfrak{g}$. Then

$$\begin{aligned} F_{A+\alpha} &= d(\boldsymbol{\omega}+\alpha) + (\boldsymbol{\omega}+\alpha)\wedge(\boldsymbol{\omega}+\boldsymbol{\omega}) \\ &= d\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} + d\alpha + [\boldsymbol{\omega},\alpha] + \alpha\wedge\alpha \\ &= F_A + D_A\alpha + \alpha\wedge\alpha \end{aligned}$$

Hence if we set $A_t = A + t\alpha$, one has

$$\begin{aligned} F(t) := \mathcal{YM}(A_t) &= \frac{1}{2}\int_M \|F_{A+t\alpha}\|^2 \\ = \frac{1}{2}\int_M \|F_A + tD_A\alpha + O(t^2)\|^2 &= F(0) + t\int_M \langle F_A, D_A\alpha\rangle \mathrm{dvol}_g + O(t^2). \end{aligned}$$

Then

$$F'(0) = \int_M \langle F_A, D_A\alpha\rangle \mathrm{dvol}_g.$$

Lemma 4.1.2. *The operator* $D_A^* = (-1)^{n+1} * D_A *$ *is a formal adjoint of* $D_A : \Omega^1(\mathrm{End}E) \to \Omega^2(\mathrm{End}E)$.

Proof. This is similar to the case of d^* on differential forms. □

Hence a critical point is a solution to

$$D_A^* F_A = (-1)^{n+1} * D_A * F_A = 0 \Longrightarrow D_A * F_A = 0.$$

Definition 4.1.3. A smooth connection $A \in \mathcal{A}(E)$ is called a **Yang–Mills connection** if

$$D_A^* F_A = 0.$$

4.1.2 Gauge transformations

There is a symmetry on the Yang–Mills functional.

Definition 4.1.4. A vector bundle automorphism $g : E \to E$ is called a **gauge transformation** on E.

Gauge transformations obviously form a group, denoted by

$$\mathcal{G}(E).$$

Since the structural group of E is reduced to G, this notation $\mathcal{G}(E)$ implicitly requires that $g : E \to E$ preserves this reduction. In other words, in local trivializations, g takes value in G.

Notice that g sends each fibre E_x isomorphically to itself. If A is a connection, with induced horizontal distribution $T^{\mathrm{hor}}E$, then g pushes $T^{\mathrm{hor}}E$ forward to another horizontal distribution, hence induces another connection. Hence there is an action

$$\mathcal{G}(E) \times \mathcal{A}(E) \to \mathcal{A}(E)$$

of the group of gauge transformations on the space of connections.

We would like to have a more explicit expression of this action. In local frame, $g \in \mathcal{G}(E)$ can be regarded as a function $g : U \to G \subset U(k)$, which can be written as $g_{\mu\nu}$ and $A = d + \boldsymbol{\omega}$. Recall that the horizontal distribution is the kernel of

$$\theta^\mu = dy^\mu + \omega_{\mu\nu} y^\nu,\ \mu = 1, \ldots, k,$$

Then $Z \in \ker\theta^\mu \subset TE|_U$ if and only if $g_* Z \in \ker(g^{-1})^*\theta^\mu$. In this coordinates, g transfers as

$$g(x, y) = (x, g_{\mu\nu} y^\nu)$$

Hence

$$\begin{aligned} g^*\theta^\mu = d(g_{\mu\nu}y^\nu) + \omega_{\mu\nu} g_{\nu\kappa} y^\kappa &= g_{\mu\nu} dy^\nu + dg_{\mu\nu} y^\nu + \omega_{\mu\nu} g_{\nu\kappa} y^\kappa \\ &= g_{\mu\nu}\Big(dy^\nu + (g^{\nu\kappa} dg_{\kappa\sigma} + g^{\nu\kappa}\omega_{\kappa\rho} g_{\rho\sigma}) y^\sigma\Big) \end{aligned}$$

so the connection matrix transforms as

$$\boldsymbol{\omega} \mapsto g^{-1}\boldsymbol{\omega} g + g^{-1} dg.$$

Lemma 4.1.5. *The connection g^*A corresponds to the operator $g^{-1} \circ d_A \circ g$.*

Lemma 4.1.6. *The curvature of $g \cdot A$ is $F_{g\cdot A} = gF_A g^{-1}$.*

Proof. The calculation is local, which is the same as the curvature matrix calculation. □

4.2 Abelian Yang–Mills connections

Consider the case that $G \subset U(k)$ is abelian, i.e. $G = U(1) \times \cdots \times U(1)$. This means E is viewed as the direct sum of Hermitian line bundles $L_1 \oplus \cdots \oplus L_k$ and connections are direct sums of connections on these line bundles. Then we only needs to consider the rank one case.

Let $L \to M$ be a Hermitian line bundle. Then $\mathrm{End}L$ is canonically identified with the trivial (real) line bundle. For any Hermitian connection $A \in \mathcal{A}(L)$, we know

$$F_A \in \Omega^2(M, \mathrm{End}L) \cong \sqrt{-1}\Omega^2(M).$$

Moreover, the induced connection $D_A : \Omega^1(\mathrm{End}E) \to \Omega^2(\mathrm{End}E)$ coincides with the standard exterior differentiation. Then the Yang–Mills equation reads

$$D_A^* F_A = d^* F_A = 0.$$

The Bianchi identity reads $dF_A = 0$. Then A is a Yang–Mills connection is equivalent to say that $-\sqrt{-1}F_A$ is a harmonic 2-form, or equivalently, that $-\frac{1}{2\pi\sqrt{-1}}F_A$ is the harmonic representative of the first Chern class of L.

Let's see that in the abelian case there are always Yang–Mills connections.

Proposition 4.2.1. *Let $L \to M$ be a Hermitian line bundle. Then there exists a Hermitian connection A such that F_A is harmonic.*

Proof. Let A_0 be a reference connection. Then any connection A can be written as $A = A_0 + \alpha$ with $\alpha \in \mathbf{i}\Omega^1(M)$. Then

$$F_A = F_{A_0} + d\alpha.$$

As F_{A_0} is closed, by Hodge decomposition, one can write

$$F_{A_0} = \gamma_0 + d\beta_0$$

in a unique way, where γ_0 is harmonic. Then for the connection $A = A_0 - \beta_0$,

$$F_A = F_{A_0} - d\beta_0$$

which is harmonic. □

On the other hand, we would like to see if Yang–Mills connections on a given Hermitian line bundle $L \to M$ are unique or not. First, notice that if $g \in \mathcal{G}(L)$ is a unitary gauge transformation, then

$$F_{g\cdot A} = gF_A g^{-1} = F_A.$$

Hence after any gauge transformation, a Yang–Mills connection becomes a gauge equivalent Yang–Mills connection. Hence it makes sense to consider the **moduli space** of Yang–Mills connections.

$$\mathcal{M}_{\mathrm{YM}}(L) = \Big\{ A \in \mathcal{A}(L) \mid d_A^* F_A = 0 \Big\} / \mathcal{G}(L).$$

Now if A_1, A_2 are both Yang–Mills connections, then we can write $A_2 = A_1 + \beta$ with $\beta \in \sqrt{-1}\Omega^1(M)$; the YM condition implies

$$F_{A_2} = F_{A_1} = F_{A_2} + d\beta \Longrightarrow d\beta = 0.$$

Hence two Yang–Mills connections differ by a closed 1-form. On the other hand, if $\beta = dh$ for $h \in \sqrt{-1}\Omega^0(M)$, then $g = e^h : M \to U(1)$ is a gauge transformation such that $F_{g\cdot A} = F_A - dh$. Hence if the difference is exact, then it comes from a gauge transformation.

We can identify this moduli space with a torus. Recall that

$$H^1_{\mathrm{dR}}(M)$$

is isomorphic to the real cohomology $H^1(M;\mathbb{R})$. Inside it there is the lattice of the free part of the integral cohomology

$$H^1(M;\mathbb{Z})_{\mathrm{free}} \subset H^1(M;\mathbb{R}).$$

The quotient

$$H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})_{\mathrm{free}} \cong (S^1)^{b_1}$$

where b_1 is the first Betti number.

Theorem 4.2.2. *There is a one-to-one correspondence*

$$\mathcal{M}_{\mathrm{YM}}(L) \cong H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})_{\mathrm{free}}.$$

Proof. Fix a reference YM connection $A_0 \in \mathcal{A}(L)$, then we know for any closed 1-form β, $A_0 + \sqrt{-1}\beta$ is another YM connection, which is gauge equivalent to A_0 if β is exact. Hence there is a well-defined map

$$f_{A_0} : H^1(M;\mathbb{R}) \to \mathcal{M}_{\mathrm{YM}}(L).$$

Moreover, it is affine linear. We would like to show that the kernel is $H^1(M;\mathbb{Z})_{\mathrm{free}}$. Using differential forms, a closed 1-form represents an integral class if its integral over any closed oriented loop is an integer. Remember such integral only depends on the homotopy class. Hence for any integral β, there is a map

$$m_\beta : \pi_1(M, p) \to \mathbb{Z}.$$

Then fix a reference point $p \in M$. For each $x \in M$, define

$$h(x) = \int_\gamma \beta$$

where γ is a path from p to x. This integral depends on the homotopy class of γ. However, if we define

$$g(x) = \exp(2\pi\sqrt{-1}h(x))$$

then it is well-defined and smooth. Hence this is a gauge transformation. Then

$$\sqrt{-1}\beta = g^{-1}dg.$$

□

Remark 4.2.3. Notice that $\mathcal{M}_{\mathrm{YM}}(L)$ has a topology induced from the topology on the space of connections and the action by $\mathcal{G}(L)$. The above identification shows that $\mathcal{M}_{\mathrm{YM}}(L)$ is in particular compact. This can be viewed as a very simple case of **Uhlenbeck compactness**.

4.3 Yang–Mills connections on surfaces

Let M be a closed oriented surface equipped with a Riemannian metric g. Let $E \to M$ be a Hermitian vector bundle. For any connection $A \in \mathcal{A}(E)$, we have $*F_A \in \Omega^0(\mathrm{End}(E))$. The Yang–Mills equation then reads

$$D_A * F_A = [D_A, *F_A] = 0.$$

Proposition 4.3.1. *Suppose A is a Yang–Mills connection. Then there exists an orthogonal splitting*

$$E = E_1 \oplus \cdots \oplus E_k,$$

Hermitian connections $A_i \in \mathcal{A}(E_i)$, and constants $\lambda_i \in \mathbb{R}$ such that

$$A = A_1 \oplus \cdots \oplus A_k$$

and

$$F_{A_i} = \sqrt{-1}\lambda_i \mathrm{dvol}_g.$$

Proof. Choose a point $p \in M$. For any point $q \in M$, choose a path γ connecting p and q. Using the parallel transport along γ, we can identify $E_{\gamma(t)}$ with E_p. Then along γ, the section $*F_A$ is a family of skew-Hermitian linear transformations $\Omega(t) : E_p \to E_p$. The Yang–Mills equation $[D_A, *F_A] = 0$ implies that

$$\Omega'(t) = 0.$$

In particular, this means the eigenvalue decompositions of E_p remains constants. So the eigenvalues of $*F_A$ are constants as well as their multiplicities. Hence for each eigenvalue λ there is a well-defined subbundle $E_\lambda \subset E$ which is the λ-eigenspace of $*F_A$. As $*F_A$ is skew-Hermitian, different eigenspaces are orthogonal.

We prove that A preserves the decomposition $E = \bigoplus E_{\lambda_i}$. Indeed, for any section s of E_λ, the Yang–Mills condition implies that

$$*F_A(D_A s) = D_A(*F_A s) = D_A(\lambda s) = \lambda D_A s$$

Hence $D_A s$ is also contained in E_λ. Therefore, the A is the direct sum of connections $A_i \in \mathcal{A}(E_i)$. Then we see

$$F_A|_{E_i} = F_{A_i} = \lambda_i \mathrm{dvol}_g. \qquad \square$$

Then Yang–Mills connections are actually direct sums of the following more basic types of connections.

Definition 4.3.2. A **Hermitian–Einstein** connection on a Hermitian vector bundle $E \to \Sigma$ is a connection A such that

$$-\frac{*F_A}{2\pi\sqrt{-1}} = \tau \mathrm{Id}_E.$$

The scalar τ is actually determined as follows. By integrating the equation over Σ one obtains

$$\deg E = \int_\Sigma \mathrm{Tr}\left(-\frac{F_A}{2\pi\sqrt{-1}}\right) = \tau \mathrm{rank} E \mathrm{Area}(\Sigma).$$

Then

$$\tau \mathrm{Area}(\Sigma) = \frac{\deg E}{\mathrm{rank} E}.$$

The right hand side, which only depends on the topological type of E, is called its **slope**.

4.4 Connections on holomorphic vector bundles

We try to relate holomorphic vector bundles with Yang–Mills connections. Let M be a complex manifold and $E \to M$ be a complex vector bundle.

4.4.1 Cauchy–Riemann operators and complex gauge transformations

Cauchy–Riemann operators

Let $E \to M$ be a complex vector bundle (not necessarily holomorphic). One can consider the space of (p,q)-forms

$$\Omega^{p,q}(E) = \Gamma(\Lambda^{p,q}(T^*M) \otimes E).$$

Definition 4.4.1. A **Cauchy–Riemann operator** on E is a linear operator

$$D'' : \Gamma(E) \to \Omega^{0,1}(E)$$

satisfying

1. $D''(fs) = fD''(s) + \overline{\partial} f \otimes s$. This property allows one to define $D'' : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ so the next condition makes sense.

2. $(D'')^2 = 0$.

Similar to connections, Cauchy–Riemann operators on E form an affine space modelled on

$$\Omega^{0,1}(\mathrm{End}E).$$

Let $\mathcal{D}''(E)$ be the space of all Cauchy–Riemann operators on E.

Proposition 4.4.2. *Let M be a complex manifold and $E \to M$ be a complex vector bundle. Let D'' be a Cauchy–Riemann operator on E. Then there is a unique holomorphic bundle structure on E, denote by $\mathcal{E}$, such that D'' coincides with the operator $\overline{\partial}^{\mathcal{E}}$.*

Sketch of Proof. The detailed proof can be found at [Kob14, Proposition 13.7]. Basically, one can define an almost complex structure on the total space E using the operator D''. The condition $(D'')^2 = 0$ implies that the almost complex structure is integralbe, making E a complex manifold with the projection $E \to M$ a holomorphic submersion. Then local holomorphic trivializations can be constructed. □

Complex gauge transformations

Recall that a gauge transformation is just a bundle isomorphism $g : E \to E$. Previously, when discussing Hermitian vector bundles, we required that g preserves the Hermitian structure. In this way, every unitary gauge transformation pulls back a Hermitian connection to a Hermitian connection.

Now suppose we do not have a Hermitian inner product. Then one can consider all complex vector bundle automorphisms $g : E \to E$. It can still pulls back Cauchy–Riemann operators. Namely,

$$g^{-1} \circ \overline{\partial}^E \circ g : \Gamma(E) \to \Omega^{0,1}(E)$$

is still a Cauchy–Riemann operator.

Corollary 4.4.3. *Let Σ be a compact Riemann surface. Let $E \to \Sigma$ be a smooth complex vector bundle of rank k and degree d. Then the set of* *isomorphism classes of holomorphic vector bundles of rank k and degree d is in one-to-one correspondence with the quotient*

$$\mathcal{A}''(E)/\mathcal{G}^{\mathbb{C}}(E).$$

Proof. The previous result tells that there is a natural map from $\mathcal{A}''(E)/\mathcal{G}^{\mathbb{C}}(E)$ to the set of isomorphism classes of holomorphic vector bundles. We prove that it is a bijection. First, if A'' and B'' defines isomorphic holomorphic bundle, then they are obviously related by a complex gauge transformation. So the map is injective. On the other hand, for any holomorphic vector bundle of the given degree and rank, it is smoothly isomorphic to E. Hence its canonical Cauchy–Riemann operator is identified with one in $\mathcal{A}''(E)$. $\square$

The Chern connection

Let A be a connection on E, which is equivalent to an operator

$$D_A : \Gamma(E) \to \Omega^1(E).$$

Notice that we can naturally decompose

$$D_A = D'_A + D''_A$$

which are the $(1,0)$ and $(0,1)$-components.

Theorem 4.4.4 (Chern)**.** *Given a holomorphic vector bundle E equipped with a Hermitian inner product h^E, there exists a unique compatible connection D_A of type* $(1,0)$.

Proof. Notice that for any $s \in \Gamma(E)$, $D_A s$ is determined by its Hermitian pairing with another $s' \in \Gamma(E)$. One has

$$d\langle s_1, s_2\rangle = \langle D_A s_1, s_2\rangle + \langle s_1, D_A s_2\rangle.$$

Take the $(1,0)$-part, one obtains

$$\partial\langle s_1, s_2\rangle = \langle D'_A s_1, s_2\rangle + \langle s_1, \overline{\partial}^E s_2\rangle.$$

This determines the operator D'_A uniquely. Define $D_A = D'_A + \overline{\partial}^E$. It is routine to check that this is a connection satisfying the requirement. $\square$

Then one can consider a natural subspace of connections. Define

$$\mathring{\mathcal{A}}(E) \subset \mathcal{A}(E)$$

be the subset of connections A on E such that D''_A is a Cauchy–Riemann operator.[1] When E is Hermitian, we also require the connection is Hermitian. In this case, one has $F_A \in \Omega^{1,1}(\mathrm{End}E)$. Then there is a natural map

$$\mathring{\mathcal{A}}(E) \to \mathcal{A}''(E).$$

Since $g \in \mathcal{G}(E)$ acts on connections by $g \cdot A = g \circ A \circ g^{-1}$, this is a $\mathcal{G}(E)$-equivariant map. On the other hand, there is an inverse given by the Chern connection construction.

Complex gauge transformation on connections

The larger group $\mathcal{G}^{\mathbb{C}}(E)$ acts on the space $\mathcal{A}''(E)$. The above one-to-one correspondence induces an action by $\mathcal{G}^{\mathbb{C}}(E)$ on the space of connections.

Let $g : E \to E$ be a complex gauge transformation. Since D_A is Hermitian, one has

$$\overline{\partial}\langle s_1, s_2\rangle = \langle D''_A s_1, s_2\rangle + \langle s_1, D'_A s_2\rangle.$$

After transforming by a complex gauge transformation g, one has

$$g \cdot D''_A = g \circ D''_A \circ g^{-1} = D''_A - (D''_A g)g^{-1}.$$

Then the change to D'_A should be equal to

$$((D''_A g)g^{-1})^\dagger.$$

Therefore, g acts on the connection by

$$D_A \mapsto D_A - (D''_A g)g^{-1} + ((D''_A g)g^{-1})^\dagger.$$

Notice that one can write any complex gauge transformation using the polar decomposition as $g = UH$ where U is unitary and H is positive-definite

[1] When $M = \Sigma$ is a Riemann surface, $\mathring{\mathcal{A}}(E) = \mathcal{A}(E)$.

Hermitian. We know how unitary gauge transformations act on connections; in particular, they do not change the magnitude of curvature. On the other hand, when $g = H$ which is positive-definite, the above formula shows

$$g \cdot D_A = D_A - (D''_A H)H^{-1} + ((D''_A H)H^{-1})^\dagger = D_A - (D''_A H)H^{-1} + H^{-1}(D'_A H).$$

A special case is when $g = H = e^h$ which is the multiplication by a positive-valued function. In this case,

$$g \cdot D_A = D_A - \overline{\partial} h + \partial h.$$

We see the curvature changes by

$$F_{g\cdot A} = F_A - 2\partial\overline{\partial} h.$$

4.4.2 Hitchin–Kobayashi for line bundles

In this subsection we prove the following theorem.

Theorem 4.4.5. *Let Σ be a compact Riemann surface equipped with a conformal Riemannian metric*[2] *and $L \to \Sigma$ be a holomorphic vector bundle. Then there exists a unique Hermitian inner product whose corresponding Chern connection A satisfies*

$$-\frac{1}{2\pi\sqrt{-1}} * F_A = \frac{\deg L}{\mathrm{vol}(\Sigma)} \mathrm{Id}_L.$$

An equivalent statement is the following.

Theorem 4.4.6. *Let Σ be a compact Riemann surface equipped with a Riemannian metric and $L \to \Sigma$ be a Hermitian line bundle equipped with a Hermitian connection A. Then there exists a complex gauge transformation g such that $g \cdot A$ solves the Hermitian Yang–Mills equation. Moreover, if g_1, g_2 are both such complex gauge transformations, then g_1 differ from g_2 by a unitary gauge transformation.*

[2] This means that the notion of "rotation by 90 degrees" defined by the metric is the same as the one defined by the complex structure.

Now let's consider the case of line bundles. A complex gauge transformation is just a function $g = e^h : M \to \mathbb{C}^*$. Notice that one can extract a unitary part. Hence, essentially we can take h to be real. Then on Cauchy–Riemann operators, the action is

$$g \cdot A = A + \overline{\partial} h - \partial h.$$

So the term $\overline{\partial} h - \partial h$ is purely imaginary. In a local holomorphic coordinate $z = s + \sqrt{-1}t$, one has

$$\begin{aligned} &\overline{\partial} h - \partial h \\ &= \frac{1}{2}\left(\frac{\partial h}{\partial s} + \sqrt{-1}\frac{\partial h}{\partial t}\right)(ds - \sqrt{-1}dt) - \frac{1}{2}\left(\frac{\partial h}{\partial s} - \sqrt{-1}\frac{\partial h}{\partial t}\right)(ds + \sqrt{-1}dt) \\ &= \sqrt{-1}\left(\frac{\partial h}{\partial t}ds - \frac{\partial h}{\partial s}dt\right). \end{aligned}$$

Then the change of curvature is

$$d(\overline{\partial} h - \partial h) = -\sqrt{-1}\left(\frac{\partial^2 h}{\partial s^2} + \frac{\partial^2 h}{\partial t^2}\right) dsdt.$$

Then the equation is

$$*F_A + \sqrt{-1}\Delta h = \tau.$$

There is a solution because of Hodge decomposition.

4.5 Narasimhan–Seshadri theorem

Let Σ be a compact Riemann surface. A general goal is to understand the moduli space of holomorphic vector bundles over Σ.

4.5.1 Slope stability

Recall that the slop of a complex vector bundle $E \to \Sigma$ is the ratio

$$\mu(E) := \frac{\deg E}{\operatorname{rank} E}.$$

Definition 4.5.1. A holomorphic vector bundle $\mathcal{E} \to \Sigma$ is called **indecomposable** if it is not isomorphic to the direct sum of two holomorphic subbundles. It is called **stable** resp. **semi-stable** if for any proper and nonzero holomorphic subbundle $\mathcal{F} \subset \mathcal{E}$, there holds

$$\mu(\mathcal{E}) < \mu(\mathcal{E}) \text{ resp. } \mu(\mathcal{E}) \leq \mu(\mathcal{E}).$$

Now let Σ be equipped with a conformal Riemannian metric. Since the complex structure Σ is given, the metric is essentially an area form $\mathrm{dvol}_\Sigma \in \Omega^2(\Sigma)$. Recall that the Hermitian–Einstein equation on Hermitian connections A reads

$$-\frac{*F_A}{2\pi\sqrt{-1}} = \frac{\deg E}{\mathrm{rank}(E)\cdot \mathrm{Area}(\Sigma)}\mathrm{Id}_E.$$

We may normalize the metric so $\mathrm{Area}(\Sigma) = 1$.

Theorem 4.5.2 (Narasimhan–Seshadri). *[NS65] An indecomposable holomorphic Hermitian vector bundle $\mathcal{E} \to M$ admits a Hermitian–Einstein connection, i.e., a unitary connection A with $\mathcal{E} \cong \mathcal{E}_A$ solving*

$$-\frac{*F_A}{2\pi\sqrt{-1}} = \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}},$$

if and only if $\mathcal{E}$ is stable.[3]

4.5.2 Harder–Narasimhan filtration and Jordan–Hölder filtration

Proposition 4.5.3. *Let $\mathcal{E} \to \Sigma$ be a holomorphic vector bundle. Then there exists* [4] *a filtration*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{E}$$

satisfying

[3] This was not the original form of the Narasimhan–Seshadri theorem.

[4] There is also a certain uniqueness part of this theorem.

1. $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable;

2. For all i there holds

$$\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i).$$

Proof. If $\mathcal{E}$ is semistable, then we use the trivial filtration. Assume that $\mathcal{E}$ is not semistable. Then there exists $\mathcal{F} \subsetneq \mathcal{E}$ such that $\mu(\mathcal{F}) > \mu(\mathcal{E})$. We claim that the slope of all subbundles has an upper bound: it suffices to show that for each rank $r < \mathrm{rank}(\mathcal{E})$, the degrees of subbundles of rank r have an upper bound. By the Riemann–Roch, one has

$$h^0(\mathcal{F}) - h^1(\mathcal{F}) = \mathrm{rank}(\mathcal{F})(1-g) + \deg(\mathcal{F}).$$

It follows that

$$\deg\mathcal{F} \leq h^0(\mathcal{F}) - \mathrm{rank}(\mathcal{F})(1-g) \leq h^0(\mathcal{E}) - \mathrm{rank}(\mathcal{F})(1-g).$$

Now take $\mathcal{F}_1$ to be a subbundle attaining this maximal slope. Moreover, if there are multiple such ones, one chooses the one with the smallest rank. Then one has the corresponding extension

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0\,.$$

By induction hypothesis, $\mathcal{Q}$ admits such a filtration by subbundles $\mathcal{Q}_j \subset \mathcal{Q}$, $j = 1, \ldots, s$. Then define $\mathcal{F}_{j+1} \subset \mathcal{E}$ to be the preimage of $\mathcal{Q}_j$ under the map $\mathcal{E} \to \mathcal{Q}$. Then the filtration by $\mathcal{F}_i$ satisfies the stated conditions. □

Proposition 4.5.4 (Jordan–Hölder filtration)**.** *Suppose $\mathcal{E}$ is a semistable vector bundle of slope μ. Then there exists a filtration of subbundles $\mathcal{F}_i$ such that $\mathcal{F}_{i+1}/\mathcal{F}_i$ is stable and has slopt μ.*

Proof. If $\mathcal{E}$ is stable, then nothing needs to be proved. If not, then there exists a subbundle $\mathcal{F} \subset \mathcal{E}$ with slope μ. We choose such $\mathcal{F}$ of the highest rank. Then $\mathcal{E}/\mathcal{F}$ is stable (otherwise we get a bigger $\mathcal{F}$). Inductively the filtration can be constructed. □

4.5.3 Curvature of subbundles and quotient bundles

Consider an exact sequence of holomorphic bundles

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0\,.$$

A Hermitian inner product on $\mathcal{E}$ can be restricted to a Hermitian inner product on $\mathcal{F}$. Then the orthogonal complement of $\mathcal{F}$ is smoothly isomorphic to $\mathcal{Q}$, hence gives a Hermitian inner product on $\mathcal{Q}$. Then this induces a C^∞ isomorphism

$$\mathcal{E} \cong \mathcal{F} \oplus \mathcal{Q}.$$

Now let D_A be the Chern connection on $\mathcal{E}$. Then one can write

$$D_A = \begin{bmatrix} D_{A^{\mathcal{F}}} & \beta \\ -\beta^\dagger & D_{A^{\mathcal{Q}}} \end{bmatrix}. \tag{4.5.1}$$ block_form

Here $\beta \in \Omega^1(\mathrm{Hom}(\mathcal{Q}, \mathcal{F}))$ and we have β and $-\beta^\dagger$ on the off-diagonal blocks because the connection is Hermitian.

Lemma 4.5.5. $\beta \in \Omega^{0,1}(\mathrm{Hom}(\mathcal{Q}, \mathcal{F}))$.

Proof. The corresponding Cauchy–Riemann operator is written as

$$D''_A = \begin{bmatrix} D''_{A^{\mathcal{F}}} & \beta^{0,1} \\ -(\beta^\dagger)^{0,1} & D''_{A_{\mathcal{Q}}} \end{bmatrix}.$$

However, as $\mathcal{F}$ is a holomorphic subbundle, for any (local) holomorphic section s of $\mathcal{F}$, $D''_A s = 0$. Hence the term $-(\beta^\dagger)^{0,1}$ has to vanish. Then

$$(\beta^\dagger)^{0,1} = (\beta^{1,0})^\dagger = 0 \Longrightarrow \beta \in \Omega^{0,1}(\mathrm{Hom}(\mathcal{Q}, \mathcal{F})).$$ □

Remark 4.5.6. In fact β represents the extension class in $\mathrm{Ext}^1(\mathcal{F}, \mathcal{Q})$.

Now we will see the following phenomenon that "subbundles have smaller curvature and quotient bundles have bigger curvature." Indeed, if we calculate the curvature of A in the block form, then one sees

$$F_A = \begin{bmatrix} F_{A^{\mathcal{F}}} - \beta \wedge \beta^\dagger & D_{A^{\mathcal{F}}} \circ \beta + \beta \circ D_{A^{\mathcal{Q}}} \\ -\beta^\dagger \circ D_{A^{\mathcal{F}}} - D_{A^{\mathcal{Q}}} \circ \beta^\dagger & F_{A^{\mathcal{Q}}} - \beta^\dagger \wedge \beta \end{bmatrix}.$$

lemma457 **Lemma 4.5.7.** *For any* $\beta \in \Omega^{0,1}(\mathrm{Hom}(\mathcal{Q},\mathcal{F}))$*, the section*

$$-\frac{*\beta\wedge\beta^\dagger}{2\pi\sqrt{-1}} \in \Gamma(\mathrm{End}(\mathcal{F})) \text{ resp. } -\frac{*\beta^\dagger\wedge\beta}{2\pi\sqrt{-1}} \in \Gamma(\mathrm{End}(\mathcal{Q}))$$

is nonpositive resp. nonnegative.

Proof. At each point $x \in \Sigma$, choose a local holomorphic coordinate $z = s+\sqrt{-1}t$ and write $\beta = d\bar{z}T(z)$ where $T(z) : \mathcal{Q}_z \to \mathcal{F}_z$ is a complex-linear map. Then

$$\beta\wedge\beta^\dagger = d\bar{z}T(z)\wedge dzT(z)^\dagger = -dz\wedge d\bar{z}T(z)T(z)^\dagger = 2\sqrt{-1}T(z)T(z)^\dagger dsdt.$$

Similarly,

$$\beta^\dagger\wedge\beta = -2\sqrt{-1}T(z)^\dagger T(z)dsdt.$$

If the volume form is $\rho dsdt$ where ρ is positive, then

$$-\frac{*\beta\wedge\beta^\dagger}{2\pi\sqrt{-1}} = -\frac{1}{\pi\rho}AA^\dagger$$

which is nonpositive and

$$-\frac{*\beta^\dagger\wedge\beta}{2\pi\sqrt{-1}} = \frac{1}{\pi\rho}T^\dagger T$$

which is nonnegative. □

ma_curvature_diagonal **Lemma 4.5.8.** *Given A of the form (4.5.1), there exists a complex gauge transformation g such that $g\cdot A$ has the same diagonal part as A and the curvature $F_{g\cdot A}$ is block-diagonal.*

Proof. Notice that the diagonal terms D_{A^F} and D_{A^Q} induce a connection D_A on $\mathrm{Hom}(Q,F)$, by

$$D_A(T) = D_{A^\mathcal{F}}\circ T - T\circ D_{A^\mathcal{Q}}.$$

We consider a complex gauge transformation of the form

$$g = \begin{bmatrix} \mathrm{Id}_\mathcal{F} & \gamma \\ 0 & \mathrm{Id}_\mathcal{Q} \end{bmatrix},\ \gamma\in\Gamma(\mathrm{Hom}(\mathcal{Q},\mathcal{F})).$$

Then the Cauchy–Riemann operator D''_A becomes

$$\begin{bmatrix} D''_{A^{\mathcal{F}}} & \beta - D''_A\gamma \\ 0 & D_{A^{\mathcal{Q}}} \end{bmatrix}.$$

So the off-diagonal curvature becomes

$$D'_A(\beta - D''_A\gamma).$$

We want to solve the equation

$$0 = D'_A D''_A\gamma - D'_A\beta.$$

This is actually possible by a variant of Hodge decomposition. □

HE $\Longrightarrow$ stability

Let A be a Hermitian–Einstein connection on the indecomposable $\mathcal{E}$. Let $\mathcal{F} \subset \mathcal{E}$ be a subbundle. We prove that $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Using the smooth orthogonal splitting $\mathcal{E} = \mathcal{F} \oplus \mathcal{Q}$, the $\mathcal{F}$-component of the HE equation is

$$-\frac{*F_{A^{\mathcal{F}}}}{2\pi\sqrt{-1}} + \frac{*\beta \wedge \beta^\dagger}{2\pi\sqrt{-1}} = \mu(\mathcal{E})\mathrm{Id}_{\mathcal{F}}.$$

Taking trace, and integrate, one can see

$$\deg\mathcal{F} = \mu(\mathcal{E})\mathrm{rank}(\mathcal{F}) + \int_\Sigma \mathrm{Tr}\left(-\frac{*\beta \wedge \beta^\dagger}{2\pi\sqrt{-1}}\right) \leq \mu(\mathcal{E})\mathrm{rank}(\mathcal{F}).$$

It follows that $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Moreover, the equality holds only when $\beta = 0$. In this case, the connection A (and hence the induced Cauchy–Riemann operator) is block-diagonal. Hence $\mathcal{E}$ becomes the holomorphic direct sum, which contradicts the assumption that $\mathcal{E}$ is indecomposable.

4.5.4 Stability $\Longrightarrow$ HE: I. Outline

The basic strategy of Donaldson's proof works as follows. Start from a stable and indecomposable holomorphic $\mathcal{E} \to \Sigma$. Then there is a unique $\mathcal{G}^{\mathbb{C}}$-orbit of unitary connections. Consider the infimum of the (modified) Yang–Mills functional

$$J(A) = \left\| \frac{*F_A}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}} \right\|^2_{L^2(\Sigma)}.$$

Solutions are of course minimizers of this functional. Then consider a sequence A_i in the same $\mathcal{G}^{\mathbb{C}}$-orbit such that $J(A_i)$ approaches to the minimal energy. We want to show that A_i actually converges to a Hermitian–Einstein connection in this orbit.

A crucial ingredient is the so-called **Uhlenbeck compactness**, which implies that a subsequenece of A_i, up to gauge transformation, converges in a certain Sobolev regularity, to a limiting connection B. However, this limit B may not be in the same orbit as A_i. But nonetheless, B also defines a holomorphic structure, potentially different from A_i, called $\mathcal{E}'$.

Then we argue that the stability condition will imply $\mathcal{E}' \cong \mathcal{E}$, meaning that the minimal energy can be achieved in the same $\mathcal{G}^{\mathbb{C}}(E)$-orbit. Then by the elliptic regularity theorem for the HE equation, it follows that B is a smooth HE connection.

To argue that $\mathcal{E}'$ is isomorphic to the original $\mathcal{E}$, one first show that there will exists a nontrivial bundle map $\alpha : \mathcal{E} \to \mathcal{E}'$. In general holomorphic bundle maps over Riemann surfaces can be factorized.

What might go wrong for unstable bundles

Suppose $\mathcal{E}$ is unstable and indecomposable vector bundle. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

such that $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$. For simplicity, assume that $\mathcal{F}$ and $\mathcal{Q}$ are line bundles. Then on $\mathcal{F}$ and $\mathcal{Q}$ there are Hermitian–Einstein connections.

Choose a Hermitian inner product on $\mathcal{E}$, leading to a C^∞ isomorphism

$\mathcal{E} \cong \mathcal{F} \oplus \mathcal{Q}$. The Cauchy–Riemann operator reads

$$\overline{\partial}^{\mathcal{E}} = \begin{bmatrix} D''_{A^{\mathcal{F}}} & \beta \\ 0 & D''_{A^{\mathcal{Q}}} \end{bmatrix}.$$

Notice that $\beta \in \Omega^{0,1}(\mathrm{Hom}(\mathcal{Q}, \mathcal{F}))$ corresponds to exactly the extension class. There is a particular family of complex gauge transformations

$$g_t = \begin{bmatrix} \mathrm{Id}_{\mathcal{F}} & 0 \\ 0 & t\mathrm{Id}_{\mathcal{Q}} \end{bmatrix}.$$

After this family of gauge transformation, one obtains

$$\begin{aligned} g_t \cdot \overline{\partial}^{\mathcal{E}} &= g_t \circ \overline{\partial}^{\mathcal{E}} \circ g_t^{-1} \\ &= \begin{bmatrix} \mathrm{Id}_{\mathcal{F}} & 0 \\ 0 & t\mathrm{Id}_{\mathcal{Q}} \end{bmatrix} \begin{bmatrix} D''_{A^{\mathcal{F}}} & \beta \\ 0 & D''_{A^{\mathcal{Q}}} \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{\mathcal{F}} & 0 \\ 0 & t^{-1}\mathrm{Id}_{\mathcal{Q}} \end{bmatrix} \\ &= \begin{bmatrix} D''_{A^{\mathcal{F}}} & t^{-1}\beta \\ 0 & D''_{A^{\mathcal{Q}}} \end{bmatrix}. \end{aligned}$$

We see that when $t \to \infty$, the sequence of Cauchy–Riemann operators, as well as the corresponding connections converge to a block-diagonal form. The associated holomorphic bundle structure then change. Moreover, if we compute the curvature, one obtains

$$F_{A_t} = \begin{bmatrix} F_{A^{\mathcal{F}}} - t^{-2}\beta \circ \beta^\dagger & O(t^{-1}) \\ O(t^{-1}) & F_{A^{\mathcal{Q}}} - t^{-2}\beta^\dagger \circ \beta \end{bmatrix}.$$

Then we see in the semi-stable case (when $\mu(\mathcal{F}) = \mu(\mathcal{E})$, we actually obtains a different bundle $\mathcal{F} \oplus \mathcal{Q}$ and a Hermitian–Einstein connection. However, not in the original $\mathcal{G}^{\mathbb{C}}(E)$-orbit.

More precisely, we can see the following estimates. This is in fact an enhancement of the previous "HE $\Longrightarrow$ stability" theorem.

prop459 **Proposition 4.5.9.** *Suppose one has an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

of holomorphic vector bundles. Suppose $A^{\mathcal{F}}$, $A^{\mathcal{Q}}$ are Hermitian connection on $\mathcal{F}$ and $\mathcal{Q}$. Then for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{A}(\mathcal{E})$ such that

$$\|A_\epsilon - A^{\mathcal{Q}} \oplus A^{\mathcal{F}}\| < \epsilon.$$

JA_estimate **Lemma 4.5.10.** *Suppose the exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$ satisfies $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$; in particular, $\mathcal{E}$ is not stable. Then for any unitary connection A, one has* [5]

$$J(A) \geq \mathrm{rank}(\mathcal{F})(\mu(\mathcal{F}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{E} - \mu(\mathcal{Q}))).$$

Proof. By the property of ν and the block form, as well as $\mathrm{Area}(\Sigma) = 1$, we see that

$$\begin{aligned}
J(A) &= \sqrt{\int_\Sigma \left[\nu\left(\frac{*F_A}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}}\right)\right]^2} \\
&\geq \int_\Sigma \left|\nu\left(\frac{*F_A}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}}\right)\right| \\
&\geq \int_\Sigma \left|\mathrm{Tr}\left(\frac{*F_{A^{\mathcal{F}}}}{2\pi\sqrt{-1}} + \frac{*\beta\wedge\beta^\dagger}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{F}}\right)\right| \\
&\quad + \int_\Sigma \left|\mathrm{Tr}\left(\frac{*F_{A^{\mathcal{Q}}}}{2\pi\sqrt{-1}} + \frac{*\beta^\dagger\wedge\beta}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{Q}}\right)\right| \\
&= \int_\Sigma \left(\mathrm{Tr}\left(\frac{*F_{A^{\mathcal{F}}}}{2\pi\sqrt{-1}} + \frac{*\beta\wedge\beta^\dagger}{2\pi\sqrt{-1}}\right) + \mu(\mathcal{E})\mathrm{rank}\mathcal{F}\right) \\
&\quad - \int_\Sigma \left(\mathrm{Tr}\left(\frac{*F_{A^{\mathcal{Q}}}}{2\pi\sqrt{-1}} + \frac{*\beta^\dagger\wedge\beta}{2\pi\sqrt{-1}}\right) + \mu(\mathcal{E})\mathrm{rank}\mathcal{Q}\right) \\
&\geq \mathrm{rank}(\mathcal{F})(\mu(\mathcal{F}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{E}) - \mu(\mathcal{Q})). \qquad \square
\end{aligned}$$

In particular, we see that if $\mathcal{E}$ is not semistable, then one can never solve the HE equation.

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[5] In fact equality implies the exact sequence split.

4.5.5 Stability $\Longrightarrow$ HE: II. Preparation

Connections with Sobolev regularities

Recall that we have the space H_k which are roughly functions whose (weak) derivatives up to order k are all L^2-integrable. It is also sometimes denoted by $W^{k,2}$. We will mainly use $L^2 = H_0$, $W^{1,2} = H_1$ and $W^{2,2} = H_2$. We can also talk about $W^{k,2}_{\text{loc}}$, meaning that the weak derivatives are L^2-integrable over compact sets. Then we can talk about $W^{k,2}$-sections of a smooth vector bundle, which means locally the sections are $W^{k,2}$-functions.

Definition 4.5.11. Let $E \to M$ be a smooth vector bundle. A $W^{1,2}$-connection on E is a connection whose local connection matrices are functions of regularity $W^{1,2}_{\text{loc}}$. Equivalently, it is a connection of the form $A_0 + \alpha$ where A_0 is smooth and α is a $W^{1,2}$-section on $\Omega^1(\text{End}E)$.

Notice that if A is a $W^{1,2}$-connection, then its curvature is an L^2-integrable form. On the other hand, the appropriate class of gauge transformations are those of class $W^{2,2}$.

Lemma 4.5.12. *Let B be a $W^{1,2}$-connection on E. Then there exists a holomorphic vector bundle $\mathcal{E}_B$ and a bundle isomorphism g (of regularity $W^{2,2}$) such that g transforms D''_B to $\overline{\partial}^{\mathcal{E}_B}$.*

Regularity modulo gauge

For a $W^{1,2}$-connection, one can also talk about the Hermitian–Einstein equation as the curvature is still L^2.

Proposition 4.5.13. *Suppose A is a $W^{1,2}$-connection soving the HE equation. Then there exists a unitary $W^{2,2}$-gauge transformation such that $g \cdot A$ is smooth.*

Sketch of Proof. Essentially this is because the HE equation is elliptic modulo gauge. We first needs to use the so-called "local slice theorem" saying that there exists a nearby connection A_0 and a unitary gauge transformation g such that

$$D^*_{A_0}(g \cdot A - A_0) = 0$$

(so called Coulomb gauge condition). Then the equation together with the HE equation allows one to use elliptic bootstrapping. The details can be found in [Weh03]. □

Uhlenbeck compactness

Theorem 4.5.14. *[Uhl82] Suppose A_i is a sequence of $W^{1,2}$-connections on E satisfying*

$$\limsup_{i\to\infty} \|F_{A_i}\|_{L^2} < \infty.$$

Then there exists a subsequence (still indexed by i) and a sequence of $W^{2,2}$ gauge transformations g_i such that $g_i \cdot A_i$ converges weakly in $W^{1,2}$.

Then by the Sobolev embedding theorem there is L^p-convergence for all p.

4.5.6 Stability $\Longrightarrow$ HE. III. The variational argument

Energy minimizers are HE

We would like to prove the following theorem.

thm_minimizer **Theorem 4.5.15.** *Suppose $A_0 \in \mathcal{A}^{1,2}(\mathcal{E})$ satisfies*

$$J(A_0) = \inf_{\mathcal{E}_A \cong \mathcal{E}} J(A)$$

then A_0 is a Hermitian–Einstein connection.

To prove this we need the following lemmas.

Lemma 4.5.16. *Suppose $\mathcal{E}$ is stable. Suppose $A \in \mathcal{A}(\mathcal{E})$ and $s \in \Gamma(\mathrm{End}\mathcal{E})$, $D_A s = 0$ if and only if s is a constant multiple of identity.*

Proof. Suppose $D_A s = 0$. Then s has a constant rank. Moreover, $D''_A s = 0$, hence s is holomorphic. Hence the image of s and the kernel of s are subbundles of $\mathcal{E}$. The stability condition implies that $\mu(\mathrm{Im} s), \mu(\ker s)$ are both smaller than $\mu(\mathcal{E})$, which is impossible. □

Lemma 4.5.17. *Suppose $s \in \mathrm{End}(E)$ is Hermitian and $\int_\Sigma \mathrm{Tr}(s) d\mathrm{vol}_\Sigma = 0$, then there exists $h \in \Gamma(\mathrm{End}E)$ such that $D^*_{A_0} D_{A_0} h = s$.*

Proof. The operator $D^*_A D_A : W^{2,2}(\mathrm{End}E) \to L^2(\mathrm{End}E)$ is Fredholm of index zero. The above lemma shows the kernel is one-dimensional. Hence the cokernel is one-dimensional. □

Proof of Theorem 4.5.15. Consider the section

$$s = \frac{*F_{A_0}}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}}.$$

Then this section has trace zero. Hence there exists $h \in \Gamma(\mathrm{End}\mathcal{E})$ such that $D^*_{A_0} D_{A_0} h = s$. Now consider the complex gauge transformation $g_t = e^{th}$. Consider $A_t = g_g \cdot A$. Then we see the derivative at $t = 0$ is

$$D''_{A_0} h - D'_{A_0} h$$

and the change of the curvature is $D^*_{A_0} D_{A_0} h$. Hence

$$\left.\frac{d}{dt}\right|_{t=0} J(A_t) = \int_\Sigma \left\langle \frac{*F_{A_0}}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}}, D^*_{A_0} D_{A_0} h \right\rangle = J(A_0).$$

The energy minimizing condition implies that $J(A_0) = 0$.

Therefore, A_0 is a Hermitian–Einstein connection of regularity $W^{1,2}$. By elliptic regularity, up to a $W^{2,2}$-gauge transformation, A_0 is equivalent to a smooth HE connection. □

Energy minimizing sequence

Consider a sequence $A_i \in \mathcal{A}(\mathcal{E})$ such that $\mathcal{E}_{A_i} \cong \mathcal{E}$ and

$$\lim_{i \to \infty} J(A_i) = \inf_{\mathcal{E}_A \cong \mathcal{E}} J(A)$$

Then by Uhlenbeck compactness, A_i converges weakly in $W^{1,2}$ to a connection B. Then we know there exists another holomorphic vector bundle $\mathcal{E}' \to \Sigma$ whose Cauchy–Riemann operator is equivalent to that of B.

Lemma 4.5.18. *There exists a nonzero holomorphic bundle map $\alpha : \mathcal{E} \to \mathcal{E}'$.*

Proof. We prove this by a certain analytical argument. Fix $A = A_0$. Then since $A_i = g_i \cdot A_0$, g is in fact a holomorphic section of $\mathrm{Hom}(\mathcal{E}_{A_0}, \mathcal{E}_{A_i})$. However, they have the same underlying complex vector bundle with different Cauchy–Riemann operators. Locally, we can write

$$D''_{A_i} = \overline{\partial} + \omega_i$$

where ω_i is a matrix with $(0,1)$-form coefficients. Then

$$\overline{\partial} g_i + \omega_i g_i - g_i \omega_0 = 0.$$

We would like to extract a convergence subsequence of g_i.

First, we can normalize g_i so that

$$\sup_{\Sigma} \|g_i\| = 1.$$

Now we appeal to the elliptic estimate. The fact that $\overline{\partial}$ is first-order and elliptic provides the "interior estimate"

$$\|u\|_{W^{k+1,2}(\mathbb{D}(\frac{1}{2}))} \leq C \left(\|u\|_{W^{k,2}(\mathbb{D})} + \|\overline{\partial} u\|_{W^{k,2}(\mathbb{D})} \right).$$

In this way, locally the $W^{2,2}$-norm of g_i is uniformly bounded. Then by the Sobolev embedding, a subsequence converges weakly to $g \in W^{2,2}$. In particular, g_i converges uniformly to g_∞. As $\sup_\Sigma \|g_i\| = 1$, it follows that $g_\infty \neq 0$. Therefore, one has

$$D''_B g_\infty = g_\infty D''_{A_0}$$

namely g_∞ is a holomorphic bundle map from $\mathcal{E}$ to $\mathcal{E}'$. □

prop_limit_morphism **Proposition 4.5.19.** *When $\mathcal{E}$ is stable, α is an isomorphism.*

We prove this proposition in the remaining subsections. Notice that this will imply our main theorem. Indeed, this shows that the minimal energy can be attained in the same complex gauge orbit (at least when allow connections and gauge transformations to have appropriate Sobolev regularities). If B is such a minimizer, then we have proved that it must be a Hermitian–Einstein connection. Hence we obtained the existence.

Proof of Proposition 4.5.19

We first need a general factorization theorem about holomorphic bundle maps.

Lemma 4.5.20. *For any holomorphic vector bundle map $\alpha : \mathcal{E} \to \mathcal{E}'$, there exists a canonical factorization in the following sense: there is a commutative diagram of holomorphic bundle maps*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{Q} \arrow[r] \arrow[d, "\overline{\alpha}"'] & 0 \\
0 & \mathcal{Q}' \arrow[l] & \mathcal{E}' \arrow[l] & \mathcal{F}' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where horizontal lines are exact sequences and the map $\overline{\alpha} : \mathcal{Q} \to \mathcal{F}'$ is almost everywhere invertible.

Proof. Let r be the maximal of ranks of α over Σ. Since α is holomorphic, we can see that α has rank smaller than r only at finitely many points $p_1, \ldots, p_m$. Moreover, if $\mathcal{E}$ has bundle rank k, then away from these points, $\ker\alpha$ has dimension $k - r$, while has higher dimensions at p_i.

We would like to define $\mathcal{F}$ to be the "largest" subbundle of $\mathcal{E}$ that is contained in $\ker\alpha$. Indeed, near p_i, choose a local holomorphic coordinate z centered at p_i and local trivializations of $\mathcal{E}$ and $\mathcal{E}'$ around. Then α is written as a matrix-valued holomorphic function in z. We see that $z \mapsto \ker\alpha(z)$ is a complex analytic map with one variables from the punctured disk into the complex Grassmannian $\mathrm{Gr}(k - r, k)$. Since the Grassmannian is compact, such holomorphic map extends. Hence we can define $\mathcal{F}$ to be the subspace generated by the images of the extended kernel map. Then we see that $\alpha|_{\mathcal{F}} \equiv 0$, hence extends to the quotient bundle $\mathcal{Q}$.

Similarly, the image bundle $\mathcal{F}'$ can be defined similarly, using locally constructed holomorphic maps to a certain Grassmannian. $\square$

Corollary 4.5.21. *In the current situation, we have*

$$\mu(\mathcal{Q}') \leq \mu(\mathcal{F}) < \mu(\mathcal{E}) = \mu(\mathcal{E}') < \mu(\mathcal{Q}) \leq \mu(\mathcal{F}').$$

In particular, $\mathcal{E}'$ is not stable.

Proof. $\mu(\mathcal{F}) < \mu(\mathcal{E}) < \mu(\mathcal{Q})$ follows from the stability condition. On the other hand, $\mathcal{E}'$ and $\mathcal{E}$ are C^∞-isomorphic, hence have the same slope. Moreover, as $\overline{\alpha} : \mathcal{Q} \to \mathcal{F}$ is almost everywhere full rank, the induced section

$$\det \alpha : \det \mathcal{Q} \to \det \mathcal{F}'$$

is a nonzero holomorphic section of the line bundle $\mathrm{Hom}(\det \mathcal{Q}, \det \mathcal{F}')$. Hence

$$\deg(\mathrm{Hom}(\det \mathcal{Q}, \det \mathcal{F}')) = \deg(\det \mathcal{F}') - \deg(\det \mathcal{Q}) = \deg \mathcal{F}' - \deg \mathcal{Q} \geq 0.$$

□

Now by Lemma 4.5.10, one has

$$\begin{aligned} \inf_{\mathcal{E}_B \cong \mathcal{E}'} J(B) &\geq \mathrm{rank}(\mathcal{F}')(\mu(\mathcal{F}') - \mu(\mathcal{E}')) + \mathrm{rank}(\mathcal{Q}')(\mu(\mathcal{E}') - \mu(\mathcal{Q}')) \\ &\geq \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{Q}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{F})(\mu(\mathcal{E}) - \mu(\mathcal{F})). \end{aligned}$$

Then the main theorem follows from the following.

Lemma 4.5.22. *Suppose we have proved the main theorem for stable vector bundles of rank at most $k-1$. Let $\mathcal{E}$ be a stable vector bundle of rank k and $\mathcal{F} \subset \mathcal{E}$ be a subbundle. Then there exists a connection $A \in \mathcal{A}(\mathcal{E})$ such that*

$$J(A) < \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{Q}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{F})(\mathrm{rank}(\mathcal{E}) - \mu(\mathcal{F})).$$

Proof. The subbundle $\mathcal{F}$ may not be stable or semistable. However, it has the Narasimhan–Seshadri filtration by $\mathcal{F}_i \subset \mathcal{F}$ and each $\mathcal{F}_i/\mathcal{F}_{i-1}$ (which has a slopt μ_i) has the Jordan–Hölder filtration by $\mathcal{C}_{i;j}$. The stability of $\mathcal{E}$ implies that $\mu_i < \mu(\mathcal{E})$. By the induction hypothesis, one has HE connection $A_{i;j}$ on $\mathcal{C}_{i;j}/C_{i;j-1}$. Then inductively, by using Proposition 4.5.9, for any $\epsilon > 0$, there exists a connection $A_{i,\epsilon} \in \mathcal{A}(\mathcal{F}_i/\mathcal{F}_{i-1})$ which is ϵ-close to the Hermitian–Einstein connection on the direct sum $\bigoplus_j \mathcal{C}_{i;j}/C_{i;j-1}$. We could further take $A_\epsilon^{\mathcal{F}}$ such that it is ϵ-close to the direct sum $\bigoplus_i A_{i,\epsilon}$. The curvature of $A_\epsilon^{\mathcal{F}}$ has the form

$$\frac{*F_{A_\epsilon^{\mathcal{F}}}}{2\pi\sqrt{-1}} = -\Lambda + O(\epsilon).$$

The same works for the quotient bundle. Let $A_\epsilon^{\mathcal{Q}}$ be a connection on $\mathcal{Q}$ which is ϵ-close to a diagonal connection. Then we can consider a connection on $\mathcal{E}$ of the form

$$\begin{bmatrix} A_\epsilon^{\mathcal{F}} & \beta \\ -\beta^\dagger & A_\epsilon^{\mathcal{Q}} \end{bmatrix}$$

where β represents the extension class. By Lemma 4.5.8, one can apply a complex gauge transformation such that to change β so that the curvature is block-diagonal. Then we see that

$$\begin{aligned}
&\frac{*F_{A_\epsilon}}{2\pi\sqrt{-1}} + \mu(\mathcal{E})\mathrm{Id}_{\mathcal{E}} \\
&\quad = \begin{bmatrix} \mu(\mathcal{E})\mathrm{Id}_{\mathcal{F}} - \Lambda^{\mathcal{F}} - \frac{*\beta\circ\beta^\dagger}{2\pi\sqrt{-1}} + O(\epsilon) & 0 \\ 0 & \mu(\mathcal{E}) - \Lambda^{\mathcal{Q}} - \frac{*\beta^\dagger\circ\beta}{2\pi\sqrt{-1}} + O(\epsilon) \end{bmatrix}
\end{aligned}$$

Notice that the leading term $\mu(\mathcal{E})\mathrm{Id}_{\mathcal{F}} - \Lambda^{\mathcal{F}}$ is positive definite and $\mu(\mathcal{E})\mathrm{Id}_{\mathcal{Q}} - \Lambda^{\mathcal{Q}}$ is negative definite.

Also remember that one can use a complex gauge transformation to rescale β without changing other parts of the connection. Hence one can choose appropriate β such that

$$\begin{aligned}
J(A_\epsilon) &< \mathrm{Tr}\left(\mu(\mathcal{E})\mathrm{Id}_{\mathcal{F}} - \Lambda^{\mathcal{F}}\right) + \mathrm{Tr}\left(\Lambda^{\mathcal{Q}} - \mu(\mathcal{E})\mathrm{Id}_{\mathcal{Q}}\right) \\
&\leq \mathrm{rank}(\mathcal{F})(\mu(\mathcal{E}) - \mu(\mathcal{F})) + \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{Q}) - \mu(\mathcal{E})).
\end{aligned}$$

□

4.6 References and Remarks

There is an excellent expository article by Uhlenbeck [Uhl88]. For Yang–Mills theory over Riemann surfaces, the paper by Atiyah–Bott [AB83] is classical. Another good reference for the involved differential geometry is [Kob14].

Certain specific Lie groups have physical meanings. For $G = S^1$, the Yang–Mills theory can describe electromagnetism. For $G = SU(2)$, it describes the weak interaction; this is the case where the original Yang–Mills theory was formulated in [YM54].

The Hitchin–Kobayashi correspondence was a main theme in gauge theory and differential geometry. After Donaldson's proof for the one-dimensional case, it was extended by Donaldson [Don85] to dimension two and Uhlenbeck–Yau [UY86] to higher dimensions. There are numerous variants of this principle in the stories such as Higgs bundles, stable pairs, Kähler–Einstein metrics, etc.

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