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# Hod Mice and the Mouse Set Conjecture

Grigor Sargsyan

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# Abstract

We develop the theory of hod mice below $AD_{\mathbb{R}}$ + "$\Theta$ is regular". We use this theory to show that HOD of the minimal model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" satisfies $GCH$. Moreover, we show that the Mouse Set Conjecture is true in the minimal model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular".

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# Introduction

The interplay between Levy's hierarchy and the canonical models of fragments of $ZFC$ has been known for many years. For instance, a real $x$ is $\Delta^1_2$ definable from a real $y$ and a countable ordinal iff $x \in L[y]$ (Solovay, [**8**]). Assuming, $\underset{\sim}{\Delta}^1_2$-determinacy, a real $x$ is $\Delta^1_3$ definable from a real $y$ and a countable ordinal iff $x \in \mathcal{M}^{\#}_1(y)$ (Steel-Woodin, see [**39**])[1]. That this phenomenon is always true is the conclusion of the *Mouse Set Conjecture*, the primary topic of this paper.

The Mouse Set Conjecture, MSC. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$[2]. Then for all reals $x$ and $y$, $x$ is ordinal definable from $y$ if and only if there is a mouse $\mathcal{M}$ over $y$ such that $x \in \mathcal{M}$.*

Below *Mouse Capturing* ($MC$) stands for the concluding statement of the Mouse Set Conjecture. Notice that ordinal definability is the most robust form of definability and hence, $MSC$ can be viewed as the ultimate generalization of the well-known phenomenon mentioned in the opening paragraph.

$AD^+$ in the statement of MSC is an axiomatic system extending $AD$. It was originally formulated by Woodin. See Definition A.8 for its statement. One of the consequences of $AD^+$ is that if in addition one assumes that $V = L(\mathcal{P}(\mathbb{R}))$ then the universe has many of the properties that $L(\mathbb{R})$ has under $AD$. For instance, under $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$, it is the case that $L_\Theta(\mathcal{P}(\mathbb{R})) \prec_{\Sigma_1} V$ ($\Theta$ here and below is the least ordinal which is not a surjective image of the reals). More is also true: the fragment of $V$ coded by Suslin, co-Suslin sets is $\Sigma_1$ elementary in $V$. See Theorem A.10 for the fundamental consequences of $AD^+$. We note that assuming $V = L(\mathbb{R})$, $AD$ is equivalent to $AD^+$. Whether the equivalence is always true is an important open problem.

Finally, notice that one cannot hope to prove $MC$ from $ZFC$ as it can be easily arranged by forcing that there are more than $\omega_1$ many ordinal definable reals while $ZFC + MC$ implies that there are $\omega_1$ many ordinal definable reals (this follows from the comparison theorem for mice).

We will prove that $MSC$ holds in the minimal model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular". Theorem 6.26 implies that $AD_{\mathbb{R}}$ + "$\Theta$ is regular" is weaker than the existence of an $\omega_1+1$-iterable mouse with a superstrong cardinal. The proof is based on the theory of hod mice below the theory $AD_{\mathbb{R}}$ + "$\Theta$ is measurable". This theory is developed in Chapter 3. Our main theorem can be summarized in the following.

Main Theorem. *Each of the following statements implies that there is a proper class model containing the reals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular".*

[1] $\mathcal{M}^{\#}_1(y)$ is the minimal sound active $y$-mouse that has a Woodin cardinal.

[2] Often times MSC is stated under the additional hypothesis that "there is no $\omega_1+1$-iterable mouse with a superstrong cardinal". This is because currently the general notion of mouse is not well-developed. See Section 0.3 for some more comments on the hypothesis of MSC.



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(1) $AD^+ + \neg MC$.
(2) *There are divergent models of* $AD^+$*, i.e., there are* $A_i \subseteq \mathbb{R}$ *(*$i = 0, 1$*) such that* $L(A_i, \mathbb{R}) \vDash AD^+$ *and* $L(A_0, A_1, \mathbb{R}) \vDash \neg AD^+$.
(3) *There is an* $\omega_1 + 1$*-iterable mouse with a Woodin limit of Woodins.*

The following is a restatement of clause 1 of the Main Theorem in a somewhat different language.

COROLLARY 0.1. *MC holds in the minimal model of* $AD_{\mathbb{R}}$ + "$\Theta$ *is regular".*

Woodin showed that $V = L(\mathcal{P}(\mathbb{R})) + AD_{\mathbb{R}}$ + "$\Theta$ is regular" implies $AD^+$. We say $M$ is the minimal model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" if $M$ is a transitive model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" containing the ordinals and reals, and any other model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" containing the ordinals and reals also contains $M$. It follows from clause 2 of the Main Theorem that if there is a model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" then there is a minimal one.

Prior to Corollary 0.1, Woodin, in unpublished work, showed that MSC holds if there is no inner model of $AD^+ + \theta_{\omega_1} < \theta$. Here, $\theta_{\omega_1}$ is the $\omega_1$th member of the Solovay sequence (see Definition 0.9). Neeman and Steel found a technical strengthening of Woodin's result and Steel obtained equiconsistencies using this work (see [**42**]). Both of these hypothesis are more stringent than the hypothesis of Theorem 0.1.

Our proof of the Main Theorem heavily relies on the analysis of HOD. Part of this analysis is to show that HOD is a hod premouse. The problem of analyzing HOD is the oldest and the most important project of *descriptive inner model theory*. In the next section we will explain its role and importance in the modern descriptive inner model theory. For an account of descriptive inner model theory aimed at non-experts see [**27**].

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## 0.1. Why analyze Hod

The study of HOD under $AD$ and later under $AD^+$ was initiated by the Cabal group and early results on HOD can be found in Cabal Volumes ([**15**], [**14**], [**9**] and [**10**]). Nowadays the study of HOD under $AD^+$ lies in the crossroads of two different subjects, pure descriptive set theory and inner model theory. It is by

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far the most important project of descriptive inner model theory. Over the years, number of deep results have been proved on the structure of HOD under $AD$ and we take the following list of theorems as our starting point.

THEOREM 0.2. *Assume $AD$. Then the following hold.*

(1) *(Folklore) Suppose $V = L(\mathbb{R})$. Then* $\mathrm{HOD} \vDash CH$.
(2) *(Solovay, [**8**]) $\omega_1$ is measurable in* HOD.
(3) *(Becker, [**2**]) $\omega_1$ is the least measurable cardinal of* HOD.

The theorem suggests that under $AD$, HOD has a rich structure. Woodin's Derived Model Theorem, Theorem A.11, opened many doors for further explorations, and the following two theorems were proved soon after. Given a set $X$, we let $trc(X)$ be the $\subseteq$-least transitive set containing $X$. A set $X$ is called *self-wellordered* if there is a well-ordering of $trc(X)$ in $\mathcal{J}_\omega(trc(X))$.[3] Given a self-wellordered set $X$ we say $\mathcal{M}$ is a mouse over $X$ if $\mathcal{M}$ has the form $\mathcal{J}_\alpha[\vec{E}][X]$ where $\vec{E}$ is a *coherent extender sequence* (see [**43**]). The distinction between "a mouse" and "a mouse over $X$" is the same as the distinction between $\mathcal{J}$ and $\mathcal{J}[X]$. Notice that every real is self-wellordered.

Let $\mathcal{M}_\omega(y)$ be the least[4] class size $y$-mouse with $\omega$ Woodin cardinals. Given a real $y$, we say "$\mathcal{M}_\omega(y)$ exists" if $\mathcal{M}_\omega(y)$ exists as a class and it is $\kappa$-iterable for all $\kappa$.

THEOREM 0.3 (Woodin, [**43**]). *Suppose $\mathcal{M}_\omega$ exists. Then AD holds in $L(\mathbb{R})$.*

THEOREM 0.4 (Steel-Woodin, [**43**]). *Suppose $\mathcal{M}_\omega$ exists. Then in $L(\mathbb{R})$, $x$ is ordinal definable iff $x$ is in some mouse. Moreover, the following statements are equivalent where $x, y \in \mathbb{R}$.*

(1) $L(\mathbb{R}) \vDash$ "*$x$ is ordinal definable from $y$*".
(2) $x \in \mathcal{M}_\omega(y)$.

COROLLARY 0.5. *Assume $\mathcal{M}_\omega$ exists. Let $\mathcal{H} = \mathrm{HOD}^{L(\mathbb{R})}$. Then*

$$\mathbb{R}^{\mathcal{H}} = \mathbb{R}^{\mathcal{M}_\omega}.$$

Notice that Theorem 0.4 is an instance of capturing definability via mice. Since $CH$ holds in every mouse, Corollary 0.5 gives another proof of 1 of Theorem 0.2. In fact it gives a deeper structural characterization of the reals of HOD prompting the question: is there such a characterization for the entire HOD? Below we collect some questions motivated by the theorems stated so far.

QUESTION 0.6. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.*

(1) *Does* $\mathrm{HOD} \vDash CH$*?*
(2) *Does* $\mathrm{HOD} \vDash GCH$*?*
(3) *Is it true that the reals of* HOD *are the reals of some mouse?*
(4) *What kind of large cardinals does* HOD *have?*
(5) *What is the structure of* HOD*?*

[3]We warn the reader that often, especially when discussing mice, we will use Jensen's hierarchy $\mathcal{J}$ rather then the customary constructible hierarchy $L$. However, the reader can easily substitute $L$ for $\mathcal{J}$ without losing much.

[4]Here "least" means that it is the hull of a club of indiscernibles.

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The second question is an important open problem, and the third question is just a different way of asking if MSC is true. Questions 4 and 5 are somewhat vague and are part of ongoing research. Theorem 0.10 is a partial answer to the fourth question. We refer the reader to [**27**] for more on these two questions. The following surprising theorem of Woodin and the proof of Theorem 0.2 imply that the answer to the first question is yes.

THEOREM 0.7 (Woodin). *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Then the set*

$$A = \{(x, y) \in \mathbb{R}^2 : x \in OD(y)\}$$

*is universal* $\Sigma^2_1$. *Moreover, for some* $\kappa$ *there is a tree* $T \in \mathrm{HOD}$ *on* $\omega \times \kappa$ *such that* $A = p[T]$.

COROLLARY 0.8. *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Then* $\mathrm{HOD} \vDash CH$.

PROOF. Let $A = \{x \in \mathbb{R} : x \in OD\}$. For $x \in A$, let $(\phi_x, \vec{\alpha}_x)$ be the lexicographically ($\leq_{lex}$) least such that $x$ is definable from $\vec{\alpha}_x$ via $\phi_x$. Let then $\leq^*$ be the $OD$ wellordering of $A$ given by $y \leq^* x$ iff $(\phi_y, \alpha_y) \leq_{lex} (\phi_x, \alpha_x)$. It is a consequence of Theorem 0.7 that there is $T \in \mathrm{HOD}$ such that $p[T] = \leq^*$. But then by a result of Mansfield and Solovay (see Theorem 14.7 of [**8**]), in HOD, for every $x \in A$, the set $\{y \in A : y \leq^* x\}$ has the perfect set property. This then easily gives that $\mathrm{HOD} \vDash CH$. □

Questions 2 and 3 above led the way towards analysis of HOD under $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Section 3 of [**27**] gives a historical account of what was proved next. In particular, we recommend that the interested reader consult Theorem 3.11 of [**27**]. In what follows we will give some modern motivations for analyzing HOD of models of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

**The inner model problem:** The project of analyzing HOD is embedded so deeply in inner model theory that it is hard to isolate the most significant impact that it will have when it is fully carried out. The following is one possible chain of implications that lead to analysis of HOD.

Suppose we would like to attack *the inner model problem* for superstrong cardinals[5], i.e., we would like to construct an $\omega_1 + 1$-iterable mouse that satisfies "there is a superstrong cardinal". It follows from the work of Mitchell and Steel (see [**20**]) that to solve the inner model problem for a superstrong cardinal it is enough to solve some form of *the iterability problem*, one form of which says that countable submodels of rank initial segments of $V$ are $\omega_1 + 1$-iterable.

Next, to solve the iterability problem we need to describe a method of choosing branches for iteration trees of length $\omega_1$. It is a well-known theorem that just in $ZFC$ there are $\omega_1$-trees with no $\omega_1$-branch. One natural idea for doing this is to construct absolutely definable $\omega_1$-iteration strategies for countable submodels of rank initial segments of $V$, since then we can use genericity to construct branches for uncountable trees. A natural attempt is to guarantee that the resulting $\omega_1$-strategy is universally Baire. To keep track of the complexity of arising iteration strategies and to guarantee that the resulting strategies are universally Baire, we need to study the descriptive set theoretic properties of these $\omega_1$-strategies, and this is how descriptive set theory gets into the picture.

[5]We only mention superstrong cardinals as the general theory of mice for supercompact cardinals at the moment isn't well understood.

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Thus, we have identified the problem we want to solve. Fix an ordinal $\gamma$ and let $\pi : M \to V_\gamma$ be a countable substructure of $V_\gamma$. We would like to construct an $\omega_1$-iteration strategy for $M$ in such a way that the set of reals coding it is universally Baire.

One particularly strong intuition is that, assuming proper class of Woodin cardinals, we should be able to construct an iteration strategy for $M$ by analyzing the kind of mice that exist in the determinacy model given by the universally Baire sets, i.e., in the model $L(\Gamma_{uB}, \mathbb{R})$ where $\Gamma_{uB}$ is the collection of the universally Baire sets of reals. This intuition is based on the idea that mice in $L(\Gamma_{uB}, \mathbb{R})$ can serve as $\mathcal{Q}$-structures for countable iteration trees that are based on $M$ (see Definition 1.22 for the definition of a $\mathcal{Q}$-structure). And this later idea is based on the fact that the $\omega_1$ fragments of the iteration strategies of all iterable mice are in $L(\Gamma_{uB}, \mathbb{R})$.

Here is then a plausible line of thought. Assume that $M$ is not $\omega_1 + 1$-iterable. Attempt to iterate it via a strategy that is guided by canonical information, such as $\mathcal{Q}$-structures, that is coded into $L(\Gamma_{uB}, \mathbb{R})$. The failure to define such a strategy should translate into the fact that $L(\Gamma_{uB}, \mathbb{R})$ has complicated descriptive set theoretic structure and hence, also complicated $\omega_1$-iteration strategies. In particular, it has an $\omega_1$-iteration strategy for a mouse with a superstrong.

To complete this vague idea we need to have methods that translate descriptive set theoretic strength into inner model theoretic strength, and this is where the Mouse Set Conjecture makes its entrance. It essentially says that complicated models of determinacy have complicated mice.

All known proofs of MSC, including the one given in this paper, are by induction. Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. We let $\Gamma$ be the largest Wadge initial segment of $\mathcal{P}(\mathbb{R})$ which satisfies MC, i.e., given any $x, y \in \mathbb{R}$, $x$ is $OD(y)$ as witnessed by a structure coded by a set of reals in $\Gamma$ if and only if there is a sound $y$-mouse $\mathcal{M}$ such that $x \in \mathcal{M}$ and $\mathcal{M}$ has an $\omega_1$-iteration strategy coded by a set in $\Gamma$. Clearly we want to show that $\Gamma = \mathcal{P}(\mathbb{R})$. Towards a contradiction assume not.

Let then $A$ be a set of reals of Wadge rank $\sup\{w(B) : B \in \Gamma\}$ where $w(B)$ is the Wadge ordinal of $B$. The idea now is to "capture" $A$ inside some mouse. Here "capture" is used in the same sense as when one captures a $\Sigma^1_2$ set by a tree in $L$. It is hard to work with $A$ as it is just devoid of any meaningful structure. Instead, we would like to produce a pair $(\mathcal{P}, \Sigma)$ such that $\mathcal{P}$ is some fine structural model, in our case a *hod mouse*, $\Sigma$ is an $\omega_1$-iteration strategy for $\mathcal{P}$ and if $C \subseteq \mathbb{R}$ is a set coding $\Sigma$ in some natural way then $w(C) \geq w(A)$. This step is what is called *the generations of pointclasses* (see the discussion before Conjecture 0.14).

Next, as now $C$ is a very meaningful set, we attempt to identify it in various mice and eventually show that it is captured by a mouse. This step is called *capturing of hod pairs* (see the discussion before Conjecture 0.15).

To produce the pair $(\mathcal{P}, \Sigma)$ we use the analysis of HOD of $\Gamma$ (i.e., the initial segment of HOD coded by sets of reals in $\Gamma$). We show that there is a pair $(\mathcal{P}, \Sigma)$ as above with the property that the direct limit of all iterates of $\mathcal{P}$ converges to HOD of $\Gamma$. It then easily follows that $\Sigma \notin \Gamma$ as $\Sigma$ gives a surjection $f : \mathbb{R} \to \Theta^\Gamma$ where $\Theta^\Gamma$ is the least ordinal that is not a surjective image of the reals. The map $f$ is constructed by considering the various iteration embeddings arising from the direct limit construction (for more details see the discussion after Theorem 0.23). This step is done by proving Conjecture 0.13. Section 0.4 has some of the technical

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details that arise while carrying out the arguments presented in this outline.

**Core model induction:** This is a method, originally due to Woodin, for constructing determinacy models from various hypothesis such as $PFA$, $\neg\square_\kappa$ for singular $\kappa$, *the unique branch hypothesis* and etc. It is the most successful tool used to calibrate the set theoretic strength of such statements. We refer the reader to [**24**], [**29**], [**31**] and [**41**] for some applications of the core model induction.

While doing core model induction one encounters two steps that can be characterized as *internal* and *external*. In the internal step of the induction, there is a concrete model $M$ and our goal is to show that $M \vDash AD^+$. $M$ is usually a fine structural model over $\mathbb{R}$. In [**41**], it is just $L(\mathbb{R})$ while in [**24**], it is the stack of all countably iterable[6] mice over $\mathbb{R}$. While doing this step of the induction, at various stages, we have to translate coarse sets of reals into iteration strategies. Just like in the case of the inner model problem, we use MSC and some weaker form of generation of pointclasses to do this. The details can be found in [**31**].

The external step of the core model induction amounts to building a new set that is beyond the sets constructed thus far. Let $M$ be the model of the determinacy constructed thus far. The goal is to produce a set of reals $A$ that is not in $M$ and such that $L(A,\mathbb{R}) \vDash AD^+$. To do this, we attempt to construct $A$ as a pair $(\mathcal{P},\Sigma)$ that has the same properties as $(P,\Sigma)$ used above to generate $\Gamma$ (in our current case $\Gamma = \mathcal{P}(\mathbb{R})$). This is exactly where the analysis of HOD is used in core model induction applications. A reader interested in the technical details may consult [**24**], [**29**] or [**31**].

Next, we give a crash course on hod mice to give some flavor of what to expect in the next few chapters. We then use our exposition to outline the proof of MSC (see Section 0.4).

## 0.2. A crash course on hod mice

Hod mice, which are specifically designed to compute HOD of models of $AD^+$, feature prominently in the proof of the Main Theorem. One of the motivations behind their definition is Theorem 0.10. A hod mouse, besides having an extender sequence, is also closed under the iteration strategies of its own initial segments. Thus it is a model of the form $\mathcal{P} = \mathcal{J}_\alpha^{\vec{E},\Sigma}$ where $\alpha$ is an ordinal, $\vec{E}$ is a fine extender sequence and $\Sigma$ is a strategy of the initial segments of $\mathcal{P}$. These initial segments, or rather the ordinals defining these initial segments, are called *layers*. Layers are used to keep track of the stages where new strategies appear in the model. More precisely, given a hod premouse $\mathcal{P}$, $\eta$ is called a *layer* of $\mathcal{P}$ if the strategy of $\mathcal{P}|\eta$ appears in the model after stage $\eta$. Here $\mathcal{P}|\eta = \mathcal{J}_\eta^{\vec{E},\Sigma}$. *There is one important exception. All hod mice have a last layer for which no strategy is activated.* All hod mice satisfy $ZFC - Replacement$ and they have exactly $\omega$-more cardinals above the last layer. See Figure 0.2.1 for a generic picture of a hod premouse.

Unlike ordinary mice, the hierarchy of hod mice grows according to *the Solovay hierarchy*, which is a determinacy hierarchy. First we let

$$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}.$$

It is not hard to show that the length of Wadge order is $\Theta$. We can now define the Solovay sequence as follows:

[6]This means that countable substructures are $\omega_1 + 1$-iterable.

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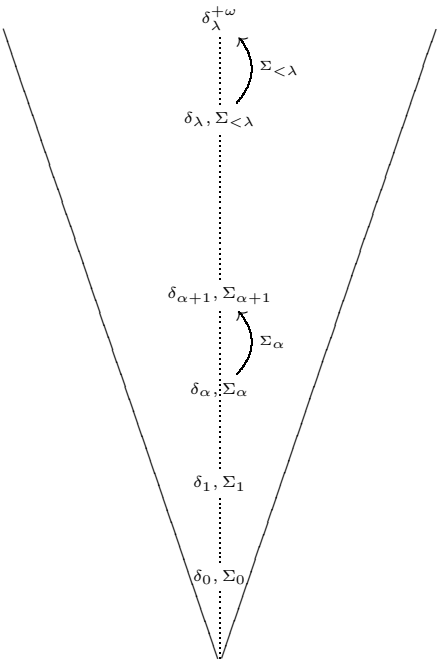


$\Sigma_{<\lambda}$ is the strategy of $\mathcal{P}|\delta_\lambda$.

FIGURE 0.2.1. Hod premouse.

DEFINITION 0.9 (Solovay sequence, AD). *The Solovay sequence is a closed increasing sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows:*

(1) $\theta_0 = \sup\{\alpha :$ *there is an ordinal definable surjection from* $\mathcal{P}(\omega)$ *onto* $\alpha\}$,
(2) *if* $\theta_\beta < \Theta$ *then fixing* $A \subseteq \mathbb{R}$ *such that* $w(A) = \theta_\beta$,
$$\theta_{\beta+1} = \sup\{\alpha : \text{there is an ordinal definable from } A \text{ surjection from } \mathbb{R} \text{ onto } \alpha\},$$
(3) *if* $\lambda$ *is a limit ordinal then* $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

It follows that $\theta_\Omega = \Theta$. The Solovay hierarchy is a hierarchy of axioms that stipulate that the Solovay sequence is long. One way to determine if the iteration strategy of hod mouse is complicated or not is to compute the pointclass that it generates in the sense described in the paragraph before Conjecture 0.14. For more information on the Solovay hierarchy see Section 2.2 of [**27**].

The following is a beautiful theorem of Woodin on the Solovay sequence.

THEOREM 0.10 (Woodin, [**16**]). *Assume $ZF+DC+AD$ and $\theta_{\alpha+1}$ exists. Then $\theta_{\alpha+1}$ is Woodin in* HOD.

Iterability for hod mice is a stronger notion than for ordinary mice. Essentially, we need to require that the external iteration strategy of a hod mouse is consistent with the internal one. First, suppose $\mathcal{M}$ is a some transitive structure and $\Sigma$ is its iteration strategy. Let $\mathcal{N}$ be a $\Sigma$-iterate of $\mathcal{N}$. We let $\Sigma_\mathcal{N}$ be the strategy of $\mathcal{N}$ that it inherits from $\Sigma$. Notice that it may well be the case that $\Sigma_\mathcal{N}$ depends

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on the particular iteration producing $\mathcal{N}$, but in this introduction, we ignore this possibility.

DEFINITION 0.11. *Suppose $\mathcal{P}$ is a hod premouse. Then $\Sigma$ is an $(\omega_1, \omega_1)$-iteration strategy for $\mathcal{P}$ if $\Sigma$ is a wining strategy for II in $\mathcal{G}(\mathcal{P}, \omega_1, \omega_1)$ and whenever $\mathcal{Q}$ is a $\Sigma$-iterate of $\mathcal{P}$, $\Sigma^{\mathcal{Q}} = \Sigma_{\mathcal{Q}} \restriction \mathcal{Q}$. If $\mathcal{P}$ is a hod mouse and $\Sigma$ is its $(\omega_1, \omega_1)$-strategy then $(\mathcal{P}, \Sigma)$ is called a hod pair.*

Hod mice have a certain peculiar pattern. All hod mice have Woodin cardinals. The layers of a hod mouse are its Woodin cardinals and their limits. Suppose now $\mathcal{P}$ is a hod premouse. We let $\langle \delta^{\mathcal{P}}_\alpha : \alpha \leq \lambda^{\mathcal{P}} \rangle$ be the enumeration of its layers in increasing order. Thus, $\langle \delta^{\mathcal{P}}_\alpha : \alpha \leq \lambda^{\mathcal{P}} \rangle$ is the sequence of Woodin cardinals and their limits. The strategies of hod premice are activated in a very careful manner. At stage $\alpha$ the strategy that is being activated is the strategy of certain $\mathcal{P}(\alpha) \trianglelefteq \mathcal{P}$. $\mathcal{P}(\alpha)$ is itself a hod premouse and *it is not the same* as $\mathcal{P}|\delta^{\mathcal{P}}_\alpha$.

### 0.3. The mouse set conjecture

In many applications, $MC$ is too weak and we need to prove a more general version of it known as the *Strong Mouse Capturing*.

**Strong Mouse Capturing,** $SMC$**:** If $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is fullness preserving, $x \in \mathbb{R}$ codes $\mathcal{P}$, and $y$ is $OD(\Sigma, x)$ then there is a $\Sigma$-mouse $\mathcal{M}$ over $x$ such that $y \in \mathcal{M}$.

Clearly SMC implies MC, just take $(\mathcal{P}, \Sigma)$ be the empty pair.

**Strong Mouse Set Conjecture,** $SMSC$**:** Assume ZF+AD$^+$+$V = L(\mathcal{P}(\mathbb{R}))$+ "there is no iteration strategy for a mouse with a superstrong". Then $SMC$ holds.

In this paper, we will prove the following theorem (see Theorem 6.19).

THEOREM 0.12 ($ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$). *Suppose there is no inner model containing the reals and ordinals and satisfying $ZF + AD_{\mathbb{R}}$ + "$\Theta$ is regular". Then $SMC$ holds.*

### 0.4. The proof of MSC

In this subsection, we outline the proof of Theorem 0.12. Below it is stated again in an equivalent form. Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. The proof of Theorem 0.12 is via proving three conjectures that collectively imply MC. These are *the HOD Conjecture (HOC)*, *the Generation of Closed Pointclasses, (GCP)* and *the Capturing of Hod Pairs, (CHP)*. As was mentioned before the notion of a hod mouse isn't well-developed much beyond $AD_{\mathbb{R}}$ + "$\Theta$ is regular". Because of this the statements of the conjectures are somewhat informal, and part of the problem is in extending the theory of hod mice to capture stronger theories from the Solovay hierarchy in a way that the conjectures still hold for such hod mice.

Given a hybrid premouse $\mathcal{Q}$, we say $\mathcal{Q}$ is a *shortening* of a hod premouse if either

(1) there is a hod premouse $\mathcal{P}$ such that letting $\kappa$ be the largest layer of $\mathcal{P}$, $\mathcal{Q} = \mathcal{P}|\kappa$ or

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(2) for some limit ordinal $\xi$, there is a sequence of hod premice $\langle \mathcal{P}_\alpha : \alpha < \xi \rangle$ such that $\mathcal{P}_\alpha \triangleleft_{hod} \mathcal{P}_\beta$ and $\mathcal{Q} = \cup_{\alpha<\xi} \mathcal{P}_\alpha$.

CONJECTURE 0.13 (The Hod Conjecture). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + MC$. Then $V_\Theta^{\mathrm{HOD}}$ is a shortening of a hod mouse. Moreover, suppose $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$ is such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Then $(H_{\Theta^{+\omega}})^{\mathrm{HOD}^{L(\Gamma,\mathbb{R})}}$ is an iterate of some countable hod mouse*[7].

The HOD Conjecture is used to show that the $V_\Theta^{\mathrm{HOD}}$ of the initial segment of the Wadge hierarchy that satisfies MC is a shortening of a hod premouse. If MC fails then letting $\Gamma$ be the largest initial segment of the Wadge hierarchy where MC holds, HOD Conjecture gives a way of characterizing a set of reals just beyond $\Gamma$ in terms of the iteration strategy of a hod mouse iterating to $(V_\Theta)^{\mathrm{HOD}^{L(\Gamma,\mathbb{R})}}$. That such a characterization is possible is the content of GCP. Given a pointclass $\Gamma$, let $w(\Gamma) = \sup\{w(A) : A \in \Gamma\}$. We say $\Gamma$ is a *closed pointclass* if $\mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) \subseteq \Gamma$.

CONJECTURE 0.14 (The Generation of Closed Pointclasses). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$ is a closed pointclass such that there is a Suslin cardinal $\kappa > w(\Gamma)$. Suppose $L(\Gamma, \mathbb{R}) \vDash MC$. Then for some hod pair $(\mathcal{P}, \Sigma)$,*

$$w(\Gamma) \leq w(Code(\Sigma)).$$

Suppose now that MC fails. It can be shown that there is $\Gamma \subset \mathcal{P}(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \vDash MC$, $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and if $A$ is such that $w(A) = w(\Gamma)$ then $L(A, \mathbb{R}) \vDash \neg MC$. Let $x$ be a real which is $OD$ but not in a mouse. It then follows from GCP that there is a hod pair $(\mathcal{P}, \Sigma)$ such that $w(\Gamma) \leq w(Code(\Sigma))$. A consequence of this is that $x$ is in some $\Sigma$-mouse. To derive a contradiction, it is shown that $\Sigma$ can be captured by mice. That such a capturing is always possible is the content of CHP.

CONJECTURE 0.15 (The Capturing of Hod Pairs). *Suppose $\delta$ is a Woodin cardinal and $V_\delta$ is $\delta+1$-iterable for trees that are in $L_\omega(V_\delta)$. Suppose further that there is no mouse with a superstrong cardinal and that $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P} \in V_\delta$ and $\Sigma$ is a $\delta^+$-iteration strategy. Let $\mathcal{N}^* = (L[\vec{E}])^{V_\delta}$, the output of the full background construction of $V_\delta$, and let $\mathcal{N} = L[\mathcal{N}^*]$. Thus, $\mathcal{N}^* \trianglelefteq \mathcal{N}$. There is then a $\Sigma$-iterate $\mathcal{Q}$ of $\mathcal{P}$ such that if $\Lambda$ is the strategy of $\mathcal{Q}$ induced by $\Sigma$ then $\mathcal{Q} \in \mathcal{N}|\delta$ and $\Lambda \restriction (V_\delta)^{\mathcal{N}} \in \mathcal{N}$.*

Continuing with the above set up, we look for $M$ such that $\mathcal{P} \in M$, $\Sigma \restriction M \in M$ and for some $\delta$, $(M, \delta, \mathcal{P}, \Sigma \restriction M)$ satisfies the hypothesis of Conjecture 0.15 (it plays the role of $V$). That there is always such an $M$ is a theorem due to Woodin. It then follows from Conjecture 0.15 that some tail of $\Sigma$ is in a mouse implying that in fact $x$ is in some mouse as well. The details of this rough sketched are worked out in great detail in the rest of this book.

What we will prove is that the three conjectures together imply MSC and all three conjectures are true under an additional assumption that there is no proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular".

THEOREM 0.16 (Theorem 4.24, Theorem 6.1 and Theorem 6.5). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and that HOC, GCP and CHP are true. Then MC holds.*

[7]Recall that $H_\kappa$ is the set of all sets of hereditarily size $< \kappa$.

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THEOREM 0.17 (Theorem 6.19). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and suppose there is no proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular". Then HOC, GCP and CHP are all true and hence, MC is true as well.*

### 0.5. The comparison theory of hod mice

While we will encounter uncountable hod premice, all hod mice of this paper are countable. Notice that comparison may not hold for arbitrary two hod pairs. For instance, if $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\lambda^{\mathcal{P}}, \lambda^{\mathcal{Q}} \geq 1$ then it is possible that in the comparison of $\mathcal{P}(0)$ and $\mathcal{Q}(0)$, $\mathcal{P}(0)$ iterates into a proper initial segment of an iterate of $\mathcal{Q}(0)$. This means that further comparison of $\mathcal{P}$ and $\mathcal{Q}$ is meaningless[8]. In general, we do not know how to compare arbitrary hod pairs. Our comparison theorem works for hod pairs $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has *branch condensation* and is *fullness preserving*. Both are technical properties that we will define at the end of this subsection. It is possible to continue with the paper without a solid understanding of what these notions are.

Here is what comparison means for hod pairs. Given two hod premice $\mathcal{P}$ and $\mathcal{Q}$, we write $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. Given a hod pair $(\mathcal{P}, \Sigma)$ we let

$$I(\mathcal{P}, \Sigma) = \{\mathcal{Q} : \mathcal{Q} \text{ is an } \Sigma\text{-iterate of } \mathcal{P}\}.$$

**Comparison for hod pairs:** Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there are $\mathcal{M} \in I(\mathcal{P}, \Sigma)$ and $\mathcal{N} \in I(\mathcal{Q}, \Lambda)$ such that one of the following holds:

(1) $\mathcal{M} \trianglelefteq_{hod} \mathcal{N}$ and $(\Lambda_{\mathcal{N}})_{\mathcal{M}} = \Sigma_{\mathcal{M}}$.
(2) $\mathcal{N} \trianglelefteq_{hod} \mathcal{M}$ and $(\Sigma_{\mathcal{M}})_{\mathcal{N}} = \Lambda_{\mathcal{N}}$.

The following is the comparison theorem proved in Section 2.5.

THEOREM 0.18 (Comparison, Theorem 2.28). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that both $\Sigma$ and $\Lambda$ have branch condensation and are fullness preserving. Then comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$.*

Branch condensation of a strategy essentially says that if any iteration is realized into an iteration via the strategy then it is also an iteration according to the strategy. Below is the definition of branch condensation for iteration trees. What is really needed for the comparison theory is a stronger notion for iterations produced via the runs of $\mathcal{G}(\omega_1, \omega_1)$ but we won't need this version in our current exposition.

DEFINITION 0.19 (Branch condensation, Definition 2.14). *Suppose $M$ is a transitive model of some fragment of ZFC and $\Sigma$ is an iteration strategy for $M$. Then $\Sigma$ has branch condensation (see Figure 0.5.1) if for any two iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $M$ and any branch $c$ of $\mathcal{U}$ if*

(1) *$\mathcal{T}$ and $\mathcal{U}$ are according to $\Sigma$,*
(2) *$lh(\mathcal{U})$ is limit and $lh(\mathcal{T}) = \gamma + 1$,*
(3) *for some $\pi : \mathcal{M}_c^{\mathcal{U}} \to_{\Sigma_1} \mathcal{M}_\gamma^{\mathcal{T}}$,*

$$\pi_{0,\gamma}^{\mathcal{T}} = \pi \circ \pi_c^{\mathcal{U}}$$

[8]To see this let $\mathcal{S}$ and $\mathcal{R}$ be this iterates. We are assuming that $\mathcal{S}(0) \triangleleft \mathcal{R}(0)$. But then on the $\mathcal{S}$ side we have the strategy of $\mathcal{S}(0)$ while on the $\mathcal{R}$ side we only have extenders.

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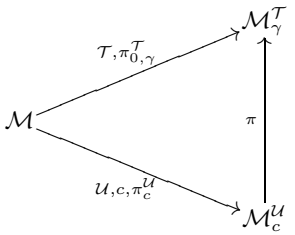


FIGURE 0.5.1. Branch condensation

*then* $c = \Sigma(\vec{\mathcal{U}})$.

Fullness preservation refers to the degree of correctness of the models. It essentially says that the iterates of the hod mouse contain all the mice present in the universe. Given a (hod) mouse $\mathcal{M}$ and $\eta$, we say $\eta$ is a strong cutpoint of $\mathcal{M}$ if there is no $\kappa \leq \eta$ which is $\eta$-strong as witnessed by the extenders on the sequence of $\mathcal{M}$.

DEFINITION 0.20 (Fullness Preservation, Definition 2.27). *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair.* $\Sigma$ *is fullness preserving if whenever* $\mathcal{Q} \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ *and* $\eta > \delta_\alpha$ *is a strong cutpoint of* $\mathcal{Q}(\alpha + 1)$, *then*

$$\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)} = Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(\mathcal{Q}|\eta).$$

*and*

$$\mathcal{Q}|(\delta_\alpha^+)^{\mathcal{Q}} = Lp^{\oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1)}}(\mathcal{Q}(\alpha)).$$

One important consequence of branch condensation and fullness preservation is that they imply that $\Sigma$ is *positional* and *commuting*. This is important for direct limit constructions. Given a hod pair $(\mathcal{P}, \Sigma)$, we say $\Sigma$ is positional if whenever $\mathcal{Q} \in I(\mathcal{P}, \Sigma)$ and $\mathcal{R} \in I(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ then the iteration embedding from $\mathcal{Q}$-to$\mathcal{R}$ is independent from the particular run of the iteration game producing $\mathcal{R}$. If $\Sigma$ is positional then we let $\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the unique iteration embedding given by $\Sigma$. It certainly depends on $\Sigma$. We say $\Sigma$ is commuting if whenever $\mathcal{Q} \in I(\mathcal{P}, \Sigma)$, $\mathcal{R} \in I(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $\mathcal{S} \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$ then

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{S}} = \pi^{\Sigma}_{\mathcal{R},\mathcal{S}} \circ \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}.$$

THEOREM 0.21 (Theorem 2.42). *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ *and suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\Sigma$ *has branch condensation and is fullness preserving. Then* $\Sigma$ *is both positional and commuting.*

Equipped with our comparison theorem we can now explain how the analysis of HOD is done.

## 0.6. Hod is a hod premouse

Hod mice were introduced in order to generalize the computation of HOD of $L(\mathbb{R})$ to larger models of $AD^+$. Specifically, they are used to prove theorems like the following.

THEOREM 0.22 (The HOD Theorem). *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Suppose that for every* $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$, $L(\Gamma, \mathbb{R}) \vDash \neg$"$AD_{\mathbb{R}} + \Theta$ *is regular". Then* $V_\Theta^{\mathrm{HOD}}$ *is a shortening of a hod premouse.*

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In this subsection, we will outline the proof of Theorem 0.22. The proof is via induction, and the following is the inductive step. Its proof is based on Woodin's computation of HOD of the minimal model of $AD^+ + \Theta = \theta_0$ (see for instance [**42**]).

LEMMA 0.23 (The Inductive Step). *Suppose $\alpha$ is such that $\theta_\alpha < \Theta$. There is then a hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation, is fullness preserving and $V_{\theta_\alpha}^{\mathrm{HOD}}$ is a shortening of the direct limit of all $\Sigma$-iterates of $\mathcal{P}$ or, using the notation developed below,*

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\alpha = V_{\theta_\alpha}^{\mathrm{HOD}}.$$

PROOF. We sketch the proof. A more detailed argument will be given later on in the paper. Fix an ordinal $\alpha$ as in the hypothesis. First we show that there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation and is fullness preserving and for any set of reals $A$,

$$w(A) < \theta_\alpha \leftrightarrow A \leq_w Code(\Sigma).$$

This is an instance of the generation of pointclasses. Let

$$\mathcal{F}(\mathcal{P}, \Sigma) = I(\mathcal{P}, \Sigma)$$

and define $\leq^\Sigma$ on $\mathcal{F}(\mathcal{P}, \Sigma)$ by letting

$$\mathcal{Q} \leq^\Sigma \mathcal{R} \text{ iff } \mathcal{R} \in I(\mathcal{Q}, \Sigma_\mathcal{Q}).$$

It follows from the comparison theorem that $\leq^\Sigma$ is directed. Also, it follows from Theorem 0.21 that $\Sigma$ is commuting. Using this, we can form a direct limit. We let

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma)$$

be the direct limit of $(\mathcal{F}(\mathcal{P}, \Sigma), \leq^\Sigma)$ under the iteration maps $\pi^\Sigma_{\mathcal{Q},\mathcal{R}}$. One then shows that

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\alpha = V_{\theta_\alpha}^{\mathrm{HOD}} \qquad (*).$$

The forward inclusion of $(*)$ namely that $\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \mathrm{HOD}$ might seem less plausible as the definition of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ seems to require $(\mathcal{P}, \Sigma)$. However, it follows from comparison, Theorem 2.28, that $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is in fact independent of $(\mathcal{P}, \Sigma)$. The full proof of $(*)$ can be found in Section 4.4. □

The next step is to prove a version of Lemma 0.23 for $\alpha$ such that $\theta_\alpha = \Theta$. This will indeed finish the proof of Theorem 0.22. However, it turns out that in this case we cannot get a pair $(\mathcal{P}, \Sigma)$ as in Lemma 0.23. The reason is that such a $\Sigma$ can be used to define a surjection from $\mathbb{R}$ onto $\Theta = \theta_\alpha$. First for $\mathcal{Q} \in \mathcal{F}(\mathcal{P}, \Sigma)$, let

$$\pi^\Sigma_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$$

be the direct limit embedding. Next let $A \subseteq \mathbb{R}$ be the set of reals coding the set

$$\{(\mathcal{Q}, \beta) : \mathcal{Q} \in \mathcal{F}(\mathcal{P}, \Sigma) \text{ and } \beta \in \mathcal{Q}\}.$$

Then define $f : A \to Ord$ by

$$f(\mathcal{Q}, \beta) = \pi^\Sigma_{\mathcal{Q},\infty}(\beta),$$

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Then clearly $\theta_\alpha \subseteq rng(f)$.

Nevertheless, one can still define a certain directed system whose direct limit is $V_\Theta^{\mathrm{HOD}}$. The proof splits into two different cases. The first case is when $\Theta = \theta_\alpha$ for some limit $\alpha$. First, it follows from comparison and $(*)$ that whenever $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is fullness preserving then for some $\beta$ we have that

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\beta = V_{\theta_\beta}^{\mathrm{HOD}} \qquad (**)$$

and $w(Code(\Sigma)) = \theta_\beta$. We then let $\beta(\mathcal{P}, \Sigma) =_{def} \beta$.

Suppose now $\alpha$ is a limit and $\Theta = \theta_\alpha$. In this case, using $(**)$ we get that

$$V_\Theta^{\mathrm{HOD}} = \cup\{\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_{\beta(\mathcal{P},\Sigma)} : (\mathcal{P}, \Sigma) \text{ is a hod pair such that } \Sigma \text{ has branch condensation and is fullness preserving}\}.$$

The second case, namely that $\Theta = \theta_{\alpha+1}$, is much harder. We do not have the space to outline it in any great detail. The main difficulty is, as mentioned above, that we cannot have a pair $(\mathcal{P}, \Sigma)$ such that $\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\Theta = V_\Theta^{\mathrm{HOD}}$. The idea, which is originally due to Woodin, is to pretend that there is such a pair $(\mathcal{P}, \Sigma)$ and approximate pieces of $\Sigma$ in $M$. We refer the interested reader to [**45**] for more details on this case. Finally, Trang, in [**45**], building on an earlier work of Woodin and the author, gave a full description of HOD. However, describing this work is beyond the scope of this paper.

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CHAPTER 1

# Hod mice

Hod mice were introduced in order to analyze HOD of models of $AD^+$. At the levels of Wadge hierarchy where the theory of hod mice is developed, it can be shown that HOD is an iterate of a hod mouse. In this chapter, our goal is to introduce the language used to discuss hod mice. In the subsequent chapters we will introduce our primary way of constructing hod mice, namely *hod pair constructions*, and use it to establish a comparison theorem. Hod mice are a brand of hybrid mice which have an extender sequence and another predicate coding initial segments of the strategy of the hod mouse. It is then important to make the notion of hybrid mice more precise. We will introduce them as a special case of *hybrid $\mathcal{J}$-structures*.

## 1.1. Hybrid $\mathcal{J}$-structures

In what follows, given a transitive set $M$ (or a structure) we set $o(M)$ be the ordinals of $M$. Also, given a set $X$, we let $trc(X)$ be the transitive closure of $X$. We also let $trc^X = (trc(X \cup \{X\}), X, \in)$.

DEFINITION 1.1. *Given a function $f$, we say $f$ is amenable if the domain of $f$ consists of transitive structures and for some formula $\phi$ and for all $a = (M, A, \in) \in dom(f)$*

1. *$f(a) \subseteq o(a)$ and $0 \in f(a)$,*
2. *letting $\beta = \sup f(a)$, $\beta < o(a)$ is the unique ordinal $\gamma$ such that $a \vDash \phi[\gamma]$,*[1]
3. *whenever $\eta < \sup f(a)$, $f(a) \cap \eta \in M$.*

*We let $\phi_f$ be the formula $\phi$ above.*

We say $f$ is a *shift of an amenable function* or a *shifted amenable function* if for all $a = (M, A, \in) \in dom(f)$, $f(a) \subseteq [min(f(a)), min(f(a)) + o(a))$ and there is an amenable function $g$ such that (i) $dom(f) = dom(g)$ and (ii) for all $a \in dom(f)$ and $\gamma < o(a)$, $f(a) = \{min(f(a)) + \gamma : \gamma \in g(a)\}$. Notice that if $f$ is a shift of an amenable function then it uniquely determines $g$. We say that $g$ is the amenable component of $f$.

Jumping ahead, we remark that iteration strategies and mouse operators provide an ample source of amenable functions. For instance, let $\mathcal{M} = \mathcal{M}_1^{\#}$ and let $\Sigma$ be its canonical iteration strategy. We define $f$ as follows. Let first $dom(f)$ be the set of structures of the form $\mathcal{J}_\omega(\mathcal{T})$ where $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ of limit length and is according to $\Sigma$. Next, define $f(\mathcal{J}_\omega(\mathcal{T})) = b$ where $b = \Sigma(\mathcal{T})$. Then $f$ is amenable. We will refer to such an $f$ as *an amenable function given by an iteration strategy*. The reason we define the domain of $f$ to be the set of

[1] We seem to need this condition in order to develop fine structure of models of the form $\mathcal{J}^{\vec{E},f}$ where $f$ is a shifted amenable function.



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$\mathcal{J}_\omega(\mathcal{T})$ instead of just the set of $trc^{\mathcal{T}}$ is that the later may not satisfy clause 2 of Definition 1.1.

Recall that a transitive structure $\mathcal{M} = (M, A)$ is called *amenable* if for every $X \in M$, $A \cap X \in M$. Following [**50**], we say $\mathcal{M}$ is a $\mathcal{J}$-structure over $X$ if $\mathcal{M} = (\mathcal{J}^A_\alpha(X), B) = (\left|\mathcal{J}^A_\alpha(X)\right|, A, B)$ is an amenable structure. Keeping the notation, we also say $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure if for all $\beta < \alpha$ and for all $\tau < \omega\beta$, if $\mathcal{P}(\tau) \cap \mathcal{J}^A_{\beta+1} \not\subseteq \mathcal{J}^A_\beta$ then there is a surjection $f : \tau \to \omega\beta$ in $\mathcal{J}^A_{\beta+1}$. Finally, we say $X$ is self-well-ordered if there is a wellordering of $X$ in $\mathcal{J}_1(X)$. We are now in a position to introduce the *hybrid $\mathcal{J}$-structures*.

Definition 1.2 (Hybrid $\mathcal{J}$-structures). *We say $\mathcal{M} = (\mathcal{J}^{A,f}_\alpha(X), B)$ is a hybrid $\mathcal{J}$-structure over self-well-ordered set $X$ with indexing scheme $\phi$ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure such that in $\mathcal{M}$, $f$ is a shift of an amenable function with amenable component $g$ such that*

1. *for all $a \in \mathcal{M}$, $a \in dom(f)$ if and only if in $\mathcal{M}$, there is $\beta$ such that $a$ is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}^{A,f}_\beta(X)$ such that*
$$\mathcal{J}^{A,f}_\beta \vDash \text{“}ZFC\text{-}Replacement + \phi[b]\text{”}$$
*and if $\gamma$ is such that $b \vDash \phi_g[\gamma]$ then $\beta + \gamma \leq \alpha$, and*
2. *for all $a \in \mathcal{M}$, if $a \in dom(f)$ then $min(f(a))$ is the least ordinal $\beta$ satisfying clause 1 above.*

Remark. *Notice that it follows from clause 1 of Definition* 1.2 *that the function $a \to min(f(a))$ is injective on $dom(f)$.*

Hod mice are a special blend of *layered hybrid $\mathcal{J}$-structures* introduced below. Before introducing them we establish some notation. Suppose that $\mathcal{M} = (\mathcal{J}^{A,f}_\alpha(X), B)$ is a hybrid $\mathcal{J}$-structure over $X$ and $\xi \leq \alpha$. Then we let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at $\xi$, i.e., we keep the predicate indexed at $\xi$. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. Also, recall that if $\beta < \alpha$ then we write $\mathcal{J}^{\mathcal{M}}_\beta$ instead of $\mathcal{J}^{A,f}_\beta$ and, we say $\mathcal{N}$ is an (a proper) initial segment of $\mathcal{M}$ and write $\mathcal{N} \trianglelefteq \mathcal{M}$ ($\mathcal{N} \triangleleft \mathcal{M}$) if there is $\beta \leq \alpha$ ($\beta < \alpha$) such that $\mathcal{N} = \mathcal{J}^{\mathcal{M}}_\beta$.

Definition 1.3 (Layered hybrid $\mathcal{J}$-structure). *We say $\mathcal{M} = (\mathcal{J}^{A,f}_\alpha(X), B)$ is a layered hybrid $\mathcal{J}$-structure over $X$ with indexing scheme $\phi(x, y)$ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure over $X$ such that in $\mathcal{M}$, $f$ is a function with domain $Y^{\mathcal{M}} \subseteq \alpha$ such that for all $\gamma \in Y^{\mathcal{M}}$, $f(\gamma)$ is a shift of an amenable function with amenable component $g_\gamma$ such that*

1. *for all $a \in \mathcal{M}$, $a \in dom(f(\gamma))$ if and only if in $\mathcal{M}$, there is $\beta$ such that $a$ is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}^{A,f}_\beta(X)$ such that*
$$\mathcal{J}^{A,f}_\beta \vDash \text{“}ZFC\text{-}Replacement + \phi[\mathcal{M}|\gamma, b]\text{”}$$
*and if $\xi$ is such that $b \vDash \phi_{g_\gamma}[\xi]$ then $\beta + \xi \leq \alpha$, and*
2. *for all $a \in \mathcal{M}$, if $a \in dom(f(\gamma))$ then $min(f(\gamma)(a))$ is the least ordinal $\beta$ satisfying clause 1 above.*

We will often omit $\phi$ when discussing a particular layered hybrid $\mathcal{J}$-structure. If $\mathcal{M}$ is a layered hybrid $\mathcal{J}$-structure then we let $f^{\mathcal{M}}$ and $Y^{\mathcal{M}}$ be as in Definition 1.3. We again have that for each $\gamma \in Y^{\mathcal{M}}$, the function $a \to min(f^{\mathcal{M}}(\gamma)(a))$ is injective on $dom(f(\gamma))$.

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Notice that hybrid $\mathcal{J}$-structures can be viewed as a special case of layered hybrid $\mathcal{J}$-structures. Because of this, in the sequel we will only establish terminology for layered hybrid $\mathcal{J}$-structures though we might use the same terminology for hybrid $\mathcal{J}$-structures.

Typically, when discussing hybrid $\mathcal{J}$-structures, $X$ will be an *iterable* structure and $f$ will be the predicate coding its strategy. As mentioned above, hod mice are a special type of layered hybrid $\mathcal{J}$-structures: the $f$ predicate of a hod mouse codes a strategy for its layers. When the $A$ predicate of a layered hybrid $\mathcal{J}$-structure is a coherent sequence of extenders then the resulting model is called a *potential hybrid layered premouse*. Just like with premice, a *hybrid layered premouse* is a potential premouse all of whose levels are sound. This is stated in Definition 1.8.

DEFINITION 1.4 (Potential layered hybrid premouse). *Suppose $\mathcal{M} = \mathcal{J}^{\vec{E},f}(X)$ is a layered hybrid $\mathcal{J}$-structure over $X$. $\mathcal{M}$ is a called a potential layered hybrid premouse (potential lhp) if $\vec{E}$ is a fine extender sequence as in Definition 2.4 of* [**43**].

## 1.2. Some fine structure

In this section, we mostly follow [**43**]. First, recall that if $\mathcal{Q}$ is any structure and $X \subseteq |\mathcal{Q}|$ then $\mathcal{H}_1^{\mathcal{Q}}$ is the transitive collapse of the substructure of $\mathcal{Q}$ whose universe consists of all $y \in |\mathcal{Q}|$ such that $\{y\}$ is $\Sigma_1^{\mathcal{Q}}$-definable from parameters in $X$. Given a potential lhp $\mathcal{M}$, we let $\mathcal{C}_0(\mathcal{M})$ be its amenable $\Sigma_0$ code (see Definition 2.11 of [**43**]). When fine structure isn't an issue, we will take $\mathcal{C}_0(\mathcal{M}) = \mathcal{M}$. The following definition combines Definition 2.12, Definition 2.14 and Definition 2.13 of [**43**]. Recall that a parameter is just a finite sequence of decreasing ordinals.

DEFINITION 1.5. *Suppose $\mathcal{M}$ is a potential lhp.*

1. *The $\Sigma_1$-projectum of $\mathcal{M}$, or $\rho_1(\mathcal{M})$, is the least ordinal $\alpha$ such that for some $\underset{\sim}{\Sigma}_1^{\mathcal{C}_0(\mathcal{M})}$ set $A \subseteq \alpha$, $A \notin \mathcal{C}_0(\mathcal{M})$.*
2. *The first standard parameter of $\mathcal{M}$, or $p_1(\mathcal{M})$, is the lexicographically least parameter $p$ such that there is a $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{p\})$ set $A$ such that $(A \cap \rho_1(\mathcal{M})) \notin \mathcal{C}_0(\mathcal{M})$.*
3. *The first core of $\mathcal{M}$, or $\mathcal{C}_1(\mathcal{M})$, is defined to be $\mathcal{C}_1(\mathcal{M}) = \mathcal{H}_1^{\mathcal{C}_0(\mathcal{M})}(\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\})$.*

Continuing our review of fine structural notions, we can follow [**43**] to introduce *solidity*, *universality*, the $n$th projectum and the $n$th core of a potential lhp. The following definition combines Definitions 2.15 and Definition 2.16 of [**43**].

DEFINITION 1.6. *Suppose $\mathcal{M}$ is a potential lhp.*

1. *We say $p_1(\mathcal{M})$ is 1-universal iff whenever $A \subseteq \rho_1(A)$ and $A \in \mathcal{C}_0(\mathcal{M})$ then $A \in \mathcal{C}_1(\mathcal{M})$.*
2. *Let $p_1(\mathcal{M}) = (\alpha_0, ..., \alpha_n)$. Then $p_1(\mathcal{M})$ is 1-solid iff whenever $i \leq n$ and $A$ is $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{\alpha_0, ..., \alpha_{i-1}\})$, then $A \cap \alpha_i \in \mathcal{C}_0(\mathcal{M})$.*
3. *$\mathcal{M}$ is said to be 1-solid if $p_1(\mathcal{M})$ is 1-solid and 1-universal.*
4. *$\mathcal{M}$ is 1-sound iff $\mathcal{M}$ is 1-solid and $\mathcal{C}_1(\mathcal{M}) = \mathcal{C}_0(\mathcal{M})$.*

Continuing in this manner, given a potential lhp $\mathcal{M}$ we can define the $n^{th}$ projectum $\rho_n(\mathcal{M})$, the $n^{th}$ standard parameter of $\mathcal{M}$ and the $n^{th}$ core $\mathcal{C}_n(\mathcal{M})$. The definitions of these notions carry over word by word from Section 2 of [**20**] and to

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avoid the technical details, we refer the reader to that paper for a complete definition of all these notions. It is worth mentioning, however, that the fine structural notions at the $n+1^{st}$ level are defined from the prospective of the $n^{th}$ level. Finally, we carry over Definition 2.17 of [**43**].

DEFINITION 1.7. *Let $\mathcal{M}$ be a potential lhp. Then $\mathcal{M}$ is $\omega$-solid iff $\mathcal{M}$ is $n$-solid for all $n$. $\mathcal{M}$ is $\omega$-sound iff $\mathcal{M}$ is $n$-sound for all $n < \omega$. If $\mathcal{M}$ is $\omega$-solid, then we let $\rho(\mathcal{M})$ be the eventual value of $\rho_n(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ be the eventual value of $\mathcal{C}_n(\mathcal{M})$ as $n \to \omega$.*

Having the notion of soundness available, we can now introduce layered hybrid premice.

DEFINITION 1.8 (Layered hybrid premouse, lhp). *A potential lhp $\mathcal{M}$ is called a layered hybrid premouse (lhp) if all of its proper initial segments are sound.*

Next, we review some of the most common forms of embeddings between lhps. The following definitions combine Definition 2.20, Definition 4.1 and Remark 4.3 of [**43**].

DEFINITION 1.9. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two lhps and $\pi : \mathcal{M} \to \mathcal{N}$ is an embedding.*

1. *$\pi$ is called an $n$-embedding iff $\mathcal{M}$ and $\mathcal{N}$ are $n$-sound, $\pi$ is $r\Sigma_{n+1}$-elementary[2], $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leq n$, $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < n$, and $\sup \pi[\rho_n(\mathcal{M})] = \rho_n(\mathcal{N})$.*
2. *$\pi$ is a weak $n$-embedding iff $\mathcal{M}$ and $\mathcal{N}$ are $n$-sound, $\pi$ is $r\Sigma_n$-elementary, $\pi$ is $r\Sigma_{n+1}$-elementary on parameters from some set $X$ cofinal in $\rho_n(\mathcal{M})$, $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leq n$, $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < n$, and $\sup \pi[\rho_n(\mathcal{M})] \leq \rho_n(\mathcal{N})$.*
3. *$\pi$ is a near $n$-embedding if it is a weak $n$-embedding which is fully $r\Sigma_{n+1}$-elementary.*

LEMMA 1.10. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{J}$-structures and $\pi : \mathcal{M} \to \mathcal{N}$ is a near 1-embedding. Then $\mathcal{M}$ is an lhp iff $\mathcal{N}$ is an lhp.*

Next we review some basic facts about extenders. Suppose first $M$ and $N$ are transitive models of fragments of $ZFC$ and $j : M \to N$ is a nontrivial embedding. Suppose $crit(j) = \kappa$ and $\lambda \leq o(N)$. Then the set $E = \{(a, A) : a \in \lambda^{<\omega} \wedge A \in M \wedge a \in j(A)\}$ is called the $(\kappa, \lambda)$-extender derived from $j$. Letting $E_a = \{A : A \in E_a\}$ it is not hard to see that $E_a$ is an ultrafilter on $\xi_a$ where $\xi_a$ is the least ordinal $\gamma$ such that $j(\gamma) > max(a)$. As is well know, we can form the ultrapower of $M$ by $E$ and get $\pi_E : M \to Ult(M, E)$ where $\pi_E$ is the ultrapower embedding given by $j_E(x) = [\kappa, c_x]_E$ where $c_x$ is the constant function with value $x$.

Extenders can be defined abstractly without an underlying embedding (see for instance [**8**]): however, if $E$ is a $(\kappa, \lambda)$ extender over $M$ such that $Ult(M, E)$ is wellfounded then the $(\kappa, \lambda)$-extender derived from $\pi_E$ is just $E$. It is then convenient to think of extenders as being derived from embeddings. Finally, if $E$ is the extender derived from $j$ then there is a canonical embedding $\sigma : Ult(M, E) \to N$ given by $\sigma([a, f]_E) = j(f)(a)$ and with the property that $j = \sigma \circ \pi_E$.

Keeping the above notations, recall that $\nu(E)$ is the natural length of $E$. More precisely, if $E$ is a $(\kappa, \lambda)$-extender an $\xi < \lambda$ then we let $E \restriction \xi = \{(a, A) \in E :$

[2]See [**20**].

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$a \in \xi^{<\omega}\}$. $\xi$ is called a *generator* of $E$ if $\xi = crit(\sigma)$ where $\sigma : Ult(M, E \restriction \xi) \to Ult(M, E)$ is the canonical factor map given by $\sigma([a, f]_{E\restriction\xi}) = [a, f]_E$. Then letting $\kappa = crit(E)$, $\nu^M(E) = \sup((\kappa^+)^M \cup \{\xi+1 : \xi \text{ is a generator of } E\})$. $\nu^M(E)$ is called the support of $E$. When $M$ is clear from context we drop it from our notation. The following definition will be used throughout this paper.

DEFINITION 1.11 (Cutpoints). *Suppose $\mathcal{M}$ is an lhp. We say $\xi < o(\mathcal{M})$ is a cutpoint of $\mathcal{M}$ if there is no extender $E$ on $\mathcal{M}$ such that $\xi \in (crit(E), lh(E))$. We say $\xi$ is a strong cutpoint if there is no $E$ on $\mathcal{M}$ such that $\xi \in [crit(E), lh(E))$. We say $\eta < o(\mathcal{M})$ is overlapped in $\mathcal{M}$ if $\eta$ isn't a cutpoint of $\mathcal{M}$.*

In a fine structural setting, one takes *fine structural ultrapowers*. More precisely, suppose $\mathcal{M}$ is an lhp and $E$ is an extender on the sequence of $\mathcal{M}$. Suppose $n$ is such that $crit(E) < \rho_n(\mathcal{M})$. Consider the ultrapower of $\mathcal{M}$ by $E$ where only the functions which are $r\Sigma_n$-definable with parameters in $\mathcal{M}$ are used. We denote this ultrapower by $Ult_n(\mathcal{M}, E)$. It is then possible to prove Loś's theorem for $r\Sigma_n$ formulae. $Ult_n(\mathcal{M}, E)$ is called the $\Sigma_n$ ultrapower of $\mathcal{M}$ (see Section 2 of [**20**]). The following is a restatement of Lemma 2.21 of [**43**].

LEMMA 1.12. *For every $n \leq \omega$, the canonical embedding associated to a $\Sigma_n$-ultrapower is an $n$-embedding.*

## 1.3. Iteration trees and iteration strategies

If $\mathcal{M}$ is a $k$-sound lhp, then a $(k, \theta)$-*iteration strategy* for $\mathcal{M}$ is a winning strategy for player II in the iteration game $G_k(\mathcal{M}, \theta)$, and a $k$-*normal iteration tree* on $\mathcal{M}$ is a play of this game in which II has not yet lost. (That is, all models are wellfounded.) ($k$-normal trees are called "$k$-maximal" in [**43**], but we shall use "maximal" for a very different property of trees.) We shall drop the reference to the fine-structural parameter $k$ whenever it seems safe to do so, and speak simply of normal trees.

We say $\mathcal{M}$ is $\theta$-iterable if $II$ has a winning strategy in $G_k(\mathcal{M}, \theta)$. We say $\mathcal{M}$ is *countably* $\theta$-iterable if any countable substructure of $\mathcal{M}$ is $\theta$-iterable. It follows from the copying construction that a $\theta$-iterable mouse is countably $\theta$-iterable (consult Section 4.1 of [**43**] for the details of the copying construction). We say $\mathcal{M}$ is *countably* iterable if all of its countable substructures are $\omega_1 + 1$-iterable.

If $\mathcal{T}$ is a normal iteration tree, then $\mathcal{T}$ has the form

$$\mathcal{T} = (T, deg, D, (E_\alpha, \mathcal{M}_{\alpha+1} | \alpha + 1 < \eta)).$$

Recall that $D$ is the set of *dropping* points. This is a natural point to make the meaning of $D$ and $k$ more precise. Recall that $\mathcal{M}_{\alpha+1} = Ult_n(\mathcal{M}^*_{\alpha+1}, E_\alpha)$ where

(1) letting $\beta = pred_T\alpha$, $\mathcal{M}^*_{\alpha+1}$ is the largest initial segment $\mathcal{M}$ of $\mathcal{M}_\beta$ such that $E_\alpha$ is an extender over $\mathcal{M}$ and
(2) $n \leq \omega$ is the largest such that (1) $crit(E_\alpha) < \rho_n(\mathcal{M}^*_{\alpha+1})$ and (2) if $D \cap [0, \alpha + 1]_T = \emptyset$ then $n \leq k$.

If $\mathcal{M}^*_{\alpha+1} \triangleleft \mathcal{M}_\beta$ then $\alpha + 1 \in D$.

Recall also that if $\eta$ is limit then

$$\vec{E}(\mathcal{T}) = \cup_{\alpha<\eta}(\dot{E}^{\mathcal{M}_\alpha \restriction lh(E_\alpha)}),$$
$$\mathcal{M}(\mathcal{T}) = \cup_{\alpha<\eta}\mathcal{M}_\alpha \restriction lh(E_\alpha),$$
$$\delta(\mathcal{T}) = \sup_{\alpha<\eta} lh(E_\alpha).$$

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If $b$ is a branch of $\mathcal{T}$ such that $D \cap b$ is finite, then $\mathcal{M}_b^{\mathcal{T}}$ is the direct limit of the models along $b$ . If $\alpha \leq_T \beta$ and $(\alpha, \beta]_{\mathcal{T}} \cap D = \emptyset$ then $\pi^{\mathcal{T}}_{\alpha,\beta} : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{T}}_\beta$ is the iteration map, and if $\xi+1 \in b \cap D$ and $(b-(\xi+1)) \cap D = \emptyset$, then $\pi^{\mathcal{T}}_{\alpha,b} : \mathcal{M}^*_{\xi+1} \to \mathcal{M}^{\mathcal{T}}_b$ is the iteration map. If $\mathcal{T}$ has a last model $\mathcal{M}^{\mathcal{T}}_\alpha$, and the branch $[0,\alpha]_T$ does not drop, then we often write $\pi^{\mathcal{T}}$ for $\pi^{\mathcal{T}}_{0,\alpha}$.

The following is a crucial lemma, essentially due to Martin and Steel (see the zipper-argument of [**18**]), which provides a strong uniqueness condition for branches of iteration trees. We will use it throughout this paper.

LEMMA 1.13. *Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ of limit length and $s$ is a cofinal subset of $\delta(\mathcal{T})$; then there is at most one cofinal branch $b$ such that there is $\alpha \in b$ with the property that $\pi^{\mathcal{T}}_{\alpha,b}$ exists and $s \subseteq rng(\pi^{\mathcal{T}}_{\alpha,b})$.*

PROOF. Towards a contradiction, suppose there are two cofinal branches $b$ and $c$ such that for some $\alpha, \beta$, both $\pi^{\mathcal{T}}_{\alpha,b}$ and $\pi^{\mathcal{T}}_{\beta,c}$ exist and $s \subseteq rng(\pi^{\mathcal{T}}_{\alpha,b}) \cap rng(\pi^{\mathcal{T}}_{\beta,c})$. Without loss of generality we can assume that $\alpha$ and $\beta$ are the least ordinals with this property, $\alpha \leq \beta$ and that $b$ and $c$ diverge at $\alpha$ or earlier, i.e., if $\gamma$ is the least ordinal in $b \cap c$ then $\gamma \leq \alpha$. By [**18**], we can choose two sequences $(\alpha_n : n < \omega)$ and $(\beta_n : n < \omega)$, such that $b = \{\gamma : \exists n < \omega(\gamma \leq_T \alpha_n)\}$, $c = \{\gamma : \exists n < \omega(\gamma \leq_T \beta_n)\}$, $\alpha_0 = \alpha$, $\beta_0 = \beta$ and for every $n$, $\alpha_n < \beta_n < \alpha_{n+1}$. Let then $\xi$ be the least ordinal in $rng(\pi^{\mathcal{T}}_{\alpha,b}) \cap rng(\pi^{\mathcal{T}}_{\beta,c})$. Let $n$ be the least such that $crit(\pi^{\mathcal{T}}_{\alpha_n,b}) > \xi$. This means that $crit(E^{\mathcal{T}}_{\alpha_{n+1}-1}) > \xi$ and that $lh(E^{\mathcal{T}}_{\alpha_n}) < \xi$. By the proof of Theorem 2.2 of [**18**], this means that for some $m \geq 1$, $\xi \in [crit(E^{\mathcal{T}}_{\beta_m-1}), lh(E^{\mathcal{T}}_{\beta_m-1}))$. This then implies that $\xi \notin rng(\pi^{\mathcal{T}}_{\beta_{m-1},c})$, which is a contradiction. □

The proof of Lemma 1.13 gives the following refinement:

LEMMA 1.14. *Suppose $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of limit length and $b, c$ are two cofinal branches of $\mathcal{T}$ such that $\pi^{\mathcal{T}}_b$ and $\pi^{\mathcal{T}}_c$ exist. Suppose that for some $\alpha$,*

$$\pi^{\mathcal{T}}_b(\alpha) = \pi^{\mathcal{T}}_c(\alpha) < \delta(\mathcal{T}).$$

*Then $\pi^{\mathcal{T}}_b \restriction \alpha = \pi^{\mathcal{T}}_c \restriction \alpha$. Moreover, if $\xi \in b$ is the least such that $crit(\pi^{\mathcal{T}}_{\xi,b}) > \pi^{\mathcal{T}}_b(\alpha)$ then $\xi \in c$, so that $b \cap (\xi+1) = c \cap (\xi+1)$.*

In addition to normal trees, we must consider linear stacks of normal trees. These are plays of the iteration game $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ in which II has not yet lost. See Definition 4.4 of [**43**] for the formal definition. We will use the vector notation $\vec{\mathcal{T}}$ for a stack of normal trees, and then $\mathcal{T}_\eta$ for its $\eta^{\text{th}}$ normal component. $\mathcal{M}_\eta$ will be used for the model at the beginning of the $\eta$th round of $\vec{\mathcal{T}}$. Also, $lh(\vec{\mathcal{T}})$ will be used for the number of rounds that $\vec{\mathcal{T}}$ has. We then have that $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \text{lh}(\vec{\mathcal{T}}))$. In the sequel, $\mathcal{T}$, without the vector sign, will always denote a normal tree. Below we establish some notation.

NOTATION. *We will use the following notation throughout this paper. Suppose $\mathcal{M}$ is an lhp and $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \text{lh}(\vec{\mathcal{T}}))$ is a stack on $\mathcal{M}$.*

1. *We let $\mathcal{M}^{\vec{\mathcal{T}}}_\alpha = \mathcal{M}_\alpha$ and if $\beta < lh(\mathcal{T}_\alpha)$ then we let $\mathcal{M}^{\vec{\mathcal{T}}}_{\alpha,\beta} = \mathcal{M}^{\mathcal{T}_\alpha}_\beta$.*
2. *If $\vec{\mathcal{T}}$ is a stack with last model then we let $\vec{\mathcal{T}}^-$ be the same stack without the last model.*
3. *Suppose that $\beta \leq \alpha < lh(\vec{\mathcal{T}})$, $\gamma < lh(\mathcal{T}_\beta)$ and $\xi < lh(\mathcal{T}_\alpha)$. If the iteration embedding $i : \mathcal{M}^{\mathcal{T}_\beta}_\gamma \to \mathcal{M}^{\mathcal{T}_\alpha}_\xi$ exists then we let $\pi^{\vec{\mathcal{T}}}_{(\beta,\gamma),(\alpha,\xi)}$ be this map.*

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*If the iteration embedding $i : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ exists then we let $\pi^{\vec{\mathcal{T}}}_{\alpha,\beta}$ be this embedding.*

(4) *If $\vec{\mathcal{T}}$ has a last model $\mathcal{N}$ and the iteration embedding $i : \mathcal{M}_\alpha \to \mathcal{N}$ exists then we let $\pi^{\vec{\mathcal{T}}}_\alpha$ be this embedding. We sometimes omit 0 when we refer to $\pi^{\vec{\mathcal{T}}}_0$.*

(5) *If $lh(\vec{\mathcal{T}}) = \alpha+1$, $\mathcal{T}_\alpha$ is of limit length, and $b$ is a branch of $\mathcal{T}_\alpha$ then we let $\vec{\mathcal{T}}^b = \vec{\mathcal{T}}^\frown \mathcal{M}_b^{\mathcal{T}_\alpha}$. If the iteration embedding $i : \mathcal{M}_0^{\vec{\mathcal{T}}} \to \mathcal{M}_b^{\vec{\mathcal{T}}}$ exists then we let $\pi^{\vec{\mathcal{T}}}_b$ be this embedding. Similarly, if the iteration embedding $i : \mathcal{M}_\xi^{\mathcal{T}_\beta} \to \mathcal{M}_b^{\vec{\mathcal{T}}}$ exists then we let $\pi^{\vec{\mathcal{T}}}_{(\beta,\xi),b}$ be this embedding, and if $i : \mathcal{M}_\alpha^{\vec{\mathcal{T}}} \to \mathcal{M}_b^{\vec{\mathcal{T}}}$ exists then we let $\pi^{\vec{\mathcal{T}}}_{\alpha,b}$ be this embedding.*

(6) *If $\vec{\mathcal{T}}$ is a stack such that its last component is of limit length then by a branch of $\vec{\mathcal{T}}$ we mean a branch of its last component. If $\vec{\mathcal{T}}$ is a stack without last component then by a branch of $\vec{\mathcal{T}}$ we mean the join of the branches of normal trees in $\vec{\mathcal{T}}$.*

Next we define the notion of "sup of the generators" for stack of trees. For a normal tree $\mathcal{T}$, this is just $\delta(\mathcal{T})$.

DEFINITION 1.15. *Suppose $\mathcal{M}$ is an lhp and $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \mathrm{lh}(\vec{\mathcal{T}}))$ is a stack of iteration trees. We define $\delta(\vec{\mathcal{T}})$ as follows;*

(1) $\delta_0(\vec{\mathcal{T}}) = \delta(\mathcal{T}_0)$

(2) *If $\alpha + 1 < lh(\vec{\mathcal{T}})$ then*

  (a) *if $\pi^{\mathcal{T}_\alpha}$ doesn't exist or $\mathcal{T}_\alpha$ has a limit length then $\delta_{\alpha+1}(\vec{\mathcal{T}}) = \delta(\mathcal{T}_\alpha)$,*

  (b) *if $\pi^{\mathcal{T}_\alpha}$ exists then*

$$\delta_{\alpha+1} = \max\{\pi^{\mathcal{T}_\alpha}(\delta_\alpha(\vec{\mathcal{T}})), \delta(\mathcal{T}_\alpha)\}.$$

(3) *If $\alpha$ is limit then $\delta_\alpha(\vec{\mathcal{T}}) = \sup_{\beta<\alpha} \delta_\beta(\vec{\mathcal{T}})$.*

(4) $\delta(\vec{\mathcal{T}}) = \sup_{\alpha<lh(\vec{\mathcal{T}})} \delta_\alpha(\vec{\mathcal{T}})$.

Suppose now that $\mathcal{M}$ is an lhp and $\Sigma$ is a $(k, \alpha, \theta)$-iteration strategy for $\mathcal{M}$. Suppose $\vec{\mathcal{T}}$ is a stack of trees on $\mathcal{M}$ played according to $\Sigma$ and having last model $\mathcal{N}$. Then we let $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ be the $(l, \alpha, \theta)$ strategy for $\mathcal{N}$ induced by $\Sigma$. ($l$ is the degree of the branch $\mathcal{M}$-to-$\mathcal{N}$ of $\vec{\mathcal{T}}$. We assume here $\alpha$ is additively closed, so that there is such a strategy.) We say $\mathcal{M}$ is countably $(\alpha, \theta)$-iterable if all of its countable submodels are $(\alpha, \theta)$-iterable. We also let

$$I(\mathcal{M}, \Sigma) = \{\mathcal{N} : \text{there is a stack } \vec{\mathcal{T}} \text{ on } \mathcal{M} \text{ according to } \Sigma \text{ with last model } \mathcal{N} \text{ and } \pi^{\vec{\mathcal{T}}} \text{ exists } \}.$$

Given an lhp $\mathcal{M}$ with a Woodin cardinal $\delta$, we let $\mathbb{B}_\delta^{\mathcal{M}}$ be the countably generated extender algebra of $\mathcal{M}$ at $\delta$. In order to have a unique choice, we stipulate that the identities determining $\mathbb{B}_\delta^{\mathcal{M}}$ are precisely those coming from extenders $E$ on the $\mathcal{M}$-sequence such that $\nu(E)$ is an inaccessible, but not a limit of inaccessibles, in $\mathcal{M}$, and $\nu(E) < \delta$. If $G \subseteq \mathbb{B}_\delta^{\mathcal{M}}$ then we let $x_G$ be the set naturally coded by $G$. For basic facts about the extender algebra, we refer the reader to [**3**] or Section 7.2 of [**43**].

We end this section with the definition of a *layered hybrid mouse* (the definition of a hybrid mouse is a special case).

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DEFINITION 1.16 (Layered hybrid mouse, lhm). *An lhp $\mathcal{M}$ is a layered hybrid mouse (lhm) if $\mathcal{M}$ is $\omega_1+1$-iterable.*

## 1.4. Layered strategy premice

In this paper, we are concerned with lhp whose $f$ predicate codes a strategy. The goal of this section is to introduce the language used to describe such structures.

Suppose that $\mathcal{M}$ is an lhp. We then say that a shifted amenable function $f$ *codes a partial strategy function* for $\mathcal{M}$ if (i) $dom(f) \subseteq \{\mathcal{J}_\omega(\vec{\mathcal{T}}) : \vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ without a last model$\}$, (ii) whenever $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ such that $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in dom(f)$ then whenever $\vec{\mathcal{U}}$ is an initial segment of $\vec{\mathcal{T}}$ without a last model, then $\mathcal{J}_\omega(\vec{\mathcal{U}}) \in dom(f)$ and (iii) if $g$ is the amenable component of $f$ then for all $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in dom(f)$, $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is a cofinal branch of $\vec{\mathcal{T}}$. Notice that we do not require that $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ be a well-founded branch of $\vec{\mathcal{T}}$ which is why we call the resulting function just a strategy function.

Suppose then a shifted amenable function $f$ codes a partial strategy function for $\mathcal{M}$. We then let $\Sigma^f$ be the partial strategy function coded by $f$. More precisely, letting $g$ be the amenable component of $f$,

(1) $dom(\Sigma^f) = dom(f)$ and
(2) for all $\vec{\mathcal{T}} \in dom(\Sigma^f)$, $\Sigma^f(\vec{\mathcal{T}}) = g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$.

We say $f$ *codes a partial strategy* if $\Sigma^f$ chooses cofinal and well-founded branches. We say $f$ *codes a total strategy* if $\Sigma^f$ is a total strategy.

Recall that if $\mathcal{M}$ is an lhp, $\mathcal{N} \trianglelefteq \mathcal{M}$ and $\Sigma$ is an iteration strategy for $\mathcal{M}$ then $\Sigma_{\mathcal{N}}$ is the strategy of $\mathcal{N}$ we get by the copy construction. More precisely, $\Sigma_{\mathcal{N}}$ is the $id$-pullback of $\Sigma$.

DEFINITION 1.17 (Layered strategy premouse, lsp). *Suppose $\mathcal{M}$ is an lhp with an indexing scheme $\phi(x,y)$. We say $\mathcal{M}$ is a layered strategy premouse (lsp) if for all $\gamma \in Y^{\mathcal{M}}$, in $\mathcal{M}$,*

(1) *$f(\gamma)$ codes a partial strategy function for $\mathcal{M}|\gamma$, and*
(2) *if $\gamma_0 < \gamma_1 \in Y^{\mathcal{M}} - \{0\}$ then letting, for $i \in 2$, $\Sigma_i$ be the partial strategy function coded by $f^{\mathcal{M}}(\gamma_i)$, then $(\Sigma_1)_{\mathcal{M}|\gamma_0} \subseteq \Sigma_0$.*

Notice that the fact that $\mathcal{M}$ is a layered strategy premouse depends on what $\phi$ says. Thus, the clauses above should be viewed as part of $\phi$. The strategy premice are a special case of layered strategy premice, and we leave the details to the reader. We let $\Sigma^{\mathcal{M}}$ be the partial strategy function coded by $f^{\mathcal{M}}$. If $\gamma \in Y^{\mathcal{M}}$ then we let $\Sigma^{\mathcal{M}}_\gamma$ be the partial strategy function coded by $f(\gamma)$.

In most applications, lsp have a very canonical indexing scheme which is originally due to Woodin. At each stage the stack whose branch is being indexed by $f$ is the least stack whose branch hasn't yet been indexed. We call this the *standard indexing scheme*.

DEFINITION 1.18 (Standard indexing scheme). *We say $\phi(x,y)$ is the standard indexing scheme if whenever $\mathcal{M}$ is an lsp and $\gamma \in Y^{\mathcal{M}}$ then $\mathcal{M} \vDash \phi[\mathcal{M}|\gamma, a]$ if and only if*

(1) *$a$ is the $\mathcal{M}$-least set of the form $\mathcal{J}_\omega(\vec{\mathcal{T}})$ where $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}|\gamma$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{M}}_\gamma$, $\vec{\mathcal{T}}$ doesn't have a last model and $f(\nu)(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined and*

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(2) *for every $\zeta \geq \gamma$ such that $a \in \mathcal{M}|\zeta$ and $\mathcal{M}|\zeta \vDash ZFC - Replacement$, in $\mathcal{M}|\zeta$, $(\gamma, a)$ isn't the lexicographically $\mathcal{M}|\zeta$-least set of the form $(\nu, \mathcal{J}_\omega(\vec{\mathcal{T}}))$ where $\nu \in Y^{\mathcal{M}|\zeta}$ and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}|\nu$ such that $\vec{\mathcal{T}}$ is according to $\Sigma_\nu^{\mathcal{M}|\zeta}$, $\vec{\mathcal{T}}$ doesn't have a last model and $f^{\mathcal{M}|\zeta}(\nu)(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined.*

Suppose $\mathcal{M}$ is an lsp and $\Sigma$ is a $(\kappa, \theta)$-iteration strategy for $\mathcal{M}||\gamma$ for some $\gamma \in Y^{\mathcal{M}}$. Then it can be the case that $\Sigma_\gamma^{\mathcal{M}} \subseteq \Sigma$. When this happens we get structures relative to $\Sigma$. Suppose $\mathcal{M}$ is an sp over $X$ and $\Sigma$ is a $(\kappa, \theta)$-iteration strategy for $X$.

DEFINITION 1.19 ($\Sigma$-premouse). *Then $\mathcal{M}$ is called a $\Sigma$-premouse if $\Sigma^{\mathcal{M}} \subseteq \Sigma \restriction \mathcal{M}$.*

DEFINITION 1.20 ($\Sigma$-mouse). *We say $\mathcal{M}$ is a $\Sigma$-mouse or just $\Sigma$-mouse if $\mathcal{M}$ has an $\omega_1 + 1$-iteration strategy $\Lambda$ such that whenever $\mathcal{N}$ is a $\Lambda$-iterate of $\mathcal{M}$ then $\mathcal{N}$ is a $\Sigma$-premouse.*

REMARK. *Suppose $\mathcal{M}$ is an lsp and $\gamma < \eta$ are two consecutive members of $Y^{\mathcal{M}}$. Then we can view $\mathcal{M}|\eta$ as a $\Sigma_\gamma^{\mathcal{M}}$-premouse over $\mathcal{M}|\gamma$.*

## 1.5. Iterations of Σ-mice

Suppose $X$ is a hybrid premouse or a layered hybrid premouse and $\Sigma$ is an $(\omega_1, \omega_1)$-iteration strategy for $X$. Given two $\Sigma$-mice, we can compare them using the usual comparison argument.

THEOREM 1.21 (Theorem 3.11 of [**43**]). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two countable $k$-sound $\Sigma$-mice with $(\omega_1+1)$-iteration strategies $\Lambda$ and $\Gamma$ respectively. Then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{M}$ and $\mathcal{N}$ respectively according to $\Lambda$ and $\Gamma$ respectively, having last models $\mathcal{M}_\alpha^{\mathcal{T}}$ and $\mathcal{N}_\eta^{\mathcal{N}}$ such that either*

(1) *the iteration embedding $\pi_{0,\alpha}^{\mathcal{T}}$-exists[3], and $\mathcal{M}_\alpha^{\mathcal{T}}$ is an initial segment of $\mathcal{M}_\eta^{\mathcal{U}}$, or*

(2) *the iteration embedding $\pi_{0,\eta}^{\mathcal{U}}$-exists, and $\mathcal{M}_\eta^{\mathcal{U}}$ is an initial segment of $\mathcal{M}_\alpha^{\mathcal{T}}$.*

Comparison for lsp is more involved and we do not know how to do it in general. Below we recall our primary method of identifying the good branches of iteration trees. Recall that the strategy for a sound mouse projecting to $\omega$ is determined by $\mathcal{Q}$-*structures*. For $\mathcal{T}$ normal, let $\Phi(\mathcal{T})$ be the phalanx of $\mathcal{T}$ (see Definition 6.6 of [**40**]).

DEFINITION 1.22. *Let $\mathcal{T}$ be a $k$-normal tree of limit length on a $k$-sound lsp, and let $b$ be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $\mathcal{M}_b^{\mathcal{T}}$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ (i.e. $\rho(\mathcal{Q}) < \delta(\mathcal{T})$) or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Next we would like to state a general result stating that branches identified by $\mathcal{Q}$-structures are unique. Suppose that $\mathcal{M}$ is a lsp and $\Sigma$ is a strategy for $\mathcal{M}$. If $\mathcal{N}$ is a $\Sigma$-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then we let $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ be the strategy of $\mathcal{N}$ given by $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

[3]In [**43**], this is stated in a somewhat stronger form, namely that $[0, \alpha]_T$ doesn't drop in model or degree.

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DEFINITION 1.23. *Suppose $\mathcal{M}$ is an lsp and $\Sigma$ is an iteration strategy for $\mathcal{M}$. We say $(\mathcal{M},\Sigma)$ is an lsm pair if $\Sigma$ has hull condensation and whenever $\mathcal{N}$ is a $\Sigma$-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then $\Sigma^{\mathcal{N}} \subseteq \Sigma_{\mathcal{N},\vec{\mathcal{T}}}$.*

We say an iteration tree $\mathcal{T}$ is above $\eta$ if all the extenders used in $\mathcal{T}$ have critical points $> \eta$.

THEOREM 1.24. *Suppose $(\mathcal{M},\Sigma)$ is an lsm pair. Suppose $\gamma < o(\mathcal{M})$ is such that $\sup(Y^{\mathcal{M}}) < \gamma$ and $\rho(\mathcal{M}) \leq \gamma$. Then $\mathcal{M}$ has at most one $(k,\omega_1+1)$ iteration strategy $\Lambda$ which acts on iteration trees that are above $\gamma$ and whenever $\mathcal{N}$ is a $\Lambda$-iterate of $\mathcal{M}$ then $\Sigma^{\mathcal{N}} \subseteq \Sigma \restriction \mathcal{N}$. Moreover, any such strategy $\Lambda$ is determined by: $\Lambda(\mathcal{T})$ is the unique cofinal $b$ such that the phalanx $\Phi(\mathcal{T})^\frown(\delta(\mathcal{T}),\deg^{\mathcal{T}}(b),\mathcal{Q}(b,\mathcal{T}))$ is $\omega_1+1$-iterable (as a $\Sigma$-phalanx).*

In some cases, however, it is enough to assume that $\mathcal{Q}(b,\mathcal{T})$ is countably iterable. This happens, for instance, when $\mathcal{M}$ has no local Woodin cardinals with extenders overlapping it. While the lsp we will consider do have local overlapped Woodin cardinals, the lsp themselves will not have such Woodin cardinals. This simplifies our situation somewhat, and below we describe exactly how this will be used.

DEFINITION 1.25 (Definition 2.1 of [**35**]). *Let $(\mathcal{M},\Sigma)$ be an lsm pair and let $\gamma < o(\mathcal{M})$ be such that $\sup(Y^{\mathcal{M}}) < \gamma$. Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ above $\gamma$; then $\mathcal{Q}(\mathcal{T})$ is the unique $\oplus_{\nu\in Y^{\mathcal{M}}}\Sigma_{\mathcal{M}|\nu}$-mouse, if there is any, extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1+1$-iterable above $\delta(\mathcal{T})$ and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Countable iterability is usually enough to guarantee there is at most one hp with the properties of $\mathcal{Q}(\mathcal{T})$. If it exists, $\mathcal{Q}(\mathcal{T})$ might identify the good branch of $\mathcal{T}$, the one any sufficiently powerful iteration strategy must choose. This is the content of the next lemma which can be proved by analyzing the proof of Theorem 6.12 of [**43**]. To state it we need to introduce fatal drops and also the following useful notation.

DEFINITION 1.26 ($\mathcal{O}^{\mathcal{P}}$-stack). *Suppose $\mathcal{P}$ is an lsp and $\eta,\gamma,\alpha < o(\mathcal{P})$. We then let*

$$\mathcal{O}^{\mathcal{P}}_{\eta,\gamma,\alpha} = \cup\{\mathcal{M}\trianglelefteq\mathcal{P} : \mathcal{P}|\eta\trianglelefteq\mathcal{M}, \rho(\mathcal{M})\leq\eta,\ Y^{\mathcal{M}}\subseteq\gamma \text{ and for all } E\in\vec{E}^M, \text{ if } \eta\in[crit(E),lh(E)) \text{ then } crit(E)\leq\alpha\}.$$

*Next we define the stack $(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha} : \xi \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha})$ according to the following recursion:*

1. $\mathcal{O}^{\mathcal{P},0}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P}}_{\eta,\gamma,\alpha}$,
2. *for $\xi+1 \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$, $\mathcal{O}^{\mathcal{P},\xi+1}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P}}_{o(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha}),\gamma,\alpha}$,*
3. *for limit $\lambda \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$, $\mathcal{O}^{\mathcal{P},\lambda}_{\eta,\gamma,\alpha} = \cup_{\xi<\lambda}\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha}$, and*
4. *$\Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$ is the least $\nu$ such that $\mathcal{O}^{\mathcal{P},\nu+1}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P},\nu}_{\eta,\gamma,\alpha}$.*

*For $\xi \leq \Omega^{\mathcal{P}}_{\eta,\eta,\alpha}$, we let $\mathcal{O}^{\mathcal{P},\xi}_{\eta} = \mathcal{O}^{\mathcal{P},\xi}_{\eta,\eta,0}$.*

We can now introduce fatal drops.

DEFINITION 1.27 (Fatal drop). *Suppose $\mathcal{P}$ is a hod-like lhp and $\mathcal{T}$ is an iteration tree on $\mathcal{P}$. We say $\mathcal{T}$ has a fatal drop if for some $\alpha < lh(\mathcal{T})$ and some $\eta < o(\mathcal{M}^{\mathcal{T}}_\alpha)$, $\mathcal{T}_{\geq\mathcal{M}^{\mathcal{T}}_\alpha}$ is a normal iteration tree on $\mathcal{O}^{\mathcal{M}^{\mathcal{T}}_\alpha}_\eta$ that is above $\eta$. We then say $\mathcal{T}$ has*

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*a fatal drop at $(\alpha, \eta)$ if the pair is the lexicographically least satisfying the above condition.*

The following is the lemma mentioned above.

LEMMA 1.28. *Let $(\mathcal{M}, \Sigma)$ be an lsm pair and let $\gamma < o(\mathcal{M})$ be such that $\sup(Y^{\mathcal{M}}) < \gamma$.*

1. *Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ above $\gamma$ of limit length and suppose $\mathcal{Q}(\mathcal{T})$ exists. Then there is at most one cofinal branch $b$ of $\mathcal{T}$ such that either $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^{\mathcal{T}}$ or $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^{\mathcal{T}}|\xi$ for some $\xi$ in the wellfounded part of $\mathcal{M}_b^{\mathcal{T}}$.*
2. *Suppose further no measurable cardinal of $\mathcal{M}$ which is $\geq \gamma$ is a limit of Woodin cardinals. If then $\mathcal{T}$ is an iteration tree according to $\Sigma$ above $\gamma$ which doesn't have a fatal drop and $b = \Sigma(\mathcal{T})$ is such that $\mathcal{Q}(b, \mathcal{T})$-exists then $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.*

$\mathcal{Q}(\mathcal{T})$ identifies $b$ because it determines a canonical cofinal subset of $rng(\pi^{\mathcal{T}}_{\alpha,b} \cap \delta(\mathcal{T}))$, for some $\alpha \in b$(which is an immediate consequence of the zipper argument from [**18**]).

REMARK. *Suppose $\mathcal{M}$ is an lsp and $\gamma \in Y^{\mathcal{M}}$. Let $\xi = o(\mathcal{M})$ if $\gamma$ is the largest member of $Y^{\mathcal{M}}$ and otherwise, let $\xi$ be the least member of $Y^{\mathcal{M}}$ which is bigger than $\gamma$. Suppose $\mathcal{T}$ is a tree on $\mathcal{M}$ which is above $\gamma$ and is based on $\mathcal{M}||\xi$. Notice that in this case we can define $\mathcal{Q}(\mathcal{T})$ just as in Definition* 4.2 *by using $\mathcal{M}||\xi$ instead of $\mathcal{M}$.*

## 1.6. Hull condensation

Suppose that $\Sigma$ is a $(\kappa, \lambda)$-iteration strategy for some lsp $\mathcal{M}$. In order to construct $\Sigma$-mice via our primary method of constructing mice, the *full background constructions*, $\Sigma$ has to have an additional property called *hull condensation*. This property is needed to guarantee that when we take cores of the levels of the resulting model the strategy indexed on the sequence collapses to itself. More precisely, during the full background constructions, at a typical stage $\xi$, letting $\mathcal{N}_\xi$ be the model at stage $\xi$, the model at stage $\xi + 1$ is $\mathcal{C}(\mathcal{N}_\xi)$. We then have the canonical embedding $\pi : \mathcal{C}(\mathcal{N}_\xi) \to \mathcal{N}_\xi$. What we would need to know in order to continue the construction is that $\mathcal{C}(\mathcal{N}_\xi)$ is still a $\Sigma$-mouse and hull condensation gives us exactly that. Our definition of hull condensation is very much influenced by the proof of Lemma 2.9.

DEFINITION 1.29 (Hull of a normal tree). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are $k$-sound lsps and $\mathcal{T}$ and $\mathcal{U}$ are trees on $\mathcal{M}$ and $\mathcal{N}$ respectively. Then $(\mathcal{N}, \mathcal{U})$ is a hull of $(\mathcal{M}, \mathcal{T})$ (see Figure* 1.6.1*) if there are (i) an embedding $\pi : \mathcal{N} \to_{\Sigma_1} \mathcal{M}$, (ii) an order preserving map $\sigma : lh(\mathcal{U}) \to lh(\mathcal{T})$ and (iii) a sequence of $\Sigma_1$-elementary embeddings $(\pi_\alpha : \alpha < lh(\mathcal{U}))$ such that letting $(T, \leq_T)$ and $(U, \leq_U)$ be the tree orders of $\mathcal{T}$ and $\mathcal{U}$ then*

1. *$\alpha \leq_U \beta \leftrightarrow \sigma(\alpha) \leq_T \sigma(\beta)$ and $[\alpha, \beta]_U \cap D^{\mathcal{U}} = \emptyset \leftrightarrow [\sigma(\alpha), \sigma(\beta)]_T \cap D^{\mathcal{T}} = \emptyset$,*
2. *$\pi_\alpha : \mathcal{M}^{\mathcal{U}}_\alpha \to \mathcal{M}^{\mathcal{T}}_{\sigma(\alpha)}$ and $\pi_\alpha(E^{\mathcal{U}}_\alpha) = E^{\mathcal{T}}_{\sigma(\alpha)}$,*
3. *if $\beta < \alpha$ then $\pi_\alpha \restriction lh(E_\beta) + 1 = \pi_\beta \restriction lh(E_\beta) + 1$,*
4. *if $\alpha \leq_U \beta$ and $[\alpha, \beta]_U \cap D^{\mathcal{U}} = \emptyset$ then $\pi_\beta \circ \pi^{\mathcal{U}}_{\alpha,\beta} = \pi^{\mathcal{T}}_{\sigma(\alpha),\sigma(\beta)} \circ \pi_\alpha$,*

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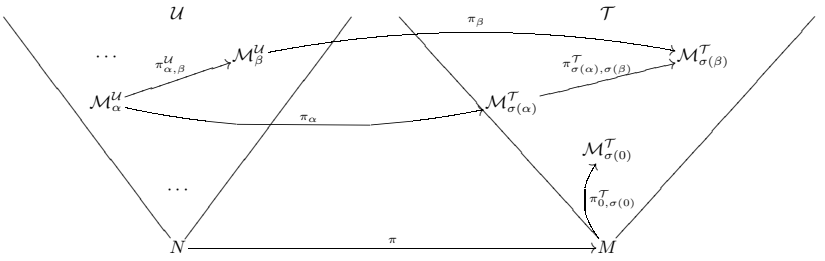


(1) $\sigma : lh(\mathcal{U}) \to lh(\mathcal{T})$ order-preserving;
(2) $\pi_0 = \pi^{\mathcal{T}}_{0,\sigma(0)} \circ \pi$;
(3) $\pi_\beta \circ \pi^{\mathcal{U}}_{\alpha,\beta} = \pi^{\mathcal{T}}_{\sigma(\alpha),\sigma(\beta)} \circ \pi_\alpha$.

FIGURE 1.6.1. Hull of a normal tree

(5) *if* $\beta = pred_U(\alpha+1)$ *then* $\sigma(\beta) = pred_T(\sigma(\alpha+1))$ *and* $\pi_{\alpha+1}([a,f]_{E^{\mathcal{U}}_\alpha}) = [\pi_\alpha(a), \pi_\beta(f)]_{E^{\mathcal{T}}_{\sigma(\alpha)}}$,
(6) $0 \leq_T \sigma(0)$, $[0,\sigma(0)] \cap D^{\mathcal{T}} = \emptyset$, *and* $\pi_0 = \pi^{\mathcal{T}}_{0,\sigma(0)} \circ \pi$.

Notice that $(\pi,\sigma)$ completely determine the embeddings $(\pi_\alpha : \alpha < lh(\mathcal{U}))$. Given $\pi : \mathcal{N} \to_{\Sigma_1} \mathcal{M}$, we say $(\mathcal{N},\mathcal{U})$ is a $(\pi,\sigma)$-hull of $(\mathcal{M},\mathcal{T})$ if $(\mathcal{N},\mathcal{U})$ is a hull of $(\mathcal{M},\mathcal{T})$ as witnessed by $\pi$, $\sigma$ and $(\pi_\alpha : \alpha < lh(\mathcal{U}))$. It is possible to define hull condensation when $\pi$ is a fine structural embedding such as in Definition 1.9. However, we will not encounter a situation when we will need hull condensation for such embeddings.

DEFINITION 1.30 (Hull of a stack). *Suppose* $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *are stacks on an lsp* $\mathcal{M}$. *Then* $\vec{\mathcal{U}}$ *is a hull of* $\vec{\mathcal{T}}$ *if there are*

(1) *an order preserving map* $\sigma : lh(\vec{\mathcal{U}}) \to lh(\vec{\mathcal{T}})$,
(2) *a sequence of functions* $(\sigma_\alpha : \alpha < lh(\vec{\mathcal{U}}))$ *such that* $\sigma_\alpha : lh(\mathcal{U}_\alpha) \to lh(\mathcal{T}_{\sigma(\alpha)})$ *is an order preserving map,*
(3) $(\pi_{\alpha,\beta} : \alpha < lh(\vec{\mathcal{U}}) \wedge \beta < lh(\mathcal{U}_\alpha))$ *such that*
   (a) *if* $\sigma(0) > 0$ *then* $\pi^{\vec{\mathcal{T}}}_{0,\sigma(0)}$*-exists and* $\pi_{0,0} = \pi^{\vec{\mathcal{T}}}_{0,\sigma(0)}$,
   (b) *for any* $\alpha < lh(\vec{\mathcal{U}})$, $\pi_{\alpha,0} : \mathcal{M}^{\vec{\mathcal{U}}}_\alpha \to_{\Sigma_1} \mathcal{M}^{\vec{\mathcal{T}}}_{\sigma(\alpha)}$ *and* $(\mathcal{M}^{\vec{\mathcal{U}}}_\alpha, \mathcal{U}_\alpha)$ *is a* $(\pi_{\alpha,0},\sigma_\alpha)$*-hull of* $(\mathcal{M}^{\vec{\mathcal{T}}}_{\sigma(\alpha)}, \mathcal{T}_{\sigma(\alpha)})$,
   (c) *if* $\beta < \alpha$, $\gamma < lh(\mathcal{U}_\beta)$, $\eta < lh(\mathcal{U}_\alpha)$, *and* $\pi^{\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$ *exists then* $\pi^{\vec{\mathcal{T}}}_{(\sigma(\beta),\sigma_\beta(\gamma)),(\sigma(\alpha),\sigma_\alpha(\eta))}$ *exists and* $\pi^{\vec{\mathcal{T}}}_{(\sigma(\beta),\sigma_\beta(\gamma)),(\sigma(\alpha),\sigma_\alpha(\eta))} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\eta} \circ \pi^{\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$.

DEFINITION 1.31 (Hull condensation). *Suppose* $\mathcal{M}$ *is an lsp and* $\Sigma$ *is a* $(\kappa,\lambda)$*-strategy for* $\mathcal{M}$. *We say* $\Sigma$ *has hull condensation if for any two stacks* $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *on* $\mathcal{M}$ *if* $\vec{\mathcal{T}}$ *is according to* $\Sigma$ *and* $(\mathcal{M},\vec{\mathcal{U}})$ *is a hull of* $(\mathcal{M},\vec{\mathcal{T}})$ *then* $\vec{\mathcal{U}}$ *is according to* $\Sigma$.

LEMMA 1.32 (Pullback of a strategy with hull condensation). *Suppose* $\mathcal{M}$ *is an lsp,* $\Sigma$ *is a* $(\kappa,\theta)$*-iteration strategy for* $\mathcal{M}$ *which has hull condensation and* $\tau : \mathcal{N} \to \mathcal{M}$ *is a* $\Sigma_1$*-elementary embedding. Then* $\Sigma^\tau$, *the* $\tau$*-pullback of* $\Sigma$, *has hull condensation.*

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PROOF. Suppose $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are stacks of trees on $\mathcal{N}$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^\tau$ and there are

(1) an order preserving map $\sigma : lh(\vec{\mathcal{U}}) \to lh(\vec{\mathcal{T}})$
(2) a sequence of functions $(\sigma_\alpha : \alpha < lh(\vec{\mathcal{U}}))$ such that $\sigma_\alpha : lh(\mathcal{U}_\alpha) \to lh(\mathcal{T}_{\sigma(\alpha)})$ is an order preserving map
(3) $(\pi_{\alpha,\beta} : \alpha < lh(\vec{\mathcal{U}}) \wedge \beta < lh(\mathcal{U}_\alpha))$ such that
   (a) For any $\alpha < lh(\vec{\mathcal{U}})$, $\pi_{\alpha,0} : \mathcal{M}_\alpha^{\vec{\mathcal{U}}} \to \mathcal{M}_{\sigma(\alpha)}^{\vec{\mathcal{T}}}$ and $(\mathcal{M}_\alpha^{\vec{\mathcal{U}}}, \mathcal{U}_\alpha)$ is a hull of $(\mathcal{M}_{\sigma(\alpha)}^{\vec{\mathcal{T}}}, \mathcal{T}_{\sigma(\alpha)})$ as witnessed by $\sigma_\alpha$ and $(\pi_{\alpha,\beta} : \beta < lh(\mathcal{U}_\alpha))$
   (b) if $\beta < \alpha$, $\gamma < lh(\mathcal{U}_\beta)$, $\eta < lh(\mathcal{U}_\alpha)$, and $\pi^{\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$ exists then $\pi^{\vec{\mathcal{T}}}_{(\sigma(\beta),\sigma_\beta(\gamma)),(\sigma(\alpha),\sigma_\alpha(\eta))}$ exists and $\pi^{\vec{\mathcal{T}}}_{(\sigma(\beta),\sigma_\beta(\gamma)),(\sigma(\alpha),\sigma_\alpha(\eta))} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\eta} \circ \pi^{\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$

We need to see that $\vec{\mathcal{U}}$ is according to $\Sigma^\tau$.

*Claim.* $\tau\vec{\mathcal{U}}$ is a hull of $\tau\vec{\mathcal{T}}$.

PROOF. We need to define

(1) an order preserving map $\sigma^* : lh(\tau\vec{\mathcal{U}}) \to lh(\tau\vec{\mathcal{T}})$
(2) a sequence of functions $(\sigma^*_\alpha : \alpha < lh(\tau\vec{\mathcal{U}}))$ such that $\sigma^*_\alpha : lh((\tau\vec{\mathcal{U}})_\alpha) \to lh((\tau\vec{\mathcal{T}})_{\sigma^*(\alpha)})$ is an order preserving map
(3) $(\pi^*_{\alpha,\beta} : \alpha < lh(\tau\vec{\mathcal{U}}) \wedge \beta < lh((\tau\vec{\mathcal{U}})_\alpha))$ such that
   (a) For any $\alpha < lh(\tau\vec{\mathcal{U}})$, $\pi^*_{\alpha,0} : \mathcal{M}_\alpha^{\tau\vec{\mathcal{U}}} \to \mathcal{M}_{\sigma^*(\alpha)}^{\tau\vec{\mathcal{T}}}$ and $(\mathcal{M}_\alpha^{\tau\vec{\mathcal{U}}}, (\tau\vec{\mathcal{U}})_\alpha)$ is a hull of $(\mathcal{M}_{\sigma^*(\alpha)}^{\tau\vec{\mathcal{T}}}, (\tau\vec{\mathcal{T}})_{\sigma(\alpha)})$ as witnessed by $\sigma^*_\alpha$ and $(\pi^*_{\alpha,\beta} : \beta < lh((\tau\vec{\mathcal{U}})_\alpha))$
   (b) if $\beta < \alpha$, $\gamma < lh((\tau\vec{\mathcal{U}})_\beta)$, $\eta < lh((\tau\vec{\mathcal{U}})_\alpha)$, and $\pi^{\tau\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$ exists then $\pi^{\tau\vec{\mathcal{T}}}_{(\sigma^*(\beta),\sigma^*_\beta(\gamma)),(\sigma^*(\alpha),\sigma^*_\alpha(\eta))}$ exists and $\pi^{\tau\vec{\mathcal{T}}}_{(\sigma^*(\beta),\sigma^*_\beta(\gamma)),(\sigma^*(\alpha),\sigma^*_\alpha(\eta))} \circ \pi^*_{\beta,\gamma} = \pi^*_{\alpha,\eta} \circ \pi^{\tau\vec{\mathcal{U}}}_{(\beta,\gamma),(\alpha,\eta)}$

Let $\sigma^* = \sigma$ and $\sigma^*_\alpha = \sigma_\alpha$. We then define $(\pi^*_{\alpha,\beta} : \alpha < lh(\tau\vec{\mathcal{U}}) \wedge \beta < lh((\tau\vec{\mathcal{U}})_\alpha))$ by induction. Suppose we have defined $(\pi^*_{\xi,\zeta} : \xi < \alpha \wedge \zeta < lh((\tau\vec{\mathcal{U}})_\xi))$, $(\pi^*_{\alpha,\xi} : \xi \leq \beta)$ and we need to define $\pi^*_{\alpha,\beta+1} : \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta+1} \to \mathcal{M}^{\tau\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta+1)}$. We let $\tau^{\vec{\mathcal{U}}}_{\alpha,\beta} : \mathcal{M}^{\vec{\mathcal{U}}}_{\alpha,\beta} \to \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta}$ and $\tau^{\vec{\mathcal{T}}}_{\alpha,\beta} : \mathcal{M}^{\vec{\mathcal{T}}}_{\alpha,\beta} \to \mathcal{M}^{\tau\vec{\mathcal{T}}}_{\alpha,\beta}$ come from the copying construction. As part of induction, we assume that for $\xi < \alpha$ and $\zeta < lh(\vec{\mathcal{U}}_\xi)$ or for $\xi = \alpha$ and $\zeta \leq \beta$

$$\tau^{\vec{\mathcal{T}}}_{\sigma(\xi),\sigma_\xi(\zeta)} \circ \pi_{\xi,\zeta} = \pi^*_{\xi,\zeta} \circ \tau^{\vec{\mathcal{U}}}_{\xi,\zeta}$$

Notice that $\tau_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta}) = E^{\tau\vec{\mathcal{U}}}_{\alpha,\beta}$. Let $E^{\tau\vec{\mathcal{U}}}_{\alpha,\beta} \in \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta}$ and let $\gamma$ be the predecessor of $\beta+1$ in $(\tau\vec{\mathcal{U}})_\alpha$. Then if $x \in \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta+1}$ then there is $f \in \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta+1}$ and $a \in [\nu(E_{\alpha,\beta})]^{<\omega}$ such that $x = [a, f]_{E_{\alpha,\beta}}$. Then we let

$$\pi^*_{\alpha,\beta+1}(x) = [\pi^*_{\alpha,\beta}(a), \pi^*_{\alpha,\gamma}(f)]_{\pi^*_{\alpha,\beta}(E^{\tau\vec{\mathcal{U}}}_{\alpha,\beta})}$$

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We need to verify that

(1) $\pi^*_{\alpha,\beta+1} \circ \pi^{\tau\vec{\mathcal{U}}}_{(\alpha,\gamma),(\alpha,\beta+1)} = \pi^{\tau\vec{\mathcal{T}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\alpha),\sigma_\alpha(\beta+1))} \circ \pi^*_{\alpha,\gamma}$ and
(2) $\tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta+1)} \circ \pi_{\alpha,\beta+1} = \pi^*_{\alpha,\beta+1} \circ \tau^{\vec{\mathcal{U}}}_{\alpha,\beta+1}$

The proof of (1) and (2) are similar and we only demonstrate (2). Recall that $\tau_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta}) = E^{\tau\vec{\mathcal{U}}}_{\alpha,\beta}$. Noting that $dom(\pi_{\alpha,\beta+1}) = \mathcal{M}^{\mathcal{U}}_{\alpha,\beta+1}$, we have that

$$\pi_{\alpha,\beta+1}(x) = [\pi_{\alpha,\beta}(a), \pi_{\alpha,\gamma}(f)]_{\pi_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta})}$$

Using the inductive hypothesis, we get that

$$\begin{aligned}
\tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta+1)} \circ \pi_{\alpha,\beta+1}(x) &= \tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta+1)}(\pi_{\alpha,\beta+1}(x)) \\
&= \tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta+1)}([\pi_{\alpha,\beta}(a), \pi_{\alpha,\gamma}(f)]_{\pi_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta})}) \\
&= [\tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta)}(\pi_{\alpha,\beta}(a)), \tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\gamma)}(\pi_{\alpha,\gamma}(f))]_{\tau^{\vec{\mathcal{T}}}_{\sigma(\alpha),\sigma_\alpha(\beta)}(\pi_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta}))} \\
&= [\pi^*_{\alpha,\beta}(\tau^{\vec{\mathcal{U}}}_{\alpha,\beta}(a)), \pi^*_{\alpha,\gamma}(\tau^{\vec{\mathcal{U}}}_{\alpha,\gamma}(f))]_{\pi^*_{\alpha,\beta}(\tau^{\vec{\mathcal{U}}}_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta}))} \\
&= \pi^*_{\alpha,\beta+1}([\tau^{\vec{\mathcal{U}}}_{\alpha,\beta}(a), \tau^{\vec{\mathcal{U}}}_{\alpha,\gamma}(f)]_{\tau^{\vec{\mathcal{U}}}_{\alpha,\beta}(E^{\vec{\mathcal{U}}}_{\alpha,\beta})}) \\
&= \pi^*_{\alpha,\beta+1} \circ \tau^{\vec{\mathcal{U}}}_{\alpha,\beta+1}(x).
\end{aligned}$$

Suppose next that $\pi_{\alpha,\beta}$ has been defined for all $\beta < \nu$ and $\nu$ is limit. Let $x \in \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\nu}$ and fix $\beta < \nu$ such that for some $\bar{x} \in \mathcal{M}^{\tau\vec{\mathcal{U}}}_{\alpha,\beta}$, $x = \pi^{\tau\vec{\mathcal{U}}}_{(\alpha,\beta),(\alpha,\nu)}$. Then we would like to set

$$\pi^*_{\alpha,\nu}(x) = \pi^{\tau\vec{\mathcal{T}}}_{(\sigma(\alpha),\sigma_\alpha(\beta)),(\sigma(\alpha),\sigma_\alpha(\nu))}(\pi^*_{\alpha,\beta}(\bar{x})).$$

However, for this to work, we need to know that if $b$ and $c$ are the branches of $(\tau\vec{\mathcal{T}})_{\sigma(\alpha)} \restriction \sigma_\alpha(\nu)$ and $(\tau\vec{\mathcal{U}})_\alpha \restriction \nu$ respectively then $\sigma_\alpha$"$c \subseteq b$. This follows from the fact that $\vec{\mathcal{U}}$ is a hull of $\vec{\mathcal{T}}$ and $c$ and $b$ are the branches of $\vec{\mathcal{U}} \restriction (\alpha,\nu)$ and $\vec{\mathcal{T}} \restriction (\sigma(\alpha),\sigma_\alpha(\nu))$. It is now straightforward to check that $\pi^*_{\alpha,\nu}$ is as desired. Thus, indeed, $\tau\vec{\mathcal{U}}$ is a hull of $\tau\vec{\mathcal{T}}$. □

It follows from the claim that if $\tau\vec{\mathcal{T}}$ is according to $\Sigma$ then $\tau\vec{\mathcal{U}}$ is according to $\Sigma$ as well. Because $\vec{\mathcal{T}}$ is according to $\Sigma^\tau$, $\tau\vec{\mathcal{T}}$ is according to $\Sigma$. Therefore, $\tau\vec{\mathcal{U}}$ is according to $\Sigma$ implying that $\vec{\mathcal{U}}$ is according to $\Sigma^\tau$. □

Lemma 1.33 (Condensation of strategies with hull condensation). *Suppose $\mathcal{M}$ is an lsp and $\Sigma$ is a $(\kappa,\lambda)$-iteration strategy for $\mathcal{M}$ which has hull condensation. Suppose $M, N$ are transitive models closed under rudimentary functions and such that $o(M)$ and $o(N)$ are limit ordinals, $\mathcal{M} \in M$ and there is a $\Sigma_1$-elementary embedding $\tau : N \to M$ with the property that $\mathcal{M} \in rng(\tau)$. Let $\mathcal{N} = \tau^{-1}(\mathcal{M})$. Then for any stack $\vec{\mathcal{T}} \in N$ on $\mathcal{N}$, if $\tau(\vec{\mathcal{T}})$ is according to $\Sigma$ then $\vec{\mathcal{T}}$ is according to $\Sigma^\tau$. In particular, for any $X \in M$ if $\Sigma \restriction X \in M$ and $\{X, \Sigma \restriction X\} \in rng(\tau)$ then $\tau^{-1}(\Sigma \restriction X) = \Sigma^\tau \restriction \tau^{-1}(X)$.*

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PROOF. The proof is similar to the proof of Lemma 1.32. Given $\vec{\mathcal{T}} \in N$ on $\mathcal{N}$ we claim that $\tau\vec{\mathcal{T}}$ is a hull of $\tau(\vec{\mathcal{T}})$. To see this, we let $\sigma : lh(\tau\vec{\mathcal{T}}) \to lh(\tau(\vec{\mathcal{T}}))$ be $\tau \restriction lh(\tau\vec{\mathcal{T}})$ and $\sigma_\alpha : lh((\tau\vec{\mathcal{T}})_\alpha) \to lh(\tau(\vec{\mathcal{T}})_{\sigma(\alpha)})$ be $\tau \restriction lh(\vec{\mathcal{T}}_\alpha)$. The construction of $\pi_{\alpha,\beta}$ is as in the proof of Lemma 1.32. The only problem here is the limit stage. Suppose $\nu < lh(\mathcal{T}_\alpha)$ is a limit ordinal and $b$ is the branch of $\mathcal{T}_\alpha \restriction \nu$. In order to carry out the construction we need to see that downward closure of $\sigma_\alpha$"$b$ in $\tau\mathcal{T}_\alpha$ is the branch of $\tau\mathcal{T}_\alpha$ chosen by $\Sigma$. This follows from the hull condensation of $\Sigma$ and the fact that $\sigma_\alpha$"$b \subseteq \tau(b)$. □

In the subsequent sections, we will show that if the background universe has a strategy with hull condensation then the full background constructions inherit a strategy with hull condensation as well (see Lemma 2.9). This fact will help us produce *hod mice*, which are the subject of the next section, whose strategies are as complicated as we want.

## 1.7. Hod mice

We are now in a position to define hod premice. Intuitively speaking a hod premouse $\mathcal{P}$ is just an lsp whose layers are the Woodin cardinals and their limits. At this point, it will be useful to review Definition 1.26. When $\mathcal{M}$ is an lsp, $\eta, \kappa < o(\mathcal{M})$ and $\alpha \in Ord$ then we let $\mathcal{O}^{\mathcal{M},\alpha}_{\eta,\kappa} = \mathcal{O}^{\mathcal{M},\alpha}_{\eta,\eta,\kappa}$. Also, if $\kappa = 0$ then we just write $\mathcal{O}^{\mathcal{M},\alpha}_{\eta}$.

DEFINITION 1.34 (Hod premouse). *Suppose $\mathcal{P} = (\mathcal{J}^{\vec{E},f}_\alpha(X), B)$ is an lsp. $\mathcal{P}$ is a hod premouse below $\mathrm{AD}_\mathbb{R}$ + "$\theta$ is measurable" (see Figure 1.7.1) if $B = \emptyset$ and letting $\lambda = o.t.(Y^\mathcal{P})$, $(\gamma_\beta : \beta < \lambda)$ enumerate $Y^\mathcal{P}$ in increasing order, and for $\beta < \lambda$, $\mathcal{P}(\beta) = \mathcal{P}||\gamma_\beta$, there is a continuous increasing sequence of $\mathcal{P}$-cardinals $(\delta_\beta : \beta \leq \lambda)$ such that letting $\mathcal{P}(\lambda) = \mathcal{P}$, the following holds in $\mathcal{P}$:*

1. *$Y^\mathcal{P} \subseteq \delta_\lambda$. For all $\beta \leq \lambda$, $\mathcal{P}(\beta) \vDash$ "ZFC-Replacement", $\mathcal{P}(\beta) = \mathcal{O}^{\mathcal{P},\omega}_{\delta_\beta}$ and if $\beta$ is limit then $\delta^+_\beta = (\delta^+_\beta)^{\mathcal{P}(\beta)}$.*
2. *$(\delta_\beta : \beta \leq \lambda)$ is the sequence of Woodin cardinals and their limits enumerated in increasing order and for all $\beta \leq \lambda$, $\delta_\beta$ is a strong cutpoint.*
3. *For all $\beta < \lambda$, $f(\beta)$ codes an $(o(\mathcal{P}), o(\mathcal{P}))$-strategy for $\mathcal{P}(\beta)$ with hull condensation[4].*
4. *If $\beta < \lambda$ then for any successor cardinal $\eta \in (\delta_\beta, \delta_{\beta+1})$, $\mathcal{P}|\eta$ is an $f(\beta)$-premouse over $\mathcal{P}(\beta)$ which is $(o(\mathcal{P}), o(\mathcal{P}))$-iterable for stacks that are above $\delta_\beta$.*

It is the requirement that $\delta_\beta$ is a strong cutpoint that makes our hod premice "below $\mathrm{AD}_\mathbb{R} + \Theta$ is measurable". From now on we drop "below $\mathrm{AD}_\mathbb{R} + \Theta$ is measurable" and simply say "$\mathcal{P}$ is a hod premouse". Also, we let "$\mathcal{P}$ is a hod premouse" stand for hod premouse over $\emptyset$, i.e., $X = \emptyset$ (in particular, $0 \notin Y^\mathcal{P}$). Clearly all notions that we will define for lightface hod premice easily generalize to non-lightface hod premice. If $X \neq \emptyset$ and $\Lambda$ is an iteration strategy of $X$ then we can define $\Lambda$-hod premice. As all the notions that we will isolate for hod premice easily generalize to $\Lambda$-hod premice, in the sequel, we will only deal with hod premice.

[4]If $\eta \in (\beta, \lambda)$ then letting $\Sigma$ and $\Lambda$ be the strategies of $\mathcal{P}(\beta)$ and $\mathcal{P}(\eta)$ coded by $f(\beta)$ and $f(\eta)$ respectively, $\Sigma = \Lambda_{\mathcal{P}(\beta)}$. This follows from Definition 1.17.

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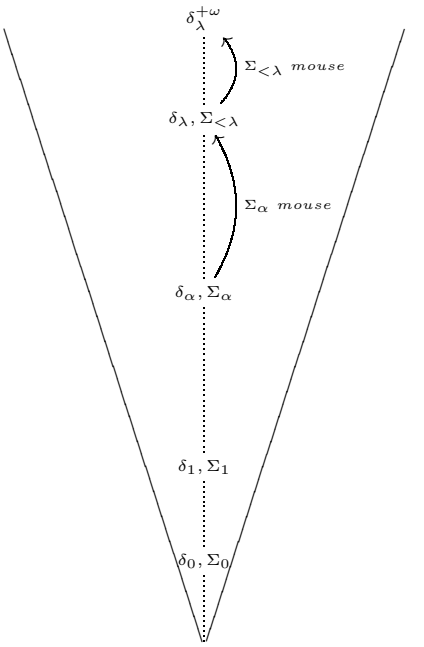


FIGURE 1.7.1. Hod premouse.

REMARK 1.35. *In general, hod premice do not satisfy condensation, i.e., if $\alpha < o(\mathcal{P})$ is a strong cutpoint of $\mathcal{P}$ then, it may not be the case that if $\mathcal{H}_1^{\mathcal{P}}(\mathcal{P}|\alpha) \trianglelefteq \mathcal{P}$. We might not even be able to compare $\mathcal{H}_1^{\mathcal{P}}(\mathcal{P}|\alpha)$ with $\mathcal{P}$.*

Below we set up our notation that we will use throughout this paper.

NOTATION. *Suppose $\mathcal{P}$ is a hod premouse.*

(1) *We let $\lambda^{\mathcal{P}} = o.t.(Y^{\mathcal{P}})$ , $(\delta_\alpha^{\mathcal{P}} : \alpha \leq \lambda^{\mathcal{P}})$ be the sequence of Woodin cardinals of $\mathcal{P}$ and their limits enumerated in increasing order, and for $\alpha \leq \lambda^{\mathcal{P}}$, $\mu_\alpha^{\mathcal{P}} = o(\mathcal{P}(\alpha))$ and $\delta^{\mathcal{P}} = \delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$.*

(2) *We also let*

$$\mathcal{P}^- = \begin{cases} \mathcal{P}|\mu_{\lambda^{\mathcal{P}}-1} & : \lambda^{\mathcal{P}} \text{ is a successor ordinal} \\ \mathcal{P}|\delta^{\mathcal{P}} & : \lambda^{\mathcal{P}} \text{ is limit.} \end{cases}$$

(3) *For $\alpha < \lambda^{\mathcal{P}}$ we let , $\Sigma_\alpha^{\mathcal{P}}$ be the internal strategy of $\mathcal{P}(\alpha)$ coded by $f(\alpha)$. If $\alpha \leq \lambda^{\mathcal{P}}$ is a limit ordinal then we let $\Sigma_{<\alpha}^{\mathcal{P}} = \oplus_{\xi<\alpha}\Sigma_\xi^{\mathcal{P}}$. It follows that for limit $\alpha$, $\Sigma_{<\alpha}^{\mathcal{P}}$ is the internal strategy of $\mathcal{P}|\delta_\alpha^{\mathcal{P}}$. We then let*

$$\Sigma^{\mathcal{P}} = \begin{cases} \Sigma_{<\lambda^{\mathcal{P}}}^{\mathcal{P}} & : \lambda^{\mathcal{P}} \text{ is limit} \\ \Sigma_{\lambda^{\mathcal{P}}-1}^{\mathcal{P}} & : \lambda^{\mathcal{P}} \text{ is a successor.} \end{cases}$$

In the chapters that follow, we will present the comparison theory of hod premice along with methods for constructing them. Before we proceed with the theory, however, we consider few examples of hod premice. The cases $\lambda^{\mathcal{P}} = 0, 1, \omega$, "$\lambda^{\mathcal{P}}$ has a measurable cofinality in $\mathcal{P}$", and "$\lambda^{\mathcal{P}}$ is a successor whose predecessor has a

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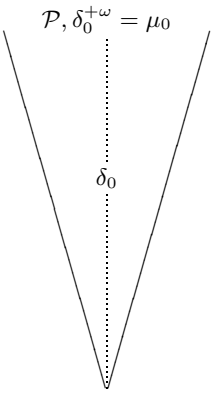


FIGURE 1.7.2. Hod premouse with $\lambda^{\mathcal{P}} = 0$, $\mathcal{P}(0) = \mathcal{P}$ and $\mu_0 = o(\mathcal{P})$.

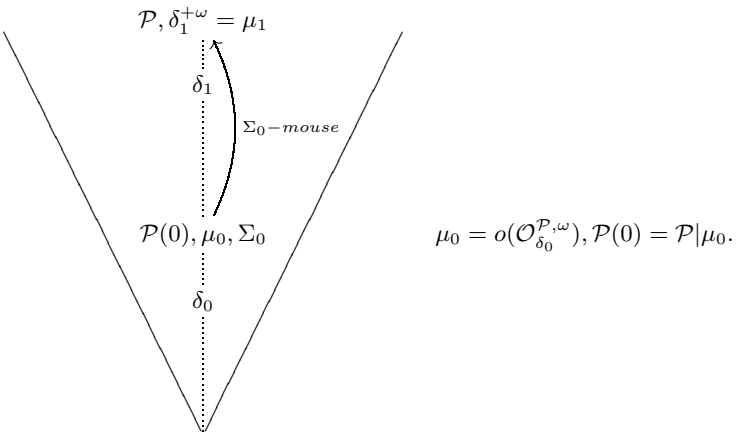


FIGURE 1.7.3. Hod premouse with $\lambda^{\mathcal{P}} = 1$.

measurable cofinality in $\mathcal{P}$" are considered below.

$\boldsymbol{\lambda^{\mathcal{P}} = \mathbf{0}}$. In this case, there is a single Woodin cardinal in $\mathcal{P}$ and there are no strategies in $\mathcal{P}$, i.e., $\mathcal{P}$ is an ordinary premouse such that $\delta_0^{\mathcal{P}}$ is the unique Woodin cardinal of $\mathcal{P}$, $\mathcal{P}(0) = \mathcal{P}$ and $\mu_0 = o(\mathcal{P})$. If $\delta$ is the Woodin cardinal of $\mathcal{M}_1^{\#}$ and $\mu = (\delta^{+\omega})^{\mathcal{M}_1^{\#}}$ then $\mathcal{P} = \mathcal{M}_1^{\#} \restriction \mu$ is a hod mouse such that $\lambda^{\mathcal{P}} = 0$. See Figure 1.7.2 for a picture.

$\boldsymbol{\lambda^{\mathcal{P}} = \mathbf{1}}$. In this case, there are two Woodin cardinals and one strategy in $\mathcal{P}$. $\delta_0^{\mathcal{P}}$ and $\delta_1^{\mathcal{P}}$ are the Woodin cardinals of $\mathcal{P}$, $\mathcal{P}(0) = \mathcal{O}_{\delta_0^{\mathcal{P}}}^{\mathcal{P},\omega}$, $\mu_0 = o(\mathcal{P}(0))$, $\Sigma_0$ is the strategy of $\mathcal{P}(0)$, $\mu_1 = o(\mathcal{P})$ and $\mathcal{P}(1) = \mathcal{P}$. Notice that $\mathcal{P}(0)$ is also a hod premouse which is actually an ordinary premouse. $\mathcal{P}_1$ is a $\Sigma_0$-premouse over $\mathcal{P}_0$. See Figure 1.7.3 for a picture.

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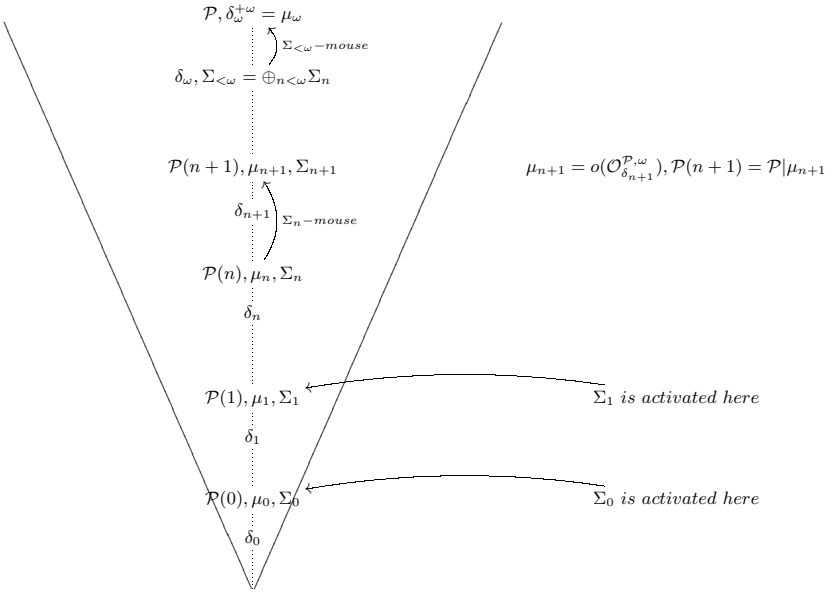


FIGURE 1.7.4. Hod premouse with $\lambda^{\mathcal{P}} = \omega$.

**$\lambda^{\mathcal{P}} = \omega$.** In this case, there are $\omega$ many Woodin cardinals and $\omega$ many strategies in $\mathcal{P}$. For $n < \omega$, $\delta^{\mathcal{P}}_n$ is Woodin. $\delta^{\mathcal{P}}_\omega$ is the sup of the Woodins of $\mathcal{P}$. For $n \leq \omega$, $\mathcal{P}(n) = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}}_n}$, $\mu_n = o(\mathcal{P}(n))$ and Notice that for every $n$, $\mathcal{P}(n)$ is a hod premouse. See Figure 1.7.4 for a picture.

**$\lambda^{\mathcal{P}}$ has a measurable cofinality.** We take the simple case when $\mathcal{P}$ is the least hod premouse with the property that $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$. Let $\kappa$ be the least measurable of $\mathcal{P}$. Then $\lambda^{\mathcal{P}} = \kappa$ and $\mathcal{P}$ has $\kappa$ many Woodin cardinals and $\kappa$ many strategies. In this case, for $\alpha < \kappa$, $\delta^{\mathcal{P}}_{\alpha+1}$ is a Woodin cardinals of $\mathcal{P}$. As before, for $\alpha \leq \kappa$, $\mathcal{P}(\alpha) = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}}_\alpha}$. Also for $\alpha \leq \kappa$, $\mathcal{P}(\alpha)$ is a hod premouse. See Figure 1.7.5 for a picture.

**$\lambda^{\mathcal{P}}$ is a successor and $\lambda^{\mathcal{P}} - 1$ is a limit of measurable cofinality.** We take the simple case when $\mathcal{P}$ is the least hod premouse with the property that $\lambda^{\mathcal{P}}$ is a successor and $\lambda^{\mathcal{P}} - 1$ is a limit of measurable cofinality. Let $\kappa$ be the least measurable of $\mathcal{P}$. Then $\lambda^{\mathcal{P}} = \kappa + 1$ and $\mathcal{P}$ has $\kappa + 1$ many Woodin cardinals and $\kappa + 1$ many strategies. In this case, for $\alpha \leq \kappa$, $\delta^{\mathcal{P}}_{\alpha+1}$'s are the Woodin cardinals of $\mathcal{P}$. As before, for $\alpha \leq \kappa$, $\mathcal{P}(\alpha) = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}}_\alpha}$. Notice that for $\alpha \leq \kappa$, $\mathcal{P}(\alpha)$ is a hod premouse. One significant way that this case is different than the other cases is that $\delta^+_\kappa = (\delta^+_\kappa)^{\mathcal{P}(\kappa)}$. See Figure 1.7.6 for a picture.

DEFINITION 1.36. *$(\mathcal{P}, \Sigma)$ is called a hod pair if it is an lsm pair, $\mathcal{P}$ is a hod premouse and $\Sigma$ is an $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{P}$ with hull condensation.*

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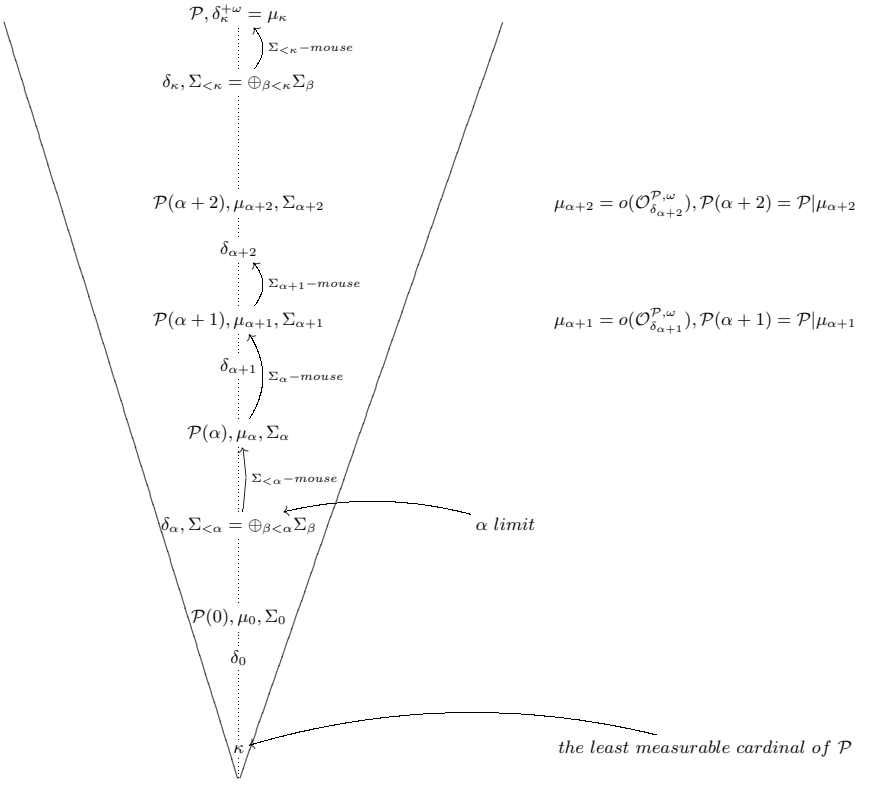


FIGURE 1.7.5. Hod premouse with $\mathcal{P} \vDash$ "$\lambda^{\mathcal{P}}$ =the least measurable cardinal $\kappa$".

DEFINITION 1.37. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$.*

If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice such that $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ then we say $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$. If $(\mathcal{P}, \Sigma)$ is a hod pair, $\mathcal{Q}$ is a non-dropping $\Sigma$-iterate of $\mathcal{P}$ and $\mathcal{R} \trianglelefteq_{hod} \mathcal{Q}$ then we let $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}$ be the iteration strategy for $\mathcal{R}$ given by $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$. In the following definition, "$B$" stands for "blow up" and "$I$" stands for "iterates".

DEFINITION 1.38. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Then*

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{P} \text{ via } \Sigma \text{ with last model } \mathcal{M} \text{ such that } \pi^{\vec{\mathcal{T}}} \text{ exists and } \mathcal{Q} \trianglelefteq_{hod} \mathcal{M}\}$$

$$I(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{P} \text{ via } \Sigma \text{ with last model } \mathcal{Q} \text{ such that } \pi^{\vec{\mathcal{T}}} \text{ exists}\}.$$

Suppose now that $\mathcal{P}$ is a hod premouse. Suppose further that $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists. Then we let $\alpha(\vec{\mathcal{T}}) \leq \lambda^{\mathcal{Q}}$ be the least $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\delta(\vec{\mathcal{T}}) \leq \delta^{\mathcal{Q}}_\alpha$. Suppose next that $\Sigma$ is a strategy such that $(\mathcal{P}, \Sigma)$ is a hod pair. Given $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma) \cup B(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$, we let

$$\pi^{\Sigma}_{(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{U}},\mathcal{R})} : \mathcal{Q} \to \mathcal{R}.$$

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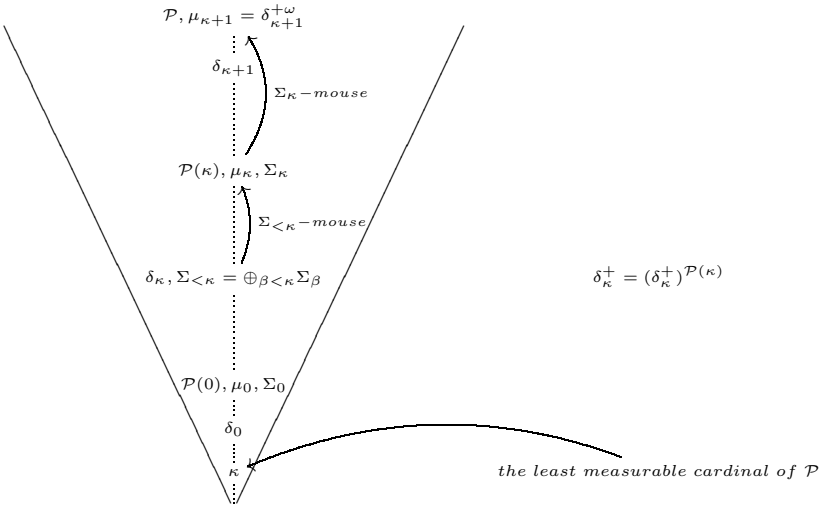


FIGURE 1.7.6. Hod premouse with $\mathcal{P} \vDash$ "$\lambda^{\mathcal{P}} = \kappa + 1$ where $\kappa$ is the least measurable cardinal".

We will later show that $\pi^{\Sigma}_{(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{U}},\mathcal{R})}$ is independent of $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ provided that $\Sigma$ has *branch condensation* (see Theorem 2.42). Once this is done, we will drop $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ from the above notation.

We end this chapter with the following easy but useful lemma. It follows immediately from the fact that being a hod premouse is a first order property.

LEMMA 1.39 (Pullback of a hod pair). *Suppose $(\mathcal{M}, \Sigma)$ is a hod pair and $\tau : \mathcal{N} \to \mathcal{M}$ is a $\Sigma_1$-elementary embedding. Then $(\mathcal{N}, \Sigma^\tau)$ is a hod pair.*

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CHAPTER 2

# Comparison theory of hod mice

In this chapter, our main goal is to develop the comparison theory of hod pairs under $AD^+$.

DEFINITION 2.1. *Given two hod pairs $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$, we say comparison holds for $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ if there are $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P},\Sigma)$ and $(\vec{\mathcal{U}},\mathcal{S})\in I(\mathcal{Q},\Lambda)$ such that one of the following holds.*

(1) *$\mathcal{R}\trianglelefteq_{hod}\mathcal{S}$ and $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}=\Lambda_{\mathcal{R},\vec{\mathcal{U}}}$.*
(2) *$S\trianglelefteq_{hod}\mathcal{R}$ and $\Lambda_{\mathcal{S},\vec{\mathcal{U}}}=\Sigma_{\mathcal{S},\vec{\mathcal{T}}}$.*

The comparison theorem of this chapter, Theorem 2.28, is useful in determinacy context. It proves that under $AD^+$ comparison holds for sufficiently similar hod pairs. In later chapters, we will develop a comparison theory that will be useful in core model induction arguments and in $ZFC$ context (see Theorem 2.47). The main idea behind our comparison argument is that if a hod pair is compared with a hod pair constructed by a fully backgrounded construction such as those introduced in Chapter 11 of [**20**] then the comparison terminates and the background side doesn't move. In order to execute this idea, we need to introduce the *hod pair constructions*.

## 2.1. Hod pair constructions

The backgrounded construction of [**20**] produces an ordinary mouse. We need to modify this construction to produce hybrid mouse. The definition given below is based on Chapter 11 of [**20**]. Following [**20**], we say that an lhp $\mathcal{M}$ is *reliable* if for every $k\leq\omega$, $\mathcal{C}_k(\mathcal{M})$ exists and is $k$-iterable.

DEFINITION 2.2. *Suppose $\mathcal{M}$ is an lhp and $\Sigma$ is a $(\delta,\delta)$-iteration strategy with hull condensation for some uncountable $\delta$. Suppose for some $\kappa<\lambda<\delta$, $E$ is a $(\kappa,\lambda)$-extender such that $\mathcal{M}\in V_\kappa$. We then say $E$ coheres $\Sigma$ if $V_\lambda\subseteq Ult(V,E)$ and $\pi_E(\Sigma)=\Sigma\restriction Ult(V,E)$.*

The backgrounded constructions that produce $\Sigma$-mice will only use extenders that cohere $\Sigma$. Recall that a $(\kappa,\lambda)$-extender $E$ reflects a set $A$ if $A\cap V_{\nu(E)}=\pi_E(A)\cap V_{\nu(E)}$.

DEFINITION 2.3. *Suppose $\mathcal{M}$ is an lhp and $\Sigma$ is a $(\delta,\delta)$-iteration strategy with hull condensation for some uncountable $\delta$. Then $((\mathcal{M}_\gamma,\mathcal{N}_\gamma:\gamma\leq\eta),(F_\gamma:\gamma<\eta))$ is the $\eta$th initial segment of the output of the fully backgrounded construction relative to $\Sigma$ if the following is true.*

(1) *$\mathcal{M}_0=trc^{\mathcal{M}}$, and $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ are $\Sigma$-mice.*
(2) *Suppose $((\mathcal{M}_\gamma,\mathcal{N}_\gamma:\gamma\leq\xi),(F_\gamma:\gamma<\xi))$ has been defined for $\xi<\eta$. Then we define $\mathcal{M}_{\xi+1}$, $\mathcal{N}_{\xi+1}$ and $F_\xi$ as follows.*



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(a) *If* $\mathcal{M}_\xi = (\mathcal{J}^{\vec{E},f}_\alpha, \in, \vec{E}, f)$ *is a passive lhp, i.e., with no last predicate, and there is an extender* $F^*$ *such that* $\nu(F^*)$ *is inaccessible and* $F^*$ *cohers* $\Sigma$ *and reflects* $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \xi), (F_\gamma : \gamma < \xi))$*, an extender* $F$ *over* $\mathcal{M}_\xi$*, and an ordinal* $\nu < \alpha$ *such that* $V_{\nu+\omega} \subseteq Ult(V, F^*)$ *and*

$$F \restriction \nu = F^* \cap ([\nu]^\omega \times \mathcal{J}^{\vec{E},f}_\alpha)$$

*then*

$$\mathcal{N}_{\xi+1} = (\mathcal{J}^{\vec{E},f}_\alpha, \in, \vec{E}, f, \tilde{F})$$

*and* $\nu = \nu^{\mathcal{N}_{\xi+1}}$ *where* $\tilde{F}$ *is the amenable code of* $F$[1]*. Also, if* $\mathcal{N}_{\xi+1}$ *is reliable then* $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$[2] *and* $F_\xi = F$*. If* $\mathcal{N}_{\xi+1}$ *is not reliable then we stop the construction.*

(b) *If* $\mathcal{M}_\xi = (\mathcal{J}^{\vec{E},f}_\alpha, \in, \vec{E}, f)$ *is a passive lhp, the hypothesis of item 3 above doesn't hold,* $\mathcal{M}_\xi \vDash$ *ZFC-Replacement, and there is a stack* $\vec{\mathcal{T}} \in \mathcal{J}^{\vec{E},f}_\alpha \cap dom(\Sigma)$ *such that* $f^{\mathcal{M}_\xi}(\vec{\mathcal{T}})$ *isn't defined then letting* $\vec{\mathcal{U}}$ *be the* $\mathcal{M}_\xi$*-least such stack,* $b = \Sigma(\vec{\mathcal{U}})$*,* $\beta = \sup b$ *and* $\mathcal{N} = \mathcal{J}_\beta(\mathcal{M}_\xi)$*, if* $\rho(\mathcal{N}) \geq \alpha$ *then*

$$\mathcal{N}_\xi = (\mathcal{J}^{\vec{E},f^+}_{\alpha+\beta}, \in, \vec{E}, f^+)$$

*where* $f^+ = f \cup (\mathcal{J}_\omega(\vec{\mathcal{U}}), \tilde{b})$ *where* $\tilde{b} \subseteq \alpha + \beta$ *is defined by* $\alpha + \nu \in \tilde{b} \leftrightarrow \nu \in b$*. Also, if* $\mathcal{N}_{\xi+1}$ *is reliable then* $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$ *and* $F_\xi = \emptyset$*. If* $\mathcal{N}_{\xi+1}$ *is not reliable then we stop the construction.*

(c) *If neither a or b happen, then* $\mathcal{N}_{\xi+1} = \mathcal{J}_1(\mathcal{M}_\xi)$ *and if* $\mathcal{N}_{\xi+1}$ *is reliable then* $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$*. If* $\mathcal{N}_{\xi+1}$ *is not reliable then we stop the construction.*

(3) *Suppose* $\xi \leq \eta$ *is a limit ordinal and* $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma < \xi), (F_\gamma : \gamma < \xi))$ *has been defined. Then we define* $\mathcal{M}_\xi$ *and* $\mathcal{N}_\xi$ *as follows*[3]*. Let* $\nu = limsup_{\lambda \to \xi}(\rho^+)^{\mathcal{M}_\xi}$*. Then we let* $\mathcal{N}_\xi$ *be the passive lhp* $\mathcal{P} = \mathcal{J}^{\mathcal{P}}_\nu$*, where for all* $\beta < \nu$ *we set* $\mathcal{J}^{\mathcal{P}}_\beta$ *be the eventual value of* $\mathcal{J}^{\mathcal{M}_\lambda}_\beta$ *as* $\lambda \to \xi$*. Also if* $\mathcal{N}_\xi$ *is reliable then* $\mathcal{M}_\xi = \mathcal{C}(\mathcal{N}_\xi)$*. If* $\mathcal{N}_\xi$ *is not reliable then we stop the construction.*

Notice that because $\mathcal{M}$ is in every core taken during the construction of

$$((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$$

and because $\Sigma$ has hull condensation, the usual proofs of solidity and universality go through without any significant changes. Hence, we will not give the proofs of these facts. Interested readers can consult [**20**] and [**43**].

The sequence $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ is called *the output of the* $\mathcal{J}^{\vec{E},\Sigma}$*-construction* or *the output of fully backgrounded construction relative to* $\Sigma$. We also say that $\mathcal{N}_\eta$ is the *last model* of $\mathcal{J}^{\vec{E},\Sigma}$-construction and write $\mathcal{N}_\eta = \mathcal{J}^{\vec{E},\Sigma}$. In the fully backgrounded constructions we may require that all the critical points of the extenders be bigger than some $\xi$. In this case we let $\mathcal{J}^{\vec{E},\Sigma}_{>\xi}$ be the resulting model. Also, we can do fully backgrounded constructions over any swo $X$ such that $\mathcal{M} \in X$. We let $\mathcal{J}^{\vec{E},\Sigma}_{>\xi}[X]$ stand for it. It is worth remarking here that

Remark. *if the backgrounded construction is carried out inside a mouse or in a structure which has a distinguished extender sequence then we tacitly assume*

[1]For the definition of the "amenable code" see the last paragraph on page 14 of [**43**].

[2]Recall that $\mathcal{C}(\mathcal{M})$ is the core of $\mathcal{M}$ (see Definition 1.7).

[3]$F_\xi$ will be defined at the next stage of the induction as in clause 2.

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*that all the extenders used in the construction actually come from that extender sequence.*

We will construct our hod mice inside sufficiently rich transitive models.

DEFINITION 2.4 (Background triples). *We say $(M, \delta, \Sigma)$ is a weak background triple if $M \vDash ZFC +$ "$\delta$ is a Woodin cardinal" and $\Sigma \in M$ is a $(\delta, \delta+1)$-iteration strategy for $V_\delta^M$ with hull condensation that acts on stacks in $L_1(V_\delta^M)$. We say $(M, \delta, \Sigma)$ is a background triple if $\Sigma$ is a $(\left|V_\delta^M\right|^+, \left|V_\delta^M\right|^+ + 1)$-strategy for $M$ that acts on stacks below $\delta$ and has hull condensation, and $(M, \delta, \Sigma \restriction L_1(V_\delta^M))$ is a weak background triple.*

REMARK 2.5. *When $M$ is a hybrid premouse or just has a distinguish extender sequence then we will tacitly assume that $\Sigma$ is a strategy in the game in which $I$ plays extenders from the distinguished sequence of extenders of the iterates of $M$.*

Theorem 2.25 provides ample source of background triples in the context of $AD^+$. We are now in a position to introduce the constructions that produce hod pairs. To describe this construction we borrow Theorem 2.7 from the next section. Thus, below when we say "$\Lambda$ is the strategy induced by $\Sigma$" we mean that $\Lambda$ is constructed from $\Sigma$ via the procedure described in Theorem 2.7.

DEFINITION 2.6 (Hod pair constructions). *Suppose $(M, \delta, \Sigma)$ is a weak background triple. Then the hod pair construction of $M$ below $\delta$ is a sequence*

$$(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta : \beta \leq \Omega)$$

*that satisfies the following properties:*

1. *For all $\beta \leq \Omega$, $(\mathcal{P}_\beta, \Sigma_\beta)$ is a hod pair, $\mathcal{N}_0 = (\mathcal{J}^{\vec{E}})^{V_\delta^M}$ and $\mathcal{N}_{\beta+1} = (\mathcal{J}^{\vec{E}, \Sigma_\beta})^{V_\delta^M}$.*
2. *If $\mathcal{N}_0$ has Woodin cardinals then $\mathcal{P}_0 = \mathcal{N}_0|(\delta_0^{+\omega})^{\mathcal{N}_0}$ where $\delta_0$ is the least Woodin of $\mathcal{N}_0$, and $\Sigma_0 \in M$ is the strategy of $\mathcal{P}_0$ induced by $\Sigma$. If $\mathcal{N}_0$ doesn't have Woodin cardinals then the construction stops with $\Omega = 0$ and the rest of the objects in the sequence with index 0 are left undefined.*
3. *If for $\beta < \Omega$*
   (a) *$\mathcal{N}_{\beta+1}$ does not have an initial segment $\mathcal{M}$ such that $\rho(\mathcal{M}) < \delta_\beta$,*
   (b) *$\mathcal{N}_{\beta+1}$ has Woodin cardinals $> \delta_\beta$,*
   (c) *if $\beta = 0$ or is a successor then $\mathcal{N}_{\beta+1} \vDash$ "$\delta_\beta$ is Woodin" and*
   (d) *if $\beta$ is limit then $(\delta_\beta^+)^{\mathcal{P}_\beta} = (\delta_\beta^+)^{\mathcal{N}_{\beta+1}}$,*

   *then letting $\delta_{\beta+1}$ be the least Woodin cardinal of $\mathcal{N}_{\beta+1}$ above $\delta_\beta$,*

$$\mathcal{P}_{\beta+1} = \mathcal{N}_{\beta+1}|(\delta_{\beta+1}^{+\omega})^{\mathcal{N}_{\beta+1}}$$

   *and $\Sigma_{\beta+1}$ is the strategy of $\mathcal{P}_{\beta+1}$ induced by $\Sigma$. If either of the above clause are not satisfied then $\Omega = \beta + 1$ and the rest of the objects in the sequence with index $\beta + 1$ are left undefined.*
4. *For limit ordinals $\beta$, letting $\mathcal{P}_\beta^* = \cup_{\gamma<\beta}\mathcal{P}_\beta$, $\Sigma_\beta^* = \oplus_{\gamma<\beta}\Sigma_\beta$ and $\delta_\beta = \sup_{\gamma<\beta}\delta_\gamma$, if $\delta_\beta$ is not measurable in $M$ and $\delta_\beta < \delta$ then $\mathcal{N}_\beta = (\mathcal{J}^{\vec{E}, \Sigma_\beta^*})^{V_\delta^M}$. If $\mathcal{N}_\beta$ doesn't project across $\delta_\beta$ then $\mathcal{P}_\beta = \mathcal{N}_\beta|(\delta_\beta^{+\omega})^{\mathcal{N}_\beta}$ and $\Sigma_\beta$ is the strategy of $\mathcal{P}_\beta$ induced by $\Sigma$. If $\delta_\beta$ is measurable in $M$ or if $\mathcal{N}_\beta$ projects across $\delta_\beta$ then $\beta = \Omega$ and the rest of the objects in the sequence with index $\beta$ are left undefined.*

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We say that the hod pair construction of $M$ below $\delta$ converges to $(\mathcal{P}_\Omega, \Sigma_\Omega)$ if $\mathcal{P}_\Omega$ and $\Sigma_\Omega$ are defined, $(\mathcal{P}_\Omega, \Sigma_\Omega)$ is a hod pair (in particular, $\mathcal{P}$ is a hod premouse) and $\mathcal{P}_{\Omega+1}$ is not defined. Notice that because of Lemma 2.9, all strategies appearing in hod pair constructions have hull condensation implying that the construction of $\mathcal{N}_\alpha$ for $\alpha \leq \Omega$ can always be done provided that $M$ has enough extenders cohering $\Sigma_\alpha$. Because $\delta$ is a Woodin cardinal of $M$, there are plenty of such extenders in $M$. We now turn to showing that the strategy of the hod pair constructed via a hod pair construction inherits hull condensation from the strategy of the background universe.

## 2.2. Iterability of hod pair constructions

Suppose $\Sigma$ is a $(\delta, \delta)$-iteration strategy with hull condensation and $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ is the $\eta$th initial segment of the output of fully backgrounded construction relative to $\Sigma$. Following Chapter 12 of [**20**], we let

$$\mathbb{C} = ((\mathcal{N}_\gamma : \gamma \leq \eta), (F^*_\gamma : \gamma < \eta \text{ and } F^*_\gamma \text{ is defined })).$$

Also recall from Chapter 12 page 115 of [**20**] that if $\mathcal{Q} = \mathcal{C}_k(\mathcal{N}_\gamma)$ and $\mathcal{Q}' = \mathcal{C}_j(\mathcal{N}_\xi)$ then $\mathcal{Q} \leq_{\mathbb{C}} \mathcal{Q}'$ iff $(\gamma, k) \leq_{lex} (\xi, j)$. We would like to describe the strategy of $\mathcal{N}_\gamma$ assuming there is a strategy for $V$. The following restatement of Theorem 12.1 of [**20**] does just that. The proof of Theorem 12.1 of [**20**] carries to current context word-by-word and therefore, we omit it.

THEOREM 2.7 (Mitchell-Steel, [**20**]). *Suppose for some $\theta$ and $\delta \leq \theta$, $\mathcal{M} \in V_\delta$ is an lhp and $\Sigma \in V_{\theta+1}$ is a $(\delta, \delta)$-iteration strategy for $\mathcal{M}$ with hull condensation. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ be the $\eta$th initial segment of $\mathcal{J}^{\vec{E},\Sigma}$ construction of $V_\theta$. Suppose $\Lambda$ is an iteration strategy for $V_\theta$. Then for each $\gamma \leq \eta$, $\Lambda$ induces a strategy $\Psi$ for $\mathcal{N}_\gamma$ with the following properties: if $\mathcal{T}$ is a tree on $\mathcal{N}_\gamma$ according to $\Psi$ then there are a tree $\mathcal{U}$ on $V_\theta$ according to $\Lambda$ which has the same tree structure as $\mathcal{T}$ (i.e., $U = T$), a sequence of models $(\mathcal{Q}_\alpha : \alpha < lh(\mathcal{T}))$ and a sequence of embeddings $(\pi_\alpha : \alpha < lh(\mathcal{T}))$ such that*

1. *For each $\alpha$ there is an ordinal $\xi_\alpha$ and natural number $n_\alpha$ such that $\mathcal{Q}_\alpha = (\mathcal{C}_{n_\alpha}(\mathcal{N}_{\xi_\alpha}))^{\mathcal{M}^{\mathcal{U}}_\alpha}$.*
2. *$\pi_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{Q}_\alpha$ is a weak $n_\alpha$-embedding.*
3. *If $\beta T \alpha$ and $(\beta, \alpha]_T \cap D^{\mathcal{T}} = \emptyset$ then $\pi_\alpha \circ \pi^{\mathcal{T}}_{\beta,\alpha} = \pi^{\mathcal{U}}_{\beta,\alpha} \circ \pi_\beta$.*
4. [4] *For each $\beta < \alpha$, if $Res^\beta$ is type I or III then $\mathcal{Q}_\alpha$ agrees with $Res^\beta$ below $\nu^{Res^\beta}$*[5]*; moreover,*
$$\pi_\alpha \restriction \nu_\beta = \sigma^\beta \circ \pi_\beta \restriction \nu_\beta \text{ and } \pi_\alpha(\nu_\beta) \geq \nu^{Res^\beta}$$
*where $\nu_\beta = \nu(E^{\mathcal{T}}_\beta)$.*
5. *For each $\beta < \alpha$, if $Res^\beta$ is type II then $\mathcal{Q}_\alpha$ agrees with $Res^\beta$ below $o(Res^\beta)$ and moreover,*

[4]Here $(\sigma^\beta, Res^\beta)$ is the *complete resurrection* of $\pi_\beta(E^{\mathcal{T}}_\beta)$ from stage $(n_\beta, \xi_\beta)$ where $n_\beta = deg^{\mathcal{T}}(\beta)$ and $\mathcal{Q}_\beta = (\mathcal{C}_j(\mathcal{N}_{\xi_\beta})^{\mathcal{M}^{\mathcal{U}}_\beta})$. For the full definition of complete resurrection consult Chapter 12 of [**20**]. Essentially the complete resurrection is the first stage in the backgrounded construction where the extender that produces $\pi_\beta(E^{\mathcal{T}}_\beta)$ appears. $Res^\beta$ is the first model that has the background certificate of $\pi_\beta(E^{\mathcal{T}}_\beta)$ and $\sigma^\beta$ is the composition of the core embeddings that produce $\pi_\beta(E^{\mathcal{T}}_\beta)$.

[5]Recall that $Res^\beta$ is just a lhp and $\nu^{Res^\beta}$ is the natural length of its last extender. See Definition 1.0.7 of [**20**].

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$$\pi_\alpha \restriction lh(E^{\mathcal{T}}_\beta) = \sigma^\beta \circ \pi_\beta \restriction lh(E^{\mathcal{T}}_\beta) \text{ and } \pi_\alpha(lh(E^{\mathcal{T}}_\beta)) \geq o(Res^\beta).$$

(6) *For each* $\beta < \alpha$, $\mathcal{M}^{\mathcal{U}}_\alpha$ *agrees with* $\mathcal{M}^{\mathcal{U}}_\beta$ *below* $\nu^{Res^\beta} + \omega$, *that is* $V_\gamma^{\mathcal{M}^{\mathcal{U}}_\alpha} = V_\gamma^{\mathcal{M}^{\mathcal{U}}_\beta}$ *for* $\gamma = \nu^{Res^\beta} + \omega$.
(7) *Let* $\beta = T - pred(\alpha+1)$ *and* $\mathbb{C}^{\alpha+1} = \pi^{\mathcal{U}}_{0,\alpha+1}(\mathbb{C})$. *Then*
  (a) $\mathcal{Q}_{\alpha+1} \leq_{\mathbb{C}^{\alpha+1}} \pi^{\mathcal{U}}_{\beta,\alpha+1}(\mathcal{Q}_\beta)$, *and*
  (b) *if* $\alpha+1 \in D^{\mathcal{T}}$, *then* $\mathcal{Q}_{\alpha+1} <_{\mathbb{C}^{\alpha+1}} \pi^{\mathcal{U}}_{\beta,\alpha+1}(\mathcal{Q}_\beta)$.
(8) *If* $\Lambda$ *is a limit ordinal then* $\pi^{\mathcal{U}}_{\alpha,\lambda}(\mathcal{Q}_\alpha) = \mathcal{Q}_\lambda$ *for all sufficiently large* $\alpha T \lambda$.

The tree $\mathcal{U}$ is called the lift up tree of $\mathcal{T}$ and is denoted by $l\mathcal{T}$. Notice that the strategy given by Theorem 2.7 doesn't necessarily give that $\mathcal{N}_\beta$ is iterable as $\Sigma$-mouse. However, this is an easy consequence of hull condensation and follows from Lemma 1.33.

LEMMA 2.8. *Suppose* $\theta$, $\delta$, $\mathcal{M}$, $\Sigma$, $\Lambda$ *and* $\Psi$ *are as in Theorem* 2.7. *Then* $\Psi$ *is a* $\Sigma$*-strategy.*

Next we show that the strategy of fully backgrounded construction inherits hull condensation.

LEMMA 2.9 (Induced strategies inherit hull condensation). *Suppose* $\theta$, $\delta$, $\Sigma$, $\Lambda$ *and* $\Psi$ *are as in Theorem* 2.7. *Suppose* $\Lambda$ *has hull condensation. Then* $\Psi$ *has hull condensation.*

PROOF. For simplicity, we verify hull condensation only for normal trees and we assume that our fully backgrounded construction is relative to $\emptyset$, i.e., the construction produces an ordinary mouse. Let $\mathcal{T}$ and $\mathcal{U}$ be two trees on $\mathcal{N}$ such that $\mathcal{T}$ is according to $\Psi$ and let $\tau : lh(\alpha) \to lh(\beta)$ witness that $\mathcal{U}$ is a hull of $\mathcal{T}$ and let $(\tau_\alpha : \alpha < lh(\mathcal{U}))$ play the role of $\pi_\alpha$'s in Definition 1.29. Let $l\mathcal{T}$ and $l\mathcal{U}$ be the lift-up trees to $V_\theta$. From the lift-up procedure we get sequences

$$(\mathcal{R}^{\mathcal{T}}_\alpha, \mathcal{Q}^{\mathcal{T}}_\alpha, \pi^{\mathcal{T}}_\alpha, \sigma^{\mathcal{T}}_\alpha, Res^{\mathcal{T}}_\alpha, \eta^{\mathcal{T}}_\alpha, F^{\mathcal{T}}_\alpha, \mathcal{S}^{\mathcal{T}}_{\alpha+1} : \alpha < lh(\mathcal{T}))$$

and

$$(\mathcal{R}^{\mathcal{U}}_\alpha, \mathcal{Q}^{\mathcal{U}}_\alpha, \pi^{\mathcal{U}}_\alpha, \sigma^{\mathcal{U}}_\alpha, Res^{\mathcal{U}}_\alpha, \eta^{\mathcal{U}}_\alpha, F^{\mathcal{U}}_\alpha, \mathcal{S}^{\mathcal{U}}_{\alpha+1} : \alpha < lh(\mathcal{T}))$$

such that (we only list the properties of the $\mathcal{T}$ sequence)

(1) $\mathcal{R}^{\mathcal{T}}_\alpha$ is the $\alpha$th model of $l\mathcal{T}$.
(2) $\pi^{\mathcal{T}}_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{Q}^{\mathcal{T}}_\alpha$
(3) $F^{\mathcal{T}}_\alpha$ is the resurrection of $\pi_\alpha(E^{\mathcal{T}}_\alpha)$.
(4) $(\sigma^{\mathcal{T}}_\alpha, Res^{\mathcal{T}}_\alpha)$ is the complete resurrection of $\pi^{\mathcal{T}}_\alpha(E^{\mathcal{T}}_\alpha)$
(5) If $\beta = pred^{\mathcal{T}}\alpha$ then $\mathcal{S}^{\mathcal{T}}_{\alpha+1}$ is either the $i$th partial resurrection of $\pi^{\mathcal{T}}_\beta(E^{\mathcal{T}}_\beta)$ and $i$ depends only $\mathcal{Q}_\beta$ and $\pi^{\mathcal{T}}_\beta(E^{\mathcal{T}}_\beta)$, or $\mathcal{Q}^{\mathcal{T}}_\beta$. If $\mathcal{S}^{\mathcal{T}}_{\alpha+1}$ is the $i$th resurrection sequence then $\eta^{\mathcal{T}}_\alpha$ is the $i$th map of this resurrection, and otherwise it is identity.
(6) $\pi^{\mathcal{T}}_{\alpha+1}([a,f]_{E^{\mathcal{T}}_\alpha}) = [\sigma^{\mathcal{T}}_\alpha(\pi^{\mathcal{T}}_\alpha(a)), \eta^{\mathcal{T}}_\alpha(\pi^{\mathcal{T}}_\beta(f))]_{F^{\mathcal{T}}_\alpha}$

*Claim.* $l\mathcal{U}$ is a hull of $l\mathcal{T}$.

PROOF. We define maps $(i_\alpha : \alpha < lh(\mathcal{U}))$ such that $i_\alpha : \mathcal{R}^{\mathcal{U}}_\alpha \to \mathcal{R}^{\mathcal{T}}_{\tau(\alpha)}$, $i_\alpha(\mathcal{Q}_\alpha) = \mathcal{Q}_{\tau(\alpha)}$, and $i_\alpha \circ \pi^{\mathcal{U}}_\alpha = \pi^{\mathcal{T}}_{\sigma(a)} \circ \tau_\alpha$. We let $i_0 = \pi^{l\mathcal{T}}_{0,\sigma(0)}$. Suppose we have defined $i_\beta$ for every $\beta \leq \alpha$ and those embeddings have the desired properties. We want to define

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$i_{\alpha+1}$. Let $\beta = pred^{\mathcal{U}}\alpha$. We let $i_{\alpha+1}([a,f]_{F^{\mathcal{U}}_\alpha}) = [i_\alpha(a), i_\beta(f)]_{i_\alpha(F^{\mathcal{U}}_\alpha)}$. This makes sense, because $\sigma(\beta) = pred^{\mathcal{T}}\sigma(\alpha)$. We need to see that $i_{\alpha+1}\circ\pi^{\mathcal{U}}_{\alpha+1} = \pi^{\mathcal{T}}_{\tau(\alpha)+1}\circ\tau_{\alpha+1}$. We have

$$\begin{aligned}
i_{\alpha+1}(\pi^{\mathcal{U}}_{\alpha+1}([a,f]_{E^{\mathcal{U}}_\alpha})) &= i_{\alpha+1}([\sigma^{\mathcal{U}}_\alpha(\pi^{\mathcal{U}}_\alpha(a)), \eta^{\mathcal{U}}_\alpha(\pi^{\mathcal{U}}_\beta(f))]_{F^{\mathcal{U}}_\alpha}) \\
&= [i_\alpha(\sigma^{\mathcal{U}}_\alpha(\pi^{\mathcal{U}}_\alpha(a))), i_\beta(\eta^{\mathcal{U}}_\alpha(\pi^{\mathcal{U}}_\beta(f)))]_{i_\alpha(F^{\mathcal{U}}_\alpha)} \\
&= [i_\alpha(\sigma^{\mathcal{U}}_\alpha)(i_\alpha(\pi^{\mathcal{U}}_\alpha(a)), i_\beta(\eta^{\mathcal{U}}_\alpha)(i_\beta(\pi^{\mathcal{U}}_\beta(f)))]_{i_\alpha(F^{\mathcal{U}}_\alpha)} \\
&= [\sigma^{\mathcal{T}}_{\tau(\alpha)}(\pi^{\mathcal{T}}_{\tau(\alpha)}(\tau_\alpha(a))), \eta^{\mathcal{T}}_{\tau(\beta)}(\pi^{\mathcal{T}}_{\tau(\beta)}(\tau_\beta(f)))]_{F^{\mathcal{T}}_{\tau(\alpha)}} \\
&= \pi^{\mathcal{T}}_{\tau(\alpha+1)}([\tau_\alpha(a), \tau_\beta(f)]_{E^{\mathcal{T}}_{\tau(\alpha)}}) \\
&= \pi^{\mathcal{T}}_{\tau(\alpha+1)}(\tau_{\alpha+1}([a,f]_{E^{\mathcal{U}}_\alpha})).
\end{aligned}$$

The construction doesn't fail at limit stages because of the hull condensation of $\Lambda$. □

□

As we mentioned before, our method of comparing two hod pairs is to show that the they both iterate to the one constructed by a hod pair construction. That this is indeed true is a consequence of universality properties of the backgrounded constructions. We now review some of the basic universality properties that backgrounded constructions have.

### 2.3. Universality of the fully backgrounded constructions

Universality of the fully backgrounded constructions refers to the fact that in many situations fully backgrounded constructions when compared with mice, win the comparison process. We start with a weaker version of such universality, which unlike the full version, doesn't assume the iterability of the backgrounded constructions. The weak version essentially states that fully backgrounded constructions are "stationary", i.e., do not move, when compared with "small" mice. Both versions are essentially due to Steel but they have never been published before. Below and throughout this paper, if $\Sigma$ is a strategy then we let $M_\Sigma$ be the structure it iterates.

LEMMA 2.10. *Suppose $\delta$ is an inaccessible cardinal and $\Sigma$ is a $\delta+1$-iteration strategy with $M_\Sigma \in V_\delta$. Suppose that $\mathcal{J}^{\vec{E},\Sigma}$-construction of $V_\delta$ converges and let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \delta), (F_\gamma : \gamma < \delta))$ be the output of this construction. Suppose further that $\mathcal{M}$ is a $\Sigma$-mouse, $\Lambda$ is a $\delta+1$-iteration strategy for $\mathcal{M}$ and if $\kappa < \delta$ is the least measurable cardinal of $\mathcal{N}_\delta$ then $\mathcal{M} \in H_\kappa$. Assume further that for every $\gamma < \delta$ if $F^*$ is the background extender giving rise to $F_\gamma$ then $F^*$ coheres $\Lambda$. Suppose also that $\mathcal{T}$ is a tree on $\mathcal{M}$ according to $\Lambda$ with last model $\mathcal{Q}$ such that for some $\xi \leq \eta$, $\mathcal{M}_\xi \trianglelefteq \mathcal{Q}$ and $F_\xi \neq \emptyset$. Then either $\mathcal{Q} = \mathcal{M}_\xi$ or $F_\xi$ is on the sequence of $\mathcal{Q}$.*

PROOF. We only sketch the proof. Towards a contradiction suppose $F_\xi$ is not on the sequence of $\mathcal{Q}$. Let $F^*$ be the resurrection of $F_\xi$ and let $\lambda = crit(F^*)$. Let $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_\lambda$. First notice that because all the extenders used in the construction cohere $\Lambda$, we have that $\mathcal{T} = j_{E^*}(\mathcal{T}) \restriction lh(\mathcal{T})$ (here we also use the property of $F^*$ given by clause 2a of Definition 2.3). Let $j = \pi_{F^*} : V \to Ult(V, F^*)$. It is not hard to see using standard arguments that $j \restriction \mathcal{P} = \pi^{\mathcal{T}}_{\lambda, j(\lambda)}$[6]. It now follows that the first

[6]This follows from the fact that if $b$ is the branch of $\mathcal{T} \restriction \lambda$ then $b = j(b) \cap \lambda$.

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extender used in $j(b)$ is compatible with $F_\xi$. This then gives a contradiction as in the usual comparison argument (see Theorem 3.11 of [**43**]). □

LEMMA 2.11 (Stationarity of fully backgrounded constructions). *Assume the hypothesis of Lemma* 2.10. *Then either*

(1) *there is $\gamma \leq \delta$ such that there is a normal tree $\mathcal{T}$ on $\mathcal{M}$ according to $\Lambda$ with last model $\mathcal{Q}$ such that $\pi^{\mathcal{T}}$ exists and $\mathcal{Q} \trianglelefteq \mathcal{N}_\gamma$, or*
(2) *for every $\gamma \leq \delta$, $\mathcal{N}_\gamma$ doesn't move in the comparison with $\mathcal{M}$.*

PROOF. We only sketch the argument. It is enough to show that if $\mathcal{N}_\gamma$ moves in the comparison with $\mathcal{M}$ then it must be the case that for some $\xi < \gamma$ clause 1 above holds. Let $\gamma$ be the least such that $\mathcal{N}_\gamma$ moves in the comparison with $\mathcal{M}$. Let $\mathcal{T}$ be the tree on $\mathcal{M}$ according to $\Lambda$ that has been constructed via the comparison process up to the stage when $\mathcal{N}_\gamma$ side has to move. Let $\mathcal{Q}$ be the last model of $\mathcal{T}$. Let $F$ be the extender on $\mathcal{N}_\gamma$ such that $F \notin \vec{E}^{\mathcal{Q}}$. We assume, towards a contradiction, that for every $\xi < \gamma$ clause 1 fails to hold for $\xi$. Notice that it follows from Lemma 2.10, for every $\xi \leq \gamma$, $F \neq F_\xi$. It follows that $F$ is obtained from a core of some $\mathcal{N}_\xi$ for $\xi < \gamma$. More precisely, there is some $\xi < \gamma$ such that $F \in \vec{E}^{\mathcal{C}(\mathcal{N}_\xi)}$.

Because $\mathcal{N}_\xi$ doesn't move in the comparison with $\mathcal{M}$ and because clause 1 fails for $\xi$, we can let $\mathcal{U}$ be a tree on $\mathcal{M}$ according to $\Lambda$ with last model $\mathcal{R}$ such that $\mathcal{N}_\xi \trianglelefteq \mathcal{R}$. But because $\mathcal{N}_\xi$ isn't sound it follows that $\mathcal{N}_\xi = \mathcal{R}$. Because clause 1 fails for $\xi$ we must have that for some $\alpha < lh(\mathcal{U})$, $\mathcal{C}(\mathcal{R}) = \mathcal{C}(\mathcal{N}_\xi) = (\mathcal{M}^*_\alpha)^{\mathcal{U}}$ (see Theorem 3.8 of [**43**]). Thus $F \in \vec{E}^{(\mathcal{M}^*_\alpha)^{\mathcal{U}}}$. Because both $\mathcal{T}$ and $\mathcal{U}$ are constructed via comparison process and because $F \in \vec{E}^{(\mathcal{M}^*_\alpha)^{\mathcal{U}}}$, we have that $\mathcal{T} = \mathcal{U} \restriction \alpha + 1$. This is then a contradiction as $F$ cannot be a disagreement between $\mathcal{N}_\gamma$ and $\mathcal{Q}$. □

Next we have the following useful lemma.

LEMMA 2.12 (Weak universality). *Suppose $\delta$ is a Woodin cardinal and $\Sigma$ is a $\delta+1$-iteration strategy with $M_\Sigma \in V_\delta$. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \delta), (F_\gamma : \gamma < \delta))$ be the output of $\mathcal{J}^{\vec{E},\Sigma}$-construction of $V_\delta$. Suppose $\mathcal{M}$ is a $\Sigma$-mouse, $\Lambda$ is a $\delta+1$-iteration strategy for $\mathcal{M}$ and if $\kappa$ is the first measurable of $\mathcal{N}$ then $\mathcal{M} \in V_\kappa$. Suppose no initial segment of $\mathcal{N}$ satisfies that there is a superstrong cardinal. Then $\mathcal{N}_\delta$ wins the coiteration with $\mathcal{M}$.*

The proof is like the proof of the next lemma. The proof we give here is Steel's adaptation (see [**42**]) of the universality proof due to Mitchell and Schindler (see [**19**]). We will need a somewhat different form of it which we will state and prove for convenience. The proof, however, is word by word the same as the proof of Lemma 11.1 of [**42**]. To see that the proof of the next lemma can be used to prove the weak universality notice that if $\mathcal{M}$ wins the comparison with $\mathcal{N}$ then because of Lemma 2.11, there is a tree $\mathcal{T}$ on $\mathcal{M}$ with last model $\mathcal{R}$ such that $\mathcal{N} \trianglelefteq \mathcal{R}$. This is then enough for applying the proof of the next lemma.

LEMMA 2.13 (Universality). *Suppose $\delta$ is a Woodin cardinal and $\Sigma$ is a $\delta+1$-iteration strategy with $M_\Sigma \in V_\delta$. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \delta), (F_\gamma : \gamma < \delta))$ be the output of $\mathcal{J}^{\vec{E},\Sigma}$-construction of $V_\delta$ and set $\mathcal{N} = \mathcal{N}_\delta$. Suppose that for some $\kappa < \delta$ and $V$-generic $g \subseteq Coll(\omega, \kappa)$ there is $\mathcal{M}, \Lambda, E \in V[g]$ such that*

(1) *$\mathcal{M} \in V_\delta[g]$ is a $\Sigma$-mouse,*

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(2) $\Lambda$ *is a* $\delta+1$*-iteration strategy for* $\mathcal{M}$,
(3) $E \in V_\delta[g]$ *is an* $\mathcal{N}$*-extender (perhaps long extender) such that* $\mathcal{N}_1 = Ult(\mathcal{N}, E)$ *is a* $\Sigma$*-mouse which is* $\delta+1$*-iterable for trees in* $L(V_\delta[g], \Lambda)$.

*Suppose, further, that no initial segment of* $\mathcal{N}$ *satisfies "there is a superstrong cardinal". Then* $\mathcal{N}_1$ *wins the coiteration with* $\mathcal{M}$.

Proof. Assume $\mathcal{M}$ iterates past $\mathcal{N}_1$. Let $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}_1$ be the trees coming from the coiteration process and $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}$ and $\mathcal{U}$. Then $\mathcal{Q} \trianglelefteq \mathcal{P}$. Let $i = \pi_E : \mathcal{N} \to \mathcal{N}_1$. We assume that the reader is familiar with the comparison argument (see Theorem 3.11 of [**43**]). Applied in our current situation, the comparison argument gives the following:

(1) $\delta = o(\mathcal{Q})$,
(2) There is some $\alpha$ and $\mu \in \mathcal{M}_\alpha^{\mathcal{T}}$ such that $\delta = \pi^{\mathcal{T}}_{\alpha,\mu}(\mu)$ and $\mu < \delta$.
(3) Letting $\kappa_\beta = \pi^{\mathcal{T}}_{\alpha,\beta}(\mu)$, there is a club $C \subseteq [0,\delta]_T \cap [0,\delta]_U$ such that
$$\beta \in C \to \beta = \kappa_\beta \wedge (\pi^{\mathcal{U}}_{0,\delta} \circ i)[\beta] \subseteq \beta.$$

Let $f(\gamma)$=least inaccessible $\beta > \gamma$ such that $\beta \in C$. Using Woodiness of $\delta$, we can find a limit point $\kappa$ of $C$ such that $E \in V_\kappa[g]$ and a $j : V[g] \to M$ such that

$$crit(j) = \kappa,\ V_{j(f)(\kappa)} \in M \text{ and } j(\vec{E}^{\mathcal{N}}) \restriction j(f)(\kappa) = \vec{E}^{\mathcal{N}} \restriction j(f)(\kappa) \qquad (*).$$

Let $F^*$ be the $j(f)(\kappa)$-extender derived from $j$. Then $F = F^* \cap \mathcal{N} \in \mathcal{N}$ (for details see the proof of Theorem 11.3 of [**20**]). We claim that some initial segment of $F$ witnesses that $\kappa$ is superstrong in $\mathcal{N}$. To show this, it is enough to show that for all $g \in \mathcal{N}$ such that $g : \kappa \to \kappa$, $\pi_F(g)(\kappa) < j(f)(\kappa)$. Given this, it then follows that $F \restriction \sup\{\pi_F^{\mathcal{N}}(g)(\kappa) : g \in \mathcal{N}\}$ is as desired[7]. Let then $g \in \mathcal{N}$. It is enough to show that for all sufficiently large $\gamma < \kappa$, $g(\gamma) < f(\gamma)$.

We have that $i(\kappa) = \kappa$, and therefore, $\pi^{\mathcal{U}}_{0,\kappa} \circ i(g) : \kappa \to \kappa$. Since $\mathcal{M}$ iterates past $\mathcal{N}_1$, we can find $\eta < \kappa$ and $h \in \mathcal{M}_\eta^{\mathcal{T}}$ such that $h : \kappa_\eta \to \kappa_\eta$, $E \in V_{\kappa_\eta}[g]$ and $\pi^{\mathcal{U}}_{0,\kappa}(i(g)) = \pi^{\mathcal{T}}_{\eta,\kappa}(h)$. Now let $\kappa_\eta < \gamma < \kappa$. Then,

$$g(\gamma) \leq \pi^{\mathcal{U}}_{0,\kappa}(i(g(\gamma)) = \pi^{\mathcal{U}}_{0,\kappa}(i(g))(\pi^{\mathcal{U}}_{0,\kappa}(i(\gamma))) = \pi^{\mathcal{T}}_{\eta,\kappa}(h)(\pi^{\mathcal{U}}_{0,\kappa}(i(\gamma))) < f(\gamma).$$

The last inequality holds because if $\beta = f(\gamma)$ then $\beta \in C$, so $\pi^{\mathcal{U}}_{0,\kappa}(i(\gamma)) < \beta$ and $\pi^{\mathcal{T}}_{\eta,\kappa}(h) \restriction \beta = \pi^{\mathcal{T}}_{\eta,\beta}(h) : \beta \to \beta$. This completes the proof.

□

There is yet another way that fully backgrounded constructions are universal. They absorb strategies with *branch condensation*. We will use this phenomenon to compare hod mice (see Theorem 2.28).

Definition 2.14 (Branch condensation). *An iteration strategy* $\Sigma$ *has branch condensation (see Figure* 2.3.1*) if for any two stacks* $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *on* $M_\Sigma$ *and branch* $c$ *of* $\vec{\mathcal{U}}$ *if*

(1) $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *are according to* $\Sigma$, $lh(\vec{\mathcal{U}}) = \gamma+1$ *and* $lh(\mathcal{U}_\gamma)$ *is limit,*
(2) $\vec{\mathcal{T}}$ *has last model* $N$ *such that* $(\vec{\mathcal{T}}, N) \in I(\Sigma)$,
(3) $\pi^{\vec{\mathcal{U}}}_c$*-exists and for some* $\pi : \mathcal{M}^{\vec{\mathcal{U}}}_c \to_{\Sigma_1} N$,
$$\pi^{\vec{\mathcal{T}}} = \pi \circ \pi^{\vec{\mathcal{U}}}_c$$

[7]Let $H = F \restriction \sup\{\pi_F^{\mathcal{N}}(g)(\kappa) : g \in \mathcal{N}\}$. Then it is not hard to see that $\pi_H^{\mathcal{N}}(\kappa) = \sup\{\pi_F^{\mathcal{N}}(g)(\kappa) : g \in \mathcal{N}\}$ implying that, using (*), $Ult(\mathcal{N}, H)|\pi_H^{\mathcal{N}}(\kappa) = \mathcal{N}|\pi_H^{\mathcal{N}}(\kappa)$.

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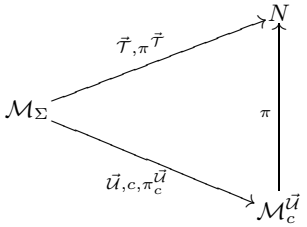


FIGURE 2.3.1. Branch condensation

*then* $c = \Sigma(\vec{\mathcal{U}})$.

LEMMA 2.15 (Absorbing strategies with branch condensation). *Suppose* $\delta$ *is an inaccessible cardinal and* $\Psi$ *is a* $(\delta, \delta+1)$*-iteration strategy for* $V_\delta$ *that acts on stacks that are in* $L_1(V_\delta)$*. Let* $\Sigma$ *be a* $\delta+1$*-iteration strategy with* $M_\Sigma \in V_\delta$*. Let* $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \delta), (F_\gamma : \gamma < \delta))$ *be the output of* $\mathcal{J}^{\vec{E},\Sigma}$*-construction of* $V_\delta$ *and set* $\mathcal{N} = \mathcal{N}_\delta$*. Suppose* $(\mathcal{P}, \Lambda)$ *is such that* $\mathcal{P}$ *is a* $\Sigma$*-mouse,* $\Lambda$ *is a* $(\delta, \delta+1)$*-iteration strategy for* $\mathcal{P}$ *with branch condensation and if* $\kappa$ *is the first measurable of* $\mathcal{N}$ *then* $\mathcal{P} \in V_\kappa$*. Then whenever* $(\vec{\mathcal{T}}, \mathcal{Q}, \xi, \Phi, \vec{\mathcal{U}}, b)$ *is such that*

(1) $\vec{\mathcal{T}}$ *is a stack on* $\mathcal{P}$ *according to* $\Lambda$ *with last model* $\mathcal{Q}$ *such that* $\pi^{\vec{\mathcal{T}}}$ *exists,*
(2) $\mathcal{N}|\xi \trianglelefteq \mathcal{Q}$ *and* $H^{\mathcal{Q}}_\xi = \mathcal{N}|\xi$*,*
(3) $\Phi$ *is the strategy of* $\mathcal{N}|\xi$ *induced from* $\Psi$*,*
(4) $\vec{\mathcal{U}}$ *is a stack on* $\mathcal{N}|\xi$ *according to both* $\Phi$ *and* $\Lambda$ *such that the last normal component of* $\vec{\mathcal{U}}$ *has limit length,*
(5) $b = \Phi(\vec{\mathcal{U}})$ *and* $\pi^{\vec{\mathcal{U}}}_b$ *exists,*

$b = \Lambda(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

PROOF. Let $i = \pi^{\vec{\mathcal{T}}} : \mathcal{P} \to \mathcal{Q}$ be the iteration embedding given by $\vec{\mathcal{T}}$. Let $\pi : V \to N$ be the result of lifting $\vec{\mathcal{U}}^\frown\mathcal{M}^{\vec{\mathcal{U}}}_b$ to the background universe. Let $\mathcal{R} = \mathcal{M}^{\vec{\mathcal{U}}}_b$. Then there is $\sigma : \mathcal{R} \to \pi(\mathcal{N}|\xi)$ such that $\pi \restriction \mathcal{N}|\xi = \sigma \circ \pi^{\vec{\mathcal{U}}}_b$. Let $\mathcal{Q}^*$ be the result of applying $\vec{\mathcal{U}}$ to $\mathcal{Q}$ and let $j : \mathcal{Q} \to \mathcal{Q}^*$ be the iteration embedding. $\sigma$ can then be lifted to act on $\mathcal{Q}^*$: let $\sigma^* : \mathcal{Q}^* \to \pi(\mathcal{Q})$ be given by

$$\sigma^*(j(f)(a)) = \pi(f)(\sigma(a))$$

where $f \in \mathcal{Q}$ and $a \in \delta(\vec{\mathcal{U}})$. Next, notice that

$$\begin{aligned} \pi^{\pi(\vec{\mathcal{T}})} &= \pi(i) \\ &= \pi \circ i \\ &= \sigma^* \circ j \circ i \\ &= \sigma^* \circ \pi^{\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}}_b. \end{aligned}$$

The above equalities imply that $\pi^{\pi(\vec{\mathcal{T}})} = \sigma^* \circ \pi^{\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}}_b$. It then follows that we can apply branch condensation of $\Lambda$ to $\pi(\vec{\mathcal{T}})$, $\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}$, $\sigma^*$ and $b$ and conclude that $b = \Lambda(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$. □

The following lemma, which is a direct consequence of Lemma 2.15, is key to the comparison argument of this paper (see Theorem 2.28).

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LEMMA 2.16. *Suppose $(M, \delta, \Sigma)$ is a weak background triple. Suppose $(\mathcal{P}, \Lambda)$ is a hod pair such that $\mathcal{P} \in V_\delta^M$, $\Lambda \restriction L_1(V_\delta^M) \in M$, $\Lambda$ has branch condensation and whenever $i : M \to N$ is an iteration embedding via $\Sigma$ in $M$, $i(\Lambda \restriction L_1(V_\delta^M)) = \Lambda \restriction (L_1(V_{i(\delta)}))^N$. Let $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta : \beta \leq \Omega)$ be the output of the hod pair construction of $M$. Suppose there is a stack $\vec{\mathcal{T}} \in M$ on $\mathcal{P}$ according to $\Lambda$ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and for some $\beta \leq \Omega$, $\mathcal{P}_\beta \trianglelefteq_{hod} \mathcal{Q}$. Suppose further that $\vec{\mathcal{S}} \in V_\delta^M$ is a stack on $\mathcal{P}_\beta$ according to $\Sigma_\beta$ with last model $\mathcal{R}$ and such that $\pi^{\vec{\mathcal{S}}}$ exists. Then $\vec{\mathcal{S}}$ is according to $\Lambda_{\mathcal{P}_\beta, \vec{\mathcal{T}}}$.*

To implement our strategy of comparing hod pairs with fully backgrounded constructions we need to find weak background triples which capture the strategy of hod pair the way $M$ captures $\Lambda$ in Lemma 2.16. Under $AD^+$ there are many such weak background triples.

## 2.4. Coarse $\Gamma$-Woodin mice

Here we introduce the background triples that we will use in the proof of the comparison theorem (Theorem 2.28). A crucial result that we will use many times is the condensation of $C_\Gamma(x)$ operator. Before we state it we need to introduce few more useful concepts. First is Suslin capturing.

DEFINITION 2.17 (Suslin Capturing). *Suppose $(N, \Sigma)$ is such that $N \vDash ZFC - Replacement$ and $\Sigma$ is an $(\omega_1, \omega_1)$-iteration strategy or just $\omega_1$-iteration strategy for $N$. Suppose that $\delta$ is countable in $V$ but is an uncountable cardinal of $N$ and suppose that $T \in N$ is a tree on $\delta^N$. Suppose $A \subseteq \mathbb{R}$. We say $T$ locally Suslin captures $A$ over $N$ if for any $\alpha < \delta$ and for any $N$-generic $g \subseteq Coll(\omega, \alpha)$, $A \cap N[g] = p[T]^{N[g]}$. We also say that $N$ locally captures $A$ at $\delta$. We say that $N$ locally captures $A$ if $N$ locally captures $A$ at any uncountable cardinal of $N$. We say $N$ locally Suslin, co-Suslin captures $A$ (at $\delta$) if $N$ locally Suslin captures $A$ and $A^c$ (at $\delta$). We say $(N, \Sigma)$ Suslin captures $A$ at $\delta$, or $(N, \delta, \Sigma)$ Suslin captures $A$, if there is a tree $T \in N$ on $\delta$ such that whenever $i : N \to M$ comes from an iteration via $\Sigma$, $i(T)$ locally Suslin captures $A$ over $M$ at $i(\delta)$. In this case we also say that $(N, \delta, \Sigma, T)$ Suslin captures $A$. We say $(\mathcal{N}, \Sigma)$ Suslin captures $A$ if for every countable $\delta$ which is an uncountable cardinal of $N$, $(\mathcal{N}, \Sigma)$ Suslin captures $A$ at $\delta$. We say $(\mathcal{N}, \Sigma)$ Suslin, co-Suslin captures $A$ if $(N, \Sigma)$ Suslin captures $A$ and $A^c$.*

A weaker notion of capturing is via term relations.

DEFINITION 2.18 (Term Capturing). *Suppose $A$, $N$, $\delta$ and $\Sigma$ are as in Definition 2.17. Suppose $\tau \in N^{Coll(\omega, \delta)}$ is a term for a set of reals. Then we say $\tau$ locally term captures $A$ at $\delta$ if whenever $g \subseteq Coll(\omega, \delta)$ is $N$-generic, $\tau_g = A \cap N[g]$. $(\mathcal{N}, \delta, \Sigma)$ term captures $A$ if there is a term $\tau \in N^{Coll(\omega, \delta)}$ such that whenever $i : N \to M$ comes from an iteration according to $\Sigma$, $i(\tau)$ locally term captures $A$ at $i(\delta)$. In this case we also say $(N, \delta, \Sigma, \tau)$ term captures $A$. $(N, \Sigma)$ term captures $A$ if for every countable $\delta$ which is an uncountable cardinal of $N$, $(\mathcal{N}, \Sigma)$ term captures $A$ at $\delta$.*

Both, Suslin capturing and term capturing are more interesting when there is a Woodin cardinal in $N$. This is because of Woodin's genericity iterations (see Theorem 7.14 of [**43**]). Term capturing is weaker than Suslin capturing. To see this let $\mathcal{M} = \mathcal{M}_1^\#$ the minimal active mouse with a Woodin cardinal and let $\Sigma$ be its

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canonical strategy. Let $\delta$ be the Woodin cardinal of $\mathcal{M}$. Let $A = \{x \in \mathbb{R} : L[x] \vDash$ "$x$ codes a $\Pi^1_2$-iterable active premouse with a Woodin cardinal$\}$. Here $\Pi^1_2$-iterability is a weakening of iterability introduced by Steel in [**39**] in order to compute the complexity of the reals appearing in projective mice. It should be clear that $A$ is term captured by $(\mathcal{M}, \delta, \Sigma)$. However, $(\mathcal{M}, \delta, \Sigma)$ doesn't Suslin capture $A$. To see this, suppose that there is $T \in \mathcal{M}$ such that $(\mathcal{M}, \delta, \Sigma, T)$ Suslin captures $A$. Let $x \in A$ and let $\mathcal{N}$ be an iterate of $\mathcal{M}$ via $\Sigma$ such that $x$ is generic for the extender algebra of $\mathcal{N}$ at $i(\delta)$ where $i : \mathcal{M} \to \mathcal{N}$ is the iteration embedding given by $\Sigma$. Then $x \in p[i(T)]$. It follows by absoluteness that $\mathcal{N} \vDash p[i(T)] \neq \emptyset$. Hence, $\mathcal{M} \vDash p[T] \neq \emptyset$. Let then $x \in \mathcal{M}$ be such that $x \in (p[T])^{\mathcal{M}}$. It follows that $x \in A$. However, Steel showed in [**39**] that for $y \in \mathbb{R}$, $y \in \mathcal{M}$ if and only if $y$ is in any active $\Pi^1_2$-iterable premouse. It then follows that $x$ is in the premouse coded by $x$, which is clearly a contradiction.

When $(M, \tau)$ locally term captures $A$ at $\kappa$ there is a canonical term relation that captures $A$. We let $\tau^M_{A,\kappa}$ be the term defined by

$$\tau^M_{A,\kappa} = \{(p, \sigma) : p \in Coll(\omega, \kappa),\ \sigma \in M^{Coll(\omega,\kappa)} \text{ is a standard name for a real, and } p \Vdash^M \sigma \in \tau\}.$$

It is not hard to see that $(M, \tau^M_{A,\kappa})$ locally term captures $A$ at $\kappa$ and it is independent of $\tau$. Moreover, $\tau^M_{A,\kappa}$ is forcing-invariant, as for any $M$-generic $g, h \subseteq Coll(\omega, \kappa)$,

$$M[g] = M[h] \to (\tau^M_{A,\kappa})_g = (\tau^M_{A,\kappa})_h.$$

We can now state the condensation properties of $C_\Gamma(a)$. We will not give the proof of the lemma. It follows from the next lemma whose proof can be found in [**32**] and [**31**].

Lemma 2.19. *Assume AD and let $\Gamma$ be a good pointclass. Let $T$ be a tree obtained from a scale on a universal $\Gamma$-set, let $(\phi_i : i < \omega)$ be a semi-scale on $\neg p[T]$ and suppose $a$ is a transitive countable set. Suppose $\pi : M \to N$ is an elementary embedding such that $\{a, T\} \in N$ and $\{a, T\} \in rng(\pi)$. Suppose further that for some $\kappa \in rng(\pi)$ for every $n$, $A_n = \leq^*_{\phi_n}$ is locally term captured by $(N, \tau^N_{A_n,\kappa})$ and $\tau^N_{A_n,\kappa} \in N \cap rng(\pi)$. Then*

$$C_\Gamma(\pi^{-1}(a)) \in M \text{ and } \pi(C_\Gamma(\pi^{-1}(a))) = C_\Gamma(a).$$

We now describe a useful generalization of the lemma to arbitrary sets of reals $A$.

Lemma 2.20. *Suppose $A \subseteq \mathbb{R}$ and that $(\phi_i : i < \omega)$ is a semi-scale on $A$. Suppose further that $N$ is a countable transitive set such that for some $\kappa$, $(N, \tau^N_{A,\kappa})$ locally term captures $A$ at $\kappa$ and $(N, \tau^N_{A_n,\kappa})$ locally term captures $A_n = \leq^*_{\phi_n}$ at $\kappa$. Suppose further that $\pi : M \to N$ is an elementary embedding such that $\kappa \in rng(\pi)$, $\tau^N_{A,\kappa} \in rng(\pi)$ and $(\tau^N_{\leq_{\phi_n},\kappa} : i < \omega) \subseteq rng(\pi)$. Then*

$$\tau^M_{A,\pi^{-1}(\kappa)} \in M \text{ and } \pi(\tau^M_{A,\pi^{-1}(\kappa)}) = \tau^N_{A,\kappa}.$$

*and $\tau^M_{A,\pi^{-1}(\kappa)}$ locally term captures $A$ over $M$.*

As a corollary, we get the following condensation property of sjs.

Corollary 2.21. *Suppose $(A_i : i < \omega)$ is a sjs or ssjs. Suppose further that $N$ is a countable transitive set such that for some $\kappa$, $(N, \tau^N_{A_n,\kappa})$ locally term captures $A_n$ at $\kappa$ for all $n$. Let $\pi : M \to N$ be an elementary embedding such that $\kappa \in rng(\pi)$, and for all $n$, $\tau^M_{A_n,\kappa} \in rng(\pi)$. Then for every $n$,*

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$$\tau^M_{A_n,\pi^{-1}(\kappa)} \in M \text{ and } \pi(\tau^M_{A_n,\pi^{-1}(\kappa)}) = \tau^M_{A,\kappa}.$$

Next we introduce the notion of a coarse $\Gamma$-Woodin mouse. Woodin showed that under $AD^+$ any Suslin, co-Suslin set $A$ can be locally Suslin captured by some $N$ (see Lemma 2.23). It can then be shown that in fact given any Suslin, co-Suslin set $A$ there is $(N, \Sigma)$ which Suslin, co-Suslin captures $A$.

DEFINITION 2.22 (Coarse $\Gamma$-Woodin mouse). *Suppose $\Gamma$ is a good pointclass and $N \vDash ZFC - Replacement$. $N$ is a* coarse $\Gamma$-Woodin mouse *if letting $T$ be a tree for the universal $\Gamma$-set*

(1) *there is $\delta \in N$ such that $N \vDash$ "$\delta$ is the only Woodin cardinal" and $o(N) = \sup_{n<\omega}(\delta^{+\omega})^N$,*
(2) $(H_{\delta^{+\omega}})^{L[T,N]} = N$.

Suppose $N$ is a coarse $\Gamma$-Woodin mouse and $\Sigma$ is an iteration strategy for $N$. Then we say $\Sigma$ is $\Gamma$-fullness preserving if all $\Sigma$-iterates of $N$ are $\Gamma$-Woodin.

LEMMA 2.23 (Woodin, [**31**] and [**32**]). *Assume $AD^+$. Suppose $\Gamma$ is a good pointclasses and there is a good pointclass $\Gamma^*$ such that $\Gamma \subseteq \Delta_{\Gamma^*}$. Let $U \in \Gamma$ be the universal $\Gamma$-set. Then there is $(N, \Sigma)$ such that $N$ is a coarse $\Gamma$-Woodin mouse, $\Sigma$ is a $\Gamma$-fullness preserving iteration strategy for $N$ and $(N, \Sigma)$ Suslin, co-Suslin captures $U$.*

We are now ready to state the most important result of this section. If $\Gamma$ is a good pointclass and $(N, \Sigma)$ Suslin captures some universal $\Gamma$-set then we say that $(N, \Sigma)$ Suslin captures $\Gamma$. Similarly we define the expression "$(N, \Sigma)$ Suslin, co-Suslin captures $\Gamma$". When introducing $\Gamma$-*hod pair constructions* (see Definition 2.32) we will need background triples satisfying additional hypothesis which is introduced in the next definition.

Suppose $(M, \delta, \Sigma)$ is a background triple and $A \subseteq \mathbb{R}$. Suppose $\lambda < \omega_1$ is an $M$-cardinal and suppose $g \subseteq Coll(\omega, \lambda)$ is $M$-generic. Then we say $(M[g], \Sigma)$ Suslin captures $A$ if for any $M$-cardinal $\nu \in (\lambda, \omega_1)$ there is a tree $T \in M[g]$ on $\nu \times \omega$ such that whenever $N$ is a $\Sigma$-iterate of $M$ obtained via an iteration above $\Lambda$, $i : M \to N$ is the iteration embedding and $i^+ : M[g] \to N[g]$ is the extension of $i$ then for any $\eta < i(\nu)$ and for any $N[g]$-generic $h \subseteq Coll(\omega, \eta)$,

$$(p[i^+(T)])^{N[g*h]} = A \cap N[g * h].$$

We say $(M[g], \Sigma)$ Suslin, co-Suslin captures $A$ if $(M[g], \Sigma)$ Suslin, co-Suslin captures both $A$ and $A^c$.

DEFINITION 2.24 (Self-capturing background triples). *Suppose $(M, \delta, \Sigma)$ is a background triple. We say $(M, \delta, \Sigma)$ is self-capturing if for every $M$-inaccessible cardinal $\lambda < \delta$ there is a set $X \in M$ such that for any $M$-generic $g \subseteq Coll(\omega, \lambda)$ and for every $M[g]$-cardinal $\eta$ which is countable in $V$, $(M[g], \Sigma)$ Suslin, co-Suslin captures $Code(\Sigma_{V^M_\lambda})$ at $\eta$ as witnessed by a pair $(T, S) \in OD^{M[g]}_X$.*

THEOREM 2.25 (Woodin, Theorem 10.3 of [**42**]). *Assume $AD^+$. Suppose $\Gamma$ is a good pointclasses and there is a good pointclass $\Gamma^*$ such that $\Gamma \subseteq \Delta_{\Gamma^*}$. Suppose $(N, \Psi)$ Suslin, co-Suslin capture $\Gamma$. There is then a function $F$ defined on $\mathbb{R}$ such that for a Turing cone of $x$, $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$ such that*

(1) $N \in L_1[x]$,
(2) $\mathcal{N}^*_x|\delta_x = \mathcal{M}_x|\delta_x$,

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(3) $\mathcal{M}_x$ *is a* $\Psi$*-mouse: in fact,* $\mathcal{M}_x = \mathcal{M}_1^{\Psi,\#}(x)|\kappa_x$ *where* $\kappa_x$ *is the least inaccessible cardinal of* $\mathcal{M}_1^{\Psi,\#}$.[8],
(4) $\mathcal{N}_x^* \vDash$ "$\delta_x$ *is the only Woodin cardinal*",
(5) $\Sigma_x$ *is the unique iteration strategy of* $\mathcal{M}_x$,
(6) $\mathcal{N}_x^* = L(\mathcal{M}_x, \Lambda)$ *where* $\Lambda$ *is the restriction of* $\Sigma_x$ *to stacks* $\vec{\mathcal{T}} \in \mathcal{M}_x$ *that have finite length and are based on* $\mathcal{M}_x \restriction \delta_x$,
(7) $(\mathcal{N}_x^*, \Sigma_x)$ *Suslin, co-Suslin captures* $Code(\Psi)$ *and hence,* $(\mathcal{N}_x^*, \Sigma_x)$ *Suslin, co-Suslin captures* $\Gamma$,
(8) $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ *is a self-capturing background triple.*

Notice that the universality theorem on fully backgrounded constructions can be applied in $\mathcal{N}_x^*$ of Theorem 2.25 as $\mathcal{N}_x^*$ satisfies the hypothesis of Lemma 2.13. The triples of the form $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ will be our background triples that we will use to build hod pairs via hod pair constructions.

## 2.5. Comparison under $AD^+$

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Say $(\mathcal{R}, \Psi)$ is a tail of $(\mathcal{P}, \Sigma)$ if there is $\vec{\mathcal{T}}$ such that $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $\Psi = \Sigma_{\mathcal{R},\vec{\mathcal{T}}}$. The ideal comparison theorem would be that given two hod pairs $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ there is a common tail $(\mathcal{R}, \Psi)$. However, this kind of comparison can fail for hod pairs. The problem is that given two hod pairs $(\mathcal{M}, \Sigma)$ and $(\mathcal{N}, \Lambda)$, it could be that $\mathcal{M}(0) \lhd \mathcal{N}(0)$. If then $\lambda^{\mathcal{M}} \geq 1$, we can have no hope of comparing $\mathcal{M}$ and $\mathcal{N}$ in any reasonable way. In a sense, as we will later see, $(\mathcal{M}, \Sigma)$ and $(\mathcal{N}, \Lambda)$ correspond to different pointclasses. To avoid such anomalies, we introduce the notion of *fullness preservation. We remark that in this section, we assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ *and all our iteration strategies are* $(\omega_1, \omega_1)$*-iteration strategies.* To introduce fullness preservation, we need to introduce lower-part stack relative to $\Sigma$.

Definition 2.26. *Suppose* $\Sigma$ *is an iteration strategy with hull-condensation,* $a$ *is a transitive set such that* $\mathcal{M}_\Sigma \in a$ *and* $\Gamma$ *is a pointclass closed under boolean operations and continuous images and preimages. Then* $Lp_{\omega_1}^{\Gamma,\Sigma}(a) = \cup_{\alpha<\kappa} Lp_\alpha^{\Gamma,\Sigma}(a)$ *where*

(1) $Lp_0^{\Gamma,\Sigma}(a) = a \cup \{a\}$
(2) $Lp_{\alpha+1}^{\Gamma,\Sigma}(a) = \cup\{\mathcal{M} : \mathcal{M}$ *is a sound* $\Sigma_{\mathcal{Q}(\alpha),\mathcal{T}}$*-mouse over* $Lp_\alpha^{\Gamma,\Sigma}(a)$ *projecting to* $\eta$ *and having an iteration strategy in* $\Gamma\}$.
(3) $Lp_\lambda^{\Gamma,\Sigma}(a) = \cup_{\alpha<\lambda} Lp_\alpha^{\Gamma,\Sigma}(a)$.

We let $Lp^{\Gamma,\Sigma}(a) = Lp_1^{\Gamma,\Sigma}(a)$

Definition 2.27 ($\Gamma$-Fullness preservation). *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair and* $\Gamma$ *is a pointclass closed under boolean operations and continuous images and preimages. Then* $\Sigma$ *is a* $\Gamma$*-fullness preserving if whenever* $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ *and* $\eta > \delta_\alpha$ *is a strong cutpoint of* $\mathcal{Q}(\alpha + 1)$*, then*

$$\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)} = Lp^{\Gamma,\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}}(\mathcal{Q}|\eta).$$

*and*

$$\mathcal{Q}|(\delta_\alpha^+)^{\mathcal{Q}} = Lp^{\Gamma,\oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1),\vec{\mathcal{T}}}}(\mathcal{Q}(\alpha)).$$

[8] $\mathcal{M}_1^{\Psi,\#}$ is the minimal $\Psi$-mouse having a Woodin cardinal and a last extender.

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When $\Gamma = \mathcal{P}(\mathbb{R})$ we omit it from our notation. In particular, a strategy is fullness preserving if it is $\mathcal{P}(\mathbb{R})$-fullness preserving. First we state the general result and then show how to modify it to prove comparison for $\Gamma$-fullness preserving iteration strategies. In the statement of Theorem 2.28, $Code(\Sigma)$ is the set of reals coding $\Sigma$. More precisely, if $n < \omega$ and $f : HC^n \to HC$ is a function then we let $Code(f)$ be the set of reals coding $f$ under some standard way of coding countable sets with reals. More precisely, given a real $x$ which is a code of a countable set, we let $M_x$ be the structure coded by $x$ and let $\pi_x : M_x \to N_x$ be the transitive collapse of $M_x$. We let $WF$ be the set of reals which code countable sets. Suppose then for some $n < \omega$, $f : HC^n \to HC$ is a function. Then $Code(f)$ is the set of triples $(x, n, m) \in \mathbb{R} \times \omega \times \omega$ such that $x \in WF$, $\pi_x(n) \in dom(f)$ and $\pi_x(m) \in f(\pi_x(n))$. If $A \subseteq \mathbb{R} \times \omega \times \omega$ then we let $f_A$ be the function, if exists, such that $Code(f_A) = A$.

At the moment we need an additional hypothesis to prove that comparison holds. Later on we will remove the additional hypothesis (see Theorem 5.9 and Theorem 5.10).

THEOREM 2.28 (Comparison of hod pairs). *Suppose that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that both $\Sigma$ and $\Lambda$ have branch condensation and are fullness preserving. Suppose further that there is a good pointclass $\Gamma$ such that $Code(\Sigma)$, $Code(\Lambda) \in \Delta_{\underset{\sim}{\Gamma}}$[9]. Then comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$.*

PROOF. Let $\Gamma$ be a good pointclass pointclass such that $Code(\Sigma), Code(\Lambda) \in \Delta_{\underset{\sim}{\Gamma}}$. Let $F$ be as in Lemma 2.25 and let $x \in \mathbb{R}$ be such that $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Code(\Sigma)$ and $Code(\Lambda)$[10]. Let $(\mathcal{N}_\alpha, \mathcal{R}_\alpha, \Sigma_\alpha, \delta_\alpha : \alpha \leq \Omega)$ be the hod pair construction of $\mathcal{N}_x^*$ below $\delta_x$.

*Claim.* There is a tree $\mathcal{T}$ on $\mathcal{P}$ according to $\Sigma$ with last model $\mathcal{R}$ such that $\pi^{\mathcal{T}}$ exists and for some $\alpha$, $\mathcal{R} = \mathcal{R}_\alpha$ and $\Sigma_\alpha = \Sigma_{\mathcal{R},\mathcal{T}}$.

PROOF. $\mathcal{T}$ is the result of comparing $\mathcal{P}$ with $\mathcal{R}_\alpha$. Notice that from the point of view of $\mathcal{N}_x^*$, because $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $\Sigma$, $\mathcal{P}$ is countable and hence, our universality theorem, Theorem 2.13, applies. We show the existence of $\mathcal{T}$ by induction. The first step of the induction and the general successor step are very much alike and this is why we only do the general successor step. Notice that the limit steps easily follow from Theorem 2.13. To start then, assume that we have constructed, via the usual comparison process of hitting the minimal disagreements, a tree $\mathcal{T}_\alpha$ on $\mathcal{P}$ according to $\Sigma$ with last model $\mathcal{R}$ such that $\pi^{\mathcal{T}_\alpha}$ exists, $\mathcal{R}_\alpha = \mathcal{R}(\alpha)$ and $\alpha < \lambda^{\mathcal{R}}$. We would like to describe $\mathcal{T}_{\alpha+1}$. Notice that it follows from Lemma 2.16 that $\Sigma_\alpha = \Sigma_{\mathcal{R}(\alpha),\mathcal{T}_\alpha}$. First we need to show that $\mathcal{R}_{\alpha+1}$ exists. To show this, we need to show that (i) $\mathcal{N}_{\alpha+1}$ does not have an initial segment $\mathcal{M}$ such that $\rho(\mathcal{M}) < \delta_\alpha$ (ii) if $\alpha$ is a successor then $\mathcal{N}_{\alpha+1} \vDash$ "$\delta_\alpha$ is a Woodin cardinal" (iii) if $\alpha$ is limit then $(\delta_\alpha^+)^{\mathcal{P}_\alpha} = (\delta_\alpha^+)^{\mathcal{N}_{\alpha+1}}$ (iv) $\mathcal{N}_{\alpha+1}$ has Woodin cardinals above $\delta_\alpha$.

(i), (ii) and (iii) are easy consequences of fullness preservation. If any of them fails then there is a least $\Sigma_\alpha$-mouse $\mathcal{M}$ which witnesses this, and such an $\mathcal{M} \trianglelefteq \mathcal{R}$. This contradicts the fullness preservation of $\Sigma$. To show (iv), we use our universality theorem, Theorem 2.13. It follows from Theorem 2.13 and fullness preservation

[9]This is the hypothesis that we will later remove. See the comments before the statement of the theorem

[10]Here we can apply Lemma 2.25 because by standard results from the theory of scales if there is one such good pointclass $\Gamma$ then there are many such good pointclasses.

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of $\Sigma_{\alpha+1}$ that for some $\eta < \delta_{\alpha+1}$ such that $\mathcal{N}_{\alpha+1} \vDash$ "$\eta$ is Woodin" and letting $\gamma = (\eta^{+\omega})^{\mathcal{N}_{\alpha+1}}$, $\mathcal{N}_{\alpha+1}|\gamma$ is a $\Sigma_{\alpha+1}$-iterate of $\mathcal{R}(\alpha+1)$. Let $\mathcal{U}$ be the tree on $\mathcal{R}(\alpha+1)$ above $\delta_\alpha$ such that $(\mathcal{U}, \mathcal{N}_{\alpha+1}|\gamma) \in I(\mathcal{R}(\alpha+1), \Sigma_{\mathcal{R}(\alpha+1),\mathcal{T}_\alpha})$. It then follows that $\mathcal{R}_{\alpha+1}$ exists. Let $\mathcal{T}_{\alpha+1} = \mathcal{T}_\alpha{}^\frown\mathcal{U}$. It follows from Lemma 2.15 that $\Sigma_{\alpha+1} = \Sigma_{\mathcal{R}_{\alpha+1},\mathcal{T}_{\alpha+1}}$.

To finish, we need to show that there is $\alpha$ such that $(\mathcal{T}_\alpha, \mathcal{R}_\alpha) \in I(\mathcal{P}, \Sigma)$. Towards a contradiction assume that there is no such $\alpha$. It follows from the argument of the previous paragraph that $\Omega$ must be limit ordinal and hence, $\Omega = \delta_x$. Let $\mathcal{S}$ be the last model of $\mathcal{T}_\Omega$. It then follows that $\delta_x = \delta^{\mathcal{S}}_\Omega$. Because $\mathcal{S}$ doesn't have measurable limit of Woodin cardinals (recall that this is the minimality condition on our hod mice), $\delta_x$ cannot be a measurable cardinal in $\mathcal{S}$. It then follows that $\delta_x$ must be a singular cardinal of $\mathcal{N}^*_x$ because it must have a non-measurable preimage in the iteration leading to $\mathcal{S}$, which is a contradiction. □

To finish the proof of the theorem, let $\mathcal{T}$ on $\mathcal{P}$ be according to $\Sigma$ with last model $\mathcal{R}$ such that $\pi^{\mathcal{T}}$ exists and for some $\alpha$, $\mathcal{R} = \mathcal{R}_\alpha$ and $\Sigma_{\mathcal{R},\mathcal{T}} = \Sigma_\alpha$. We can repeat the construction and get $\mathcal{U}$ on $\mathcal{Q}$ according to $\Lambda$ with last model $\mathcal{S}$ such that $\pi^{\mathcal{U}}$ exists and for some $\beta$, $\mathcal{S} = \mathcal{R}_\beta$ and $\Lambda_{\mathcal{S},\mathcal{U}} = \Sigma_\beta$. Because $\Sigma_\alpha$ and $\Sigma_\beta$ are compatible, this finishes our proof. □

Unfortunately, we will need a somewhat local form of this comparison argument. When we construct hod mice there will be a step where we will need to know how to compare hod mice whose strategies are not fullness preserving but are $\Gamma$-fullness preserving with respect to some pointclass. The proof of this general comparison theorem is just like Theorem 2.28. The only missing ingredient is that we do not yet have the notion of hod pair constructions that produce $\Gamma$-full hod premice. Below we introduce such constructions.

Fix a pointclass $\Gamma$ which is closed under continuous preimages. Below if $\Lambda$ is an iteration strategy then we let $M_\Lambda$ be the structure that $\Lambda$ iterates. Let

$$HP^\Gamma = \{(\mathcal{P}, \Lambda) : (\mathcal{P}, \Lambda) \text{ is a hod pair and } Code(\Lambda) \in \Gamma\}$$

and

$$Mice^\Gamma = \{(a, \Lambda, \mathcal{M}) : a \in HC,\ a \text{ is self-wellordered transitive set, } \Lambda \text{ is an iteration strategy such that } (M_\Lambda, \Lambda) \in HP^\Gamma,\ M_\Lambda \in a, \text{ and } \mathcal{M} \trianglelefteq Lp^{\Gamma,\Lambda}(a)\}.$$

Suppose $(\mathcal{P}, \Sigma) \in HP^\Gamma$. We then let

$$Mice^\Gamma_\Sigma = \{(a, \mathcal{M}) : (a, \Sigma, \mathcal{M}) \in Mice^\Gamma\}.$$

Suppose now $A \subseteq \mathbb{R}$ is such that $w(A) \geq w(\Gamma)$. We let $A_\Gamma$ be the set of reals $\sigma$ that code a pair $(\sigma_0, \sigma_1)$ of continuous functions such that $\sigma_0^{-1}[A]$ is a code for some $(\mathcal{P}, \Lambda) \in HP^\Gamma$ and $\sigma_1^{-1}[A]$ is a code for some $(a, \mathcal{M}, \Psi)$ such that $(a, \Lambda, \mathcal{M}) \in Mice^\Gamma$ and $\Psi$ is the unique strategy of $\mathcal{M}$. If $\Gamma = \mathcal{P}(\mathbb{R})$ then we let $HP = HP^\Gamma$ and $Mice = Mice^\Gamma$.

We will introduce the notion of $\Gamma$-hod pair construction while working inside self-capturing background triples. Suppose now that $(M, \delta, \Sigma)$ is a self-capturing background triple such that $(M, \Sigma)$ Suslin, co-Suslin captures the pair $(A_\Gamma, A)$. Suppose $B \subseteq \mathbb{R}$ and suppose $\lambda < \delta$ is an $M$-inaccessible cardinal such that whenever $g \subseteq Coll(\omega, \lambda)$, $(M[g], \Sigma)$ Suslin, co-Suslin captures $B$. We then write $(M, \lambda, \Sigma) \vDash B \in \Gamma$ if whenever $g \subseteq Coll(\omega, \lambda)$ is $M$-generic, there is $\sigma \in M[g] \cap A_\Gamma$ such that if $h$ is $M[g]$-generic and $(\sigma_0, \sigma_1)$ is the pair coded by $\sigma$ then, letting $B^* = B \cap M[g * h]$ and $A^* = A \cap M[g * h]$, $M[g * h] \vDash B^* = \sigma_0^{-1}[A^*]$. Notice that the definition of

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$(M, \lambda, \Sigma) \vDash B \in \Gamma$ depends on the pair $(A_\Gamma, A)$. In our exposition the pair $(A_\Gamma, A)$ will always be clear. The following lemma isn't difficult to show.

LEMMA 2.29. *Suppose $(M, \delta, \Sigma)$ is a self-capturing background triple such that $(M, \Sigma)$ Suslin, co-Suslin captures the pair $(A_\Gamma, A)$. Suppose $B \subseteq \mathbb{R}$ and suppose $\lambda < \delta$ is an $M$-inaccessible cardinal such that whenever $g \subseteq Coll(\omega, \lambda)$, $(M[g], \Sigma)$ Suslin, co-Suslin captures $B$. Then $(M, \lambda, \Sigma) \vDash B \in \Gamma$ if and only if $B \in HP^\Gamma$.*

PROOF. Suppose first that $(M, \lambda, \Sigma) \vDash B \in \Gamma$. Then easy genericity argument shows that $B \in HP^\Gamma$. Indeed, let $g \subseteq Coll(\omega, \lambda)$ be $M$-generic. Let $\sigma \in M[g] \cap A_\Gamma$ be such that whenever $h$ is $M[g]$-generic, letting $B^* = B \cap M[g * h]$ and $A^* = A \cap M[g * h]$, $B^* = \sigma_0^{-1}[A^*]$. It then follows that $B = \sigma_0^{-1}[A]$ and hence, $B \in HP^\Gamma$.

Suppose then $B \in HP^\Gamma$. Let $g \subseteq Coll(\omega, \lambda)$ be $M$-generic. Let $B^* = M[g] \cap B$ and $A^* = M[g] \cap A$. Because $B^*$ and $A^*$ are $\delta^+$-uB in $M[g]$, it follows from a theorem of Neeman (see Theorem 6.17 of [**23**]) that in $M[g]$, either $A^* \leq_W B^*$ or $B^* \leq_W A^*$. We claim that

$$M[g] \vDash A^* \leq_W B^* \text{ if and only if } A \leq_W B$$
$$M[g[\vDash B^* \leq_W A^* \text{ if and only if } B \leq_W A.$$

By symmetry, we only show the first equality. Also, by symmetry it is enough to show that $A^* \leq_W B^*$ implies that $A \leq_W B$.

Fix then $\sigma \in M[g]$ such that $A^* = \sigma^{-1}[B^*]$. Because $A^*$ and $B^*$ are $\delta^+$-uB it follows that whenever $h$ is $Coll(\omega, \delta)$-generic over $M[g]$,

$$A \cap M[g * h] = \sigma^{-1}[M[g * h] \cap B].$$

It the follows from an easy genericity argument that $A = \sigma^{-1}[B]$. This finishes the proof of the above equivalence.

Because we are assuming that $B \in HP^\Gamma$, we must have that $M[g] \vDash B^* \leq_W A^*$. It then follows from the proof given above that $(M, \lambda, \Sigma) \vDash B \in \Gamma$. □

Next we show that mouse operators are definable over self-capturing background triples.

LEMMA 2.30. *Suppose that $(M, \delta, \Sigma)$ is a self-capturing background triple such that $(M, \Sigma)$ Suslin, co-Suslin captures $(A_\Gamma, A)$. Suppose further that $(\mathcal{P}, \Lambda) \in HP^\Gamma$ and $\lambda < \delta$ is an $M$-inaccessible cardinal such that $(M, \lambda, \Sigma) \vDash Code(\Lambda) \in \Gamma$. Let $F : HC \to HC$ be given by $F(a) = Lp^{\Gamma,\Lambda}(a)$. Suppose $g \subseteq Coll(\omega, \lambda)$ is $M$-generic and $h \in HC$ is $V_\delta^M[g]$-generic. Then $F \restriction V_\delta^M[g][h]$ is definable over $M[g][h]$ from the pair $(A_\Gamma \cap M[g], A \cap M[g])$ uniformly in $h$.*

PROOF. Fix $\sigma \in A_\Gamma \cap M[g]$ such that $Code(\Lambda) \cap M[g] = \sigma_0^{-1}[A \cap M[g]]$ where $(\sigma_0, \sigma_1)$ is the pair coded by $\sigma$. Notice that it follows that $\sigma_0^{-1}[A] = Code(\Lambda)$. Given $a \in M[g][h]$, let $G(a)$ be the union of all $\mathcal{M}$ such that whenever $k \subseteq Coll(\omega, \delta)$ is $M[g][h][k]$-generic, for some $x \in A \cap M[g][h][k]$ and some $\pi$ such that $(\sigma_0, \pi) \in A_\Gamma \cap M[g][h][k]$, $\pi^{-1}(x)$ codes the triple $(a, \mathcal{M}, \emptyset)$. By a standard absoluteness and genericity arguments using the fact that $(M, \Sigma)$ Suslin, co-Suslin captures $(A_\Gamma, A)$, we get that $G(a) = F(a)$. □

LEMMA 2.31. *Assume the hypothesis of Lemma 2.30. Suppose further that the function $\eta \to \Lambda \restriction V_\eta^M$ is definable over $M$ (from parameters) and that there is some set $X \in M$ such that there is an invariant $\tau \in M^{Coll(\omega,\lambda)}$ such that $\tau \in OD_X^M$*

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*and whenever $g \subseteq Coll(\omega, \lambda)$ is $M$-generic then $\tau_g = \{(x, y) \in \mathbb{R}^2 : x$ codes $\mathcal{P}$ and $y \in Code(\Lambda)\}$. Then $F \restriction V_\delta^M$ is definable over $V_\delta^M$.*

PROOF. Fix $a \in M$. Let $G(a)$ be the union of all $\mathcal{M}$ such that whenever $g \subseteq Coll(\omega, \lambda)$ is $M$-generic, $h \subseteq Coll(\omega, \delta)$ is $M[g]$-generic, $x \in M[g]$ codes $\mathcal{P}$ and $\sigma \in A_\Gamma \cap M[g]$ is such that letting $(\sigma_0, \sigma_1)$ be the pair coded by $\sigma$, $(\tau_g)_x = \sigma_0^{-1}[A \cap M[g]]$, then there is $y \in \mathbb{R}^{M[g*h]}$ such that $\sigma_1^{-1}(y)$ codes the triple $(a, \mathcal{M}, \emptyset)$. It again follows that $G(a) = F(a)$. □

We can now introduce $\Gamma$-hod pair constructions.

DEFINITION 2.32 ($\Gamma$-hod pair constructions). *Suppose $\Gamma$ is a pointclass closed under continuous preimages and images and suppose that $A \subseteq \mathbb{R}$ is such that $w(A) = w(\Gamma)$. Suppose further $(M, \delta, \Sigma)$ is a self-capturing background triple such that $M$ locally Suslin, co-Suslin captures $(A_\Gamma, A)$. Then the* $\Gamma$-hod pair construction *of $M$ below $\delta$ is a sequence $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta : \beta \leq \Omega)$ that satisfies the following properties.*

1. *$M \vDash$ " for all $\beta < \Omega, (\mathcal{P}_\beta, \Sigma_\beta)$ is a hod pair such that $\Sigma_\beta \in \Gamma$" [11].*
2. *For $\beta \in [-1, \delta)$, letting $\Sigma_{-1} = \emptyset$, $\mathbb{C}_{\beta+1} = ((\mathcal{M}_\xi^{\beta+1}, \mathcal{N}_\xi^{\beta+1} : \xi \leq \delta), (F_\xi^{\beta+1} : \xi < \delta))$ is the output of $\mathcal{J}^{\vec{E}, \Sigma_\beta}$-construction of $V_\delta^M$. Also, $\delta_{\beta+1}$ is the least $\gamma$ such that $o(\mathcal{N}_\gamma^{\beta+1}) = \gamma$ and $Lp^{\Gamma, \Sigma_\beta}(\mathcal{N}_\gamma^{\beta+1}) \vDash$ "$\gamma$ is Woodin".*
3. *For $\beta \in [-1, \delta)$, letting $\Sigma_{-1} = \emptyset$, if*
   (a) *$\delta_{\beta+1}$ exists,*
   (b) *$\mathcal{N}_{\delta_{\beta+1}}^{\beta+1}$ doesn't have initial segments projecting across $\delta_\beta$,*
   (c) *if $\beta$ is a successor then $\mathcal{N}_{\delta_{\beta+1}}^{\beta+1} \vDash$ "$\delta_\beta$ is Woodin" and*
   (d) *if $\beta$ is limit then $(\delta_\beta^+)^{\mathcal{P}_\beta} = (\delta_\beta^+)^{\mathcal{N}_{\delta_{\beta+1}}^{\beta+1}}$,*

   *then $\mathcal{P}_{\beta+1} = Lp_\omega^{\Gamma, \Sigma_\beta}(\mathcal{N}_{\delta_{\beta+1}}^{\beta+1} | \delta_{\beta+1})$ and $\Sigma_{\beta+1}$ is the strategy of $\mathcal{P}_{\beta+1}$ induced by $\Sigma$.*
4. *For limit ordinals $\beta$, letting $\mathcal{P}_\beta^* = \cup_{\gamma<\beta} \mathcal{P}_\beta$, $\Sigma_\beta^* = \oplus_{\gamma<\beta} \Sigma_\beta$ and $\delta_\beta = \sup_{\gamma<\beta} \delta_\gamma$, if $\delta_\beta$ is not measurable in $M$ and $\delta_\beta < \delta$ then $\mathbb{C}_\beta = ((\mathcal{M}_\xi^\beta, \mathcal{N}_\xi^\beta : \xi \leq \delta), (F_\xi^\beta : \xi < \delta))$ is the output of $\mathcal{J}^{\vec{E}, \Sigma_\beta^*}$-construction of $V_\delta^M$. If there is no $\gamma$ such that $o(\mathcal{N}_\gamma^\beta) = \gamma$ and $Lp^{\Gamma, \Sigma_\beta^*}(\mathcal{N}_\gamma^\beta) \vDash$ "$\gamma$ is Woodin" then we let $\mathcal{P}_\beta$ be undefined. Otherwise, let $\gamma$ be the least such that $o(\mathcal{N}_\gamma^\beta) = \gamma$ and $Lp^{\Gamma, \Sigma_\beta^*}(\mathcal{N}_\gamma^\beta) \vDash$ "$\gamma$ is Woodin". If $\mathcal{N}_\gamma^\beta$ doesn't have an initial segment projecting across $\delta_\beta$ then $\mathcal{P}_\beta = \mathcal{N}_\beta | (\delta_\beta^{+\omega})^{\mathcal{N}_\beta}$ and $\Sigma_\beta$ is the iteration strategy for $\mathcal{P}_\beta$ induced by $\Sigma$. Otherwise, $\mathcal{P}_\beta$ is undefined.*

An alternative definition of a $\Gamma$-full hod pair construction is one in which we let $\delta_{\beta+1}$ be the least $\gamma > \delta_\beta$ which is a cardinal of $M$ and is such that $Lp^{\Gamma, \Sigma_\beta}(V_\gamma^M) \vDash$ "$\gamma$ is Woodin" and we let $\mathcal{N}_{\beta+1}$ be the output of $\mathcal{J}^{\vec{E}, \Sigma_\beta}$-construction of $V_{\delta_{\beta+1}}^M$. This is done in Definition 3.48. We then let $\mathcal{P}_{\beta+1} = Lp_\omega^{\Gamma, \Sigma_\beta}(\mathcal{N}_{\beta+1})$. That the two definitions are equivalent follows from the $S$-construction method of Section 3.8. The problem that one needs to address is that (i) Why is $\delta_{\beta+1}$ Woodin in $\mathcal{P}_{\beta+1}$?

[11] We abuse our notation and let $\Sigma_\beta$ stand for both the induced strategy which acts on all trees in $V$ and for the strategy which acts on trees that are in $M$. We will do this throughout this paper.

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and (ii) Why is $\delta_{\beta+1}$ the least Woodin above $\delta_\beta$ in $\mathcal{P}_{\beta+1}$?. To show both we will use $S$-constructions: because $V_\gamma^M$ is generic over $\mathcal{N}_{\beta+1}$, Lemma 3.42 implies that $\mathcal{P}_{\beta+1}[V_\gamma^M] = Lp_\omega^{\Gamma,\Sigma_\beta}(V_\gamma^M)$. To prove that the $S$-constructions work, however, we will need to prove theorems about how to interpret the strategy of a hod mouse in generic extensions. For this we will first need to investigate the internal theory of hod mice which we will do in Section 3.1.

We end this section by stating the comparison theorem for $\Gamma$-fullness preserving hod pairs the proof of which is just like the proof of Theorem 2.28.

THEOREM 2.33 (Comparison of hod pairs). *Suppose that $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are two hod pairs such that both $\Sigma$ and $\Lambda$ have branch condensation and are $\Gamma$-fullness preserving for some pointclass $\Gamma$ closed under continuous images and preimages. Suppose further that there is a good pointclass $\Gamma^*$ such that $\Gamma \cup \{Code(\Sigma), Code(\Lambda)\} \subseteq \Delta_{\underset{\sim}{\Gamma}^*}$. Then there are $(\mathcal{T},\mathcal{R}) \in I(\mathcal{P},\Sigma)$ and $(\mathcal{U},\mathcal{S}) \in I(\mathcal{Q},\Lambda)$ such that either*

(1) $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ *and* $\Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}$
*or*
(2) $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ *and* $\Lambda_{\mathcal{S},\mathcal{U}} = \Sigma_{\mathcal{S},\mathcal{T}}$.

REMARK 2.34. *The comparison theorem of this section has an important consequence namely that the comparison can be achieved via a normal tree, i.e., if $(P,\Sigma)$ and $(\mathcal{Q},\Lambda)$ are two hod pairs such that $\Sigma$ and $\Lambda$ are fullness preserving and have branch condensation then there are $(\mathcal{T},\mathcal{R}) \in I(\mathcal{P},\Sigma)$ and $(\mathcal{U},\mathcal{S}) \in I(\mathcal{Q},\Lambda)$ such that either $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Lambda_{\mathcal{R},\mathcal{U}} = \Sigma_{\mathcal{R},\mathcal{T}}$ or $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Sigma_{\mathcal{S},\mathcal{T}} = \Lambda_{\mathcal{S},\mathcal{U}}$. This fact will be used in Lemma* 2.42.

## 2.6. Positional and commuting iteration strategies

In this section, we introduce several key properties of iteration strategies that we will use in this paper. One of the main goals of this paper is to show that initial segments of HOD of models of determinacy are iterates of hod mice (see Section 4.4 and in particular, Theorem 4.24). The typical proofs of such theorems are as follows. We fix an initial segment of HOD, say $\mathcal{Q}$, and show that there is a hod pair $(\mathcal{P},\Sigma)$ such that the direct limit of all iterates of $\mathcal{P}$ via $\Sigma$ converges to $\mathcal{Q}$. In order to make sense of such direct limit construction our strategy has to be *commuting*, i.e., if $\mathcal{R}$ is an iterate of $\mathcal{P}$ via two different iterations according to $\Sigma$, then we must have that the iteration embedding doesn't depend on the particular iteration producing $\mathcal{R}$. In this section, we use our comparison argument, more precisely the fact that we achieved comparison via normal trees, to show that strategies that are fullness preserving and have branch condensation are also commuting (see Theorem 2.42).

Suppose $\Sigma$ is some iteration strategy and $M = M_\Sigma$. Note that if $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$ then in general, $\Sigma_{N,\vec{\mathcal{T}}}$ depends on $\vec{\mathcal{T}}$, i.e., there could be some other stack $\vec{\mathcal{U}}$ with $N$ as its last model and such that $\Sigma_{N,\vec{\mathcal{U}}} \neq \Sigma_{N,\vec{\mathcal{T}}}$. We call $\Sigma_{N,\vec{\mathcal{T}}}$ the $\vec{\mathcal{T}}$-tail of $\Sigma$. If now $\Gamma$ is the $\vec{\mathcal{T}}$-tail of $\Sigma$ then we let

$$\Sigma^{\vec{\mathcal{T}}} = \Gamma^{\pi^{\vec{\mathcal{T}}}} =_{def} \pi^{\vec{\mathcal{T}}}\text{-pullback of } \Gamma^{12}.$$

[12]See Definition 4.5 of [**43**]. The pullback strategies are defined using the copying construction described in Section 4.1 of [**43**].

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If $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are two different trees according to $\Sigma$ then $\Sigma^{\vec{\mathcal{T}}}$ might be different from $\Sigma^{\vec{\mathcal{U}}}$.

DEFINITION 2.35. *Suppose $\Sigma$ is an iteration strategy and $M = M_\Sigma$. Then,*

(1) (**Dodd-Jensen Property**) *$\Sigma$ has the Dodd-Jensen property if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$ then for any $\pi : M \to_{\Sigma_1} N$ and for any ordinal $\alpha$, $\pi^{\vec{\mathcal{T}}}(\alpha) \leq \pi(\alpha)$.*

(2) (**Positional Dodd-Jensen Property**) *$\Sigma$ has the positional Dodd-Jensen property if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$ then $\Sigma_{N,\vec{\mathcal{T}}}$ has the Dodd-Jensen property.*

(3) (**Weakly positional**) *$\Sigma$ is weakly positional if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$ and $(\vec{\mathcal{U}}, N) \in I(M, \Sigma)$ then $\Sigma_{N,\vec{\mathcal{T}}} = \Sigma_{N,\vec{\mathcal{U}}}$.*

(4) (**Positional**) *$\Sigma$ is positional if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$, $\Sigma_{N,\vec{\mathcal{T}}}$ is positional.*

(5) (**Near weakly positional**) *Assume $M$ is fine structural. $\Sigma$ is near weakly positional if whenever $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(M, \Sigma)$ and $\mathcal{S}$ is a tree on $N$ of limit length which is according to both $\Sigma_{N,\vec{\mathcal{T}}}$ and $\Sigma_{N,\vec{\mathcal{U}}}$ then letting $\Sigma_{N,\vec{\mathcal{T}}}(\mathcal{S}) = b$ and $\Sigma_{\mathcal{N},\vec{\mathcal{U}}} = c$, if $b \neq c$ then both $\pi_b^{\mathcal{S}}$ and $\pi_c^{\mathcal{S}}$ don't exist.*

(6) (**Weakly pullback consistent**) *$\Sigma$ is weakly pullback consistent if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ whenever $\vec{\mathcal{T}}$ is a stack according to $\Sigma$ such that $\pi^{\vec{\mathcal{T}}}$ exists.*

(7) (**Pullback consistent**) *$\Sigma$ is pullback consistent if for any $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$, $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent.*

(8) (**Weakly commuting**) *$\Sigma$ is weakly commuting if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$ and $(\vec{\mathcal{U}}, N) \in I(M, \Sigma)$ then $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$.*

(9) (**Commuting**) *$\Sigma$ is commuting if whenever $(\vec{\mathcal{T}}, N) \in I(M, \Sigma)$ then $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly commuting.*

Clearly the Dodd-Jensen property implies weakly commuting and the positional Dodd-Jensen property implies commuting. Readers familiar with the proof of Dodd-Jensen lemma are perhaps deceived by the appearance of the branch condensation and are probably thinking that the proof can be used to show that strategies with branch condensation are at least commuting. As a consolation, we confess that for a long time the author too thought that such an implication is trivial and only some time later he realized that this fact needs a proof. In general, we do not know whether branch condensation implies Dodd-Jensen property or even commuting. We can prove that for hod mice branch condensation does imply commuting (see Theorem 2.42). The following proposition summarizes the relationship between branch condensation, commuting and the Dodd-Jensen property. Its proof uses the copying construction (see Section 4.1 of [**43**]).

PROPOSITION 2.36. *Suppose $\Sigma$ is an iteration strategy. Then the following holds.*

(1) *If $\Sigma$ has hull condensation then it is pullback consistent.*

(2) *If $\Sigma$ is positional and pullback consistent then $\Sigma$ is commuting.*

(3) *If $\Sigma$ has branch condensation and is weakly commuting then $\Sigma$ is near weakly positional.*

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PROOF. We only sketch the argument. To see hull condensation implies pullback consistency let $\vec{\mathcal{T}}$ be a stack according to $\Sigma$. We need to show that $\Sigma^{\vec{\mathcal{T}}} = \Sigma$. To see this, fix a stack $\vec{\mathcal{U}}$ according to both $\Sigma$ and $\Sigma^{\vec{\mathcal{T}}}$. We need to show that $\Sigma(\vec{\mathcal{U}}) = \Sigma^{\vec{\mathcal{T}}}(\vec{\mathcal{U}})$. Let $c = \Sigma^{\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}})$ where $j$ is the iteration map given by $\vec{\mathcal{T}}$ (recall that $j\vec{\mathcal{U}}$ is the stack on the last model of $\vec{\mathcal{T}}$ constructed by the copying construction). For simplicity, we assume that $\pi^{\vec{\mathcal{U}}}_c$-exists. We then have

$$\begin{aligned} \pi^{\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}}}_c : M_\Sigma &\to \mathcal{M}^{\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}}}_c; \\ \pi^{\vec{\mathcal{U}}}_c : M_\Sigma &\to \mathcal{M}^{\vec{\mathcal{U}}}_c; \\ \pi : \mathcal{M}^{\vec{\mathcal{U}}}_c &\to \mathcal{M}^{\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}}}_c \end{aligned}$$

such that

$$\pi^{\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}}}_c = \pi \circ \pi^{\vec{\mathcal{U}}}_c.$$

where $\pi$ comes from the copying construction. Notice then that $\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_c\}$ is a hull of $\vec{\mathcal{T}}^\frown j\vec{\mathcal{U}}^\frown\{\mathcal{M}_c^{\mathcal{T}^\frown j\mathcal{U}}\}$[13]. Then by hull condensation, $c$ is according to $\Sigma$.

The usual proof of the Dodd-Jensen property (see Theorem 4.8 of [**43**]) shows that if $\Sigma$ is positional and pullback-consistent then it is commuting. To see that if $\Sigma$ has branch condensation and is commuting then $\Sigma$ is near weakly positional fix $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{T}}_2$ on $M_\Sigma$ with common last model $N$. Let $\vec{\mathcal{U}}$ be a stack on $N$ without last model which is according to both $\Sigma_{N,\vec{\mathcal{T}}_1}$ and $\Sigma_{N,\vec{\mathcal{T}}_2}$. Let $\Sigma_{N,\vec{\mathcal{T}}_1}(\vec{\mathcal{U}}) = b$ and $\Sigma_{N,\vec{\mathcal{T}}_2}(\vec{\mathcal{U}}) = c$. We need to show that if $b \neq c$ then both $\pi^{\vec{\mathcal{U}}}_b$ and $\pi^{\vec{\mathcal{U}}}_c$ don't exist. Towards a contradiction suppose that $b \neq c$ but $\pi^{\vec{\mathcal{U}}}_b$ exists. By commuting, we have that

$$\pi^{\vec{\mathcal{T}}_1} = \pi^{\vec{\mathcal{T}}_2}.$$

Therefore,

$$\pi^{\vec{\mathcal{U}}}_b \circ \pi^{\vec{\mathcal{T}}_2} = \pi^{\vec{\mathcal{U}}}_b \circ \pi^{\vec{\mathcal{T}}_1}.$$

We can then apply branch condensation to $\vec{\mathcal{T}}_2^\frown\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_b\}$ and $\vec{\mathcal{T}}_1^\frown\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_b\}$ where $\pi$ of the definition of branch condensation is just the identity. This shows that $\vec{\mathcal{T}}_2^\frown\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_b\}$ is according to $\Sigma$ and therefore, $b = c$, contradiction! □

We remark that the last argument does not give the full form of positional because branch condensation applies to branches for which the branch embedding exists. We do not know if hull condensation by itself implies weakly commuting. The difficulty is that we seem to need that if $N$ is a $\Sigma$-iterate of $\mathcal{M}_\Sigma$ via $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ and $i = \pi^{\vec{\mathcal{T}}}$ and $j = \pi^{\vec{\mathcal{U}}}$ exist then

$$\Sigma = \Sigma^j_{N,\vec{\mathcal{T}}} = \Sigma^i_{N,\vec{\mathcal{U}}}.$$

The equality holds if $\Sigma$ has branch condensation but we do not know if it holds when $\Sigma$ has just hull condensation. We now work towards showing that branch condensation for hod mice implies positional. Our first task is to introduce several useful notions.

Given a hod premouse $\mathcal{P}$ and a stack $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < lh(\vec{\mathcal{T}}))$ on $\mathcal{P}$, we would like to rearrange $\vec{\mathcal{T}}$ into components in a way that if $\mathcal{M}$ is a model appearing

[13]In the definition of hull condensation, $\sigma(0)$ was allowed to be not equal to 0.

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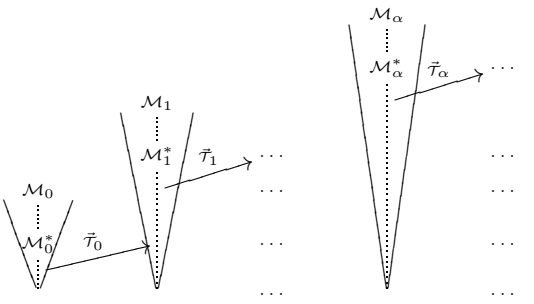


FIGURE 2.6.1. Essential components.

at the end of some round or, equivalently at the beginning of some round, and $\alpha \leq \lambda^{\mathcal{M}}$ is the least such that $I$'s first move of the round is an extender from $\mathcal{M}(\alpha)$ then the next component of $\vec{\mathcal{T}}$ played on $\mathcal{M}$ is the largest initial segment of $\vec{\mathcal{T}}$ that is based entirely on $\mathcal{M}(\alpha)$. Such re-arrangements of stacks will be used throughout this paper.

DEFINITION 2.37 (Essential components of stacks: limit case). *Suppose $\mathcal{P}$ is a hod premouse, $\lambda^{\mathcal{P}}$ is limit and $\vec{\mathcal{T}}$ is a stack of iteration trees on $\mathcal{P}$. Then the sequence $(\mathcal{M}_\alpha, \mathcal{M}_\alpha^*, \vec{\mathcal{T}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \eta)$ that has the following properties is called the sequence of essential components of $\vec{\mathcal{T}}$ or just the essential components of $\vec{\mathcal{T}}$ (see Figure 2.6.1)*

1. *$\mathcal{M}_0 = \mathcal{P}$, $\mathcal{M}_0^*$ is the least hod premouse initial segment of $\mathcal{P}$ such that $E_0^{\vec{\mathcal{T}}} \in \mathcal{M}_0^*$, and $\vec{\mathcal{T}}_0$ is the largest initial segment of $\vec{\mathcal{T}}$ that is entirely on $\mathcal{M}_0^*$.*
2. *For any $\alpha$, if $\vec{\mathcal{T}}_\beta$ is defined for all $\beta < \alpha$ then $\oplus_{\beta<\alpha}\vec{\mathcal{T}}_\beta$ is an initial segment of $\vec{\mathcal{T}}$.*
3. *If for some $\alpha$, $\oplus_{\beta<\alpha}\vec{\mathcal{T}}_\alpha = \vec{\mathcal{T}}$ then $\eta = \alpha$, and $\mathcal{M}_\alpha^*$, $\mathcal{M}_\alpha$, $\vec{\mathcal{T}}_\alpha$ are undefined.*
4. *If $\oplus_{\beta<\alpha}\vec{\mathcal{T}}_\beta \lhd \vec{\mathcal{T}}$[14] then $\mathcal{M}_\alpha$ is the last model of $\oplus_{\beta<\alpha}\vec{\mathcal{T}}_\beta$ in $\vec{\mathcal{T}}$, and $\mathcal{M}_\alpha^*$ is the least hod initial segment of $\mathcal{M}_\alpha$ such that the next extender used in $\vec{\mathcal{T}}$ is in $\mathcal{M}_\alpha^*$.*
5. *If $\mathcal{M}_\alpha^*$ is defined then $\vec{\mathcal{T}}_\alpha$ is the largest initial segment of $\vec{\mathcal{T}}$ that is based entirely on $\mathcal{M}_\alpha^*$,*
6. *$\pi_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the iteration embedding.*

DEFINITION 2.38 (Essential components of stacks: successor case). *Suppose $\mathcal{P}$ is a hod premouse, $\lambda^{\mathcal{P}}$ is a successor ordinal and $\vec{\mathcal{T}}$ is a stack of iteration trees on $\mathcal{P}$. Then the sequence $(\mathcal{M}_\alpha^0, \mathcal{M}_\alpha^1, \vec{\mathcal{U}}_\alpha^0, \vec{\mathcal{U}}_\alpha^1 : \alpha \leq \eta)$ that has the following properties is called the sequence of essential components of $\vec{\mathcal{T}}$ or just the essential components of $\vec{\mathcal{T}}$.*

1. *$\mathcal{M}_0^0 = \mathcal{P}$ and $\vec{\mathcal{U}}_0^0$ is the largest initial segment of $\vec{\mathcal{T}}$ that is entirely based on $\mathcal{P}^-$, $\mathcal{M}_0^1$ is the last model of $\vec{\mathcal{U}}_0^0$ if it exists, and $\vec{\mathcal{U}}_0^1$ is the largest initial segment of $\vec{\mathcal{T}}$ after $\vec{\mathcal{U}}_0^0$ that is entirely above $(\mathcal{M}_0^1)^-$,*

[14] I.e., the rest of $\vec{\mathcal{T}}$ doesn't use $\oplus_{\beta<\alpha}\vec{\mathcal{T}}_\beta$.

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(2) *for any $\alpha$, if for all $\beta<\alpha$, $\vec{\mathcal{U}}^0_\beta$ and $\vec{\mathcal{U}}^1_\beta$ are defined then, $\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}^1_\beta)$ is an initial segment of $\vec{\mathcal{T}}$,*

(3) *if for some $\alpha$, for all $\beta<\alpha$, $\vec{\mathcal{U}}^0_\beta$ and $\vec{\mathcal{U}}^1_\beta$ are defined and $\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}^1_\beta)=\vec{\mathcal{T}}$ then $\eta=\alpha$ and $\mathcal{M}^1_\alpha$, $\mathcal{M}_\alpha$, $\vec{\mathcal{U}}^0_\alpha$ and $\vec{\mathcal{U}}^1_\alpha$ are undefined,*

(4) *if $\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}^1_\beta)\lhd\vec{\mathcal{T}}$ then $\mathcal{M}^0_\alpha$ is the last model of $\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}_\beta{}^1)$, $\vec{\mathcal{U}}^0_\alpha$ is the largest initial segment of $\vec{\mathcal{T}}$ that is based entirely on $(\mathcal{M}^0_\alpha)^-$. If*

$$(\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}^1_\beta))^\frown\vec{\mathcal{U}}^0_\alpha=\vec{\mathcal{T}}$$

*then $\alpha=\eta$, $\mathcal{M}^1_\eta$ is the last model of $\vec{\mathcal{U}}^0_\alpha$ if it exists and is undefined otherwise, and $\vec{\mathcal{U}}^1_\alpha$ is left undefined. If $(\oplus_{\beta<\alpha}(\vec{\mathcal{U}}^0_\beta{}^\frown\vec{\mathcal{U}}^1_\beta))^\frown\vec{\mathcal{U}}^0_\alpha\lhd\vec{\mathcal{T}}$ then $\mathcal{M}^1_\alpha$ is the last model of $\vec{\mathcal{U}}^0_\alpha$ and $\vec{\mathcal{U}}^1_\alpha$ is the largest initial segment of $\vec{\mathcal{T}}$ that is entirely above $\mathcal{M}^-_\alpha$.*

Suppose $\mathcal{P}$ is a hod premouse and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. Suppose $\lambda^{\mathcal{P}}$ is limit and

$$(\mathcal{M}_\alpha,\mathcal{M}^*_\alpha,\vec{\mathcal{T}}_\alpha,\pi_{\alpha,\beta}:\alpha<\beta\leq\eta)$$

are the essential components of $\vec{\mathcal{T}}$. Then for $\alpha\leq\eta$, we let $\vec{\mathcal{T}}\restriction\alpha=\oplus_{\gamma<\alpha}\vec{\mathcal{T}}_\gamma$. Suppose now $\lambda^{\mathcal{P}}$ is a successor ordinal and $(\mathcal{M}^0_\alpha,\mathcal{M}^1_\alpha,\vec{\mathcal{U}}^0_\alpha,\vec{\mathcal{U}}^1_\alpha:\alpha\leq\eta)$ are the essential components of $\vec{\mathcal{T}}$. Then for $\alpha\leq\eta$, we let $\vec{\mathcal{T}}\restriction\alpha=\oplus_{\gamma<\alpha}(\vec{\mathcal{U}}^0_\gamma{}^\frown\vec{\mathcal{U}}^1_\gamma)$.

Next we introduce an important definition and prove a lemma. Both will be used in the next section as well as in the proof of Theorem 2.42. Suppose we have two hod pairs $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ such that both $\Sigma$ and $\Lambda$ are fullness preserving (or $\Gamma$-fullness preserving) yet $\Sigma\neq\Lambda$. Is there in some sense a minimal disagreement between $\Sigma$ and $\Lambda$? It turns out that there is.

DEFINITION 2.39 (Minimal disagreement; limit case). *Suppose $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ are two hod pairs such that $\lambda^{\mathcal{P}}$ is limit and for any $\alpha<\lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}=\Lambda_{\mathcal{P}(\alpha)}$. If $\Sigma\neq\Lambda$, then $\vec{\mathcal{T}}$ is a minimal disagreement between $\Sigma$ and $\Lambda$ if letting $(\mathcal{M}_\alpha,\mathcal{M}^*_\alpha,\vec{\mathcal{T}}_\alpha,\pi_{\alpha,\beta}:\alpha<\beta\leq\eta)$ be the essential components of $\vec{\mathcal{T}}$,*

(1) *$\eta=\gamma+1$ for some $\gamma$,*

(2) *for any $\alpha<\gamma$, $\Sigma_{\mathcal{M}_\alpha,\vec{\mathcal{T}}\restriction\alpha}=\Lambda_{\mathcal{M}_\alpha,\vec{\mathcal{T}}\restriction\alpha}$,*

(3) *$\Sigma_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}\neq\Lambda_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}$ and for any $\mathcal{Q}\lhd_{hod}\mathcal{M}^*_\gamma$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}\restriction\gamma}=\Lambda_{\mathcal{Q},\vec{\mathcal{T}}\restriction\gamma}$,*

(4) *$\lambda^{\mathcal{M}^*_\gamma}$ is a successor ordinal,*

(5) *$\vec{\mathcal{T}}_\gamma$ is a stack on $\mathcal{M}^*_\gamma$ of successor length such that its last component has limit length, $\vec{\mathcal{T}}_\gamma$ is according to both $\Sigma_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}$ and $\Lambda_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}$ but*

$$\Sigma_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}(\vec{\mathcal{T}}_\gamma)\neq\Lambda_{\mathcal{M}^*_\gamma,\vec{\mathcal{T}}\restriction\gamma}(\vec{\mathcal{T}}_\gamma).$$

DEFINITION 2.40 (Minimal disagreement; successor case). *Suppose $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ are two hod pairs such that $\lambda^{\mathcal{P}}$ is a successor and for any $\alpha<\lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}=\Lambda_{\mathcal{P}(\alpha)}$. If $\Sigma\neq\Lambda$, then $\vec{\mathcal{T}}$ is a minimal disagreement between $\Sigma$ and $\Lambda$ if letting $(\mathcal{M}^0_\alpha,\mathcal{M}^1_\alpha,\vec{\mathcal{T}}^0_\alpha,\vec{\mathcal{T}}^1_\alpha:\alpha\leq\eta)$ be the essential components of $\vec{\mathcal{T}}$, then $\mathcal{M}^1_\eta$ and $\vec{\mathcal{T}}^1_\eta$ are defined and*

$$\Sigma_{\mathcal{M}^1_\eta,\vec{\mathcal{T}}\restriction\eta{}^\frown\vec{\mathcal{T}}^0_\eta}(\vec{\mathcal{T}}^1_\eta)\neq\Lambda_{\mathcal{M}^1_\eta,\vec{\mathcal{T}}\restriction\eta{}^\frown\vec{\mathcal{T}}^0_\eta}(\vec{\mathcal{T}}^1_\eta).$$

Note that if $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ are as in Definition 2.40, then any $\vec{\mathcal{T}}$ which is according to both $\Sigma$ and $\Lambda$, doesn't have a last model and $\Sigma(\vec{\mathcal{T}})\neq\Lambda(\vec{\mathcal{T}})$, constitutes

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a minimal disagreement. The existence of minimal disagreements in the limit case is not so trivial.

PROPOSITION 2.41 (The existence of minimal disagreements). *Suppose* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{P},\Lambda)$ *are two hod pairs such that* $\lambda^{\mathcal{P}}$ *is limit and for any* $\alpha<\lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}=\Lambda_{\mathcal{P}(\alpha)}$. *If* $\Sigma\neq\Lambda$ *then there is a minimal disagreement between* $\Sigma$ *and* $\Lambda$.

PROOF. Let $\vec{\mathcal{T}}$ be such that $\Sigma(\vec{\mathcal{T}})\neq\Lambda(\vec{\mathcal{T}})$ and let $b=\Sigma(\vec{\mathcal{T}})$ and $c=\Lambda(\vec{\mathcal{T}})$. Let $(\mathcal{M}_\alpha,\mathcal{M}^*_\alpha,\vec{\mathcal{T}}_\alpha,\pi_{\alpha,\beta}:\alpha<\beta\leq\eta)$ be the essential components of $\vec{\mathcal{T}}$. There must be $\alpha$ such that for some $\beta<\lambda^{\mathcal{M}_\alpha}$, $\Sigma_{\mathcal{M}_\alpha(\beta),\vec{\mathcal{T}}\restriction\alpha}\neq\Lambda_{\mathcal{M}_\alpha(\beta),\vec{\mathcal{T}}\restriction\alpha}$ (for instance $\alpha=\eta$ works). Let $(\alpha,\beta)$ be the least pair such that $\Sigma_{\mathcal{M}_\alpha(\beta),\vec{\mathcal{T}}\restriction\alpha}\neq\Lambda_{\mathcal{M}_\alpha(\beta),\vec{\mathcal{T}}\restriction\alpha}$. Let $\mathcal{Q}^*=\mathcal{M}_\alpha$ and $\mathcal{Q}=\mathcal{M}_\alpha(\beta)$. Note that if $\lambda^{\mathcal{Q}}$ is a successor then we are done, because if we let $\vec{\mathcal{U}}$ be any stack on $\mathcal{Q}$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}\restriction\alpha}(\vec{\mathcal{U}})\neq\Lambda_{\mathcal{Q},\vec{\mathcal{T}}\restriction\alpha}(\vec{\mathcal{U}})$, we have that $\vec{\mathcal{T}}\restriction\alpha^\frown\vec{\mathcal{U}}$ is as desired.

Thus, suppose $\lambda^{\mathcal{Q}}$ is limit. In this case, let $\vec{\mathcal{T}}^*=\vec{\mathcal{T}}\restriction\alpha$ and let $\vec{\mathcal{U}}$ be a stack on $\mathcal{Q}$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}^*}(\vec{\mathcal{U}})\neq\Lambda_{\mathcal{Q},\vec{\mathcal{T}}^*}(\vec{\mathcal{U}})$. Let $(\mathcal{N}_\alpha,\mathcal{N}^*_\alpha,\vec{\mathcal{U}}_\alpha,\pi_{\alpha,\beta}:\alpha<\beta\leq\nu)$ be the essential components of $\vec{\mathcal{U}}$. Then as before we can let $(\alpha_1,\beta_1)$ be the least $(\alpha,\beta)$ such that $\Sigma_{\mathcal{N}_\alpha(\beta),\vec{\mathcal{T}}^*{}^\frown\vec{\mathcal{U}}\restriction\alpha}\neq\Lambda_{\mathcal{N}_\alpha(\beta),\vec{\mathcal{T}}^*{}^\frown\vec{\mathcal{U}}\restriction\alpha}$. Then let $\mathcal{Q}^*_1=\mathcal{N}_{\alpha_1}$ and $\mathcal{Q}_1=\mathcal{N}_{\alpha_1}(\beta_1)$. We again have two cases. If $\lambda^{\mathcal{Q}_1}$ is a successor ordinal then we finish as before. Thus, suppose $\lambda^{\mathcal{Q}_1}$ is limit. We can now repeat the above process and get $\mathcal{Q}_2$. Again, if $\lambda^{\mathcal{Q}_2}$ is a successor ordinal then we finish as before and otherwise we repeat the above process and get $\mathcal{Q}_3$. If during this construction we never produce $\mathcal{Q}_n$ such that $\lambda^{\mathcal{Q}_n}$ is a successor then we get a sequence $(\mathcal{Q}^*_j,\mathcal{Q}_j,i_{j,k}:j<k<\omega)$ where $i_{j,k}:\mathcal{Q}^*_j\to\mathcal{Q}^*_k$ is the iteration embedding. Then, because $\mathcal{Q}_{j+1}\trianglelefteq_{hod}i_{j,j+1}(\mathcal{Q}_j)$, the direct limit of $(\mathcal{Q}^*_j,i_{j,k}:j<k<\omega)$ is ill-founded. Because this cannot be, it must be the case that there is some $n\in\omega$ such that $\lambda^{\mathcal{Q}_n}$ is a successor ordinal, in which case we can finish as before. □

THEOREM 2.42 (From condensation to commuting). *Assume* $AD^+$. *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair such that* $\Sigma$ *has branch condensation and is fullness preserving. Suppose there is a good pointclass* $\Gamma$ *such that* $Code(\Sigma)\in\Delta_{\underset{\sim}{\Gamma}}$. *Suppose further that there is a good pointclass* $\Gamma$ *such that* $Code(\Sigma)\in\Delta_{\underset{\sim}{\Gamma}}$. *Then* $\Sigma$ *is positional and hence, commuting (see Lemma* 2.36*).*

PROOF. We start with the following claim.

*Claim.* Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that for any $(\vec{\mathcal{S}}_i,\mathcal{Q}_i)\in I(\mathcal{P},\Sigma)$ $(i=0,1,2)$, there are $(\mathcal{S}_i,\mathcal{R})\in I(\mathcal{Q}_i,\Sigma_{\mathcal{Q}_i,\vec{\mathcal{S}}_i})$ such that

(1) $\Sigma_{\mathcal{R},\vec{\mathcal{S}}_0}=\Sigma_{\mathcal{R},\vec{\mathcal{S}}_1}=\Sigma_{\mathcal{R},\vec{\mathcal{S}}_2}$.
(2) if for some $\alpha<\min(\lambda^{\mathcal{Q}_1},\lambda^{\mathcal{Q}_2})$, $\mathcal{Q}_1(\alpha+1)=\mathcal{Q}_2(\alpha+1)$ and $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{S}}_1}=\Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{S}}_2}$ then letting $\mathcal{S}^k$ be the part of $\mathcal{S}_k$ $(k=0,1,2)$ that is based on $\mathcal{Q}_k(\alpha+1)$, $lh(\mathcal{S}^0)=lh(\mathcal{S}^1)=lh(\mathcal{S}^2)$ and for every $\xi<lh(\mathcal{S}^0)$, $\mathcal{S}^0\restriction\xi=\mathcal{S}^1\restriction\xi=\mathcal{S}^2\restriction\xi$.

Then $\Sigma$ is positional and hence, commuting.

PROOF. Notice that by the proof of the Dodd-Jensen property (see Theorem 4.8 of [**43**]) if $(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{U}},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ and $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}=\Sigma_{\mathcal{Q},\vec{\mathcal{U}}}$ then $\pi^{\vec{\mathcal{T}}}=\pi^{\vec{\mathcal{U}}}$. Suppose then $\mathcal{P}$ isn't positional. By induction we can assume that if $(\vec{\mathcal{T}},\mathcal{Q})\in B(\mathcal{P},\Sigma)$

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then $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ is positional. We have two cases. First assume that $\lambda^{\mathcal{P}}$ is a successor ordinal. Fix $(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{U}},\mathcal{Q}) \in I(\mathcal{P},\Sigma)$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q},\vec{\mathcal{U}}}$. Notice that $\Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1),\vec{\mathcal{U}}}$. By our assumption there are $(\mathcal{S}_1,\mathcal{R}) \in I(\mathcal{Q},\Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$, $(\mathcal{S}_2,\mathcal{R}) \in I(\mathcal{Q},\Sigma_{\mathcal{Q},\vec{\mathcal{U}}})$ and $(\mathcal{S}_3,\mathcal{R}) \in I(\mathcal{P},\Sigma)$ such that

$$\Sigma_{\mathcal{R},\vec{\mathcal{U}}^\frown\mathcal{S}_1} = \Sigma_{\mathcal{R},\vec{\mathcal{U}}^\frown\mathcal{S}_2} = \Sigma_{\mathcal{R},\vec{\mathcal{U}}^\frown\mathcal{S}_3}.$$

From the discussion above we get that

$$\pi^{\vec{\mathcal{T}}^\frown\mathcal{S}_1} = \pi^{\vec{\mathcal{U}}^\frown\mathcal{S}_2} = \pi^{\mathcal{S}_3}.$$

Moreover, we have that in fact $lh(\mathcal{S}_1) = lh(\mathcal{S}_2)$ and for any $\alpha < lh(\mathcal{S}_1)$, $\mathcal{S}_1 \restriction \alpha = \mathcal{S}_2 \restriction \alpha$. This then implies that

$$rng(\pi^{\mathcal{S}_1}) \cap rng(\pi^{\mathcal{S}_2}) \cap \delta^{\mathcal{R}} \text{ is cofinal in } \delta^{\mathcal{R}}$$

implying that in fact $\mathcal{S}_1 = \mathcal{S}_2$ (see Lemma 1.13). But then

$$\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma^{\pi^{\mathcal{S}_1}}_{\mathcal{R},\mathcal{S}_3} = \Sigma^{\pi^{\mathcal{S}_2}}_{\mathcal{R},\mathcal{S}_3} = \Sigma_{\mathcal{Q},\vec{\mathcal{U}}}.$$

The case when $\lambda^{\mathcal{P}}$ is limit uses the same idea. First we claim the following:

*Subclaim.* Whenever $(\vec{\mathcal{T}},\mathcal{Q}_1),(\vec{\mathcal{U}},\mathcal{Q}_2) \in I(\mathcal{P},\Sigma)$ are such that for some $\alpha$, (i) $\alpha(\vec{\mathcal{T}}),\alpha(\vec{\mathcal{U}}) \leq \alpha+1$, (ii) $\mathcal{Q}_1(\alpha+1) = \mathcal{Q}_2(\alpha+1)$, and (iii) $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{U}}}$ then $\Sigma_{\mathcal{Q}_1(\alpha+1),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}_2(\alpha+1),\vec{\mathcal{U}}}$.

Proof. The proof is as above. Towards a contradiction assume that $(\vec{\mathcal{T}},\mathcal{Q}_1)$, $(\vec{\mathcal{U}},\mathcal{Q}_2)$ are such that (i), (ii), and (iii) hold yet, $\Sigma_{\mathcal{Q}_1(\alpha+1),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}_2(\alpha+1),\vec{\mathcal{U}}}$. We can again fix $(\mathcal{S}_1,\mathcal{R}) \in I(\mathcal{Q}_1,\Sigma_{\mathcal{Q}_1,\vec{\mathcal{T}}})$, $(\mathcal{S}_2,\mathcal{R}) \in I(\mathcal{Q}_2,\Sigma_{\mathcal{Q},\vec{\mathcal{U}}})$ and $(\mathcal{S}_3,\mathcal{R}) \in I(\mathcal{P},\Sigma)$ such that

$$\Sigma_{\mathcal{R},\vec{\mathcal{T}}^\frown\mathcal{S}_1} = \Sigma_{\mathcal{R},\vec{\mathcal{U}}^\frown\mathcal{S}_2} = \Sigma_{\mathcal{R},\mathcal{S}_3}.$$

This then gives that

$$\pi^{\vec{\mathcal{T}}^\frown\mathcal{S}_1} = \pi^{\vec{\mathcal{U}}^\frown\mathcal{S}_2} = \pi^{\mathcal{S}_3}.$$

As before, we can assume that if $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ are the parts of $\mathcal{S}_1$ and $\mathcal{S}_2$ which are based on $\mathcal{Q}_1(\alpha+1) = \mathcal{Q}_2(\alpha+1)$ then $lh(\mathcal{S}_1^*) = lh(\mathcal{S}_2^*)$ and for every $\beta < lh(\mathcal{S}_1^*)$, $\mathcal{S}_1^* \restriction \beta = \mathcal{S}_2^* \restriction \beta$. Let $\mathcal{S} = \mathcal{S}_1^* \restriction lh(\mathcal{S}_1^*)-1 = \mathcal{S}_2^* \restriction lh(\mathcal{S}_2^*)-1$. We now claim that

$$rng(\pi^{\mathcal{S}_1^*}) \cap rng(\pi^{\mathcal{S}_2^*}) \text{ is cofinal in } \delta(\mathcal{S}).$$

To see this, notice that if

$$A_1 = \{\pi^{\vec{\mathcal{T}}}(f)(a) : f \in \mathcal{P} \wedge a \in [\delta_\alpha^{\mathcal{Q}_1}]^{<\omega}\} \cap \delta_{\alpha+1}^{\mathcal{Q}_1}$$

and

$$A_2 = \{\pi^{\vec{\mathcal{U}}}(f)(a) : f \in \mathcal{P} \wedge a \in [\delta_\alpha^{\mathcal{Q}_2}]^{<\omega}\} \cap \delta_{\alpha+1}^{\mathcal{Q}_2}$$

then $A_1$ and $A_2$ are cofinal subsets of $\delta_{\alpha+1}^{\mathcal{Q}_1} = \delta_{\alpha+1}^{\mathcal{Q}_2}$ and

$$\pi^{\mathcal{S}_1^*}[A_1] = \pi^{\mathcal{S}_1^*}[A_2].$$

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Hence, $rng(\pi^{\mathcal{S}_1^*}) \cap rng(\pi^{\mathcal{S}_2^*})$ is cofinal in $\delta(\mathcal{S})$ implying that in fact, $\mathcal{S}_1^* = \mathcal{S}_2^*$. This gives a contradiction because

$$\Sigma_{\mathcal{Q}_1(\alpha+1),\vec{\mathcal{T}}} = \Sigma^{\pi^{\mathcal{S}_1^*}}_{\mathcal{R}(\pi^{\mathcal{S}_1^*}(\alpha)+1),\mathcal{S}_3} = \Sigma^{\pi^{\mathcal{S}_2^*}}_{\mathcal{R}(\pi^{\mathcal{S}_2^*}(\alpha)+1),\mathcal{S}_3} = \Sigma_{\mathcal{Q}_2(\alpha+1),\vec{\mathcal{U}}}.$$

□

Now towards a contradiction, we assume that $\Sigma$ isn't positional. We would like to produce two pairs $(\vec{\mathcal{T}},\mathcal{Q}_1),(\vec{\mathcal{U}},\mathcal{Q}_2)$ such that for some $\alpha$, (i) $\delta(\vec{\mathcal{T}}),\delta(\vec{\mathcal{U}}) \leq \delta^{\mathcal{Q}_1}_{\alpha+1} = \delta^{\mathcal{Q}_2}_{\alpha+1}$, (ii) $\mathcal{Q}_1(\alpha+1) = \mathcal{Q}_2(\alpha+1)$, and (iii) $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{U}}}$ and $\Sigma_{\mathcal{Q}_1(\alpha+1),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}_2(\alpha+1),\vec{\mathcal{U}}}$. This then gives the contradiction. Let $(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{U}},\mathcal{Q}) \in I(\mathcal{P},\Sigma)$ be two pairs such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q},\vec{\mathcal{U}}}$. Suppose first that for some $\alpha < \lambda^{\mathcal{Q}}$, $\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{U}}}$ yet $\Sigma_{\mathcal{Q}(\alpha+1),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}(\alpha+1),\vec{\mathcal{U}}}$. It is then easy to modify $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ so that the new stacks satisfy (i)-(iii): just through away the parts that do not contribute to producing $\mathcal{Q}(\alpha+1)$. Suppose then that there is no such $\alpha$. Let $\alpha$ be the least $\gamma \leq \lambda^{\mathcal{Q}}$ such that $\Sigma_{\mathcal{Q}(\gamma),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}(\gamma),\vec{\mathcal{U}}}$. It then follows that $\gamma$ is a limit ordinal. Let then $\vec{\mathcal{S}} = (\mathcal{M}_\xi,\mathcal{M}^*_\xi,\vec{\mathcal{S}}_\xi,\pi_{\xi,\nu} : \xi < \nu \leq \eta)$ be a minimal disagreement between $\Sigma_{\mathcal{Q}(\gamma),\vec{\mathcal{T}}}$ and $\Sigma_{\mathcal{Q}(\gamma),\vec{\mathcal{U}}}$. Then it is easy to modify $\vec{\mathcal{T}}^\frown \oplus_{\xi<\eta}\vec{\mathcal{S}}_\xi$ and $\vec{\mathcal{U}}^\frown \oplus_{\xi<\eta}\vec{\mathcal{S}}_\xi$ in such a way that the resulting stacks satisfy (i)-(iii): just through away the parts that do not contribute to producing $\mathcal{M}^*_\eta$. □

Notice that the hypothesis of the claim is always satisfied as it follows from our comparison theorem Theorem 2.28. It then follows that $\Sigma$ is positional.

□

The following is a corollary of the proof of Theorem 2.42, especially of the proof of the claim in the proof of Theorem 2.42.

COROLLARY 2.43. *Suppose $(\mathcal{P},\Sigma)$ is as in Theorem 2.42. Then whenever $(\vec{\mathcal{T}},\mathcal{Q}_1),(\vec{\mathcal{U}},\mathcal{Q}_2) \in I(\mathcal{P},\Sigma)$ are such that for some $\alpha \leq \min(\lambda^{\mathcal{Q}_1},\lambda^{\mathcal{Q}_2})$, $\mathcal{Q}_1(\alpha) = \mathcal{Q}_2(\alpha)$ then $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{T}}}$.*

PROOF. The proof is like the end of the proof of Theorem 2.42. Suppose towards a contradiction that there are $(\vec{\mathcal{T}},\mathcal{Q}_1),(\vec{\mathcal{U}},\mathcal{Q}_2) \in I(\mathcal{P},\Sigma)$ such that for some $\alpha \leq \min(\lambda^{\mathcal{Q}_1},\lambda^{\mathcal{Q}_2})$, $\mathcal{Q}_1(\alpha) = \mathcal{Q}_2(\alpha)$ and $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{T}}}$. Let $\beta$ be the least $\alpha \leq \min(\lambda^{\mathcal{Q}_1},\lambda^{\mathcal{Q}_2})$ such that $\mathcal{Q}_1(\alpha) = \mathcal{Q}_2(\alpha)$ and $\Sigma_{\mathcal{Q}_1(\alpha),\vec{\mathcal{T}}} \neq \Sigma_{\mathcal{Q}_2(\alpha),\vec{\mathcal{T}}}$. If $\beta = \gamma+1$ then we can get into the situation of the claim in the proof of Theorem 2.42: just through away the parts of $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ that do not contribute to producing $\mathcal{Q}_1(\beta) = \mathcal{Q}_2(\beta)$. Thus, we must have that $\beta$ is limit. In this case, to get into the situation of the claim, repeat the argument from the last paragraph of the proof of Theorem 2.42. □

It follows from Corollary 2.43 that under $AD^+$ if $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation, is fullness preserving and there is a good pointclass $\Gamma$ such that $Code(\Sigma) \in \Delta_{\underset{\sim}{\Gamma}}$ then whenever $(\vec{\mathcal{T}},\mathcal{Q}),(\vec{\mathcal{S}},\mathcal{Q}) \in B(\mathcal{P},\Sigma) \cup I(\mathcal{P},\Sigma)$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q},\vec{\mathcal{S}}}$. It is not ambiguous to omit $\vec{\mathcal{T}}$ from the subscript of $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ and we do that from now on. If $\mathcal{Q}$ is any iterate of $\mathcal{P}$ via $\Sigma$ such that for some $\vec{\mathcal{T}}$, $(\mathcal{Q},\vec{\mathcal{T}}) \in I(\mathcal{P},\Sigma)$ then we let $\Sigma_{\mathcal{Q}}$ be the $\vec{\mathcal{T}}$-tail of $\Sigma$. We also let

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$$pB(\mathcal{P},\Sigma)=\{\mathcal{Q}:\exists\vec{\mathcal{S}}(\vec{\mathcal{S}},\mathcal{Q})\in B(\mathcal{P},\Sigma)\},$$
$$pI(\mathcal{P},\Sigma)=\{\mathcal{Q}:\exists\vec{\mathcal{S}}(\vec{\mathcal{S}},\mathcal{Q})\in I(\mathcal{P},\Sigma)\}.$$

Also, if $\mathcal{Q}\in pI(\mathcal{P},\Sigma)$ and $\mathcal{R}\in pI(\mathcal{R},\Sigma_{\mathcal{R}})$ then we let $\pi^{\mathcal{P},\Sigma}_{\mathcal{Q},\mathcal{R}}:\mathcal{Q}\to\mathcal{R}$ be the iteration embedding, and given $\mathcal{Q}\in pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma)$ and $\mathcal{R}\in pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma)$ we write $\mathcal{Q}\preceq^{\mathcal{P},\Sigma}\mathcal{R}$ if $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ doesn't win the comparison with $(\mathcal{R},\Sigma_{\mathcal{R}})$.

Notice that our proof of Theorem 2.42 generalizes to the case when $\Sigma$ is $\Gamma$-fullness preserving instead of fullness preserving.

COROLLARY 2.44. *Assume $AD^+$. Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is $\Gamma$-fullness preserving for some $\Gamma$ which is closed under continuous images and preimages. Suppose further that there is a good pointclass $\Gamma^*$ such that $Code(\Sigma)\in\underset{\sim}{\Delta}_{\Gamma^*}$ and $\Gamma\subseteq\underset{\sim}{\Delta}_{\Gamma^*}$. Then $\Sigma$ is positional and hence, commuting.*

## 2.7. The diamond comparison argument

Our comparison theorem, Theorem 2.28, is applicable in $AD^+$-context. However, in the proof of Theorem 6.19, we will need a way of comparing hod pairs in models of $ZFC$. Also, sometimes, as in Section 3.5, one doesn't know a priori, that the strategies of hod pairs involved have branch condensation. The comparison argument we present in this section does not need branch condensation in the limit case. This makes the argument useful not only in the aforementioned context but also in core model induction applications. An important disadvantage of the argument presented below is that in general, working under $ZFC$, its hard to make sense of fullness preservation. We present the comparison argument for pairs that are *close* and are *of the same kind*. We expect that in many situations including those mentioned above it is possible to define a meaningful notion of fullness preservation and show that two fullness preserving strategies are of the same kind, and hence, the comparison argument of this section can be applied.

First we introduce closeness. Intuitively speaking two strategies are close if branches chosen by them for the same stack yield exactly the same model.

DEFINITION 2.45. *Suppose $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ are hod mice such that $\lambda^{\mathcal{P}}$ is a successor ordinal and $\Sigma_{\mathcal{P}^-}=\Lambda_{\mathcal{P}^-}$. We say $\Sigma$ and $\Lambda$ are close if II doesn't lose any run of the game $\mathcal{G}(\mathcal{P},\Sigma,\Lambda)$ defined as follows. In $\mathcal{G}(\mathcal{P},\Sigma,\Lambda)$, just like the usual iteration game I plays extenders and unlike the usual iteration game, II plays two branches, one according to $\Sigma$ and one according to $\Lambda$. II loses, without having much to do during the game, if the play of the game produces two stacks with normal components $\vec{\mathcal{T}}=(\mathcal{M}_\alpha,\mathcal{T}_\alpha:\alpha<\eta)$ and $\vec{\mathcal{U}}=(\mathcal{N}_\alpha,\mathcal{U}_\alpha:\alpha<\eta)$ such that*

1. *$\vec{\mathcal{T}}$ is according to $\Sigma$, and $\vec{\mathcal{U}}$ is according to $\Lambda$,*
2. *for every $\alpha\leq\eta$, $lh(\mathcal{T}_\alpha)=lh(\mathcal{U}_\alpha)$ and $\mathcal{M}_\alpha=\mathcal{N}_\alpha$,*
3. *for every $\alpha,\beta$ such that $\alpha\leq\eta$, $\beta<lh(\mathcal{T}_\alpha)$, $\mathcal{T}_\alpha\restriction\beta=\mathcal{U}_\alpha\restriction\beta$,*
4. *if $\eta$ is a successor ordinal then $lh(\mathcal{T}_\eta)$ is a limit ordinal, and if $b=\Sigma(\vec{\mathcal{T}})$ and $c=\Lambda(\vec{\mathcal{U}})$ then $\mathcal{M}_b^{\vec{\mathcal{T}}}\neq\mathcal{M}_c^{\vec{\mathcal{U}}}$.*

DEFINITION 2.46. *Two hod pairs $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are of the same kind if whenever $(\mathcal{R},\Psi)$ and $(\mathcal{S},\Phi)$ are tails of $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ respectively then*

1. *for all $\alpha<\min(\lambda^{\mathcal{R}},\lambda^{\mathcal{S}})$ such that $\mathcal{R}(\alpha)=\mathcal{S}(\alpha)$ and $\Psi_{\mathcal{R}(\alpha)}=\Phi_{\mathcal{S}(\alpha)}$ if $\mathcal{R}(\alpha+1)\trianglelefteq\mathcal{S}(\alpha+1)$ or $\mathcal{S}(\alpha+1)\trianglelefteq\mathcal{R}(\alpha+1)$ then*

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(a) $\mathcal{R}(\alpha+1)=\mathcal{S}(\alpha+1)$;
(b) *there is a tail* $(\mathcal{M},\Psi^*)$ *of* $(\mathcal{R}(\alpha+1),\Psi_{\mathcal{R}(\alpha+1)})$ *and a tail* $(\mathcal{M},\Phi^*)$ *of* $(\mathcal{S}(\alpha+1),\Phi_{\mathcal{S}(\alpha+1)})$ *such that* $\Psi^*$ *and* $\Phi^*$ *are close;*

(2) *for all limit* $\alpha\leq\min(\lambda^{\mathcal{R}},\lambda^{\mathcal{S}})$ *if for all* $\beta<\alpha$, $\mathcal{R}(\beta)=\mathcal{S}(\beta)$ *and* $\Psi_{\mathcal{R}(\beta)}=\Phi_{\mathcal{S}(\beta)}$ *then* $\mathcal{R}(\alpha)=\mathcal{S}(\alpha)$.

Here is the comparison theorem.

THEOREM 2.47 (The diamond comparison). *Suppose* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$ *are two hod pairs such that*

(1) *for any* $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P},\Sigma)$ *and* $(\vec{\mathcal{S}},\mathcal{W})\in I(\mathcal{Q},\Lambda)$ *and any* $\alpha<\lambda^{\mathcal{R}}$ *and* $\beta<\lambda^{\mathcal{W}}$, *both* $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}$ *and* $\Lambda_{\mathcal{W}(\beta+1),\vec{\mathcal{S}}}$ *have branch condensation,*
(2) $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$ *are of the same kind,*
(3) *there is a cardinal* $\kappa$ *such that both* $\Sigma$ *and* $\Lambda$ *are* $(\kappa^+,\kappa^+)$*-iteration strategies,* $\mathcal{P},\mathcal{Q}\in H_{\kappa^+}$ *and for any* $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P},\Sigma)\cup B(\mathcal{P},\Sigma)$ *and* $(\vec{\mathcal{U}},\mathcal{S})\in I(\mathcal{Q},\Lambda)\cup B(\mathcal{Q},\Lambda)$ *such that* $\lambda^{\mathcal{R}}$ *and* $\lambda^{\mathcal{S}}$ *are successor ordinals, both* $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}$ *and* $\Lambda_{\mathcal{S},\vec{\mathcal{U}}}$ *are* $(\kappa^++1,\kappa^++1)$*-iteration strategies.*

*Then there are* $(\mathcal{R},\Psi)$ *and* $(\mathcal{S},\Phi)$ *such that*

(1) $(\mathcal{R},\Psi)$ *is a tail of* $(\mathcal{P},\Sigma)$,
(2) $(\mathcal{S},\Phi)$ *is a tail of* $(\mathcal{Q},\Lambda)$,
(3) $\mathcal{R},\mathcal{S}\in H_{\kappa^+}$,
(4) *either*
   (a) $\mathcal{R}\trianglelefteq_{hod}\mathcal{S}$ *and* $\Psi=\Phi_{\mathcal{R}}$,
   *or*
   (b) $\mathcal{S}\trianglelefteq_{hod}\mathcal{R}$ *and* $\Phi=\Psi_{\mathcal{S}}$.

This entire section is devoted to the proof of Theorem 2.47. Given two hod pairs $(\mathcal{P},\Sigma)$ and $(\mathcal{P},\Lambda)$ such that for all $\alpha<\lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}=\Lambda_{\mathcal{P}(\alpha)}$ we can try to compare $(\mathcal{P},\Sigma)$ with $(\mathcal{Q},\Lambda)$ by using minimal disagreements (see Definition 2.39). Then the hope is that if we do this long enough then eventually we will end up with a common tail. The following definition is motivated by this discussion. Suppose $\vec{\mathcal{T}}$ is a stack on a hod premouse $\mathcal{P}$ with last model $\mathcal{R}$ such that $\pi^{\vec{\mathcal{T}}}$ exists. Recall that we let $\alpha(\vec{\mathcal{T}})$ be the least $\gamma\leq\lambda^{\mathcal{R}}$ such that $\delta(\vec{\mathcal{T}})\leq\delta^{\mathcal{R}}_{\gamma}$.

DEFINITION 2.48 (Bad block). *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair. Then* $((\mathcal{P}_i:i\leq 4),(\vec{\mathcal{T}}_i:i\leq 3),\Lambda,(\pi_i:i\leq 4),\xi,\vec{\mathcal{U}})$ *is a bad block for* $(\mathcal{P},\Sigma)$ *(see Figure* 2.7.1*) if*

(1) $\mathcal{P}_0=\mathcal{P}$, $\vec{\mathcal{T}}_0$ *is a stack on* $\mathcal{P}_0$ *according to* $\Sigma$ *with last model* $\mathcal{P}_1$, *and* $\pi_0=\pi^{\vec{\mathcal{T}}_0}$,
(2) $\Lambda$ *is a strategy for* $\mathcal{P}_1$ *such that* $(\mathcal{P}_1,\Lambda)$ *is a hod pair, and* $\Lambda_{\mathcal{P}_1(\alpha(\vec{\mathcal{T}}_0))}=\Sigma_{\mathcal{P}_1(\alpha(\vec{\mathcal{T}}_0)),\vec{\mathcal{T}}_0}$,
(3) *there is* $\alpha\in(\alpha(\vec{\mathcal{T}}_0),\lambda^{\mathcal{P}_1})$ *such that* $\Lambda_{\mathcal{P}_1(\alpha+1)}\neq\Sigma_{\mathcal{P}_1(\alpha+1),\vec{\mathcal{T}}_0}$ *but* $\Lambda_{\mathcal{P}_1(\alpha)}=\Sigma_{\mathcal{P}_1(\alpha),\vec{\mathcal{T}}_0}$, *and* $\xi$ *is the least such* $\alpha$,
(4) $\vec{\mathcal{T}}_1$ *is a stack on* $\mathcal{P}_1(\xi+1)$ *such that*
   (a) $\vec{\mathcal{T}}_1$ *has successor length and its last component has limit length,*
   (b) $\vec{\mathcal{T}}_1$ *is according to both* $\Lambda_{\mathcal{P}_1(\xi+1)}$ *and* $\Sigma_{\mathcal{P}_1(\xi+1),\vec{\mathcal{T}}_0}$,
   (c) $\mathcal{P}_2$ *and* $\mathcal{P}_3$ *are the last models of* $\vec{\mathcal{T}}_1$ *according to respectively* $\Lambda_{\mathcal{P}_1(\xi+1)}$ *and* $\Sigma_{\mathcal{P}_1(\xi+1),\vec{\mathcal{T}}_0}$,

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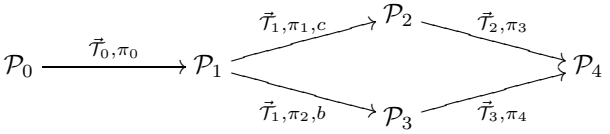


(1) $c = \Lambda(\vec{\mathcal{T}}_1), b = \Sigma_{\mathcal{P}_1,\vec{\mathcal{T}}_0}(\vec{\mathcal{T}}_1), b \neq c$,
(2) $\pi_i$ are the iteration embeddings,
(3) $\xi$ is the least $\alpha \in (\alpha(\vec{\mathcal{T}}_0), \lambda^{\mathcal{P}_1})$ such that $\mathcal{T}_1$ is a stack on $\mathcal{P}_1(\alpha+1)$, $\Lambda_{\mathcal{P}_1(\alpha+1)} \neq \Sigma_{\mathcal{P}_1(\alpha+1),\vec{\mathcal{T}}_0}$, and $\Lambda_{\mathcal{P}_1(\alpha)} = \Sigma_{\mathcal{P}_1(\alpha),\vec{\mathcal{T}}_0}$,
(4) for any $\alpha < \lambda^{\mathcal{P}_4}$, $\Lambda_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_2{}^\frown\vec{\mathcal{T}}_2} = \Sigma_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_3{}^\frown\vec{\mathcal{T}}_3}$,
(5) $\vec{\mathcal{U}} = \vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_3{}^\frown\vec{\mathcal{T}}_3$.

FIGURE 2.7.1. Bad block.

(d) $\pi_1 : \mathcal{P}_1 \to \mathcal{P}_2$ *and* $\pi_2 : \mathcal{P}_1 \to \mathcal{P}_3$ *are the iteration embeddings,*
(5) $\vec{\mathcal{T}}_2$ *is a stack on* $\mathcal{P}_2$, $\vec{\mathcal{T}}_3$ *is a stack on* $\mathcal{P}_3$, $\vec{\mathcal{T}}_2$ *and* $\vec{\mathcal{T}}_3$ *have a common last model* $\mathcal{P}_4$, $\pi_3 = \pi^{\vec{\mathcal{T}}_2}$ *and* $\pi_4 = \pi^{\vec{\mathcal{T}}_3}$,
(6) *for any* $\alpha < \lambda^{\mathcal{P}_4}$, $\Lambda_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_2{}^\frown\vec{\mathcal{T}}_2} = \Sigma_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_3{}^\frown\vec{\mathcal{T}}_3}$,
(7) $\vec{\mathcal{U}} = \vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{T}}_1{}^\frown\mathcal{P}_3{}^\frown\vec{\mathcal{T}}_3$.

A consequence of the next lemma is that if we compare hod mice via hitting minimal disagreements then we will succeed as otherwise we will produce a long sequence of bad blocks.

LEMMA 2.49 (No bad sequence). *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\Sigma$ *is a* $(\kappa^+, \kappa^+)$*-iteration strategy for some infinite* $\kappa$. *Then there is no sequence* $(B^\alpha, j^t_{\beta,\alpha}, j^b_{\beta,\alpha} : \beta < \alpha < \kappa^+)$ *such that*

(1) $B^\alpha = ((\mathcal{P}^\alpha_i : i \leq 4), (\vec{\mathcal{T}}^\alpha_i : i \leq 3), \Lambda^\alpha, (\pi^\alpha_i : i \leq 4), \xi^\alpha, \vec{\mathcal{U}}^\alpha)$, $B^\alpha \in H_{\kappa^+}$ *and* $(\mathcal{P}^\alpha_1, \Lambda^\alpha)$ *is a hod pair such that* $\Lambda^\alpha$ *is a* $(\kappa^+, \kappa^+)$*-iteration strategy,*
(2) $B^0$ *is a bad block for* $(\mathcal{P}, \Sigma)$,
(3) $B^{\alpha+1}$ *is a bad block for* $(\mathcal{P}^\alpha_4, \Sigma_{\mathcal{P}^\alpha_4, \oplus_{\beta\leq\alpha}\mathcal{U}^\beta})$ *(thus,* $\mathcal{P}^\alpha_4 = \mathcal{P}^{\alpha+1}_0$*),*
(4) $j^t_{\alpha,\beta} : \mathcal{P}^\alpha_0 \to \mathcal{P}^\beta_0$ *is the iteration embedding along the "top", i.e.,*
$$j^t_{\alpha,\beta} = \oplus_{\xi\in[\alpha,\beta)}(\pi^\xi_3 \circ \pi^\xi_1 \circ \pi^\xi_0),$$
(5) $j^b_{\alpha,\beta} : \mathcal{P}^\alpha_0 \to \mathcal{P}^\beta_0$ *is the iteration embedding along the "bottom", i.e.,*
$$j^b_{\alpha,\beta} = \oplus_{\xi\in[\alpha,\beta)}(\pi^\xi_4 \circ \pi^\xi_2 \circ \pi^\xi_0),$$
(6) *If* $\alpha$ *is limit then* $\mathcal{P}^\alpha_0 = dirlim(\mathcal{P}^\beta_0, j^t_{\beta,\alpha} : \beta < \alpha < \kappa^+) = dirlim(\mathcal{P}^\beta_0, j^b_{\beta,\alpha} : \beta < \alpha < \kappa^+)$.

PROOF. Towards a contradiction suppose that there is such a sequence and let $\vec{B} = (B^\alpha, j^t_{\alpha,\beta}, j^b_{\alpha,\beta} : \alpha < \beta < \kappa^+)$ be one. Let $X_0 \prec X_1 \prec H_{\kappa^{++}}$ be such that $\vec{B} \in X_0$ and for $i = 0, 1$, $|X_i| = \kappa$. Let $\pi_0 : H_0 \to X_0$ and $\pi_1 : H_1 \to X_1$ be the inverses of the transitive collapses of $X_0$ and $X_1$. We then get $\pi : H_0 \to H_1$. Consulting Figure 2.7.2 will be helpful. Let $\kappa_0$ and $\kappa_1$ be the collapses of $\kappa^+$ in respectively $H_0$ and $H_1$. Note that all sets in $B^\alpha = ((\mathcal{P}^\alpha_i : i \leq 4), (\vec{\mathcal{T}}^\alpha_i : i \leq 3), \Lambda^\alpha, (\pi^\alpha_i : i \leq 4), \xi^\alpha, \vec{\mathcal{U}}^\alpha)$ except $\Lambda^\alpha$ are of size $< \kappa^+$, and therefore if $\alpha < \kappa_k$ then

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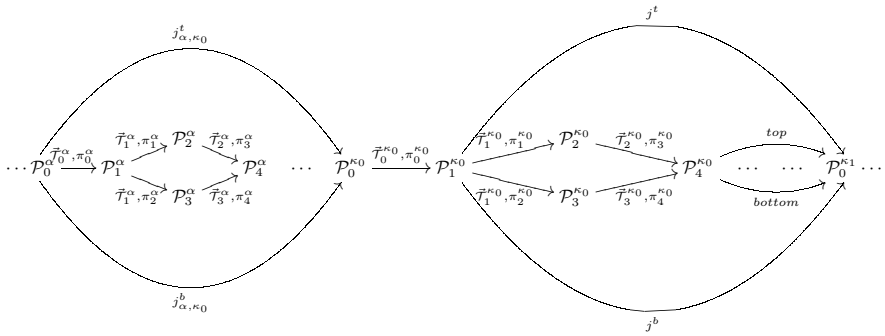


FIGURE 2.7.2. A bad sequence of length $\kappa^+$

$$((\mathcal{P}_i^\alpha : i \leq 4), (\vec{\mathcal{T}}_i^\alpha : i \leq 3), (\pi_i^\alpha : i \leq 4), \xi^\alpha, \vec{\mathcal{U}}^\alpha) \in H_k,\ k = 0, 1,$$

and if $\alpha < \kappa_0$

$$\pi(((\mathcal{P}_i^\alpha : i \leq 4), (\vec{\mathcal{T}}_i^\alpha : i \leq 3), (\pi_i^\alpha : i \leq 4), \xi^\alpha, \vec{\mathcal{U}}^\alpha)) = ((\mathcal{P}_i^\alpha : i \leq 4), (\vec{\mathcal{T}}_i^\alpha : i \leq 3), (\pi_i^\alpha : i \leq 4), \xi^\alpha, \vec{\mathcal{U}}^\alpha).$$

*Claim.* $j^t_{\kappa_0,\kappa_1} = \pi \restriction \mathcal{P}_0^{\kappa_0} = j^b_{\kappa_0,\kappa_1}$.

PROOF. The proof is both easy and standard. Let $x \in \mathcal{P}_0^{\kappa_0}$. Then there is $\alpha < \kappa_0$ such that for some $y \in \mathcal{P}_0^\alpha$, $j^t_{\alpha,\kappa_0}(y) = x$. Then

$$\pi(x) = \pi(j^t_{\alpha,\kappa_0}(y)) = \pi(j^t_{\alpha,\kappa_0})(\pi(y)) = j^t_{\alpha,\kappa_1}(y) = j^t_{\kappa_0,\kappa_1}(j^t_{\alpha,\kappa_0}(y)) = j^t_{\kappa_0,\kappa_1}(x).$$

The rest is similar. □

Let $\vec{\mathcal{T}} = \oplus_{\xi<\kappa_0}\vec{\mathcal{U}}^\xi$. We have that $\vec{\mathcal{T}}$ is according to $\Sigma$. Let $j^t : \mathcal{P}_1^{\kappa_0} \to \mathcal{P}_0^{\kappa_1}$ be the embedding along the "top" and let $j^b : \mathcal{P}_1^{\kappa_0} \to \mathcal{P}_0^{\kappa_1}$ be the embedding along the "bottom". Then we claim that $j^t = j^b$. To see this, let $x \in \mathcal{P}_1^{\kappa_0}$. There is then a function $f \in \mathcal{P}_0^{\kappa_0}$ and $a \in \delta(\vec{\mathcal{T}}_0^{\kappa_0})^{<\omega}$ such that $x = \pi_0^{\kappa_0}(f)(a)$. Then

$$j^t(x) = j^t(\pi_0^{\kappa_0}(f))(j^t(a)) = j^t_{\kappa_0,\kappa_1}(f)(j^t(a)) = \pi(f)(j^t(a)).$$

Similarly, $j^b(x) = \pi(f)(j^b(a))$. Thus, it is enough to show that $j^t(a) = j^b(a)$. This follows from the fact that further iterations never disagree on how $a$ is moved (this is guaranteed by conditions 2 and 3 of Definition 2.48).

It is now easy to get a contradiction. Let $j = j^t = j^b$. We have that $\vec{\mathcal{T}}_1^{\kappa_0}$ is a stack on $\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1)$ and

$$j \restriction \mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1) : \mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1) \to \mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1)$$

is the iteration embedding according to both

$$\Lambda^{\kappa_0}_{\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1)}$$

and

$$\Sigma_{\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1),(\oplus_{\alpha<\kappa_0}\vec{\mathcal{U}}^\alpha)^\frown\vec{\mathcal{T}}_0^{\kappa_0}}.$$

Let $\vec{\mathcal{S}}$ be the stack on $\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1)$ along the "top" with last model $\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1)$. Because of clause 6 of the definition of bad block, we have that

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$$\Lambda_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),\vec{\mathcal{S}}} = \Sigma_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),(\oplus_{\alpha<\kappa_1}\vec{\mathcal{U}}^\alpha)}.$$

By hull condensation of both $\Lambda$ and $\Sigma$ we have that

$$\Lambda_{\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1)}(\vec{\mathcal{T}}_1^{\kappa_0}) = \Lambda_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),\vec{\mathcal{S}}}(j\vec{\mathcal{T}}_1^{\kappa_0})$$
$$\Sigma_{\mathcal{P}_1^{\kappa_0},(\oplus_{\alpha<\kappa_0}\vec{\mathcal{U}}^\alpha)^\frown\vec{\mathcal{T}}_0^{\kappa_0}}(\vec{\mathcal{T}}_1^{\kappa_0}) = \Sigma_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),(\oplus_{\xi<\kappa_1}\vec{\mathcal{U}}^\xi)}(j\vec{\mathcal{T}}_1^{\kappa_0}).$$

But,

$$\Lambda_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),\vec{\mathcal{S}}}(j\vec{\mathcal{T}}_1^{\kappa_0}) = \Sigma_{\mathcal{P}_0^{\kappa_1}(j(\xi^{\kappa_0})+1),(\oplus_{\alpha<\kappa_1}\vec{\mathcal{U}}^\alpha)}(j\vec{\mathcal{T}}_1^{\kappa_0}).$$

Hence,

$$\Lambda_{\mathcal{P}_1^{\kappa_0}(\xi^{\kappa_0}+1)}(\vec{\mathcal{T}}_1^{\kappa_0}) = \Sigma_{\mathcal{P}_1^{\kappa_0},(\oplus_{\alpha<\kappa_0}\vec{\mathcal{U}}^\alpha)^\frown\vec{\mathcal{T}}_0^{\kappa_0}}(\vec{\mathcal{T}}_1^{\kappa_0}).$$

This last equality is a contradiction showing that $B^{\kappa_0}$ is not a bad block of the first kind for $(\mathcal{P}_0^{\kappa_0}, \Sigma_{\mathcal{P}_0^{\kappa_0},\vec{\mathcal{T}}})$. □

We now start proving Theorem 2.47. Fix $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ as in the hypothesis of Theorem 2.47. Suppose comparison is false, i.e., there are no $(\mathcal{R},\Psi)$ and $(\mathcal{S},\Phi)$ such that

(1) $(\mathcal{R},\Psi)$ is a tail of $(\mathcal{P},\Sigma)$,
(2) $(\mathcal{S},\Phi)$ is a tail of $(\mathcal{Q},\Lambda)$,
(3) $\mathcal{R},\mathcal{S} \in H_{\kappa^+}$,
(4) either
  (a) $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Psi = \Phi_{\mathcal{R}}$,
  or
  (b) $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Phi = \Psi_{\mathcal{S}}$.

We can then assume that $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are "minimal counterexamples" in the sense that given any two $(\mathcal{R},\vec{\mathcal{T}}) \in B(\mathcal{P},\Sigma)\cap H_{\kappa^+}$ and $(\mathcal{S},\vec{\mathcal{U}}) \in B(\mathcal{Q},\Lambda)\cap H_{\kappa^+}$, the comparison holds for the following pairs:

(1) $(\mathcal{R},\Sigma_{\mathcal{R},\vec{\mathcal{T}}})$ and any $(\mathcal{M},\Lambda_{\mathcal{M},\vec{\mathcal{W}}})$ where $(\vec{\mathcal{W}},\mathcal{M}) \in I(\mathcal{Q},\Lambda)\cup B(\mathcal{Q},\Lambda)$,
(2) $(\mathcal{S},\Sigma_{\mathcal{S},\vec{\mathcal{U}}})$ and any $(\mathcal{M},\Sigma_{\mathcal{M},\vec{\mathcal{W}}})$ where $(\vec{\mathcal{W}},\mathcal{M}) \in I(\mathcal{P},\Sigma)\cup B(P,\Sigma)$.

We then have two cases.

*Case 1.* $\lambda^{\mathcal{P}}$ is a successor
and
*Case 2.* $\lambda^{\mathcal{P}}$ is limit.

We start with *Case 1.* In this case, by using our minimality assumption and by comparing $(\mathcal{P}^-,\Sigma)$ and $(\mathcal{Q},\Lambda)$, we can find $(\mathcal{P}_1,\Sigma_1)$ and $(\mathcal{Q}_1,\Lambda_1)$ such that $(\mathcal{P}_1,\Sigma_1)$ is a tail of $(\mathcal{P},\Sigma)$, $(\mathcal{Q}_1,\Lambda_1)$ is a tail of $(\mathcal{Q},\Lambda)$, and either

(1) $\mathcal{P}_1^- \triangleleft_{hod} \mathcal{Q}_1$ and $(\Sigma_1)_{\mathcal{P}_1^-} = (\Lambda_1)_{\mathcal{P}_1^-}$
  or
(2) $\mathcal{Q}_1 \trianglelefteq_{hod} \mathcal{P}_1^-$ and $\Lambda_1 = (\Sigma_1)_{\mathcal{Q}_1}$.

If clause 2 holds then we get a contradiction because $(\mathcal{P}_1,\Sigma_1)$ and $(\mathcal{Q}_1,\Lambda_1)$ constitute a successful comparison of $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$. Thus, suppose clause 1 holds. Without loss of generality, then, we can assume that $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ already satisfy clause 1, i.e., $\mathcal{P}^- \triangleleft_{hod} \mathcal{Q}$ and $\Sigma_{\mathcal{P}^-} = \Lambda_{\mathcal{P}^-}$. Let $\alpha < \lambda^{\mathcal{Q}}$ be such that $\mathcal{P}^- = \mathcal{Q}(\alpha)$. It is then enough to show that we can successfully compare $(\mathcal{P},\Sigma)$ and $(\mathcal{Q}(\alpha+1),\Lambda_{\mathcal{Q}(\alpha+1)})$. Again, without loss of generality, we can assume that in fact $\mathcal{Q} = \mathcal{Q}(\alpha+1)$. It then follows from our hypothesis that both $\Sigma$ and $\Lambda$ have

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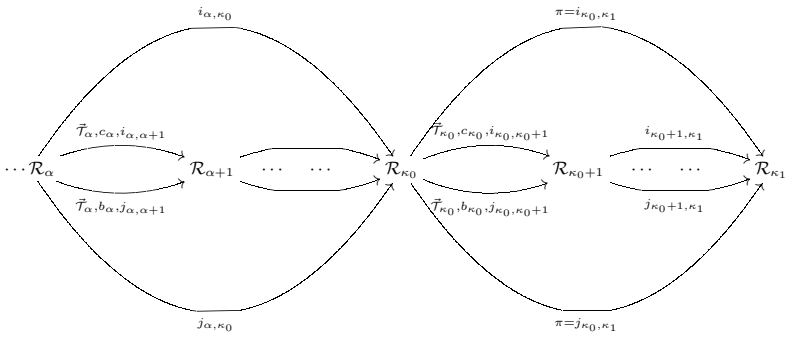


FIGURE 2.7.3. Comparison in case 1.

branch condensation. Because $\mathcal{P}^- = \mathcal{Q}^-$, $\Sigma_{\mathcal{P}^-} = \Lambda_{\mathcal{Q}^-}$, and $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are of the same kind, by just using the usual extender comparison, we can get $(\mathcal{R}, \Psi)$ and $(\mathcal{R}, \Phi)$ such that

(1) $\mathcal{R}$ is a normal $\Sigma$-iterate of $\mathcal{P}$ above $\mathcal{P}^-$ via $\mathcal{T}$ and $\Psi = \Sigma_{\mathcal{R},\mathcal{T}}$,
(2) $\mathcal{R}$ is a normal $\Lambda$-iterate of $\mathcal{Q}$ above $\mathcal{Q}^-$ via $\mathcal{U}$ and $\Phi = \Lambda_{\mathcal{R},\mathcal{U}}$.

The fact that we have the same last model on both sides of the coiteration is a consequence of our requirement that the two pairs are close.

Notice that our hypothesis implies that the usual comparison of $\mathcal{P}$ and $\mathcal{Q}$ above $\mathcal{P}^-$ terminates. It is then enough to compare $(\mathcal{R}, \Psi)$ and $(\mathcal{R}, \Phi)$. We, moreover, have that $\Psi_{\mathcal{R}^-} = \Phi_{\mathcal{R}^-}$. Because we assumed we cannot compare $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$, there cannot be a successful comparison of $(\mathcal{R}, \Psi)$ and $(\mathcal{R}, \Phi)$. Without loss of generality, by clause 1b of Definition 2.46, we can assume that $\Psi$ and $\Phi$ are close (if not then certain tails of the two are close and we can work with these tails instead). It is then easy to obtain a sequence $(\mathcal{R}_\alpha, \vec{\mathcal{T}}_\alpha, b_\alpha, c_\alpha, i_{\alpha,\beta}, j_{\alpha,\beta} : \alpha < \beta < \kappa^+)$ with the following properties (see Figure 2.7.3)

(1) $\mathcal{R}_0 = \mathcal{R}$, $\vec{\mathcal{T}}_0$ is a minimal disagreement between $(\mathcal{R}_0, \Psi)$ and $(\mathcal{R}_0, \Phi)$, $b_0$ is the branch of $\vec{\mathcal{T}}_0$ according $\Psi$ and $c_0$ is the branch of $\vec{\mathcal{T}}_0$ according to $\Phi$, $\mathcal{M}_{b_0}^{\vec{\mathcal{T}}_0} = \mathcal{R}_1 = \mathcal{M}_{c_0}^{\vec{\mathcal{T}}_0}$, $i_{0,1} : \mathcal{R}_0 \to \mathcal{R}_1$ is the iteration embedding according to $\Psi$ and $j_{0,1} : \mathcal{R}_0 \to \mathcal{R}_1$ is the iteration embedding according to $\Phi$,
(2) $\vec{\mathcal{T}}_\alpha$ is a minimal disagreement between
$$(\mathcal{R}_\alpha, \Psi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}) \text{ and } (\mathcal{R}_\alpha, \Phi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown c_\xi)}),$$
$b_\alpha$ is the branch of $\vec{\mathcal{T}}_\alpha$ according $\Psi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}$ and $c_\alpha$ is the branch of $\vec{\mathcal{T}}_\alpha$ according to $\Phi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown c_\xi)}$, $\mathcal{M}_{b_\alpha}^{\vec{\mathcal{T}}_\alpha} = \mathcal{R}_{\alpha+1} = \mathcal{M}_{c_\alpha}^{\vec{\mathcal{T}}_\alpha}$, $i_{\alpha,\alpha+1} : \mathcal{R}_\alpha \to \mathcal{R}_{\alpha+1}$ is the iteration embedding according to $\Psi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}$ and $j_{\alpha,\alpha+1} : \mathcal{R}_\alpha \to \mathcal{R}_{\alpha+1}$ is the iteration embedding according to $\Phi_{\mathcal{R}_\alpha, \oplus_{\xi<\alpha}(\vec{\mathcal{T}}_\xi^\frown c_\xi)}$;
(3) $i_{\alpha,\beta} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$ is the join of all $i$ embeddings and $j_{\alpha,\beta} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$ is the join of all $j$ embeddings,
(4) for limit $\alpha$, $\mathcal{R}_\alpha = dirlim(\mathcal{R}_\beta, i_{\beta,\gamma} : \beta < \gamma < \alpha) = dirlim(\mathcal{R}_\beta, j_{\beta,\gamma} : \beta < \gamma)$.

Getting such a sequence isn't difficult. One just needs to keep hitting minimal disagreements. The fact that we always end up with the same last model $\mathcal{R}_\alpha$ even

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though we use different strategies is a consequence of our assumption that $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are close. Now, let $B=(\mathcal{R}_\alpha,\vec{\mathcal{T}}_\alpha:\alpha<\kappa^+)$ and let $X_0\prec X_1\prec H_{\kappa^{++}}$ be such that $B\in X_0$ and for $i=0,1$, $|X_i|=\kappa$. Let $H_0$ and $H_1$ be the transitive collapses of $X_0$ and $X_1$ and let $\pi:H_0\to H_1$ be the canonical map. Let $\kappa_i$ be the collapse of $\kappa^+$ in $H_i$ and let $B_i$ be the collapse of $B$ in $H_i$. Then $B_i\restriction\kappa_i=B\restriction\kappa_i$. Moreover, by standard arguments

$$i_{\kappa_0,\kappa_1}=\pi\restriction\mathcal{R}_{\kappa_0}=j_{\kappa_0,\kappa_1} \tag{1}$$

We claim that $\vec{\mathcal{T}}_{\kappa_0}$ isn't a disagreement. Using (1) we get that

$$j_{\kappa_0+1,\kappa_1}\circ j_{\kappa_0,\kappa_0+1}=i_{\kappa_0,\kappa_1},$$

Notice that if $c_{\kappa_0}$ is not according to $\Psi_{\mathcal{R}_{\kappa_0},\oplus_{\xi<\kappa_0}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}$ then

$$\vec{\mathcal{T}}_{\kappa_0},c_{\kappa_0},\oplus_{\xi\in[\kappa_0,\kappa_1)}(\vec{\mathcal{T}}_\xi^\frown b_\xi),\quad j_{\kappa_0+1,\kappa_1}\quad\text{and}\quad j_{\kappa_0,\kappa_0+1}$$

witness that $\Psi_{\mathcal{R}_{\kappa_0},\oplus_{\xi<\kappa_0}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}$ doesn't have branch condensation, which is a contradiction. Thus it must be the case that $c_{\kappa_0}$ is according to $\Psi_{\mathcal{R}_{\kappa_0},\oplus_{\xi<\kappa_0}(\vec{\mathcal{T}}_\xi^\frown b_\xi)}$, and we are done with *Case 1*.

Before we move to *Case 2*, we make the following observation. In comparing $(\mathcal{R},\Psi)$ with $(\mathcal{R},\Lambda)$ we produced stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ on $\mathcal{R}$ that resulted in the successful comparison with the property that $\vec{\mathcal{T}}^-=\vec{\mathcal{U}}^-$, i.e., the parts of the stacks based on $\mathcal{R}^-$ were the same. This was a consequence of always hitting the minimal disagreement. Thus, as part of our minimality assumption we can assume the following: whenever $(\vec{\mathcal{T}},\mathcal{R})\in B(\mathcal{P},\Sigma)\cup I(\mathcal{P},\Sigma)$ and $(\vec{\mathcal{U}},\mathcal{S})\in B(\mathcal{Q},\Lambda)\cup I(\mathcal{Q},\Lambda)$ are such that $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P},\Sigma)\leftrightarrow(\vec{\mathcal{U}},\mathcal{S})\in B(\mathcal{Q},\Lambda)$ and for some $\alpha\leq\min(\lambda^{\mathcal{R}},\lambda^{\mathcal{S}})$, $\mathcal{R}(\alpha)=\mathcal{S}(\alpha)$ and $\Sigma_{\mathcal{R}(\alpha),\vec{\mathcal{T}}}=\Lambda_{\mathcal{S}(\alpha),\vec{\mathcal{U}}}$ then there are stacks $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{U}}_1$ such that

(1) $\vec{\mathcal{T}}_1$ is on $\mathcal{R}$ and is according to $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}$,
(2) $\vec{\mathcal{U}}_1$ is on $\mathcal{Q}$ and is according to $\Lambda_{\mathcal{S},\vec{\mathcal{U}}}$,
(3) the part of $\vec{\mathcal{T}}_1$ that is based on $\mathcal{R}(\alpha)$ is the same as the part of $\vec{\mathcal{U}}_1$ that is based on $\mathcal{R}(\alpha)$,
(4) $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{U}}_1$, viewed as stacks on $\mathcal{R}(\alpha)$, have last models $\mathcal{R}_1$ and $\mathcal{S}_1$ such that either
   (a) $\mathcal{R}_1\trianglelefteq_{hod}\mathcal{S}_1$ and $\Sigma_{\mathcal{R}_1,\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1}=\Lambda_{\mathcal{R}_1,\vec{\mathcal{U}}^\frown\vec{\mathcal{U}}_1}$
   or
   (b) $\mathcal{S}_1\trianglelefteq_{hod}\mathcal{R}_1$ and $\Lambda_{\mathcal{S}_1,\vec{\mathcal{U}}^\frown\vec{\mathcal{U}}_1}=\Sigma_{\mathcal{S}_1,\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1}$.

Suppose now that $\lambda^{\mathcal{P}}$ is a limit ordinal. By symmetry, we can assume that $\lambda^{\mathcal{Q}}$ is a limit ordinal as well. Suppose first that $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ isn't a measurable cardinal of $\mathcal{P}$. Then let $\eta=cf^V(\lambda^{\mathcal{P}})$ and fix a function $f:\eta\to\mathcal{P}$ such that for each $\xi<\eta$, $f(\xi)=\mathcal{P}(\alpha)$ for some $\alpha<\lambda^{\mathcal{P}}$ and $\mathcal{P}(\alpha)|\delta^{\mathcal{P}}=\cup_{\xi<\eta}f(\xi)$. We then have that $\Sigma=\cup_{\xi<\eta}\Sigma_{f(\xi)}$. Comparing $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ is not hard now as we can just successively compare $(f(\xi),\Sigma_{f(\xi)})$ with $(\mathcal{Q},\Lambda)$ and use our minimality assumption to show that the comparison terminates. The case when $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is a measurable cardinal is more involved and includes all the ideas needed to complete the case when $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ isn't measurable cardinal. We therefore assume that $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$ and by symmetry, we also assume that $\mathrm{cf}^{\mathcal{Q}}(\lambda^{\mathcal{Q}})$ is measurable in $\mathcal{Q}$.

*Claim.* There are stacks $\vec{\mathcal{S}}$ and $\vec{\mathcal{U}}$ such that

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(1) $\vec{\mathcal{U}}$ is on $\mathcal{P}$ and is according to $\Sigma$,
(2) $\vec{\mathcal{S}}$ is on $\mathcal{Q}$ and is according to $\Lambda$,
(3) $\vec{\mathcal{S}}$ and $\vec{\mathcal{U}}$ have common last model $\mathcal{M}$,
(4) for any $\alpha < \lambda^{\mathcal{M}}$, $\Sigma_{\mathcal{M}(\alpha),\vec{\mathcal{U}}} = \Lambda_{\mathcal{M}(\alpha),\vec{\mathcal{S}}}$.

PROOF. By our minimality assumption, whenever $(\vec{\mathcal{T}},\mathcal{R}) \in B(\mathcal{P},\Sigma)$ and $(\vec{\mathcal{U}},\mathcal{S}) \in B(\mathcal{Q},\Lambda)$, we can compare $(\mathcal{R},\Sigma_{\mathcal{R},\vec{\mathcal{T}}})$ with $(\mathcal{S},\Lambda_{\mathcal{S},\vec{\mathcal{U}}})$[15]. To prove the theorem, we use this observation repetitively. Let $\lambda = (\mathrm{cf}(\lambda^{\mathcal{P}}))^{\mathcal{P}}$ and $\kappa = (\mathrm{cf}(\lambda^{\mathcal{Q}}))^{\mathcal{Q}}$, and also, let $\mu \in \vec{E}^{\mathcal{P}}$ and $\nu \in \vec{E}^{\mathcal{Q}}$ be the normal Mitchell order 0 extenders on respectively $\lambda$ and $\kappa$. We build two sequences $(\mathcal{P}_m,\mathcal{P}^*_m,\vec{\mathcal{U}}_m,\mu_m,j_m,j_{m,n} : m < n < \omega)$ and $(\mathcal{Q}_m,\mathcal{Q}^*_m,\vec{\mathcal{S}}_m,\nu_m,i_m,i_{m,n} : m < n < \omega)$ as follows (see Figure 2.7.4). Start by setting $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$. $\mu_m$ is the image of $\mu$ in $\mathcal{P}_m$ and $\nu_m$ is the image of $\nu$ in $\mathcal{Q}_m$. If $m$ is even then $\mathcal{P}^*_m = Ult(\mathcal{P}_m,\mu_m)$, $j_m = j_{\mu_m}$, $\mathcal{Q}^*_m = \mathcal{Q}_m$ and $i_m = id$. If $m$ is odd then $\mathcal{P}^*_m = \mathcal{P}_m$, $j_m = id$, $\mathcal{Q}^*_m = Ult(\mathcal{Q}_m,\nu_m)$ and $i_m = j_{\nu_m}$. The embeddings $j_{m,n} : \mathcal{P}^*_m \to \mathcal{P}_n$ and $i_{m,n} : \mathcal{Q}^*_m \to \mathcal{Q}_n$ are the iteration embeddings according to respectively $\Sigma$ and $\Lambda$. As we build the sequence, we maintain the following conditions.

(1) $\vec{\mathcal{U}}_m$ is a stack on $\mathcal{P}^*_m$ according to $\Sigma_{\mathcal{P}^*_m,\oplus_{k<m}(\vec{\mathcal{U}}_k{}^\frown\mathcal{P}^*_k)}$ with last model $\mathcal{P}_{m+1}$.
(2) $\vec{\mathcal{S}}_m$ is a stack on $\mathcal{Q}^*_m$ according to $\Lambda_{\mathcal{Q}^*_m,\oplus_{i<m}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)}$ with last model $\mathcal{Q}_{m+1}$.
(3) If $m$ is even and $n > m$ then $\mathcal{P}_n(j_{m,n}(\lambda^{\mathcal{P}_m})) = \mathcal{Q}_n(j_{m,n}(\lambda^{\mathcal{P}_m}))$ and
$$\Sigma_{\mathcal{P}_n(j_{m,n}(\lambda^{\mathcal{P}_m})),\oplus_{k<n}(\vec{\mathcal{U}}_k{}^\frown\mathcal{P}^*_k)} = \Lambda_{\mathcal{Q}_n(j_{m,n}(\lambda^{\mathcal{P}_m})),\oplus_{k<n}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)}.$$
(4) If $m$ is odd and $n > m$ then $\mathcal{P}_n(i_{m,n}(\lambda^{\mathcal{Q}_m})) = \mathcal{Q}_n(i_{m,n}(\lambda^{\mathcal{Q}_m}))$ and
$$\Sigma_{\mathcal{P}_n(i_{m,n}(\lambda^{\mathcal{Q}_m})),\oplus_{i<n}(\vec{\mathcal{U}}_k{}^\frown\mathcal{P}^*_k)} = \Lambda_{\mathcal{Q}_n(i_{m,n}(\lambda^{\mathcal{Q}_m})),\oplus_{i<n}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)}.$$

Suppose $m$ is even and we have constructed $\mathcal{Q}_m$ and $\mathcal{P}_m$ and the above conditions have been maintained. Let $\alpha = \max(j_{m-2,m}(\lambda^{\mathcal{P}_{m-2}}), i_{m-1,m}(\lambda^{\mathcal{Q}_{m-1}}))$. Notice that $\alpha < \min(\lambda^{\mathcal{P}_m},\lambda^{\mathcal{Q}_m})$. Then we must have that $\mathcal{Q}_m(\alpha) = \mathcal{P}_m(\alpha)$ and $\Sigma_{\mathcal{P}_m(\alpha),\oplus_{i<m}\vec{\mathcal{U}}_i} = \Lambda_{\mathcal{Q}_m(\alpha),\oplus_{k<m}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)}$. To maintain clauses 3 and 4 above for $k < m$ we will make sure that the parts of $\vec{\mathcal{U}}_m$ and $\vec{\mathcal{S}}_m$ that are based on $\mathcal{P}_m(\alpha) = \mathcal{Q}_m(\alpha)$ are the same.

We let $\mathcal{P}^*_m = Ult(\mathcal{P}_m,\mu_m)$. We can then compare
$$(\mathcal{P}^*_m(\lambda^{\mathcal{P}_m}),\Sigma_{\mathcal{P}^*_m(\lambda^{\mathcal{P}_m}),(\oplus_{k<m}(\vec{\mathcal{U}}_k{}^\frown\mathcal{P}^*_k))^\frown\mathcal{P}^*_m}) \text{ and } (\mathcal{Q}_m,\Lambda_{\mathcal{Q}_m,(\oplus_{k<m}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k))})$$
in such a way that the stacks of the successful comparison agree on $\mathcal{P}^*_m(\alpha) = \mathcal{Q}_m(\alpha)$. Let the stacks be $\vec{\mathcal{U}}_m$ and $\vec{\mathcal{S}}_m$ and the last models be $\mathcal{P}_{m+1}$ and $\mathcal{Q}_{m+1}$. Similarly we take care of the case when $m$ is an odd integer.

Now, let $\mathcal{M}_1 = dirlim(\mathcal{P}_m, j_{m,n} : m < n < \omega)$ and $\mathcal{M}_2 = dirlim(\mathcal{Q}_m, i_{m,n} : m < n < \omega)$. Let $j_m : \mathcal{P}_m \to \mathcal{M}_1$ and $i_m : \mathcal{Q}_m \to \mathcal{M}_2$ be the direct limit embeddings. It is not hard to see that because of our construction $\mathcal{M}_1 = \mathcal{M}_2$. The point is that if for instance $\alpha < \lambda^{\mathcal{M}_1}$ then there is some $m$ such that for some $\beta < \lambda^{\mathcal{P}_m}$, $\alpha = i_m(\beta)$. But because of our construction, we can choose $m$ and $\beta$ such that $\mathcal{P}_m(\beta) = \mathcal{Q}_m(\beta)$ and $\Sigma_{\mathcal{P}_m(\beta),\oplus_{k<m}(\vec{\mathcal{U}}_i{}^\frown\mathcal{P}^*_k)} = \Lambda_{\mathcal{Q}_m(\beta),\oplus_{k<m}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)}$. Then $j_m(\beta) = i_m(\beta) = \alpha$. Let then $\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2$, $\vec{\mathcal{U}} = \oplus_{k<\omega}(\vec{\mathcal{U}}_k{}^\frown\mathcal{P}^*_k)$ and $\vec{\mathcal{S}} = \oplus_{k<\omega}(\vec{\mathcal{S}}_k{}^\frown\mathcal{Q}^*_k)$. We need to

[15]Recall that our minimality assumption essentially states the claim of this sentence.

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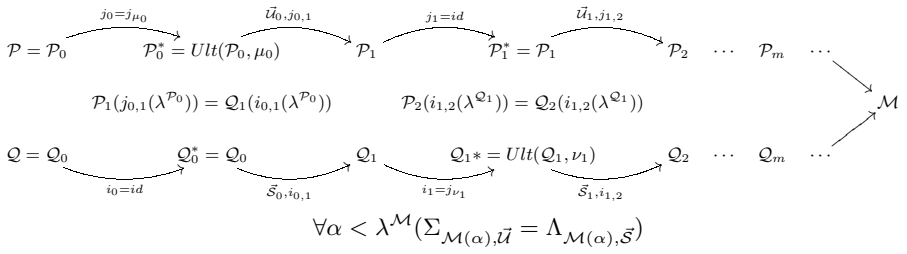


FIGURE 2.7.4. The inductive step.

see that if $\alpha < \lambda^{\mathcal{M}}$ then $\Sigma_{\mathcal{M}(\alpha),\vec{\mathcal{U}}} = \Lambda_{\mathcal{M}(\alpha),\vec{\mathcal{S}}}$. This again follows from our construction. We can fix $m$ large enough so that for some $\beta$ we have that $\mathcal{P}_m(\beta) = \mathcal{Q}_m(\beta)$, $j_m(\beta) = i_m(b) = \alpha$ and $\Sigma_{\mathcal{P}_m(\beta),\oplus_{k<m}(\vec{\mathcal{U}}_k^\frown \mathcal{P}_k^*)} = \Lambda_{\mathcal{Q}_m(\beta),\oplus_{k<m}(\vec{\mathcal{S}}_k^\frown \mathcal{Q}_k^*)}$. Moreover, we can choose $m$ so large that the parts of $\oplus_{k\in[m,\omega)}(\vec{\mathcal{S}}_k^\frown \mathcal{Q}_k^*)$ and $\oplus_{k\in[m,\omega)}(\vec{\mathcal{U}}_k^\frown \mathcal{P}_k^*)$ that are based on $\mathcal{P}_m(\beta) = \mathcal{Q}_m(\beta)$ are the same. This then easily implies that $\Sigma_{\mathcal{M}(\alpha),\vec{\mathcal{U}}} = \Lambda_{\mathcal{M}(\alpha),\vec{\mathcal{S}}}$. □

Without loss of generality then we assume that $\mathcal{P} = \mathcal{Q}$ and for every $\alpha < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)} = \Lambda_{\mathcal{Q}(\alpha)}$. Since $\Sigma \neq \Lambda$, it is easy to produce a bad block on $(\mathcal{P}, \Sigma)$. Let $\vec{\mathcal{T}}$ be a minimal disagreement between $\Sigma$ and $\Lambda$. Let $(\mathcal{M}_\alpha, \mathcal{M}_\alpha^*, \vec{\mathcal{W}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \eta)$ be the essential components of $\vec{\mathcal{T}}$. Let then $\mathcal{P}_0 = \mathcal{P}$, $\vec{\mathcal{T}}_0 =_{def} \vec{\mathcal{T}} \restriction \eta$ and $\vec{\mathcal{T}}_1 =_{def} \vec{\mathcal{W}}_\eta$. Let $\mathcal{P}_1 = \mathcal{M}_\eta$ and $\Psi = \Lambda_{\mathcal{P}_1,\vec{\mathcal{T}}_0}$. Let $c = \Psi(\vec{\mathcal{T}}_1)$, $b = \Sigma_{\mathcal{P}_1,\vec{\mathcal{T}}_0}(\vec{\mathcal{T}}_1)$, $\mathcal{P}_2 = \mathcal{M}_c^{\vec{\mathcal{T}}_1}$ and $\mathcal{P}_3 = \mathcal{M}_b^{\vec{\mathcal{T}}_1}$. Let $\vec{\mathcal{T}}_2$ and $\vec{\mathcal{T}}_3$ be stacks on $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively that according to $\Lambda_{\mathcal{P}_2,\vec{\mathcal{T}}_1^\frown\{\mathcal{P}_2\}}$ and $\Sigma_{\mathcal{P}_3,\vec{\mathcal{T}}^\frown\{\mathcal{P}_3\}}$ constructed via the procedure described in the claim such that they have a common last model $\mathcal{P}_4$ such that for every $\alpha < \lambda^{\mathcal{P}_4}$, $\Lambda_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}_1^\frown\{\mathcal{P}_2\}^\frown\vec{\mathcal{T}}_2} = \Sigma_{\mathcal{P}_4(\alpha),\vec{\mathcal{T}}^\frown\{\mathcal{P}_3\}^\frown\vec{\mathcal{T}}_3}$. Let $\pi_0 = \pi^{\vec{\mathcal{T}}_0}$, $\pi_1 = \pi_c^{\vec{\mathcal{T}}_1}$, $\pi_2 = \pi_b^{\vec{\mathcal{T}}_1}$, $\pi_3 = \pi^{\vec{\mathcal{T}}_2}$ and $\pi_4 = \pi^{\vec{\mathcal{T}}_3}$. Let $\vec{\mathcal{U}} = \vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\{P_3\}^\frown\vec{\mathcal{T}}_3$ and let $\xi = \lambda^{\mathcal{M}_\eta^*}$. Then it is not hard to see that $B_0 = ((\mathcal{P}_i : i \leq 4), (\vec{\mathcal{T}}_i : i \leq 3), \Psi, (\pi_i : i \leq 4), \xi, \vec{\mathcal{U}})$ is a bad block on $\mathcal{P}_0$. Repeating the above process and taking direct limits at limit stages we can get $(B^\alpha, j_{\beta,\alpha}^t, j_{\beta,\alpha}^b : \beta < \alpha < \omega_1)$ as in Lemma 2.49, which gives a contradiction. This completes the proof of Theorem 2.47.

Before we move on, we state a version of the comparison argument presented in this section that is applicable in $AD^+$ context.

THEOREM 2.50 (The diamond comparison of hod pairs). *Assume $AD^+$. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that*

1. *for any $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{S}}, \mathcal{W}) \in I(\mathcal{Q}, \Lambda)$ and any $\alpha < \lambda^{\mathcal{R}}$ and $\beta < \lambda^{\mathcal{W}}$, both $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}$ and $\Lambda_{\mathcal{W}(\beta+1),\vec{\mathcal{S}}}$ have branch condensation,*
2. *$(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are of the same kind,*
3. *both $\Sigma$ and $\Lambda$ are $(\omega_1, \omega_1)$-iteration strategies,*
4. *there is a good pointclass $\Gamma$ such that $code(\Sigma), Code(\Lambda) \in \Delta_{\underset{\sim}{\Gamma}}$.*

*Then there are $(\mathcal{R}, \Psi)$ and $(\mathcal{S}, \Phi)$ such that*

1. *$(\mathcal{R}, \Psi)$ is a tail of $(\mathcal{P}, \Sigma)$,*
2. *$(\mathcal{S}, \Phi)$ is a tail of $(\mathcal{Q}, \Lambda)$,*
3. *either*

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(a) $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ *and* $\Psi = \Phi_{\mathcal{R}}$,
*or*
(b) $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ *and* $\Phi = \Psi_{\mathcal{S}}$.

The proof is the same as before with one minor difference. Let $T, S$ be trees that project to $Code(\Sigma)$ and $Code(\Lambda)$ respectively. We can then carry out the proof of Theorem 2.47 in the model $L[T, S]$. The existence of $T$ and $S$ follows from the fact that because $Code(\Sigma), Code(\Lambda) \in \Gamma$, both $Code(\Sigma)$ and $Code(\Lambda)$ are Suslin.

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CHAPTER 3

# Hod mice revisited

The main goal of this chapter is to analyze the internal structure of hod mice and use this analysis to investigate the derived models of hod mice. In Section 3.1 we show that the internal strategies of hod mice are unique (Theorem 3.3) and use this to show that the internal strategies of hod mice can be interpreted in generic extensions (see Theorem 3.10). We will then use the generic interpretability result to prove that the Solovay sequence of the derived model of a hod mouse is completely characterized by its internal strategies (Theorem 3.19). In particular, strategies for different layers of the hod mouse correspond to different members of the Solovay sequence. Such an analysis of the derived models of hod mice is important because it allows us to "pullback" information, as in Theorem 3.26, from the iterates of the hod mouse. In Section 3.5, we show that if $(\mathcal{P}, \Sigma)$ is a hod pair with some additional properties then some tail of $\Sigma$ has branch condensation (see Theorem 3.28). This is useful in core model induction applications. At the end of this section we propose a reorganization of hod mice into a different hierarchy. The main reason for this is that with the hierarchy currently in use we do not know how to do $S$-constructions and also, how to define hod mice over sets that are not self-well-ordered. In Section 3.8, we show that the new hierarchy can be used to carry out $S$-constructions. Such constructions are very useful in applications (see for instance Definition 3.48).

## 3.1. The internal theory of hod premice

In this section we analyze the internal theory of hod premice. Given a hod premouse $\mathcal{P}$ and $\alpha < \lambda^{\mathcal{P}}$, it is natural to ask whether $\mathcal{P} \vDash$ "$\mathcal{P}(\alpha)$ has a unique iteration strategy". We show that it is indeed the case (see Theorem 3.3). Later in this section we also show that the internal strategies of $\mathcal{P}$ can be interpreted in generic extensions (see Theorem 3.10).

Notice first that for any $\mathcal{P}$ such that $\lambda^{\mathcal{P}} \geq 1$, $\mathcal{P} \vDash$ "$\mathcal{P}(0)$ has a unique iteration strategy". This follows from the fact that $\delta_0$ is a regular cardinal of $\mathcal{P}$ and hence, the following corollary of Theorem 2.2 of [**18**] can be used.

LEMMA 3.1 (Martin, Steel [**18**]). *Suppose $\delta$ is a cardinal and $\mathcal{T}$ is an iteration tree of limit length on $V_\delta$. Suppose $\mathcal{T}$ has two well-founded branches $b \neq c$. Then* $\mathrm{cf}(\delta(\mathcal{T})) = \omega$.

The proof of Theorem 3.3 is just a generalization of this lemma. To prove that the internal strategies are unique, we first show that if $\mathcal{P}$ is a hod premouse then all the Woodin cardinals of an internal iterate of $\mathcal{P}$ have uncountable cofinality.

LEMMA 3.2. *Suppose $\mathcal{P}$ is a hod premouse and $\vec{\mathcal{T}} \in \mathcal{P}$ is a stack on $\mathcal{P}$ with a last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists. Suppose $\delta$ is a Woodin cardinal of $\mathcal{Q}$. Then* $\mathrm{cf}^{\mathcal{P}}(\delta) > \omega$.



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PROOF. The proof is a straightforward computation involving ultrapowers. Let $\mathcal{R}$ in $\vec{\mathcal{T}}$ be the first place where a preimage of $\delta$ appears. Thus we have $j : \mathcal{P} \to \mathcal{R}$, $k : \mathcal{R} \to \mathcal{Q}$ and $\delta^* \in \mathcal{R}$ such that $j, k$ are the iteration embeddings, $k \circ j = \pi^{\vec{\mathcal{T}}}$ and $k(\delta^*) = \delta$. Because $\mathcal{R}$ is the least place where a preimage of $\delta$ appears, it must be the case that either $\mathcal{R} = \mathcal{P}$ or $\mathcal{R}$ has a predecessor in $\vec{\mathcal{T}}$. In the first case, we have that $\pi^{\vec{\mathcal{T}}}[\delta^*]$ is cofinal in $\delta$. Suppose then $\mathcal{R} \neq \mathcal{P}$ and let $\mathcal{S}$ be the predecessor of $\mathcal{R}$. Then $\mathcal{R} = Ult(\mathcal{S}, E)$ for some extender $E$. Notice that $\delta^*$ doesn't have a preimage in $\mathcal{S}$. Now, towards a contradiction assume that $\mathrm{cf}^{\mathcal{P}}(\delta)$ and hence, $\mathrm{cf}^{\mathcal{P}}(\delta^*)$ is $\omega$. We can then fix $(f_n, a_n : n < \omega)$ in $\mathcal{P}$ such that if $i : \mathcal{P} \to \mathcal{S}$ is the iteration embedding then $a_n \in \nu(E)^{<\omega}$ and $\delta^* = \sup\{j_E(i(f_n))(a_n) : n < \omega\}$. Notice that $(i(f_n) : n < \omega) \in \mathcal{S}$ implying that $(j_E(i(f_n)) : n < \omega) \in \mathcal{R}$. Let $\xi_n = \{j_E(i(f_n))(b) : b < \nu(E)^{<\omega}\}$. Then $\xi_n < \delta^*$ and $\delta^* = \sup_{n<\omega} \xi_n$. But we have that $(\xi_n : n < \omega) \in \mathcal{R}$, which is a contradiction as $\delta^*$ is Woodin in $\mathcal{R}$. □

THEOREM 3.3 (Uniqueness of the internal strategies). *Suppose $\mathcal{P}$ is a hod premouse and let $\alpha < \lambda^{\mathcal{P}}$. Then $\mathcal{P} \vDash$ "$\mathcal{P}(\alpha)$ has a unique iteration strategy".*

PROOF. The idea behind the proof is that we never create genuinely new Woodin cardinals that have cofinality $\omega$. Towards a contradiction, suppose there is an $\alpha < \lambda^{\mathcal{P}}$ such that $\mathcal{P}(\alpha)$ has two different iteration strategies inside $\mathcal{P}$. Let $\alpha$ be the least such. In the following, we let $V = \mathcal{P}$, $\delta_\beta^{\mathcal{P}} = \delta_\beta$ and $\Sigma_\beta^{\mathcal{P}} = \Sigma_\beta$.

*Case 1.* $\alpha = \gamma + 1$.

Here we allow $\gamma = -1$. Let $\Lambda$ be a strategy for $\mathcal{P}(\alpha)$ which is different from $\Sigma_\alpha$. We then have that $\Sigma_\gamma = \Lambda_{\mathcal{P}(\gamma)}$. Let $\vec{\mathcal{T}}$ be a stack according to both $\Sigma_\alpha$ and $\Lambda$ such that $\Sigma_\alpha(\vec{\mathcal{T}}) \neq \Lambda(\vec{\mathcal{T}})$. Let $(\mathcal{M}_\beta, \mathcal{T}_\beta : \beta \leq \eta)$ be the normal components of $\vec{\mathcal{T}}$. It follows that $\mathcal{T}_\eta$ is entirely above $\mathcal{M}_\eta^-$ and that $\mathrm{cf}(\delta(\mathcal{T}_\eta)) = \omega$. Let $b = \Sigma_\alpha(\vec{\mathcal{T}})$ and $c = \Lambda(\vec{\mathcal{T}})$. Suppose first that $\mathcal{Q}(b, \mathcal{T}_\eta)$ exists (see Definition 1.22). This then implies that $\mathcal{Q}(c, \mathcal{T}_\eta)$ exists because otherwise $\pi_c^{\vec{\mathcal{T}}}$ exists and because $\pi_c^{\vec{\mathcal{T}}}[\delta_\alpha]$ is cofinal in $\pi_c^{\vec{\mathcal{T}}}(\delta_\alpha)$, we must have that $\mathrm{cf}(\delta_\alpha) = \omega$. Thus, $\mathcal{Q}(c, \mathcal{T}_\eta)$ must exist which implies that $\mathcal{Q}(c, \mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$ and hence, $b = c$, contradiction! It follows that we must have that $\mathcal{Q}(b, \mathcal{T}_\eta)$ doesn't exist. By the symmetric argument, $\mathcal{Q}(c, \mathcal{T}_\eta)$ doesn't exist. Hence, $\pi_b^{\vec{\mathcal{T}}}$ and $\pi_c^{\vec{\mathcal{T}}}$ exist and $\pi_b^{\vec{\mathcal{T}}}(\delta_\alpha) = \delta(\vec{\mathcal{T}}) = \pi_c^{\vec{\mathcal{T}}}(\delta_\alpha)$. But because $\pi_b^{\vec{\mathcal{T}}}[\delta_\alpha]$ is cofinal in $\delta(\vec{\mathcal{T}})$, this implies that $\mathrm{cf}(\delta_\alpha) = \omega$, contradiction!.

*Case 2.* $\alpha$ is limit.

Fix a strategy $\Lambda \neq \Sigma_\alpha$ and let $\vec{\mathcal{T}}$ be according to both $\Sigma_\alpha$ and $\Lambda$ such that $\Lambda(\vec{\mathcal{T}}) \neq \Sigma(\vec{\mathcal{T}})$. Let $(\mathcal{M}_\beta, \mathcal{M}_\beta^*, \vec{\mathcal{T}}_\beta, \pi_{\beta,\gamma} : \beta < \gamma \leq \eta)$ be the essential components of $\vec{\mathcal{T}}$. It must be that $\vec{\mathcal{T}}_\eta$ is defined but it doesn't have a last model. In fact, there are two different branches of $\vec{\mathcal{T}}_\eta$, one according to $\Sigma$ and one according to $\Lambda$. We make the following minimality assumption on $\vec{\mathcal{T}}$:

(1) for each $\beta \leq \eta$, $\lambda^{\mathcal{M}_\beta^*}$ is a successor ordinal,
(2) for all $\beta < \eta$ and for all $\xi < \lambda^{\mathcal{M}_\beta}$, $\mathcal{M}_\beta(\xi)$ has a unique iteration strategy which is $(\Sigma_\alpha)_{\mathcal{M}_\beta(\xi), \vec{\mathcal{T}} \restriction \beta}$,

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(3) $\mathcal{M}_\eta^* \triangleleft_{hod} \mathcal{M}_\eta$ is the least hod initial segment $\mathcal{Q}$ of $\mathcal{M}_\eta$ which has two different strategies one of which is $(\Sigma_\alpha)_{\mathcal{Q},\vec{\mathcal{T}}\restriction\eta}$.

It is not hard to show, using the proof of Lemma 2.41, that our assumption doesn't harm the generality of our argument. Note that because of our minimality assumption, $\delta(\vec{\mathcal{T}} \restriction \eta) < \lambda^{\mathcal{M}_\eta^*}$. Let $\Psi = \Lambda_{\mathcal{M}_\eta^*,\vec{\mathcal{T}}\restriction\eta}$ and let $\Phi = (\Sigma_\alpha)_{\mathcal{M}_\eta^*,\vec{\mathcal{T}}\restriction\eta}$. But now we can derive a contradiction as in the successor case. By our minimality assumption, letting $(\mathcal{N}_\nu, \mathcal{U}_\nu : \nu \leq \xi)$ be the normal components of $\vec{\mathcal{T}}_\eta$, it must be that $\mathcal{U}_\nu$ is on $\mathcal{N}_\nu$ and is entirely above $\mathcal{N}_\nu^-$. This then, as in the successor case, implies that both strategies have to choose branches for $\mathcal{U}_\nu$ that do not have $\mathcal{Q}$-structures (otherwise these branches will have to be the same) and therefore, they witness that $\delta^{\mathcal{M}_\eta^*}$ (which is mapped to $\delta(\mathcal{U}_\nu)$ by both branches) has cofinality $\omega$ (in $\mathcal{P}$), which by Lemma 3.2 cannot happen. □

Before we go on we list several easy but useful facts. We omit the proofs as they are rather standard.

FACT 3.4. *Suppose $\mathcal{P}$ is a hod premouse and $\nu < \delta^{\mathcal{P}}$ isn't a Woodin cardinal of $\mathcal{P}$. Suppose there is an extender $E$ such that $\nu \in (crit(E), lh(E))$. Then $Ult(\mathcal{P}, E) \vDash$ "$\nu$ isn't a Woodin cardinal".*

FACT 3.5. *Suppose $\mathcal{P}$ is a premouse or a hybrid premouse possibly over some set such that $\mathcal{P} \vDash$ "$ZFC - Powerset$+there is a unique Woodin cardinal". Then if $\mathcal{T}$ is a normal tree on $\mathcal{P}$ of limit length and $b$ is a branch of $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is well founded, $\mathcal{Q}(b,\mathcal{T})$-exists (thus, $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}(b,\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$) and $\delta(\mathcal{T})$ isn't a strong cutpoint of $\mathcal{Q}(b,\mathcal{T})$ then $\mathcal{T}$ has a fatal drop at some $(\beta,\nu)$ (see Definition* 1.27*).*

As a corollary to our facts we obtain the following very useful fact about coiterations of mice that look like $\mathcal{P}$ of our facts above.

FACT 3.6. *Suppose $\kappa$ is some cardinal and $\mathcal{P}, \mathcal{M} \in H_{\kappa^+}$ are $\kappa^+ + 1$-iterable mice or hybrid mice with respect to the same strategy. Suppose further that they are both over the same set $X$ and that $\mathcal{P}$ is as in Fact* 3.5*. If $\mathcal{M}$ wins the coiteration with $\mathcal{P}$ then no $\mathcal{Q}$-structure appearing either on $\mathcal{M}$-side or on $\mathcal{P}$-side has overlaps, i.e., letting $\mathcal{T}$ be the tree on $\mathcal{M}$ side and $\mathcal{U}$ be the tree on $\mathcal{P}$ side then for every limit $\alpha < lh(\mathcal{T}) = lh(\mathcal{U})$, if $\mathcal{Q}$ is a $\mathcal{Q}$-structure for $\mathcal{T} \restriction \alpha$ or for $\mathcal{U} \restriction \alpha$ then there is no $E \in \vec{E}^{\mathcal{Q}}$ such that $\delta(\mathcal{T} \restriction \alpha) = \delta(\mathcal{U} \restriction \alpha) \in (crit(E), \nu(E))$.*

PROOF. Let $\mathcal{T}$ on $\mathcal{P}$ and $\mathcal{U}$ on $\mathcal{M}$ be two trees coming from the successful coiteration of $\mathcal{P}$ and $\mathcal{M}$ and suppose $\mathcal{M}$ wins the coiteration, i.e., $\pi^{\mathcal{T}}$ exists and $\mathcal{M}_{lh(\mathcal{T})-1}^{\mathcal{T}} \trianglelefteq \mathcal{M}_{lh(\mathcal{U})-1}^{\mathcal{U}}$. Suppose that some $\mathcal{Q}$-structure in the coiteration process has an overlap. Notice that because both $\mathcal{P}$ and $\mathcal{M}$ are iterable and $\mathcal{M}$ wins the coiteration, if $\mathcal{P}$ side chooses a branch with a $\mathcal{Q}$-structure without overlaps then the $\mathcal{M}$ side chooses a branch with exactly the same $\mathcal{Q}$-structure. Therefore, if some $\mathcal{Q}$-structure in the coiteration process has an overlap then it must be on the $\mathcal{P}$-side. But in this case, $\mathcal{P}$-side drops in a way that it cannot later undo this drop (see Fact 3.5), which means that $\mathcal{P}$ cannot lose the coiteration, i.e., $\pi^{\mathcal{T}}$ cannot exist. □

Our theorem on the uniqueness of the internal strategies implies that we can interpret the internal strategy of hod premice on the generic extensions of $\mathcal{P}$. This result is important because it implies that the derived models of hod mice are

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closed under the iteration strategies of their own initial segments. We will need such generic interpretability results while studying *prehod pairs*.

DEFINITION 3.7 (Prehod pair). $(\mathcal{P},\Sigma)$ *is a prehod pair if*

(1) $\mathcal{P}$ *is a countable hod premouse,*
(2) $\lambda^{\mathcal{P}}$ *is a successor,*
(3) $\Sigma$ *is an* $(\omega_1,\omega_1+1)$*-strategy for* $\mathcal{P}$ *acting on stacks based on* $\mathcal{P}^-$ *such that* $(\mathcal{P}^-,\Sigma)$ *is a hod pair and that whenever* $i:\mathcal{P}\to\mathcal{Q}$ *comes from an iteration by* $\Sigma$ *then for every* $\alpha<\lambda^{\mathcal{P}}$, $\Sigma^{\mathcal{Q}}_{i(\alpha)}=\mathcal{Q}\cap\Sigma$,
(4) $\mathcal{P}$ *is a* $\Sigma$*-premouse over* $\mathcal{P}^-$,
(5) *for any* $\mathcal{P}$*-cardinal* $\eta\in(\delta_{\lambda^{\mathcal{P}}-1},\delta^{\mathcal{P}}_{\lambda})^{\mathcal{P}}$, *considering* $\mathcal{P}|\eta$ *as a* $\Sigma$*-premouse over* $\mathcal{P}^-$, *there is an* $\omega_1$*-strategy* $\Lambda$ *for* $\mathcal{P}|\eta$.

Notice that because $\mathcal{P}\eta$ has no Woodins above $\mathcal{P}^-$, there must be a unique strategy $\Lambda$ as in 5 of Definition 3.7.

DEFINITION 3.8 (Generic interpretability). *Suppose now* $(\mathcal{P},\Sigma)$ *is a prehod pair or a hod pair with* $\lambda^{\mathcal{P}}$ *limit. We say generic interpretability holds for* $(\mathcal{P},\Sigma)$ *if there is a function* $F$ *such that*

(1) $F$ *is definable over* $\mathcal{P}$ *with no parameters,*
(2) $dom(F)=\{(\alpha,\kappa):\alpha<\lambda^{\mathcal{P}}\wedge\kappa>\delta^{\mathcal{P}}_{\alpha}$ *is a cardinal of* $\mathcal{P}\}$,
(3) *for all* $(\alpha,\kappa)\in dom(F)$, $F(\alpha,\kappa)=(\dot{T},\dot{S})$ *such that*
   (a) $\dot{T},\dot{S}\in\mathcal{P}^{Coll(\omega,\mathcal{P}(\alpha))}$,
   (b) $\mathcal{P}\vDash$ "$\Vdash_{Coll(\omega,\mathcal{P}(\alpha))}\dot{T}$ *and* $\dot{S}$ *are* $\kappa$*-complementing*",
   (c) *for any* $\nu\in(\delta^{\mathcal{P}}_{\alpha},\kappa)$ *and any* $\mathcal{P}$*-generic* $g\subseteq Coll(\omega,\nu)$, $\mathcal{P}[g]\vDash$ "$p[\dot{T}_g]$ *is a* $(\nu^+,\nu^+)$*-iteration strategy for* $\mathcal{P}(\alpha)$ *which extends* $\Sigma^{\mathcal{P}}_{\alpha}$" *and* $\Sigma_{\mathcal{P}(\alpha)}\cap\mathcal{P}[g]=(p[\dot{T}_g])^{\mathcal{P}[g]}$.

Before we prove our result on generic interpretability, we show that if $(\mathcal{P},\Sigma)$ is a prehod pair then clause 4 of Definition 1.34 can be strengthened in the following way.

LEMMA 3.9. *Suppose* $(\mathcal{P},\Sigma)$ *is a prehod pair and* $\alpha+1=\lambda^{\mathcal{P}}$. *Let* $\eta\in(\delta_{\alpha},\delta_{\alpha+1})^{\mathcal{P}}$ *be a cardinal of* $\mathcal{P}$ *and let* $\Lambda^*$ *be the iteration strategy of* $\mathcal{P}|\eta$ *as in 5 of Definition* 3.7. *Let* $\Lambda$ *be the fragment of* $\Lambda^*$ *that acts on non-dropping stacks. Let* $g\subseteq Coll(\omega,\eta)$ *be* $\mathcal{P}$*-generic. Then* $\mathcal{P}[g]$ *locally Suslin captures* $Code(\Lambda)$ *and its complement at any cardinal of* $\mathcal{P}$ *greater than* $\eta$.

PROOF. We just describe a procedure that defines $\Lambda$ over $\mathcal{P}[g]$. Then club many hulls will be genericaly correct implying the claim (see the proof of Theorem 3.3.15 of [**17**] or Lemma 4.1 of [**34**]). Also, we only consider normal trees. Our argument can be easily modified to handle stacks. Notice that since $\mathcal{P}|\eta$ has no Woodin cardinals above $\delta_{\alpha}$, $\Lambda$ always chooses branches with $\mathcal{Q}$-structures. Because of this, it is enough to prove the theorem for those $\eta$ that are successor cardinals. From now on we assume that $\eta$ is a successor cardinal of $\mathcal{P}$. Notice also that it follows that $\Lambda$ has the Dodd-Jensen property.

For $\kappa\in(\eta,\delta^{\mathcal{P}}_{\alpha+1})$, let $((\mathcal{M}^{\kappa}_{\gamma},\mathcal{N}^{\kappa}_{\gamma}:\gamma\leq\delta^{\mathcal{P}}_{\alpha+1}),(F^{\kappa}_{\gamma}:\gamma<\delta^{\mathcal{P}}_{\alpha+1}))$ be the output of $\mathcal{J}^{\vec{E},\Sigma^{\mathcal{P}}_{\alpha}}$-construction of $\mathcal{P}$ where the extenders used have critical points $>\kappa$. By clause 4 of Definition 1.7.1, $\mathcal{P}|\eta$, as a $\Sigma^{\mathcal{P}}_{\alpha}$-mouse, has a $<o(\mathcal{P})$ iteration strategy in $\mathcal{P}$. Let $\Phi\in\mathcal{P}$ be this strategy. It follows from Lemma 2.11 that for some $\beta\leq\delta^{\mathcal{P}}_{\alpha+1}$ and some $\xi^{\kappa}$, $\mathcal{N}^{\kappa}_{\beta}|\xi^{\kappa}$ is a normal iterate of $\mathcal{P}|\eta$. We then let $\mathcal{N}_{\kappa}=\mathcal{N}^{\kappa}|\xi^{\kappa}$ and also

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$\mathcal{T}_\kappa$ be the tree on $\mathcal{P}|\eta$ according to $\Phi$ with last model $\mathcal{N}_\kappa$. We must have that $\pi^{\mathcal{T}_\kappa}$ exists. Let $\pi_\kappa = \pi^{\mathcal{T}_\kappa}$.

Also it follows that $\mathcal{T}_\kappa$ is according to $\Lambda$. To see this, fix some limit ordinal $\beta < lh(\mathcal{T}_\kappa)$. Let $b = \Phi(\mathcal{T}_\kappa \restriction \beta)$. We want to see that $b = \Lambda(\mathcal{T}_\kappa)$. Let $\nu$ be such that $\mathcal{M}(\mathcal{T}_\kappa \restriction \beta) = \mathcal{N}_\kappa|\nu$. Notice that since there are no Woodin cardinals in $\mathcal{P}|\eta$ above $\delta_\alpha$, $\mathcal{Q}(\mathcal{T}_\kappa, b)$ is defined and, because of Fact 3.6, $\mathcal{Q}(\mathcal{T}_\kappa, b)$ cannot have overlaps (otherwise $\mathcal{T}_\kappa$ would drop in a way that we can never undo this drop and therefore, $\mathcal{P}|\eta$ cannot lose the coiteration with $\mathcal{N}_\kappa$). Also, $\mathcal{Q}(\mathcal{T}_\kappa, b) \trianglelefteq \mathcal{N}_\kappa$ and because of clause 5 of Definition 3.7, $\mathcal{Q}(\mathcal{T}_\kappa, b)$ is $\omega_1$-iterable. It then follows that $b = \Lambda(\mathcal{T}_\kappa)$.

Let now $g \subseteq Coll(\omega, \eta)$ be $\mathcal{P}$-generic and let $\mathcal{U} \in \mathcal{P}[g]$ be a tree on $\mathcal{P}|\eta$ above $\delta_\alpha^{\mathcal{P}}$ which is non-dropping. Working inside $\mathcal{P}[g]$ we need a prescription which decides whether $\mathcal{U}$ is according to $\Lambda$. For this, all we need to do is to identify the correct $\mathcal{Q}$-structures.

We claim that the following definition works: $\mathcal{U}$ is according to $\Lambda$ if for any limit $\beta \leq lh(\mathcal{U})$ and any $\kappa \in (\eta, \delta_{\alpha+1})$ such that $\mathcal{U} \in \mathcal{P}|\kappa[g]$, there is $i_\beta : \mathcal{M}_\beta^{\mathcal{U}} \to \mathcal{N}_\kappa$ such that

$$i_\kappa = i_\beta \circ \pi^{\mathcal{U}}_{0,\beta} \qquad (*).$$

Clearly if (*) holds then $\mathcal{U}$ is via $\Lambda$. To finish our proof, we need to show that if $\mathcal{U} \in \mathcal{P}[g]$ is a tree on $\mathcal{P}|\eta$ as above such that (*) holds for every limit $\alpha < lh(\mathcal{U})$ and $lh(\mathcal{U})$ is of limit length then there is $b \in \mathcal{P}[g]$ such that (*) holds for $\mathcal{U}^\frown b$. To see this, fix such a tree $\mathcal{U}$ and let $\kappa \in (\eta, \delta_{\alpha+1})$ be such that $\mathcal{U} \in \mathcal{P}|\kappa[g]$. It follows that $\mathcal{U}$ is according to $\Lambda$. We now compare $\mathcal{M}(\mathcal{U})$ with $\mathcal{N}_\kappa$. Let $b = \Lambda(\mathcal{U})$ and let $\mathcal{Q} = \mathcal{Q}(\mathcal{U}, b)$. By Lemma 2.11, in the comparison of $\mathcal{M}(\mathcal{U})$ and $\mathcal{N}_\kappa$, only $\mathcal{M}(\mathcal{U})$ moves. To finish, we need to show that we can handle the $\mathcal{M}(\mathcal{U})$-side.

Suppose $\mathcal{S}$ is the tree on $\mathcal{M}(\mathcal{U})$ produced by the comparison process and $lh(\mathcal{S})$ is a limit ordinal. We need to guess the correct branch of $\mathcal{S}$. We have that for some $\xi$, $\mathcal{M}(\mathcal{S}) = \mathcal{N}_\kappa|\xi$. We also have that $\mathcal{N}_\kappa \vDash$ "$\delta(\mathcal{S})$ isn't Woodin". Let $\mathcal{W} \trianglelefteq \mathcal{N}_\kappa$ be the $\mathcal{Q}$-structure of $\mathcal{M}(\mathcal{S})$. Then if $\Lambda(\mathcal{U}^\frown b^\frown \mathcal{S}) = c$ then regarding $\mathcal{S}$ as a tree on $\mathcal{Q}$ (note that this is possible because $\rho(\mathcal{Q}) = \delta(\mathcal{U})$), we either have that $\mathcal{W} = \mathcal{Q}(\mathcal{S}, c) \triangleleft \mathcal{M}_c^{\mathcal{S}}$ or $\mathcal{W} = \mathcal{Q}(\mathcal{S}, c) = \mathcal{M}_c^{\mathcal{S}}$. Either way, $c \in \mathcal{P}[g]$.

The argument just presented shows that the comparison produces $\mathcal{S}$ on $\mathcal{M}(\mathcal{U})$, a final branch $c$ of $\mathcal{S}$ and $\xi$ such that regarding $\mathcal{S}$ a tree on $\mathcal{Q}$, $\mathcal{M}_c^{\mathcal{S}} = \mathcal{N}_\kappa|\xi$ (notice that we use the fact that $\Lambda$ has the Dodd-Jensen property to conclude that $\mathcal{Q}$ doesn't iterate past $\mathcal{N}_\kappa$). Let $i = \pi^{\mathcal{S}} \restriction \mathcal{M}(\mathcal{S})$. It then follows that $i \in \mathcal{P}[g]$. It is not hard now to find $\mathcal{Q}$ while working in $\mathcal{P}$ using $i$: $\mathcal{Q}$ is essentially coded as a subset $A$ of $\delta(\mathcal{U})$ and we have that $i[A] \in \mathcal{P}[g]$.

What we have shown is that we can find $\mathcal{Q}$ in $\mathcal{P}[g]$ and by doing so we have that $b \in \mathcal{P}[g]$. By repeating the above comparison argument, we get that $\mathcal{M}_b^{\mathcal{U}}$ iterates to $\mathcal{N}_\kappa$ and moreover, the iteration embedding $\pi : \mathcal{M}_b^{\mathcal{U}} \to \mathcal{N}_\kappa$ is in $\mathcal{P}[g]$. But, because $\Lambda$ has the Dodd-Jensen property, this implies that (*) holds for $\mathcal{U}^\frown b$. □

THEOREM 3.10 (The generic interpretability). *Suppose $(\mathcal{P}, \Sigma)$ is a prehod pair or is a hod pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal. Assume that for every $\alpha < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}$ has branch condensation. Then generic interpretability holds for $(\mathcal{P}, \Sigma)$.*

PROOF. We need to define a function $F$ satisfying the conditions of Definition 3.8. We simplify our task and assume that $(\mathcal{P}, \Sigma)$ is a prehod pair. Let $\alpha$ be such that $\alpha + 1 = \lambda^{\mathcal{P}}$. We only define $F$ at points $(\alpha, \kappa)$ and leave its definition on

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pairs $(\beta,\kappa)$ for $\beta<\alpha$ to the reader. Let $g\subseteq Coll(\omega,\mathcal{P}(\alpha))$ be $\mathcal{P}$-generic. We need to describe a uniform way of deciding within $\mathcal{P}[g]$ whether a given stack $\vec{\mathcal{T}}\in\mathcal{P}[g]$ is according to $\Sigma_{\mathcal{P}(\alpha)}$. Once this is done club many hulls will be generically correct producing $(\dot{T},\dot{S})$ as in Definition 3.8 (see the proof of Theorem 3.3.15 of [**17**] or Lemma 4.1 of [**34**]).

For each cardinal $\kappa\in(\delta^{\mathcal{P}}_{\alpha},\delta^{\mathcal{P}}_{\alpha+1})$, using the extenders of $\mathcal{P}$ with critical points $>\kappa$, we can construct, using hod pair constructions, a pair $(\mathcal{N}_\kappa,\Sigma_\kappa)\in\mathcal{P}$ such that, in $\mathcal{P}$, $(\mathcal{N}_\kappa,\Sigma_\kappa)$ is a tail of $(\mathcal{P}(\alpha),\Sigma^{\mathcal{P}}_{\alpha})$ (see the proof of Theorem 2.28). Let $\mathcal{W}_\kappa\in\mathcal{P}$ be the tree according to $\Sigma^{\mathcal{P}}_{\alpha}$ on $\mathcal{P}(\alpha)$ with last model $\mathcal{N}_\kappa$. We have $\mathcal{P}\vDash$ "$\Sigma_\kappa=\Sigma_{\mathcal{N}_\kappa,\mathcal{W}_\kappa}$". Let $i_\kappa:\mathcal{P}(\alpha)\to\mathcal{N}_\kappa$ be the iteration map according to $\Sigma$. Clearly, $i_\kappa\in\mathcal{P}$.

Notice now that $\Sigma_\kappa$ can be extended to act on all trees in $V$. This is because by clause 5 of Definition 3.7, $\mathcal{N}_\kappa$ is iterable in $V$. Moreover, because $\Sigma_{\mathcal{P}(\alpha)}$ has branch condensation, it follows from Lemma 2.15 that the interpretation of $\Sigma_\kappa$ on all trees in $V$ coincides with $\mathcal{W}_\kappa$ tail of $\Sigma_{\mathcal{P}(\alpha)}$. We abuse our notation and let $\Sigma_\kappa$ be the strategy acting on all trees. By Theorem 3.9, if $g$ is $<\kappa$-generic then in $\mathcal{P}[g]$, $\mathcal{N}_\kappa$ has a $<o(\mathcal{P})$-iteration strategy $\Psi$ such that $\Sigma_\kappa\cap\mathcal{P}[g]=\Psi$ ($\Psi$ is just the induced strategy from the background universe). We can then find $\Sigma_{\mathcal{P}(\alpha)}\cap\mathcal{P}[g]$ as the $i_\kappa$-pullback of $\Psi$. □

We continue our analysis of the internal theory of hod pairs. We show, as is expected, that internal strategies of hod premice are fullness preserving.

Definition 3.11. *Suppose $\mathcal{P}$ is a hod premouse and $\alpha<\lambda^{\mathcal{P}}$. $\Sigma^{\mathcal{P}}_{\alpha}$ is* internally fullness preserving *if whenever $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P}(\alpha),\Sigma)\cap\mathcal{P}$, $\gamma<\pi^{\vec{\mathcal{T}}}(\alpha)$ and $\eta\in(\delta^{\mathcal{R}}_{\gamma},\delta^{\mathcal{R}}_{\gamma+1}]$[1] is a cutpoint of $\mathcal{R}$ then*

1. *if $\eta\geq o(\mathcal{R}(\gamma))$ and $\mathcal{M}\in\mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}}+1,(|\vec{\mathcal{T}}|^+)^{\mathcal{P}}+1)$-iterable $\Sigma_\gamma$-mouse over $\mathcal{R}|\eta$ then $\mathcal{M}\trianglelefteq\mathcal{R}$, and*
2. *if $\eta<o(\mathcal{R}(\gamma))$ and $\mathcal{M}\in\mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}}+1,(|\vec{\mathcal{T}}|^+)^{\mathcal{P}}+1)$-iterable $\oplus_{\xi<\gamma}\Sigma_\xi$-mouse over $\mathcal{R}(\gamma)|\eta$ then $\mathcal{M}\trianglelefteq\mathcal{R}$,*

Notice that, because of clause 4 of Definition 1.34, if $\mathcal{P}$ is a hod premouse then for any $\alpha+1<\lambda$, any backgrounded construction of $\mathcal{P}$ using extenders with critical points in the interval $(\delta^{\mathcal{P}}_{\alpha},\delta^{\mathcal{P}}_{\alpha+1})$, hybrid or else, doesn't break down. This then implies, using Lemma 2.12, that if $\xi+1\leq\lambda^{\mathcal{P}}$ and $\vec{\mathcal{T}}$ is as in Definition 3.11 and is such that $\vec{\mathcal{T}}\in\mathcal{P}|\delta^{\mathcal{P}}_{\xi+1}$, then clause 1 of Definition 3.11 is equivalent to requiring that if $\mathcal{M}$ is $(\Sigma_\alpha)_{\mathcal{R}(\gamma),\vec{\mathcal{T}}}$-premouse over $\mathcal{R}|\eta$ constructed by the fully backgrounded construction of $\mathcal{P}|\delta^{\mathcal{P}}_{\xi+1}$ done with respect to $(\Sigma_\alpha)_{\mathcal{R}(\gamma),\vec{\mathcal{T}}}$ and over $\mathcal{R}|\eta$ using extenders with critical point $>\max(\eta,\delta^{\mathcal{P}}_{\xi})$ then $\mathcal{M}\trianglelefteq\mathcal{R}$. Similar fact also holds for clause 2 of Definition 3.11.

Theorem 3.12 (Internal fullness preservation). *Suppose $\mathcal{P}$ is a hod premouse and $\alpha<\lambda^{\mathcal{P}}$. Then $\Sigma^{\mathcal{P}}_{\alpha}$ is internally fullness preserving.*

Proof. We assume that $V=\mathcal{P}$, in particular we omit all superscripts and subscripts that involve $\mathcal{P}$. Thus, $\delta^{\mathcal{P}}_{\xi}$ is just $\delta_\xi$ and $\delta=\delta^{\mathcal{P}}$. We only consider the first clause of the definition of internal fullness preservation. The verification of the second clause is very similar.

[1] We let $\delta^{\mathcal{P}}_{-1}=0$.

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It is enough to show that $\Sigma_\alpha$ is internally fullness preserving for stacks that are in $\mathcal{P}|\delta^+_\alpha$. To see this, assume it and let $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}(\alpha), \Sigma_\alpha) \cap \mathcal{P}$ be such that $\vec{\mathcal{T}} \notin \mathcal{P}|\delta^+_\alpha$. Suppose for some $\gamma < \pi^{\vec{\mathcal{T}}}(\alpha)$ there is $\eta \in (\delta_\gamma, \delta_{\gamma+1}]^{\mathcal{R}}$ and a $(\Sigma_\alpha)_{\mathcal{R}(\gamma),\vec{\mathcal{T}}}$-mouse $\mathcal{M}$ over $\mathcal{R}|\eta$ which is not in $\mathcal{R}$ but is $\max(\delta+1, |\vec{\mathcal{T}}|^+ + 1)$-iterable. We let $\phi(\vec{\mathcal{T}}, \mathcal{R}, \mathcal{M}, \alpha, \gamma, \eta)$ be the formula that expresses the above statement. Fix then some cardinal $\mu > \max(\delta, |\vec{\mathcal{T}}|^+)$ and let $\pi : \mathcal{H} \to \mathcal{P}|\mu$ be an elementary embedding such that $crit(\pi) > \delta_\alpha$, $\vec{\mathcal{T}}, \mathcal{M} \in \mathcal{P}|\mu$, $\mathcal{H} \in \mathcal{P}|\delta^+_\alpha$ and $\vec{\mathcal{T}}, \mathcal{R}, \mathcal{M}, \gamma, \eta \in ran(\pi)$. Let $(\vec{\mathcal{S}}, \mathcal{Q}, \mathcal{N}, \beta, \nu) = \pi^{-1}((\vec{\mathcal{T}}, \mathcal{R}, \mathcal{M}, \gamma, \eta))$. Then

$$\mathcal{H} \vDash \phi[\vec{\mathcal{S}}, \mathcal{Q}, \mathcal{N}, \beta, \nu].$$

$\mathcal{N}$ is now $\max(\delta^{\mathcal{H}}+1, (|\vec{\mathcal{S}}|^+)^{\mathcal{H}})+1$-iterable inside $\mathcal{H}$ and because of $\pi \restriction \mathcal{N} : \mathcal{N} \to \mathcal{M}$, $\mathcal{N}$ is $\max(\delta, |\vec{\mathcal{T}}|^+)+1$-iterable in $\mathcal{P}$. Moreover, by hull condensation of $\Sigma_\alpha$, $\vec{\mathcal{S}}$ is according to $\Sigma_\alpha$. Therefore, it must be, by our assumption, that $\mathcal{N} \trianglelefteq \mathcal{Q}$, while $\mathcal{H} \vDash \mathcal{N} \not\trianglelefteq \mathcal{Q}$, contradiction.

In the light of this observation we can assume, without losing generality, that $\lambda^{\mathcal{P}} = \alpha + 1$. This is because if we organize our proof as an induction on $\lambda^{\mathcal{P}}$ then limit case would follow from successor case quite easily. The successor cases, on the other hand, are all the same as one can see from the proof. We therefore assume that $\lambda^{\mathcal{P}} = \alpha + 1$ and we will indicate the place where the general successor case is somewhat different than our case though the modifications needed to accommodate the general successor case are trivial.

*Claim.* It is enough to show that if $(\mathcal{P}, \Sigma)$ is a hod pair and $\alpha < \lambda^{\mathcal{P}}$ then $\Sigma^{\mathcal{P}}_\alpha$ is internally fullness preserving.

Proof. To see this, fix a hod premouse $\mathcal{P}$, $\alpha < \lambda^{\mathcal{P}}$ and a stack $\vec{\mathcal{T}} \in \mathcal{P}|\delta^+_\alpha$ on $\mathcal{P}(\alpha)$ with last model $\mathcal{R}$, $\gamma < \pi^{\vec{\mathcal{T}}}(\alpha)$, $\eta \in (\delta_\gamma, \delta_{\gamma+1}]$ and a $\Sigma_{\mathcal{R}(\gamma),\vec{\mathcal{T}}}$-premouse $\mathcal{M} \in \mathcal{P}$ over $\mathcal{R}|\eta$ which is not in $\mathcal{R}$ but satisfies one of the clauses of Definition 3.11. Let $\pi : \mathcal{H} \to \mathcal{P}|\mu$ be an elementary embedding such that

(1) $\mu$ is a cardinal $> \delta^{\mathcal{P}}$,
(2) $\mathcal{P}(\alpha) \subseteq \mathcal{H}$,
(3) $\mathcal{H} \cap \delta^{\mathcal{P}} \in \delta^{\mathcal{P}}$,
(4) $\alpha, \vec{\mathcal{T}}, \mathcal{R}, \gamma, \eta, \mathcal{M} \in \mathcal{H}$.

Let $\phi$ be the formula above and let $(\bar{\vec{\mathcal{T}}}, \bar{\mathcal{R}}, \bar{\mathcal{M}}, \bar{\gamma}, \bar{\eta}) = \pi^{-1}(\vec{\mathcal{T}}, \mathcal{R}, \mathcal{M}, \gamma, \eta)$. We have that

$$\mathcal{H} \vDash \phi[\vec{\mathcal{T}}, \mathcal{R}, \mathcal{M}, \gamma, \eta].$$

By clause 4 of Definition 1.34, $\mathcal{H}$ is iterable inside $\mathcal{P}$ as $ran(\pi) \cap \delta^{\mathcal{P}}$ is bounded in $\delta^{\mathcal{P}}$. Let now $\Phi$ be the strategy of $\mathcal{H}$. Then $(\mathcal{H}, \Phi)$ is a hod pair and therefore, $\mathcal{M} \trianglelefteq \mathcal{R}$, contradiction. □

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Thus[2], it is enough to verify that the theorem holds for hod pairs and we assume that $\mathcal{P}$ has an iteration strategy $\Sigma$ such that $(\mathcal{P}, \Sigma)$ is a hod pair. Again towards a contradiction assume $\Sigma_\alpha$ isn't fullness preserving (recall that $V = \mathcal{P}$, which is somewhat awkward now because of $\Sigma$, and we omit all the subscripts and superscripts that involve $\mathcal{P}$) as witnessed by a stack in $\mathcal{P}|\delta^+_\alpha$. Let $\vec{\mathcal{T}}_0$, $\gamma_0$, $\eta_0$, $\mathcal{R}_1$, $i_0$, $\kappa_0$, and $\mathcal{M}_0$ be such that

(1) $\vec{\mathcal{T}}_0$ is a stack on $\mathcal{P}(\alpha)$ with last model $\mathcal{R}_1$ and $i_0 = \pi^{\vec{\mathcal{T}}_0}$,
(2) $\eta_0 \in (\delta_{\gamma_0}, \delta_{\gamma_0+1}]^{\mathcal{R}_1}$ and $\mathcal{M}_0$ is a $(\Sigma_\alpha)_{\mathcal{R}_1(\gamma_0),\vec{\mathcal{T}}_0}$-mouse over $\mathcal{R}_1|\eta_0$ such that $\mathcal{M}_0$ is constructed by the background construction of $\mathcal{P}$ done over $\mathcal{R}_1|\eta_0$ with respect to $(\Sigma_\alpha)_{\mathcal{R}_1(\gamma_0),\vec{\mathcal{T}}_0}$ using extenders with critical point $\kappa_0 > \delta^+_\alpha$, and $\mathcal{M}_0 \not\trianglelefteq \mathcal{R}_1$,
(3) $\kappa_0 < \delta^{\mathcal{P}}$ and $(\vec{\mathcal{T}}_0, \gamma_0, \eta_0, \mathcal{R}_1, i_0, \mathcal{M}_0) \in \mathcal{P}|\delta^+_\alpha$.

We now define a sequence $((\mathcal{P}_n, \vec{\mathcal{T}}_n, \gamma_n, \eta_n, \mathcal{R}_n, i_n, \kappa_n, \mathcal{M}_n) : n < \omega)$ as follows:

(1) $\mathcal{P}_0 = \mathcal{P}$, $i_0 = \pi^{\vec{\mathcal{T}}_0}$ and $\mathcal{P}_1$ is the last model of $\vec{\mathcal{T}}_0$ when it is regarded as a stack on $\mathcal{P}$ (thus, $\mathcal{R}_0 \trianglelefteq_{hod} \mathcal{P}_1$),
(2)

$$(\vec{\mathcal{T}}_{n+1}, \gamma_{n+1}, \eta_{n+1}, \mathcal{R}_{n+1}, i_{n+1}, \kappa_{n+1}, \mathcal{M}_{n+1}) = i_n((\vec{\mathcal{T}}_n, \gamma_n, \eta_n, \mathcal{R}_n, i_n, \kappa_n, \mathcal{M}_n))$$

and

$$i_n = \pi^{\vec{\mathcal{T}}_n} : \mathcal{P}_n \to \mathcal{P}_{n+1}.$$

Let $\mathcal{N}_n$ be the fully backgrounded construction of $\mathcal{P}_n$ relative to $\Sigma_{\mathcal{R}_n(\gamma_n),\oplus_{k<n}\vec{\mathcal{T}}_k}$ over $\mathcal{R}_n|\eta_n$ using extenders with critical point $> \kappa_n$. Let $\mathcal{N}^+_n$ be the fully backgrounded construction of $\mathcal{P}_n$ relative to $\Sigma_{\mathcal{R}_{n+1}(\gamma_{n+1}),\oplus_{k<n+1}\vec{\mathcal{T}}_k}$ over $\mathcal{R}_{n+1}|\eta_{n+1}$ using extenders with critical point $> \kappa_n$. We have that $\mathcal{M}_n \trianglelefteq \mathcal{N}^+_n$. Notice that by choosing $\kappa_0$ large enough, we can assume that $\kappa_0 = \kappa_n$ for all $n$. We make this assumption and let $\kappa$ be the common value of $\kappa_n$.

What we like to do now is to simultaneously compare all $\mathcal{N}_n$ and $\mathcal{N}^+_n$. We do this by comparing the fully backgrounded constructions of $\mathcal{P}_n$. However, notice that $\mathcal{N}_n$ and $\mathcal{N}^+_n$ are two different kind of hybrid structures. What we can do is compare the construction of $\mathcal{P}_n$ producing $\mathcal{N}^+_n$ with the construction of $\mathcal{P}_{n+1}$ producing $\mathcal{N}_{n+1}$ and do all the comparisons simultaneously. Notice that the comparison involving $\mathcal{N}^+_n$ and $\mathcal{N}_{n+1}$ when lifted to $\mathcal{P}_n$ and to $\mathcal{P}_{\nu+1}$ respectively, fix $\mathcal{R}_n$ and $\mathcal{R}_{n+1}$. This means that different comparisons do not interfere with one another.

We let $\mathcal{P}^*_n$ be the result of this comparison, i.e., $\mathcal{P}^*_n$ is the iterate of $\mathcal{P}_n$ produced by the comparison process and if $j_n : \mathcal{P}_n \to \mathcal{P}^*_n$ is the iteration embedding then $j_n(\mathcal{N}^+_n)$ and $j_{n+1}(\mathcal{N}_{n+1})$ are compatible. Because $\mathcal{M}_n \not\trianglelefteq \mathcal{R}_{n+1}$, we must have $\mathcal{M}_n \not\trianglelefteq \mathcal{N}_{n+1}$. Because $j_n$ fixes $\mathcal{M}_n$ and $\mathcal{M}_n \trianglelefteq \mathcal{N}^+_n$, we must have that $\mathcal{M}_n \trianglelefteq j_n(\mathcal{N}^+_n)$. Thus, $j_n(\mathcal{N}^+_n)$ must be strictly longer than $j_{n+1}(\mathcal{N}_{n+1})$. This means that $o(\mathcal{P}^*_n) > o(\mathcal{P}^*_{n+1})$, and we get an infinite decreasing sequence which is our

[2]There is an alternative proof which works in the case of $\lambda^{\mathcal{P}} = 0$ quite easily but in general it needs the interpretability result of Theorem 3.10. The proof is based on the following fact. Suppose $W \subseteq V$ are two models of $ZFC^-$, $X \in W$ and there is a mouse $\mathcal{M}$ over $X$ such that for some sentence $\phi$, $\mathcal{M}$ is the least mouse $\mathcal{N}$ over $X$ such that $\mathcal{N} \models \phi$. Suppose also that there is some mouse $\mathcal{S} \in W$ perhaps over a different set such that in $V$ there is $\sigma : \mathcal{M} \to \mathcal{S}$. Then $\mathcal{M} \in W$ as in $W$, one can build a tree searching for $\mathcal{M}$. An attentive reader will find the place where this observation could be used in the case $\lambda^{\mathcal{P}} = 0$ and we leave the general case as an exercise.

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contradiction. This contradiction shows that indeed $\Sigma_\alpha$ must be internally fullness preserving. □

## 3.2. OD-full pointclasses

In this section we introduce $OD$-full and mouse-full pointclasses. In Section 6.1, we will show that the mouse-full pointclasses can be generated by hod pairs in the following sense (in particular, see Theorem 6.1). Suppose first that $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit. For $\alpha < \beta \leq \lambda^{\mathcal{P}}$ let,

$$\Gamma(\mathcal{P}, \Sigma) = \{A : \exists(\vec{\mathcal{S}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A <_w Code(\Sigma_{\mathcal{Q},\vec{\mathcal{S}}}))\}.$$

In Section 5.6, we also define $\Gamma(\mathcal{P}, \Sigma)$ in the case $\lambda^{\mathcal{P}}$ is a successor ordinal. We say that a pointclass $\Gamma$ is *generated* by $(\mathcal{P}, \Sigma)$ if $\Gamma = \Gamma(\mathcal{P}, \Sigma)$. Establishing that mouse-full pointclasses are generated by hod pairs is one of the key steps towards proving MSC.

Given $A \subseteq \mathbb{R}$, we let $W_A = \{C : C <_w A\}$. The motivation behind the following terminology comes from the core model induction applications. In such applications, one builds a certain collection of "nice" sets of reals by induction. At various points of the induction, one has to identify the next *new* set.

Definition 3.13 (New sets). *$A \subseteq \mathbb{R}$ is new if*

(1) $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$,
(2) $\mathcal{P}(\mathbb{R}) \cap L(W_A, \mathbb{R}) = W_A$,
(3) $\Theta^{\mathrm{L}(W_A,\mathbb{R})}$ *is a Suslin cardinal of* $L(A, \mathbb{R})$.

For instance, $\mathbb{R}^\#$ is a "new" set beyond $L(\mathbb{R})$. It is not hard to see that $\mathbb{R}^\# = \oplus_{i<\omega} A_i$ where $A_i$ is the theory of the first $i$ many $\mathbb{R}$-indiscernibles. Clearly for every $i < \omega$, $A_i \in L(\mathbb{R})$ implying that $w(\mathbb{R}^\#) = \Theta^{L(\mathbb{R})}$. Notice also that $\mathbb{R}^\#$ is a minimal new set in a sense that any transitive model of determinacy properly containing $L(\mathbb{R})$ satisfies that $\mathbb{R}^\#$ exists. In general, new sets don't have to be minimal in this sense as there can be divergent models of $AD^+$.

Also, notice that new set of reals $A$ may not contain new information about $L(W_A, \mathbb{R})$. Clearly $\mathbb{R}^\#$ contains information not coded by any set in $L(\mathbb{R})$. However, if the new set $A$ is such that $\theta^{L(W_A,\mathbb{R})}$ is a member of the Solovay sequence of $L(A, \mathbb{R})$ then $W_A$ is already full with respect to ordinal definability relative to any set $B \in W_A$. In such situations, new sets do not contain new information about $L(W_A, \mathbb{R})$.

Call a set $A$ *almost new* if it only satisfies clauses 1 and 2 of Definition 3.13. It is consistent that there is an almost new set which is not new. The minimal model of $\mathrm{AD}^+$ + "there is an almost new set which is not new" has the form $L(U^\#, \mathbb{R})$ where $U$ is the universal $\Sigma^2_1$ set of $L(U^\#, \mathbb{R})$. In this case, $U^\#$ is the almost new set.

Notice that any set whose Wadge rank is a member of the Solovay sequence is a new set. It follows that $AD_{\mathbb{R}}$ implies that the new sets are Wadge cofinal in $\Theta$. More locally, if $\theta_\alpha < \theta$ then almost new sets are Wadge cofinal in $\theta_\alpha$, and if $\alpha$ is limit then the new sets are Wadge cofinal in $\theta_\alpha$.

Notice that being new or almost new is absolute among the models containing the reals and ordinals. Also, notice that if $M \vDash AD^+$ and $A, B \in M$ are such that $B$ is almost new and $A <_w B$ then $A^\#$ exists and $A^\# \in M$. This follows from a general fact that under $AD$, if $B \notin L(A, \mathbb{R})$ then $A^\#$-exists.

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Below we define *completely OD-full* and OD-full pointclasses. These pointclasses are closed under Wadge reducibility and complements. Hence, both hierarchies are well-ordered under inclusion. If $\Gamma$ is a pointclass closed under Wadge reducibility then we let

$$\theta^\Gamma = \{|\leq^*| : \leq^* \in \Gamma \text{ is a prewellordering } \}.$$

DEFINITION 3.14. *Suppose $\Gamma$ is a pointclass closed under Wadge reducibility. Then $\Gamma$ is completely OD-full if either $\Gamma = \mathcal{P}(\mathbb{R})$ or there is a new set $A$ such that $\Gamma = W_A$ and $\theta^\Gamma$ is a member of the Solovay sequence of $L(A, \mathbb{R})$.*

Given a completely $OD$-full pointclass $\Gamma$, we let $(\theta^\Gamma_\beta : \beta \leq \Omega^\Gamma)$ be the Solovay sequence of $L(\Gamma, \mathbb{R})$. Under $\mathrm{AD}^+$ there is a characterization of completely $OD$-full pointclasses.

PROPOSITION 3.15. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and let $\Gamma$ be a pointclass such that $\mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma$ and $L(\Gamma, \mathbb{R}) \vDash$ "if for some $\alpha$, $\theta_{\alpha+1}$ exists then there is a Suslin cardinal in the interval $(\theta_\alpha, \theta_{\alpha+1})$". Then*

(1) *$\Gamma$ is completely OD-full iff either $\Gamma = \mathcal{P}(\mathbb{R})$ or there is a new set $A \subseteq \mathbb{R}$ such that $\Gamma = W_A$ and for any $x \in \mathbb{R}$ and for any $B \in \Gamma$ such that there is a Suslin cardinal in the interval $(w(B), \theta_B^{L(\Gamma,\mathbb{R})})$*
$$(\Sigma^2_1(B, x))^{L(\Gamma,\mathbb{R})} = (\Sigma^2_1(B, x))^{L(A,\mathbb{R})}.$$
(2) *$\Gamma$ is completely OD-full iff either $\Gamma = \mathcal{P}(\mathbb{R})$ or there is a new set $A$ such that $\Gamma = W_A$ and for any $x, y \in \mathbb{R}$ and for any $B \in \Gamma$ such that there is a Suslin cardinal in the interval $(w(B), \theta_B^{L(\Gamma,\mathbb{R})})$,*
$$L(A, \mathbb{R}) \vDash \text{"}x \text{ is } OD(B, y)\text{" iff } L(\Gamma, \mathbb{R}) \vDash \text{"}x \text{ is } OD(B, y)\text{"}.$$

PROOF. We first show that clause 1 implies clause 2 and then we establish clause 1. Fix $\Gamma$ as in the hypothesis of the proposition and suppose first the right side of clause 2 holds. If $\Gamma = \mathcal{P}(\mathbb{R})$ then we have nothing to prove. Assume then that $\Gamma \neq \mathcal{P}(\mathbb{R})$ and let $A$ be as in the right side of clause 2. Then we need to show that $\Gamma = W_A$ and that $\theta^\Gamma$ is a member of the Solovay sequence of $L(A, \mathbb{R})$. The first is immediate. To see the second, assume that $\theta^\Gamma$ isn't a member of the Solovay sequence of $L(A, \mathbb{R})$. Let then $B \in \Gamma$ be such that $\theta_B^{L(A,\mathbb{R})} > \theta^\Gamma$. Let $C = \{(y, x) \in \mathbb{R}^2 : L(\Gamma, \mathbb{R}) \vDash x \notin OD(B, y)\}$. Then because $\theta^\Gamma$ is a Suslin cardinal in $L(A, \mathbb{R})$, $C$ can be uniformized by a set $D$ that is ordinal definable from a real and $B$ in $L(A, \mathbb{R})$. Let $y \in \mathbb{R}$ be such that $L(A, \mathbb{R}) \vDash D \in OD(B, y)$. Let then $y \in \mathbb{R}$ be such that $(y, x) \in D$. We then have that $L(A, \mathbb{R}) \vDash x \in OD(B, y)$ but $L(\Gamma, \mathbb{R}) \vDash x \notin OD(B, y)$.

Let now $A$ be as in clause 1. We claim that $A$ is as in clause 2. To see this, fix $B \in \Gamma$ such that there is a Suslin cardinal in the interval $(w(B), \theta_B^{L(\Gamma,\mathbb{R})})$ and suppose for some $x, y \in \mathbb{R}$, $L(A, \mathbb{R}) \vDash$ "$x$ is $OD(B, y)$". Then let

$$C = \{z : L(A, \mathbb{R}) \vDash \text{"}z \text{ is } OD(B, y)\text{"}\} = C_{(\Sigma^2_1(B,y))^{L(A,\mathbb{R})}}.$$

Then $C$ is countable in $L(A, \mathbb{R})$ and, because we are assuming that clause 1 is true, $C \in (\Sigma^2_1(B, y))^{L(\Gamma,\mathbb{R})}$. Because $C$ is countable, we must have that

$$C \subseteq C_{(\Sigma^2_1(B,y))^{L(\Gamma,\mathbb{R})}}{}^3.$$

[3]Recall that for good $\Phi$, $C_\Phi$ is the largest countable $\Phi$ set, i.e., if $C \in \Phi$ and $C$ is countable then $C \subseteq C_\Phi$.

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Therefore, $x$ is $OD(B, y)$ in $L(\Gamma, \mathbb{R})$. The other direction is trivial. We now prove clause 1.

First let $\Gamma$ be a completely OD-full pointclass. If $\Gamma = \mathcal{P}(\mathbb{R})$ then there is nothing to prove. Assume then that $\Gamma \neq \mathcal{P}(\mathbb{R})$. Fix then $A \subseteq \mathbb{R}$ such that $\Gamma = W_A$ and $\theta^\Gamma$ is a member of the Solovay sequence of $L(A, \mathbb{R})$. Let $B \in \Gamma$ be such that there is a Suslin cardinal in the interval $(w(B), \theta_B^{L(\Gamma,\mathbb{R})})$ and fix a real $x$. We want to show that

$$(\Sigma^2_1(B, x))^{L(\Gamma,\mathbb{R})} = (\Sigma^2_1(B, x))^{L(A,\mathbb{R})}.$$

and we have that $\subseteq$-direction is immediate. The other direction follows from clause 4 of Theorem A.10.

Suppose now that $\Gamma, A, B$ witness that right to left direction of clause 1 is false. Thus $\Gamma$ isn't completely $OD$-full. This means that $\theta^\Gamma$ isn't a member of the Solovay sequence of $L(A, \mathbb{R})$ yet it is a Suslin cardinal in $L(A, \mathbb{R})$. Let then $C \in \Gamma$ be such that

$$\theta^\Gamma < \theta_C^{L(A,\mathbb{R})} \qquad (*).$$

We have that $(\Sigma^2_1(C, x))^{L(A,\mathbb{R})} = (\Sigma^2_1(C, x))^{L(\Gamma,\mathbb{R})}$ which means that

$$(\underset{\sim}{\delta}^2_1(C))^{L(A,\mathbb{R})} = (\underset{\sim}{\delta}^2_1(C))^{L(\Gamma,\mathbb{R})}.$$

Let $\delta$ then be this common value. Because of $(*)$ and because $\theta^\gamma$ is a Suslin cardinal in $L(A, \mathbb{R})$, we must have that $\delta \geq \theta^\Gamma$. But, because $\delta$ is the largest Suslin cardinal less than $(\theta_C)^{L(\Gamma,\mathbb{R})}$, we must have that $\delta < \theta^\Gamma$, which is a contradiction. □

Given two completely OD-full pointclasses $\Gamma_1$ and $\Gamma_2$, we say $\Gamma_1$ is a $\theta$-initial segment of $\Gamma_2$ and write

$$\Gamma_1 \trianglelefteq_\theta \Gamma_2$$

if there is $\alpha \leq \Omega^{\Gamma_2}$ such that $\Gamma_1 = \{B : w(B) < \theta_\alpha^{\Gamma_2}\}$.

Definition 3.16 (OD-full pointclass). *$\Gamma$ is an OD-full pointclass if either $\Gamma$ is a completely OD-full pointclass or there is a sequence $(\Gamma_\alpha : \alpha \leq \Omega^\Gamma)$ such that*

1. *$\Omega^\Gamma$ is a limit ordinal,*
2. *$\Gamma_\alpha$ is a completely OD-full pointclass,*
3. *if $\alpha < \beta < \Omega^\Gamma$ then $\Gamma_\alpha \triangleleft_\theta \Gamma_\beta$,*
4. *$\Gamma = \Gamma_\Omega = \cup_{\alpha<\Omega^\Gamma}\Gamma_\alpha$.*

For $\alpha < \Theta$, we let $\Pi_\alpha$ be the $\alpha$th OD-full pointclass (in the Wadge hierarchy). Suppose $\Gamma$ is an $OD$-full. If it is completely $OD$-full pointclass then we have already explained how to define its Solovay sequence. Notice that if $\Pi_\alpha$ and $\Pi_\beta$ are such that $\Pi_\alpha \trianglelefteq_\theta \Pi_\beta$ then the Solovay sequence of $\Pi_\alpha$ is an initial segment of the Solovay sequence of $\Pi_\beta$. It follows that we can let the Solovay sequence of $\Gamma$ be the union of the Solovay sequences of $\Gamma_\alpha$ where $(\Gamma_\alpha : \alpha \leq \Omega^\Gamma)$ is as in Definition 3.16. Note that we can require that $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ satisfy the following two additional conditions:

1. there is no completely OD-full pointclass $\Psi \subseteq \Gamma$ such that for some $\alpha$, $\Gamma_\alpha \triangleleft_\theta \Psi \triangleleft_\theta \Gamma_{\alpha+1}$,
2. if $\alpha \leq \Omega^\Gamma$ is limit then $\Gamma_\alpha = \cup_{\beta<\alpha}\Gamma_\beta$.

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We call $(\Gamma_\alpha : \alpha \leq \Omega^\Gamma)$ that satisfies the four conditions of Definition 3.16 and the two conditions above the *$\theta$-initial segments* of $\Gamma$. It is not hard to see that if $(\Gamma_\alpha : \alpha \leq \Omega^\Gamma)$ are the $\theta$-initial segments of $\Gamma$ then $\Omega^{\Gamma_\alpha} = \alpha$. Therefore, we have that

$$\theta^\Gamma_\alpha = \theta^{\Gamma_\alpha}.$$

Next we define mouse-full pointclasses.

DEFINITION 3.17. *Suppose $\Gamma$ is a pointclass closed under Wadge reducibility. Then $\Gamma$ is completely mouse-full if either $\Gamma = \mathcal{P}(\mathbb{R})$ or there is a new set $A$ such that*

1. *$\Gamma = W_A$,*
2. *if $(\mathcal{P}, \Sigma)$ is a hod pair such that $Code(\Sigma) \in \Gamma$ and $L(A, \mathbb{R}) \vDash$ "$\Sigma$ has branch condensation and is fullness preserving" then for every $a \in HC$, $Lp^{\Gamma,\Sigma}(a) = (Lp^\Sigma(a))^{L(A,\mathbb{R})}$.*

Given two pointclasses $\Gamma_1$ and $\Gamma_2$ such that both are completely mouse-full, we say $\Gamma_1$ is a mouse-initial segment of $\Gamma_2$ and write

$$\Gamma_1 \trianglelefteq_{mouse} \Gamma_2$$

if whenever $(\mathcal{P}, \Sigma)$ is a hod pair such that $Code(\Sigma) \in \Gamma_1$ and $L(\Gamma_2, \mathbb{R}) \vDash$ "$\Sigma$ has branch condensation and is fullness preserving", then for every $a \in HC$, $Lp^{\Gamma_1,\Sigma}(a) = Lp^{\Gamma_2,\Sigma}(a)$.

DEFINITION 3.18 (Mouse-full pointclass). *$\Gamma$ is a mouse-full pointclass if either $\Gamma$ is a completely mouse-full pointclass or there is a sequence $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ such that*

1. *$\Omega$ is a limit ordinal,*
2. *$\Gamma_\alpha$ is a completely mouse-full,*
3. *if $\alpha < \beta < \Omega^\Gamma$ then $\Gamma_\alpha \triangleleft_{mouse} \Gamma_\beta$,*
4. *$\Gamma = \cup_{\alpha<\Omega^\Gamma} \Gamma_\alpha$.*

We let $\Phi^m_\gamma$ be the $\gamma$-th mouse-full pointclass (in the Wadge hierarchy). Suppose $\Gamma$ is mouse-full but not completely mouse-full. Then let $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ be as in Definition 3.18. Note that we can also require that $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ satisfies the following two additional conditions:

1. there is no completely mouse-full pointclass $\Psi \subseteq \Gamma$ such that for some $\alpha$, $\Gamma_\alpha \triangleleft_{mouse} \Psi \triangleleft_{mouse} \Gamma_{\alpha+1}$,
2. if $\alpha \leq \Omega^\Gamma$ is limit then $\Gamma_\alpha = \cup_{\beta<\alpha} \Gamma_\beta$.

If $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ satisfies the conditions of Definition 3.18 and the 2 conditions above, then we say $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ are the mouse-initial segments of $\Gamma$.

If $\Gamma$ is completely OD-full then we write $\Gamma \vDash SMC$ if $\mathrm{L}(\Gamma, \mathbb{R}) \vDash SMC$. If $\Gamma$ is OD-full but not completely OD-full then we write $\Gamma \vDash SMC$ if letting $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ be the $\theta$-initial segments of $\Gamma$, for every $\alpha < \Omega^\Gamma$, $\Gamma_\alpha \vDash SMC$.

Notice that if $A \subseteq \mathbb{R}$ is a new set such that $L(A, \mathbb{R}) \vDash SMC$ and $W_A = \Phi^m_\gamma$ for some $\gamma$ then $\Phi^m_\gamma = \Phi_\gamma$.

## 3.3. The derived models of hod mice

We assume $AD^+$ throughout this section. Let $(\mathcal{P}, \Sigma)$ be a hod pair such that $\lambda^\mathcal{P}$ is limit and $\Sigma$ has branch condensation and is fullness preserving. Our motivational

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question here is the following. What are the sets of reals that appear in the derived model of $\mathcal{P}$ at $\delta^{\mathcal{P}}$ as computed by $\Sigma$?

Fix now a hod pair $(\mathcal{P},\Sigma)$ such that $\lambda^{\mathcal{P}}$ is limit. Suppose $\alpha \leq \lambda^{\mathcal{P}}$ is a limit ordinal such that $\mathrm{cf}^{\mathcal{P}}(\alpha)$ isn't a measurable cardinal in $\mathcal{P}$. We let $D^*(\mathcal{P},\Sigma,\alpha)$ be the set of all $A \subseteq \mathbb{R}$ such that for some $\beta < \alpha$ and $g \subseteq Coll(\omega,\delta^{\mathcal{P}}_\beta)$ generic over $\mathcal{P}(\alpha)$ there are $\delta^{\mathcal{P}}_\alpha$-complementing trees $T,U \in \mathcal{P}(\alpha)[g]$ such that $x \in A$ if and only if there is $(\vec{\mathcal{S}},\mathcal{R}) \in I(\mathcal{P}(\alpha),\Sigma_{\mathcal{P}(\alpha)})$ such that $\pi^{\vec{\mathcal{S}}}$ is above $\delta^{\mathcal{P}}_\beta$ and for some $\gamma < \lambda^{\mathcal{R}}$, $x$ is generic for the extender algebra of $\mathcal{R}[g]$ at $\delta^{\mathcal{R}}_{\gamma+1}$ and $\mathcal{R}[g,x] \vDash x \in p[\pi^{\vec{\mathcal{S}}}(T)]$. We let $D(\mathcal{P},\Sigma,\alpha)$ be the derived model of $\mathcal{P}(\alpha)$ as computed by $\Sigma$, i.e., for $A \subseteq \mathbb{R}$, $A \in D(\mathcal{P},\Sigma,\alpha)$ if there is $(\vec{\mathcal{S}},\mathcal{Q}) \in I(\mathcal{P}(\alpha),\Sigma)$ such that $A \in D^*(\mathcal{Q},\Sigma_{\mathcal{Q}},\pi^{\vec{\mathcal{S}}}(\alpha))$. The following is mainly a corollary to Theorem 2.42.

THEOREM 3.19. *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is fullness preserving. Suppose further that there is a good pointclass $\Gamma$ such that $Code(\Sigma) \in \Delta_{\underset{\sim}{\Gamma}}$. Then*

1. $\Gamma(\mathcal{P},\Sigma) = \cup_{\mathcal{Q} \in pI(\mathcal{P},\Sigma),\alpha<\lambda^{\mathcal{Q}}} D(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha)$.
2. *For any $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$, if $\alpha+\omega < \lambda^{\mathcal{P}}$ then $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha)$ is completely mouse full and if $\alpha+\omega = \lambda^{\mathcal{P}}$ then $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha)$ is mouse full.*
3. *For any $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$, if $\alpha < \lambda^{\mathcal{P}}$ then letting $\Gamma = D(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha+\omega)$, if $\xi$ is such that $\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\alpha)})} = \theta^{\Gamma}_{\xi}$ then for every $n$, $\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\alpha+n)})} = \theta^{\Gamma}_{\xi+n}$ and $\Omega^{\Gamma} = \xi+\omega$.*
4. *$\Gamma(\mathcal{P},\Sigma)$ is a mouse full pointclass.*

PROOF. We start with clause 1.

*Claim 1.* Suppose $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ and $\alpha \leq \lambda^{\mathcal{Q}}$ is a limit ordinal such that $\mathrm{cf}^{\mathcal{Q}}(\alpha)$ isn't a measurable cardinal in $\mathcal{Q}$. Then $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha) = \Gamma(\mathcal{Q}(\alpha),\Sigma_{\mathcal{Q}(\alpha)})$.

PROOF. We start by showing that $D^*(\mathcal{Q},\Sigma_{\mathcal{Q}},\alpha)$ is well-defined. To simplify our notation, we assume that $\mathcal{Q} = \mathcal{P}$ (clearly this assumption doesn't harm the generality of the proof). It is enough to show that if $T,S \in \mathcal{P}$ are $\delta^{\mathcal{P}}_\alpha$-complementing and

$$(\vec{\mathcal{S}}_0,\mathcal{R}_0),(\vec{\mathcal{S}}_1,\mathcal{R}_1) \in I(\mathcal{P}(\alpha),\Sigma_{\mathcal{P}(\alpha)}),$$

then for any $x$ if for some $\beta_i < \pi^{\vec{\mathcal{S}}_i}(\alpha)$ $(i=0,1)$, $x$ is generic for the extender algebra of $\mathcal{R}_i$ at $\delta^{\mathcal{R}_i}_{\beta_i+1}$, then

$$\mathcal{R}_0[x] \vDash \text{“}x \in p[T_0]\text{”} \leftrightarrow \mathcal{R}_1[x] \vDash \text{“}x \in p[T_1]\text{”}$$

where $(T_i,S_i) = \pi^{\vec{\mathcal{S}}_i}(T,S)$. To see this, note that by Theorem 2.28 and Theorem 2.42, we can compare $(\mathcal{R}_0,\Sigma_{\mathcal{R}_0})$ with $(\mathcal{R}_1,\Sigma_{\mathcal{R}_1})$ and get a common tail $(\mathcal{R},\Psi)$. Examining the proof of Theorem 2.28, we see that we can also make the comparison in such a way that $x$ is generic at $\delta^{\mathcal{R}}_0$. Because $\pi^{\Sigma}_{\mathcal{R}_0,\mathcal{R}}(T_0,S_0) = \pi^{\Sigma}_{\mathcal{R}_1,\mathcal{R}}(T_1,S_1)$, we get our desired equality.

Note that by Theorem 3.10, for all $\beta < \alpha$, $Code(\Sigma_{\mathcal{P}(\beta)}) \in D^*(\mathcal{P},\Sigma,\alpha)$. Now, suppose $(\vec{\mathcal{S}},\mathcal{R}) \in I(\mathcal{P}(\alpha),\Sigma_{\mathcal{P}(\alpha)})$. We claim that

*Subclaim.* $D^*(\mathcal{R},\Sigma_{\mathcal{R}},\pi^{\vec{\mathcal{S}}}(\alpha)) \subseteq \Gamma(\mathcal{P}(\alpha),\Sigma_{\mathcal{P}(\alpha)})$.

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PROOF. Because $\Gamma(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)}) = \Gamma(\mathcal{R}, \Sigma_{\mathcal{R}})$, it is enough to show that

$$D^*(\mathcal{R}, \Sigma_{\mathcal{R}}, \pi^{\vec{\mathcal{S}}}(\alpha)) \subseteq \Gamma(\mathcal{R}, \Sigma_{\mathcal{R}}).$$

Fix then two trees $T, S$ in $\mathcal{R}[g]$ where $g \subseteq Coll(\omega, \delta^{\mathcal{R}}_{\beta})$ is $\mathcal{R}$-generic for some $\beta < \pi^{\vec{\mathcal{S}}}(\alpha)$ and $T, S$ are $\delta^{\mathcal{R}}$-complementing in $\mathcal{R}[g]$. Let $A$ be the set Suslin captured by $(\mathcal{R}, \Sigma_{\mathcal{R}}, T)$. Then $x \in A$ iff for any normal tree $\mathcal{T}$ on $\mathcal{R}$ such that

(1) $\mathcal{T}$ is based on the window $[\delta_{\beta}, \delta_{\beta+1}]^{\mathcal{R}}$,
(2) $\mathcal{T}$ is according to $\Sigma_{\mathcal{R}}$,
(3) $\mathcal{T}$ has a last model $\mathcal{Q}$,
(4) $x$ is generic over $\mathcal{Q}$ for the extender algebra at $\delta^{\mathcal{Q}}_{\beta+1}$,

$$x \in p[\pi^{\mathcal{T}}(T)]^{\mathcal{Q}[x]}.$$

Thus, $A \leq_w Code(\Sigma_{\mathcal{R}(\beta+1)})$ and therefore, $A \in \Gamma(\mathcal{R}, \Sigma_{\mathcal{R}})$. □

To finish the proof of Claim 1, notice that

$$D(\mathcal{P}, \Sigma, \alpha) = \cup_{(\vec{\mathcal{S}}, \mathcal{R}) \in I(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})} D^*(\mathcal{R}, \Sigma_{\mathcal{R}, \vec{\mathcal{S}}}, \pi^{\vec{\mathcal{S}}}(\alpha)).$$

It then follows from the Subclaim that $D(\mathcal{P}, \Sigma, \alpha) \subseteq \Gamma(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$. On the other hand, Theorem 3.8 implies that $\Gamma(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)}) \subseteq D(\mathcal{P}, \Sigma, \alpha)$. Hence, $D(\mathcal{P}, \Sigma, \alpha) = \Gamma(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$ □

Claim 1 then easily implies that clause 1 holds. This is because if $A = \{(\mathcal{Q}, \alpha) : \mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $\mathcal{Q} \vDash$ "$\alpha < \lambda^{\mathcal{Q}}$ and $\mathrm{cf}(\alpha)$ isn't measurable"$\}$ then

$$\Gamma(\mathcal{P}, \Sigma) = \cup_{(\mathcal{Q}, \alpha) \in A} \Gamma(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}).$$

and by Claim 1, $\Gamma(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}) = D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$. We now start proving clause 2.

*Claim 2.* Suppose $\alpha \leq \lambda^{\mathcal{P}}$ is such that $\mathrm{cf}^{\mathcal{P}}(\alpha)$ isn't a measurable cardinal in $\mathcal{P}$. Then for all $\beta < \gamma < \alpha$, in $D(\mathcal{P}, \Sigma, \alpha)$, $\Sigma_{\mathcal{P}(\gamma)}$ isn't ordinal definable from a real and $\Sigma_{\mathcal{P}(\beta)}$.

PROOF. Suppose not and let $x$ be such that $\Sigma_{\mathcal{P}(\gamma)}$ is ordinal definable from $\Sigma_{\mathcal{P}(\beta)}$ and $x$ in $D(\mathcal{P}, \Sigma, \alpha)$. Then we can iterate $\mathcal{P}$ below $\delta^{\mathcal{P}}_{\beta+1}$ but above $\delta^{\mathcal{P}}_{\beta}$ to make $x$ generic. Let $\mathcal{R}$ be this iterate and let $i : \mathcal{P} \to \mathcal{R}$ be the iteration map. Let $\Lambda = \Sigma_{\mathcal{R}}$ be the strategy of $\mathcal{R}$. Since $i(\beta) = \beta$, we have that $\Lambda_{\mathcal{R}(i(\beta))} = \Sigma_{\mathcal{P}(\beta)}$. Because $\Sigma_{\mathcal{P}(\gamma)}$ is ordinal definable from $\Sigma_{\mathcal{P}(\beta)}$ and $x$ in $D(\mathcal{P}, \Sigma, \alpha)$, we have that $\mathcal{P}(\gamma) \in \mathcal{R}[x]$ and $\mathcal{R}[x] \vDash$ "$\mathcal{P}(\gamma)$ is countable". Moreover, it also follows that $\Sigma_{\mathcal{P}(\gamma)} \restriction \mathcal{R}[x]$ is definable over $\mathcal{R}[x]$. Because of this, $\mathcal{R}[x]$ can reconstruct the iteration from $\mathcal{P}(\gamma)$ to $\mathcal{R}(i(\gamma))$ implying that $\delta^{\mathcal{R}}_{i(\gamma)}$ isn't a cardinal in $\mathcal{R}[x]$. Because $i(\gamma) \geq \beta + 1$, we get a contradiction. □

Next claim shows that $D(\mathcal{P}, \Sigma, \alpha)$ is a completely mouse-full pointclass.

*Claim 3.* Suppose $(\vec{\mathcal{S}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\alpha < \lambda^{\mathcal{P}}$. Suppose $\alpha < \lambda^{\mathcal{Q}}$ is a limit such that $\mathrm{cf}^{\mathcal{Q}}(\alpha)$ isn't a measurable cardinal in $\mathcal{Q}$. Suppose $\Lambda \in D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$ is an $\omega_1$-iteration strategy with hull condensation and suppose for some real $x$, $\mathcal{M}$ is a sound $\Lambda$-mouse over $x$ projecting to $x$. Suppose further that if $\Psi$ is the unique iteration strategy of $\mathcal{M}$ then $Code(\Psi) \in \Gamma(\mathcal{P}, \Sigma)$. Then $Code(\Psi) \in D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$.

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PROOF. Towards a contradiction, suppose $\Lambda$, $\mathcal{M}$ and $\Psi$ are as above but $Code(\Psi) \notin D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$. Without loss of generality, using Claim 1, we can assume that $Code(\Lambda) \in D^*(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$. Let then $\beta < \alpha$ and $g \subseteq Coll(\omega, \delta^{\mathcal{Q}}_{\beta})$ be such that $g$ is $\mathcal{Q}$-generic, there are trees $(T, U) \in \mathcal{Q}[g]$ witnessing that $Code(\Lambda) \in D^*(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$ and $x \in \mathcal{Q}[g]$. We then have that $\Lambda \restriction \mathcal{Q}[g]$ is definable over $\mathcal{Q}[g]$. Notice that $Code(\Lambda) <_w Code(\Sigma_{\mathcal{Q}(\beta+1)})$ and therefore,

$$Code(\Psi) \in (\Sigma^2_1(Code(\Sigma_{\mathcal{Q}(\beta+1)}), x))^{D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)}.$$

Now, fix $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $\gamma < \lambda^{\mathcal{R}}$ such that $Code(\Psi) \leq_w Code(\Sigma_{\mathcal{R}(\gamma)})$. Let $\vec{\mathcal{U}} = \pi^{\vec{\mathcal{S}}}\vec{\mathcal{T}}$, i.e., the copy of $\vec{\mathcal{T}}$ to $\mathcal{Q}$. Let $\mathcal{W}$ be the last model of $\vec{\mathcal{U}}$ and let $\pi : \mathcal{R} \to \mathcal{W}$ come from the copying construction. Then because $Code(\Sigma_{\mathcal{R}(\gamma)}) \leq_w Code(\Sigma_{\mathcal{W}(\pi(\gamma))})$, we have that $Code(\Psi) \leq_w Code(\Sigma_{\mathcal{W}(\pi(\gamma))})$. But because $Code(\Psi) \notin D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$, we must have that $Code(\Psi) \notin D(\mathcal{W}, \Sigma_{\mathcal{W}}, \pi^{\vec{\mathcal{U}}}(\alpha))$. Because $Code(\Psi) \in (\Sigma^2_1(Code(\Sigma_{\mathcal{W}(\pi^{\vec{\mathcal{U}}}(\beta+1))}, x))^{D(\mathcal{W}, \Sigma_{\mathcal{W}}, \pi(\gamma))}$, it must be the case that in $D(\mathcal{W}, \Sigma_{\mathcal{W}}, \pi(\gamma))$, $\Sigma_{\mathcal{W}(\pi^{\vec{\mathcal{U}}}(\beta+2))}$ is $OD$ from a real and $\Sigma_{\mathcal{W}(\pi^{\vec{\mathcal{U}}}(\beta+1))}$. This, because of Claim 2, is a contradiction. □

Claim 3 then implies that if $\mathcal{Q}_i \in pI(\mathcal{P}, \Sigma)$, $(i = 0, 1)$ and $\alpha_i \leq \lambda^{\mathcal{Q}_i}$ are limit ordinals such that $\mathrm{cf}^{\mathcal{Q}_i}(\alpha_i)$ isn't measurable in $\mathcal{Q}_i$ then either $D(\mathcal{Q}_0, \Sigma_{\mathcal{Q}_0}, \alpha_0) \trianglelefteq_{mouse} D(\mathcal{Q}_1, \Sigma_{\mathcal{Q}_1}, \alpha_1)$ or $D(\mathcal{Q}_1, \Sigma_{\mathcal{Q}_1}, \alpha_1) \trianglelefteq_{mouse} D(\mathcal{Q}_0, \Sigma_{\mathcal{Q}_0}, \alpha_0)$. This then implies that if $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $\alpha \leq \lambda^{\mathcal{Q}}$ is a limit ordinal such that $\mathrm{cf}^{\mathcal{Q}}(\alpha)$ isn't measurable then $D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha)$ is completely mouse-full as witnessed by $D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \alpha + \omega)$. This then finishes the proof of clause 2 in the case when for each $\alpha < \lambda^{\mathcal{P}}$, $\alpha + \omega < \lambda^{\mathcal{P}}$. If there is $\alpha < \lambda^{\mathcal{P}}$ such that $\alpha + \omega = \lambda^{\mathcal{P}}$, then Claim 2 easily implies clause 2. Claim 2 and 3 easily imply clause 3. To show clause 4, notice that because of clause 1, $\Gamma(\mathcal{P}, \Sigma)$ is a mouse-full pointclass. □

Next we show that $\Gamma(\mathcal{P}, \Sigma)$ satisfies mouse capturing relative to any $\Sigma_{\mathcal{Q}}$ where $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$.

THEOREM 3.20. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and $\Sigma$ has branch condensation and is fullness preserving. Suppose further that there is a good pointclass $\Gamma$ such that $Code(\Sigma) \in \Delta_{\underset{\sim}{\Gamma}}$. Then for every $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, $\Gamma(\mathcal{P}, \Sigma) \vDash$ "$MC$ for $\Sigma_{\mathcal{Q}}$".*

PROOF. Towards a contradiction suppose that there is $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) \vDash$ "$MC$ fails for $\Sigma_{\mathcal{Q}}$". Changing $\mathcal{P}$ with an iterate if necessary, we can assume that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{P}$. Let then $\beta < \lambda^{\mathcal{P}}$ be such that $\mathcal{Q} = \mathcal{P}(\beta)$. By clause 1 and 2 of Theorem 3.19, there are $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ and $\alpha \leq \lambda^{\mathcal{R}}$ such that $\alpha$ is a limit ordinal and $\mathrm{cf}^{\mathcal{R}}(\alpha)$ isn't measurable in $\mathcal{R}$, and there are reals $x, y \in \mathbb{R}$ such that $D(\mathcal{R}, \Sigma_{\mathcal{R}}, \alpha) \vDash$ "$y \in OD(Code(\Sigma_{\mathcal{Q}}), x)$ but $y$ isn't in a $\Sigma_{\mathcal{Q}}$-mouse". Let $i = \pi^{\Sigma}_{\mathcal{P}, \mathcal{R}} : \mathcal{P} \to \mathcal{R}$ and let $\xi$ be such that $i(\mathcal{Q}) = \mathcal{R}(\xi)$. Then, by Claim 2 of Theorem 3.19, we must have that $D(\mathcal{R}, \Sigma_{\mathcal{R}}, \xi + \omega) \vDash$ "$y \in OD(Code(\Sigma_{\mathcal{Q}}), x)$ but $y$ isn't in a $\Sigma_{\mathcal{Q}}$-mouse". Then because, by Claim 1 of Theorem 3.19,

(1) $D(\mathcal{R}, \Sigma_{\mathcal{R}}, \xi + \omega) = D(\mathcal{P}, \Sigma, \beta + \omega)$, $D(\mathcal{P}, \Sigma, \beta + \omega) \vDash$ "$y \in OD(Code(\Sigma_{\mathcal{Q}}), x)$ but $y$ isn't in a $\Sigma_{\mathcal{Q}}$-mouse".

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It is a theorem of Steel and Woodin that mouse capturing on a cone implies mouse capturing (see Chapter 4 of [**37**]). Relativizing this result to $\Sigma_{\mathcal{Q}}$, we get from (1) that if

$$A = \{(z, \mathcal{M}) : z \in \mathbb{R} \text{ and } \mathcal{M} \text{ is a sound } \Sigma_{\mathcal{Q}}\text{-mouse over } z \text{ projecting to } z \text{ with an iteration strategy in } D(\mathcal{P}, \Sigma, \beta + \omega)\}$$

then $A \in (\underset{\sim}{\Delta}^2_1(Code(\Sigma_{\mathcal{Q}})))^{D(\mathcal{P},\Sigma,\beta+\omega)}$ and hence, $A$ and $A^c$ carry scales in $D(\mathcal{P}, \Sigma, \beta + \omega)$ which are $OD$ from a real and $\Sigma_{\mathcal{Q}}$. Moreover, in $D(\mathcal{P}, \Sigma, \beta + \omega)$, we can get a sjs $\vec{A} = (A_i : i < \omega) \in \underset{\sim}{\Delta}^2_1(Code(\Sigma_{\mathcal{Q}}))$ such that $A_0 = A$, $A_1 = A^c$ and $\vec{A}$ is $OD$ from a real and $\Sigma_{\mathcal{Q}}$. Let $z$ be a real such that $\vec{A} \in (OD(\Sigma_{\mathcal{Q}}, z))^{D(\mathcal{P},\Sigma,\beta+\omega)}$. Assume now that, in $D(\mathcal{P}, \Sigma, \beta + \omega)$, $\vec{A}$ is the least $OD(\Sigma_{\mathcal{Q}}, z)$ sjs with the above properties.

Let $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ be a normal iterate of $\mathcal{P}$ above $\delta^{\mathcal{Q}}$ such that $z$ is generic over $\mathcal{R}$ for the extender algebra at $\delta^{\mathcal{R}}_{\beta+1}$. We then still have that

$$\vec{A} \in (OD(\Sigma_{\mathcal{Q}}, z))^{D(\mathcal{R},\Sigma_{\mathcal{R}},\beta+\omega)}$$

and $D(\mathcal{R}, \Sigma_{\mathcal{R}}, \beta+\omega) \vDash$ "$y \in OD(Code(\Sigma_{\mathcal{Q}}), x)$ but $y$ isn't in a $\Sigma_{\mathcal{Q}}$-mouse". Without loss of generality, we then assume that $\mathcal{R} = \mathcal{P}$ and $z$ is generic over $\mathcal{P}$ for the extender algebra at $\delta^{\mathcal{P}}_{\beta+1}$.

Let $g \subseteq Coll(\omega, < \delta^{\mathcal{P}}_{\beta+1})$ be $\mathcal{P}[z]$-generic. Working in $\mathcal{P}[z][g]$, we let $\tau_n \in \mathcal{P}[z, g]^{Coll(\omega,\delta_{\beta+n})}$ be an invariant term such that $(\mathcal{P}[z, g], \Sigma_{\mathcal{P}(\beta+n)}, \tau_n)$ term captures $\vec{A}$ at $\delta^{\mathcal{P}}_{\beta+n}$. (It is clear that there is a term $\tau_n$ such that $(\mathcal{P}[z, g], \tau_n)$ locally term captures $\vec{A}$ at $\delta^{\mathcal{P}}_{\beta+n}$. To see that whenever $(\vec{\mathcal{S}}, \mathcal{R}) \in I(\mathcal{P}(\beta + \omega), \Sigma_{\mathcal{P}(\beta+\omega)})$ is such that $\vec{\mathcal{S}}$ is above $\delta^{\mathcal{P}}_{\beta+1}$ , then $(\mathcal{R}[z, g], \pi^{\vec{\mathcal{S}}}(\tau_n))$ locally term captures $\vec{A}$ at $\delta^{\mathcal{R}}_{\beta+n}$, notice that it follows by elementarity that, $\pi^{\vec{\mathcal{S}}}(\tau_n)$ is a term denoting a sjs in $D(\mathcal{R}, \Sigma_{\mathcal{R}}, \beta+\omega)$ which is $(OD(\Sigma_{\mathcal{Q}}, z))^{D(\mathcal{R},\Sigma_{\mathcal{R}},\beta+\omega)}$. By our minimality assumption, then, $\pi^{\vec{\mathcal{S}}}(\tau_n)$ must be a term for $\vec{A}$.) Notice that $\tau =_{def} (\tau_n : n < \omega) \in \mathcal{P}$.

Now, let $\pi : \mathcal{R}^* \to \mathcal{P}(\beta + \omega)[z, g]|(\delta^+_{\beta+\omega})^{\mathcal{P}(\beta+\omega)}$ be such that $(\tau_n : n < \omega) \in ran(\pi)$, $\mathcal{P}(\beta), z, g \in \mathcal{R}^*$ and $|\mathcal{R}^*| < \delta^{\mathcal{P}}_{\beta+1} = \omega_1^{\mathcal{P}[z,g]}$. Let $\mathcal{R}$ be such that $\mathcal{R}^* = \mathcal{R}[z, \pi^{-1}(g)]$. Then $\pi \restriction \mathcal{R} : \mathcal{R} \to \mathcal{P}(\beta + \omega)|(\delta^+_{\beta+\omega})^{\mathcal{P}(\beta+\omega)}$. Let $\Lambda = \Sigma^{\pi}_{\mathcal{P}(\beta+\omega)}$. Then, it follows from Lemma 2.21 that, in $D(\mathcal{P}, \Sigma, \beta + \omega)$, $\Lambda_{\mathcal{R}(\beta+1)}$ is fullness preserving. This then implies that if we compare $\mathcal{R}(\beta + 1)$ with $\mathcal{P}(\beta + 1)$ by using $\Lambda_{\mathcal{R}(\beta+1)}$ on $\mathcal{R}(\beta + 1)$ and $\Sigma_{\mathcal{P}(\beta+1)}$ on $\mathcal{P}(\beta + 1)$-side then the comparison will end at a common last model $\mathcal{S}$ (the comparison done here is just a comparison to line up the models not the strategies and hence, it only involves iterating above $\delta^{\mathcal{P}}_{\beta}$ and only extender disagreements show up). Notice also that we can assume that the comparison is done inside $\mathcal{P}[z, g]$. Let $i : \mathcal{R}(\beta + 1) \to \mathcal{S}$ and $j : \mathcal{P}(\beta + 1) \to \mathcal{S}$ be the iteration embeddings. Then since $i[\delta^{\mathcal{R}(\beta+1)}]$ and $j[\delta^{\mathcal{P}(\beta+1)}]$ are cofinal in $\delta^{\mathcal{S}}$, we have that $\delta^{\mathcal{P}}_{\beta+1}$ has countable cofinality in $\mathcal{P}[z, g]$. This is a contradiction, as $\delta^{\mathcal{P}}_{\beta+1} = \omega_1^{\mathcal{P}[z,g]}$. □

### 3.4. An anomaly

There is one anomalous case to which the theory developed in the previous section doesn't apply. Our main method of constructing hod mice is via a hod mouse construction inside some $\mathcal{N}^*_x$. For this to work, we need to be able to prove

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that $\mathcal{N}_{\alpha+1}$ doesn't project across $\mathcal{P}_\alpha$. The proof of this fact is usually done as follows.

The difficult case is when $\alpha$ has a measurable cofinality in $\mathcal{P}_\alpha$. Let then $\kappa$ be the cofinality of $\alpha$, $E \in \vec{E}^{\mathcal{P}}$ be the measure on $\kappa$ of Mitchell order 0 and $\mathcal{Q} = Ult(\mathcal{P}_\alpha, \mu)(\alpha)$. Then we prove by induction that the direct limit of all $\Sigma_{\mathcal{Q}}$-iterates of $\mathcal{Q}$ contains $V^{\mathrm{HOD}}_{\theta_\gamma}$ for some $\gamma$ such that $\theta_\gamma$ is the image of $\delta^{\mathcal{Q}}$ under the direct limit embedding.

Suppose now that $\mathcal{N}_{\alpha+1}$ projects across $\mathcal{P}_\alpha$. Then let $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha+1}$ be the least such that $\rho(\mathcal{M}) < \delta_\alpha$. We then would like to form the direct limit of all such $(\mathcal{M}, \mathcal{P}, \Sigma)$ where $\mathcal{P} = \mathcal{P}_\alpha$ and $\Sigma = \Sigma_\alpha$. If we can form such a direct limit then letting $\pi$ be the embedding from $\mathcal{P}$ into this direct limit, $\pi(\mathcal{M})$ will be independent of $\pi$. However, $\pi(\mathcal{M}) \in \mathrm{HOD}$ and is essentially a bounded subset of some $\theta_{\xi+1}$ for $\xi < \gamma$. Hence, $\pi(\mathcal{M}) \in V^{\mathrm{HOD}}_{\theta_{\xi+1}}$. It then follows that $\mathcal{M} \in \mathcal{P}$ as $\pi(\mathcal{M})$ is in a proper initial segment of an iterate of $\mathcal{P}$. This then is a contradiction.

In order for this to work, we need to have a comparison theory for pairs that look like $(\mathcal{M}, \Sigma)$. Here, $\mathcal{M}$ may not be a hod mouse and $\Sigma$ may not be fullness preserving. It is this comparison problem that we would like to solve in this section. We start by giving a name to hod pairs like $(\mathcal{M}, \Sigma)$.

Definition 3.21 (Anomalous hod premouse of type I). *$\mathcal{P}$ is an anomalous hod premouse of type I if there is a hod premouse $\mathcal{Q} \trianglelefteq \mathcal{P}$ such that $\lambda^{\mathcal{Q}}$ is a successor, $\mathcal{P} \vDash$ "$\delta^{\mathcal{Q}}$ is Woodin", $\mathcal{P}$ can be organized as $\mathcal{J}^{\vec{E},f}(\mathcal{Q})$ where $f$ codes a fragment of a strategy for $\mathcal{Q}$ and either $\rho(\mathcal{P}) < \delta^{\mathcal{Q}}$ or there is a function witnessing non-Woodiness of $\delta^{\mathcal{Q}}$ via the extenders in $\vec{E}^{\mathcal{Q}}$ definable over $\mathcal{P}$.*

Definition 3.22 (Anomalous hod premouse of type II). *$\mathcal{P}$ is an anomalous hod premouse of type II if for some limit ordinal $\lambda$ and some $\delta$ there is a sequence $(\mathcal{P}_\alpha : \alpha < \lambda)$ such that*

1. *$\mathcal{P}_\alpha$ is a hod premouse such that $\lambda^{\mathcal{P}_\alpha} = \alpha$,*
2. *for $\alpha < \beta < \lambda$, $\mathcal{P}_\alpha \trianglelefteq_{hod} \mathcal{P}_\beta$ and $\mathcal{P}_\alpha = \mathcal{P}_\beta(\alpha)$,*
3. *$\mathcal{P}|\delta = \cup_{\alpha<\lambda} \mathcal{P}_\alpha$,*
4. *$\mathcal{P}$ is a $\oplus_{\alpha<\lambda} \Sigma_{\mathcal{P}(\alpha)}$-premouse over $\mathcal{P}|\delta$,*
5. *$\rho(\mathcal{P}) < \delta^{\mathcal{P}}$ but for every $\xi \in (\delta, o(\mathcal{P}))$, $\rho(\mathcal{P}|\xi) \geq \delta$.*

Definition 3.23 (Anomalous hod premouse of type III). *$\mathcal{P}$ is an anomalous hod premouse of type III if for some limit ordinal $\lambda$ and some $\delta$ there is a sequence $(\mathcal{P}_\alpha : \alpha \leq \lambda)$ such that*

1. *$\mathcal{P}_\alpha$ is a hod premouse such that $\lambda^{\mathcal{P}_\alpha} = \alpha$,*
2. *for $\alpha < \beta \leq \lambda$, $\mathcal{P}_\alpha \trianglelefteq_{hod} \mathcal{P}_\beta$ and $\mathcal{P}_\alpha = \mathcal{P}_\beta(\alpha)$,*
3. *$\mathcal{P}$ is a $\Sigma_{\mathcal{P}(\lambda)}$-premouse over $\mathcal{P}(\lambda)$,*
4. *$\delta = \sup_{\alpha<\lambda} \delta^{\mathcal{P}(\alpha)}$,*
5. *$\rho(\mathcal{P}) < \delta^{\mathcal{P}}$ but for every $\xi \in (\delta, o(\mathcal{P}))$, $\rho(\mathcal{P}|\xi) \geq \delta$,*

We say $\mathcal{P}$ is an anomalous hod premouse if it is an anomalous hod premouse of some type. If $\mathcal{P}$ is an anomalous hod premouse then we let $\delta^{\mathcal{P}}$ and $\lambda^{\mathcal{P}}$ be as in the above definitions. We then let $\Sigma^{\mathcal{P}}$ be the strategy that is on the sequence of $\mathcal{P}$.

Definition 3.24 (Anomalous hod pair). *$(\mathcal{P}, \Sigma)$ is an anomalous hod pair if $\mathcal{P}$ is an anomalous hod premouse, $\Sigma$ is an iteration strategy with hull condensation and whenever $\mathcal{Q}$ is a $\Sigma$ iterate of $\mathcal{P}$, $\Sigma^{\mathcal{Q}} = \Sigma \cap \mathcal{Q}$.*

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There are two different ways that anomalous hod pairs can be iterated. Suppose $(\mathcal{P},\Sigma)$ is an anomalous hod pair and suppose $n$ is the least such that $\rho_n(\mathcal{P})<\delta^{\mathcal{P}}$. Then it is important to keep track of the fine structural ultrapowers we take. We say $\vec{\mathcal{T}}$ is a stack of type I on $\mathcal{P}$ if the degree of the embeddings in it is the maximal it can be. We say $\vec{\mathcal{T}}$ is a stack of type II if letting $(\mathcal{M}_\alpha,\mathcal{T}_\alpha:\alpha<lh(\vec{\mathcal{T}}))$ be the normal components of $\vec{\mathcal{T}}$ then if $\alpha<lh(\vec{\mathcal{T}})$ and $\beta<lh(\mathcal{T}_\alpha)$ are such $\pi^{\vec{\mathcal{T}}}_{\alpha,\beta}:\mathcal{P}\to\mathcal{M}^{\mathcal{T}_\alpha}_\beta$ exists and $E$ is the extender that is applied to $\mathcal{M}^{\mathcal{T}_\alpha}_\beta$ then if $crit(E)<\delta^{\mathcal{M}^{\mathcal{T}_\alpha}_\beta}$ and $crit(E)$ is a cardinal of $\mathcal{M}^{\mathcal{T}_\alpha}_\beta$ then the ultrapower taken by $E$ is of degree $n-1$. We will only need to consider type II iterations, and therefore, from now on we assume that whenever $(\mathcal{P},\Sigma)$ is an anomalous hod pair then $\Sigma$ acts on stacks of type II.

The following lemma is important for the iterations of anomalous hod mice. It is essentially due to Mitchell and Steel though they stated it quite differently. We omit the proof but it is essentially the proof of Claim 5 that appears in the proof of Theorem 6.2 of [**20**]. We call our lemma "the resurrection of strong uniqueness" because Theorem 6.2 is called "strong uniqueness" and, prior to the discovery of weak Dodd-Jensen property, its main use had been to prove solidity and other fine structural facts. In modern inner model theory, one uses Neeman-Steel theorem (see Theorem 4.10 of [**43**]), which shows that any mouse has a strategy with weak Dodd-Jensen property, to prove various fine structural facts. Because of this the importance of "strong uniqueness" along with Claim 5 has diminished. However, for us it is of crucial importance as it is one of the main ingredients for establishing that hod mouse constructions converge. We thank John Steel for suggesting its use.

LEMMA 3.25 (The resurrection of strong uniqueness). *Suppose $(\mathcal{P},\Sigma)$ is a an anomalous hod pair, $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ and $n$ is least such that $\rho_n(\mathcal{P})<\delta^{\mathcal{P}}$. Then $\rho_n(\mathcal{Q})<\delta^{\mathcal{Q}}$.*

We define $I(\mathcal{P},\Sigma)$, $B(\mathcal{P},\Sigma)$ and $\Gamma(\mathcal{P},\Sigma)$ as before. Suppose $(\mathcal{P},\Sigma)$ is a anomalous hod pair. Notice that we do not require that $Lp^{\Sigma_{\mathcal{P}|\delta^{\mathcal{P}}}}(\mathcal{P}|\delta^{\mathcal{P}})\trianglelefteq\mathcal{P}$ and therefore, we cannot hope that $\Sigma$ is fullness preserving. We will show that $\Sigma$ is $\Gamma(\mathcal{P},\Sigma)$-fullness preserving. Given this Theorem 2.33 implies that two anomalous hod pairs $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ can be compared provided that $\Gamma(\mathcal{P},\Sigma)=\Gamma(\mathcal{Q},\Lambda)$. However, we do not need such comparison arguments in a complete abstract setting. The initial segments of anomalous hod pairs we will consider later on are embedded into hod pairs whose strategies are fullness preserving and have branch condensation. Our goal then is to somehow use elementarity to prove that many of the nice properties of hod pairs goes down to anomalous hod pairs.

THEOREM 3.26. *Suppose $(\mathcal{P},\Sigma)$ is an anomalous hod pair of type II or III. Suppose that for any $(\vec{\mathcal{T}},\mathcal{Q})\in B(\mathcal{P},\Sigma)$ there is a hod pair $(\mathcal{R},\Lambda)$ such that $\Lambda$ has branch condensation and is fullness preserving, and there is $\pi:\mathcal{Q}\to\mathcal{R}$ such that $\Lambda^\pi=\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$. Then*

1. *For every $(\vec{\mathcal{T}},\mathcal{Q})\in B(\mathcal{P},\Sigma)$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation, is positional and is commuting.*
2. *$\Sigma$ is $\Gamma(\mathcal{P},\Sigma)$-fullness preserving and $\Gamma(\mathcal{P},\Sigma)$ is a mouse full pointclass.*

PROOF. We give the proof in the case $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$. The other case is easier. We start by proving clause 1. Fix $(\vec{\mathcal{T}},\mathcal{Q})\in B(\mathcal{P},\Sigma)$. We want to show that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation, is positional and is commuting. Let

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$\mathcal{Q}^*$ be such that $(\vec{\mathcal{T}}, \mathcal{Q}^*) \in I(\mathcal{P}, \Sigma)$ and let $\alpha$ be such that $\mathcal{Q}^*(\alpha) = \mathcal{Q}$. Let $\mathcal{Q}^+ = \mathcal{Q}^*(\alpha + \omega)$. Let $(\mathcal{R}, \Lambda)$ be a hod pair such that $\Lambda$ has branch condensation and is fullness preserving, and there is an elementary embedding $\pi : \mathcal{Q}^+ \to \mathcal{R}$ such that $\Sigma_{\mathcal{Q}^+, \vec{\mathcal{T}}} = \Lambda^\pi$ (notice that $(\vec{\mathcal{T}}, \mathcal{Q}^+) \in B(\mathcal{P}, \Sigma)$ because $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$).

It follows from Theorem 3.10 that the interpretation of $\Sigma^{\mathcal{R}}_{\pi(\alpha)}$ onto generic extensions of $\mathcal{R}$ is $\Lambda_{\mathcal{R}(\alpha)}$. Hence, it is true in $\mathcal{R}$ that the generic interpretability procedure of Theorem 3.10 produces an iteration strategy that has branch condensation, is positional and is commuting. By elementarity of $\pi$, in $\mathcal{Q}^+$, the same procedure yields a strategy for $\mathcal{Q}$ that has branch condensation, is commuting and is positional. All we have to do then is to show that this strategy and $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ are the same.

Fix then a stack $\vec{\mathcal{S}}$ on $\mathcal{Q}$ which is according to $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}} = \Lambda^\pi$. Let $\mathcal{W}$ be the iterate of $\mathcal{Q}^+$ according to $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ obtained by iterating in the window $[\delta^{\mathcal{Q}^+}_\alpha, \delta^{\mathcal{Q}^+}_{\alpha+1}]$ to make $\vec{\mathcal{S}}$ generic for the extender algebra at $\delta^{\mathcal{W}}_{\alpha+1}$. Let $\sigma : \mathcal{W} \to \mathcal{M}$ come from copying construction (thus, $\mathcal{M} \in pI(\mathcal{R}, \Lambda)$). Let $\mathcal{N}$ be the iterate of $\mathcal{Q}$ constructed by the hod mouse construction of $\mathcal{W}(\alpha + 2)$ using extenders with critical point $> \delta^{\mathcal{W}}_{\alpha+1}$. Let $\mathcal{U}$ be the tree on $\mathcal{Q}$ with last model $\mathcal{N}$. Let $\eta \in (\delta^{\mathcal{W}}_{\alpha+1}, \delta^{\mathcal{W}}_{\alpha+2})$ be a successor cardinal such that $\mathcal{N}$ is constructed via a hod mouse construction of $\mathcal{W}|\eta$.

*Claim.* Let $\Phi$ be the strategy of $\mathcal{M}|\sigma(\eta)$ that acts on non-dropping trees that are above $\sigma(\delta^{\mathcal{W}}_\alpha)$. Then for any $< \delta^{\mathcal{W}}_{\alpha+1}$ generic $g$, $\Phi^\sigma \restriction \mathcal{W}[g]$ is definable over $\mathcal{W}[g]$ via the procedure described in the proof of Theorem 3.9.

Proof. We verify the claim for normal trees. Let $g$ be a $< \delta^{\mathcal{W}}_{a+1}$-generic over $\mathcal{W}$. Let $\mathcal{P} \subseteq \mathcal{W}[g]$ be the strategy of $\mathcal{W}|\eta$ that is definable over $\mathcal{W}[g]$ via the procedure described in the proof of Theorem 3.9. Let $\mathcal{T} \in \mathcal{W}[g]$ be a normal tree on $\mathcal{W}|\eta$ which has limit length, is above $\delta^{\mathcal{W}}_{\alpha+1}$ and is according to both $\Psi$ and $\Phi^\sigma$. We need to see that $\Psi(\mathcal{T}) = \Phi^\sigma(\mathcal{T})$.

Notice that $\Sigma^{\mathcal{W}}_{\alpha+1} = \Lambda^\sigma_{\mathcal{M}(\sigma(\alpha+1))} \restriction \mathcal{W}$. Let $\mathcal{S}$ be the $\Psi$-iterate of $\mathcal{W}|\eta$ that is constructed by $\mathcal{J}^{\vec{E}, \Sigma^{\mathcal{W}}_{a+1}}$-construction of $\mathcal{W}$ using critical points $> \eta$. Let $\mathcal{U}$ be the tree according to $\Psi$ with last model $\mathcal{S}$ and let $i : \mathcal{W}|\eta \to \mathcal{S}$ be the iteration embedding given by $\mathcal{U}$. Also, let $\Psi(\mathcal{T}) = b$ and $\Phi^\sigma(\mathcal{T}) = c$. Notice that $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ exist and what might prevent them from being the same is that they are hybrid mice in different hierarchy. However, it follows from the procedure described in Theorem 3.9 that there is $\tau : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{S}$ such that $i = \tau \circ \pi^{\mathcal{T}}_b$. Because $\mathcal{S}$ is a $\Lambda^\sigma_{\mathcal{M}(\sigma(\alpha+1))}$-mouse, we have that $\mathcal{M}^{\mathcal{T}}_b$ is a $\Lambda^\sigma_{\mathcal{M}(\sigma(\alpha+1))}$-mouse. Because $\mathcal{Q}(c, \mathcal{T})$ is also $\Lambda^\sigma_{\mathcal{M}(\sigma(\alpha+1))}$-mouse, we have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(c, \mathcal{T})$, and hence, $b = c$. □

We continue with the notation of the claim. Let $\Psi \subseteq \mathcal{W}[\vec{\mathcal{S}}]$ be the strategy of $\mathcal{N}$ induced by $\Phi^\sigma$. Then the strategy of $\mathcal{Q}$ that is according to the procedure of the proof of Theorem 3.10 first copies $\vec{\mathcal{S}}$ onto $\mathcal{N}$ via $\pi^{\mathcal{U}}$ and then uses $\Psi$ to choose branches. Without loss of generality, we can assume that $\vec{\mathcal{S}}$ doesn't have a last branch, its last normal component is of limit length and $\pi^{\mathcal{U}}\vec{\mathcal{S}}$ is according to $\Psi$. We need to see that $\Psi(\pi^{\mathcal{U}}\vec{\mathcal{S}}) = \Lambda^\sigma(\vec{\mathcal{S}})$. Let $i = \pi^{\mathcal{U}}$, $j = \sigma(i)$ and $\vec{\mathcal{S}}^* = \sigma\vec{\mathcal{S}}$. Let $\Psi^*$ be the strategy of $\sigma(\mathcal{N})$ induced by $\Phi$.

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It follows from the claim that $\Psi(i\vec{\mathcal{S}}) = \Psi^*(j\vec{\mathcal{S}}^*)$. It follows from Theorem 3.10 that $\Psi^*(j\vec{\mathcal{S}}^*) = \Lambda(\pi\vec{\mathcal{S}})$ (we have that $\pi\vec{\mathcal{S}} = \sigma\vec{\mathcal{S}}$). It then follows that

$$\Psi(i\vec{\mathcal{S}}) = \Psi^*(j\vec{\mathcal{S}}^*) = \Lambda(\pi\vec{\mathcal{S}}) = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}}(\vec{\mathcal{S}}).$$

This then finishes our proof of clause 1.

We now prove clause 2. Suppose $\Sigma$ isn't $\Gamma(\mathcal{P},\Sigma)$-fullness preserving. We only consider clause 1 of fullness preservation. There is then $(\vec{\mathcal{T}},\mathcal{Q}) \in B(\mathcal{P},\Sigma)$ such that $\lambda^{\mathcal{Q}}$ is a successor ordinal, and letting $\alpha = \lambda^{\mathcal{Q}} - 1$, for some strong cutpoint $\eta \geq \delta^{\mathcal{Q}}_{\alpha}$ of $\mathcal{Q}$ there is a sound $\Sigma_{\mathcal{Q}^-,\vec{\mathcal{T}}}$-mouse $\mathcal{M}$ over $\mathcal{Q}|\eta$ such that $\rho(\mathcal{M}) = \eta$ and $\mathcal{M} \not\trianglelefteq \mathcal{Q}$ yet $\mathcal{M}$ has an iteration strategy in $\Gamma(\mathcal{P},\Sigma)$.

Fix $(\vec{\mathcal{S}},\mathcal{R}) \in B(\mathcal{P},\Sigma)$ such that $\mathcal{M}$'s unique strategy is Wadge reducible to $Code(\Sigma_{\mathcal{R},\vec{\mathcal{S}}})$. Let $\mathcal{Q}^*,\mathcal{R}^*$ be such that $(\vec{\mathcal{T}},\mathcal{Q}^*),(\vec{\mathcal{S}},\mathcal{R}^*) \in I(\mathcal{P},\Sigma)$. Let $\mathcal{W}$ be the last model of $\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}$ and let $\pi : \mathcal{R}^* \to \mathcal{W}$ come from the copying construction. Let $\beta$ be such that $\pi(\mathcal{R}) = \mathcal{W}(\beta)$. Then $\mathcal{M}$'s strategy is Wadge reducible to $Code(\Sigma_{\mathcal{W}(\beta),\vec{\mathcal{T}}^\frown\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}})$.

Let $j : \mathcal{Q}^* \to \mathcal{W}$ be the iteration embedding according to $\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}$. Let $x$ be the real that codes the Wadge reduction of $\mathcal{M}$'s strategy to $Code(\Sigma_{\mathcal{W}(\beta),\vec{\mathcal{T}}^\frown\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}})$. Let $\mathcal{W}^*$ be the iterate of $\mathcal{W}(\beta+\omega)$ according to $\Sigma_{\mathcal{W}(\beta+\omega),\vec{\mathcal{T}}^\frown\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}}$ obtained by iterating $\mathcal{W}$ in the window $(\delta^{\mathcal{W}}_{\beta},\delta^{\mathcal{W}}_{\beta+1}]$ to make $x$ and $j \restriction \mathcal{Q}$ generic for the extender algebra at $\delta^{\mathcal{W}}_{\beta+1}$. Let $g \subseteq Coll(\omega,<\delta^{\mathcal{W}^*})$ be $\mathcal{W}^*$-generic. Let $M$ be the derived model of $\mathcal{W}^*$ as computed by $g$. Then it follows from the proof of clause 1 that letting $\mathcal{U}$ be the tree on $\mathcal{W}(\beta+\omega)$ with last model $\mathcal{W}^*$ and $\Psi$ be the $\vec{\mathcal{T}}^\frown\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}^\frown\mathcal{U}$-tail of $\Sigma$, for every $n<\omega$, $\Psi_{\mathcal{W}^*(\beta+n)} \restriction M \in M$. Moreover, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \restriction M \in M$ and $\mathcal{M}$'s strategy is in $M$.

However, because $\mathcal{M}$'s strategy is in $(\underset{\sim}{\Sigma}^2_1(Code(\Psi_{j(\mathcal{Q})})))^M$, the proofs of Theorem 3.19 and Theorem 3.20 and our hypothesis imply that $\mathcal{M}$'s strategy is in the derived model of $\mathcal{W}^*(j(\alpha)+\omega)$. But by Theorem 3.19 and clause 1,

$$D(\mathcal{W}^*,\Sigma_{\mathcal{W}^*,\vec{\mathcal{T}}^\frown\pi^{\vec{\mathcal{T}}}\vec{\mathcal{S}}},j(\alpha)+\omega) = D(\mathcal{Q}^*,\Sigma_{\mathcal{Q}^*,\vec{\mathcal{T}}},\alpha+\omega)$$

It follows then that in $D(\mathcal{Q}^*,\Sigma_{\mathcal{Q}^*,\vec{\mathcal{T}}},\alpha+\omega)$ there is a sound $\Sigma_{\mathcal{Q}(\alpha)}$-mouse over $\mathcal{Q}|\eta$ which is not in $\mathcal{Q}$. This contradicts Theorem 3.12. We omit the proof that $\Gamma(\mathcal{P},\Sigma)$ is a mouse full pointclass. It follows from the proof of Theorem 3.20 and clause 1. □

Notice that instead of requiring fullness preservation in the statement of Theorem 3.26 we can require $\Gamma$-fullness preservation.

THEOREM 3.27. *Suppose $(\mathcal{P},\Sigma)$ is an anomalous hod pair of type II or III. Suppose that there is a pointclass $\Gamma$ such that for any $(\vec{\mathcal{T}},\mathcal{Q}) \in B(\mathcal{P},\Sigma)$ there is a hod pair $(\mathcal{R},\Lambda)$ such that $\Lambda$ has branch condensation and is $\Gamma$-fullness fullness preserving, and there is $\pi : \mathcal{Q} \to \mathcal{R}$ such that $\Lambda^\pi = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$. Then*

1. *For every $(\vec{\mathcal{T}},\mathcal{Q}) \in B(\mathcal{P},\Sigma)$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation, is positional and is commuting.*
2. *$\Sigma$ is $\Gamma(\mathcal{P},\Sigma)$-fullness preserving and $\Gamma(\mathcal{P},\Sigma)$ is a mouse full pointclass.*

What we haven't shown is that the strategy of an anomalous hod pair has branch condensation. There are two ways of doing this. What we will show in the

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next section is that if $(\mathcal{P},\Sigma)$ is an anomalous hod pair such that $\lambda^{\mathcal{P}}$ is limit then some tail of $\Sigma$ has branch condensation. Notice that if $\lambda^{\mathcal{P}}$ is a successor or is limit but $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ isn't measurable in $\mathcal{P}$ then $\Sigma$ indeed has branch condensation. This is because in the first case, $\Sigma$ is guided by a $\mathcal{Q}$-structure and in the second case we can use clause 1 of Theorem 3.26. In the case when $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$, one can show that $\Sigma$ is still guided by $\mathcal{Q}$-structures but it is much harder and we leave it as an exercise. One has to show various fine structural facts about the iterations of such $\mathcal{Q}$-structures and the correct proof seems to need reorganizing our entire exposition. Instead, we will prove that some tail of $\Sigma$ has branch condensation. This fact is also useful in core model induction applications.

## 3.5. Getting branch condensation

In this section we would like to show that given any hod pair $(\mathcal{P},\Sigma)$ such that $\Sigma$ has hull condensation there is a tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$ such that $\Lambda$ has branch condensation. We do not know how to prove such a result without additional assumptions on $\Sigma$ although we strongly believe that the result is true in general. There are two applications of this result. The first is that the strategies of anomalous hod pairs have tails that are positional and commuting. This will be used in direct limit constructions and in showing that hod mouse constructions converge. The second use is in core model induction and that is beyond the scope of this paper.

THEOREM 3.28 (Getting branch condensation). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair or an anomalous hod pair of type II or III. Suppose further that $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$, and that whenever $(\vec{\mathcal{T}},\mathcal{Q})\in B(\mathcal{P},\Sigma)$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. Then there is $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation.*

Let us summarize a lemma that we proved as part of proving Theorem 3.26.

LEMMA 3.29 (Branch condensation pulls back). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and $\Sigma$ has branch condensation. Suppose $\pi:\mathcal{Q}\to\mathcal{P}$ is elementary. Then for every $\alpha<\lambda^{\mathcal{Q}}$, $(\Sigma^{\pi})_{\mathcal{Q}(\alpha)}$ has branch condensation.*

We now start proving our theorem and this entire section will be devoted to it. The basic idea behind the proof is the same as the one behind the diamond comparison argument (see Theorem 2.47): if we keep using the bad stacks we will end up with a bad sequence of length $\omega_1$ that cannot exist. We start by explaining the kind of bad sequences that we will encounter while proving Theorem 3.28. We call them *bad blocks* again and we warn the reader not to confuse them with the bad blocks of Theorem 2.47.

DEFINITION 3.30 (Bad block). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair. Then $((\mathcal{P}_i:i\leq 5),(\vec{\mathcal{T}}_i:i\leq 4),\Lambda,(\pi_i:i\leq 5),\xi,\vec{\mathcal{U}})$ is a* bad block *for $(\mathcal{P},\Sigma)$ if (see Figure 3.5.1)*

1. *$\mathcal{P}_0=\mathcal{P}$, $\vec{\mathcal{T}}_0$ is a stack on $\mathcal{P}_0$ according to $\Sigma$, $\mathcal{P}_1$ is the last model of $\vec{\mathcal{T}}_0$, and $\pi_0=\pi^{\vec{\mathcal{T}}_0}$,*
2. *$\xi\in(\delta(\vec{\mathcal{T}}_0),\lambda^{\mathcal{P}_1})$ is a successor ordinal, $\vec{\mathcal{T}}_1$ is a stack on $\mathcal{P}_1$ based on $\mathcal{P}_1(\xi)$ and according to $\Sigma_{\mathcal{P}_1,\vec{\mathcal{T}}_0}$, $\mathcal{P}_2$ is the last model of $\vec{\mathcal{T}}_1$ and $\pi_1=\pi^{\vec{\mathcal{T}}_1}$,*
3. *$\Lambda$ is a strategy for $\mathcal{P}_2$ such that $(\mathcal{P}_2,\Lambda)$ is a hod pair, and*

$$\Lambda_{\mathcal{P}_2(\pi_1(\xi))}\neq\Sigma_{\mathcal{P}_2(\pi_1(\xi)),\vec{\mathcal{T}}_0^{\frown}\vec{\mathcal{T}}_1}\text{ but }\Lambda_{\mathcal{P}_2(\pi_1(\xi)-1)}=\Sigma_{\mathcal{P}_2(\pi_1(\xi)-1),\vec{\mathcal{T}}_0^{\frown}\vec{\mathcal{T}}_1},$$

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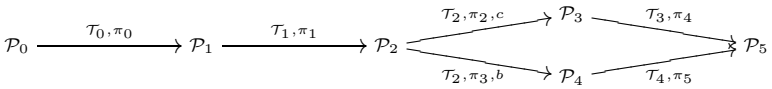


(1) $c = \Lambda(\vec{\mathcal{T}}_2), b = \Sigma_{\mathcal{P}_2,\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1}(\vec{\mathcal{T}}_2), b \neq c$,
(2) $\pi_i$'s are the iteration embeddings,
(3) $\xi \in (\delta(\vec{\mathcal{T}}_0), \lambda^{\mathcal{P}_1})$, $\vec{\mathcal{T}}_1$ is a stack on $\mathcal{P}_1(\xi+1)$, $\Lambda_{\mathcal{P}_2(\pi_1(\xi))} \neq \Sigma_{\mathcal{P}_2(\pi_1(\xi)),\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1}$, and $\Lambda_{\mathcal{P}_1(\pi_1(\xi)-1)} = \Sigma_{\mathcal{P}_1(\pi_1(\xi)-1),\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1}$,
(4) for any $\alpha < \lambda^{\mathcal{P}_5}$, $\Lambda_{\mathcal{P}_5(\alpha),\vec{\mathcal{T}}_2^\frown\mathcal{P}_3^\frown\vec{\mathcal{T}}_3} = \Sigma_{\mathcal{P}_5(\alpha),\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\vec{\mathcal{T}}_2^\frown\mathcal{P}_4^\frown\vec{\mathcal{T}}_4}$,
(5) $\vec{\mathcal{U}} = \vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\vec{\mathcal{T}}_2^\frown\mathcal{P}_4^\frown\vec{\mathcal{T}}_4$.

FIGURE 3.5.1. Bad block.

(4) *$\vec{\mathcal{T}}_2$ is a minimal disagreement between $\Lambda_{\mathcal{P}_2(\pi_1(\xi))}$ and $\Sigma_{\mathcal{P}_2(\pi_1(\xi)),\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1}$, and $\mathcal{P}_3$ and $\mathcal{P}_4$ are its last models according to respectively $\Lambda_{\mathcal{P}_2(\pi_1(\xi))}$ and $\Sigma_{\mathcal{P}_2(\pi_1(\xi)),\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1}$,*
(5) *$\pi_2 = \pi^{\vec{\mathcal{T}}_2^\frown\mathcal{P}_3}$ and $\pi_3 = \pi^{\vec{\mathcal{T}}_2^\frown\mathcal{P}_4}$,*
(6) *$\vec{\mathcal{T}}_3$ is a stack on $\mathcal{P}_3$ according to $\Lambda_{\mathcal{P}_3,\vec{\mathcal{T}}_2^\frown\mathcal{P}_3}$ and $\vec{\mathcal{T}}_4$ is a stack on $\mathcal{P}_4$ according to $\Sigma_{\mathcal{P}_4,\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\vec{\mathcal{T}}_2^\frown\mathcal{P}_4}$, $\vec{\mathcal{T}}_3$ and $\vec{\mathcal{T}}_4$ have a common last model $\mathcal{P}_5$, $\pi_4 = \pi^{\vec{\mathcal{T}}_3}$, and $\pi_5 = \pi^{\vec{\mathcal{T}}_4}$,*
(7) *$\vec{\mathcal{U}} = \vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\vec{\mathcal{T}}_2^\frown\mathcal{P}_4^\frown\vec{\mathcal{T}}_4$,*
(8) *for all $\alpha < \lambda^{\mathcal{P}_5}$, $\Sigma_{\mathcal{P}_5(\alpha),\vec{\mathcal{U}}} = \Lambda_{\mathcal{P}_5(\alpha),\vec{\mathcal{T}}_2^\frown\mathcal{P}_3^\frown\vec{\mathcal{T}}_3}$.*

LEMMA 3.31 (No bad sequence). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair with branch condensation at successors. Then there is no sequence*

$$(B^\alpha, j^t_{\beta,\alpha}, j^b_{\beta,\alpha} : \beta < \alpha < \omega_1)$$

*such that*

(1) *$B^\alpha = ((\mathcal{P}^\alpha_i : i \leq 5), (\vec{\mathcal{T}}^\alpha_i : i \leq 4), \Lambda^\alpha, (\pi^\alpha_i : i \leq 5), \xi^\alpha, \vec{\mathcal{U}}^\alpha)$,*
(2) *$B^0$ is a bad block for $(\mathcal{P},\Sigma)$,*
(3) *$B^{\alpha+1}$ is a bad block of the second kind for $(\mathcal{P}^\alpha_5, \Sigma_{\mathcal{P}^\alpha_5,\oplus_{\beta\leq\alpha}\mathcal{U}^\beta})$ (thus, $\mathcal{P}^\alpha_5 = \mathcal{P}^{\alpha+1}_0$),*
(4) *$j^t_{\alpha,\beta} : \mathcal{P}^\alpha_0 \to \mathcal{P}^\beta_0$ is the iteration embedding along the "top", i.e.,*

$$j^t_{\alpha,\beta} = \oplus_{\xi\in[\alpha,\beta)}(\pi^\xi_4 \circ \pi^\xi_2 \circ \pi^\xi_1 \circ \pi^\xi_0),$$

(5) *$j^b_{\alpha,\beta} : \mathcal{P}^\alpha_0 \to \mathcal{P}^\beta_0$ is the iteration embedding along the "bottom", i.e.,*

$$j^b_{\alpha,\beta} = \oplus_{\xi\in[\alpha,\beta)}(\pi^\xi_5 \circ \pi^\xi_3 \circ \pi^\xi_1 \circ \pi^\xi_0),$$

(6) *If $\alpha$ is limit then $\mathcal{P}^\alpha_0 = dirlim(\mathcal{P}^\beta_0, j^t_{\beta,\alpha} : \beta < \alpha < \omega_1) = dirlim(\mathcal{P}^\beta_0, j^b_{\beta,\alpha} : \beta < \alpha < \omega_1)$.*

PROOF. Towards a contradiction suppose there is such a sequence and let $\vec{B} = (B^\alpha, j^t_{\alpha,\beta}, j^b_{\alpha,\beta} : \alpha < \beta < \omega_1)$ be one. Let $X_0 \prec X_1 \prec H_{\omega_2}$ be such that $\vec{B} \in X_0$. Let $\pi_0 : H_0 \to X_0$ and $\pi_1 : H_1 \to X_1$ be the collapses of $X_0$ and $X_1$. We then get $\pi : H_0 \to H_1$. Let $\kappa_0 = \omega_1^{H_0}$ and $\kappa_1 = \omega_1^{H_1}$. Note that all sets in $B^\alpha = ((\mathcal{P}^\alpha_i : i \leq 5), (\vec{\mathcal{T}}^\alpha_i : i \leq 4), \Lambda^\alpha, (\pi^\alpha_i : i \leq 5), \xi^\alpha, \vec{\mathcal{U}}^\alpha)$ except $\Lambda^\alpha$ are countable, and therefore if $\alpha < \kappa_i$ then

$$((\mathcal{P}^\alpha_i : i \leq 5), (\vec{\mathcal{T}}^\alpha_i : i \leq 4), (\pi^\alpha_i : i \leq 5), \xi^\alpha, \vec{\mathcal{U}}^\alpha) \in H_k,\ k = 0, 1$$

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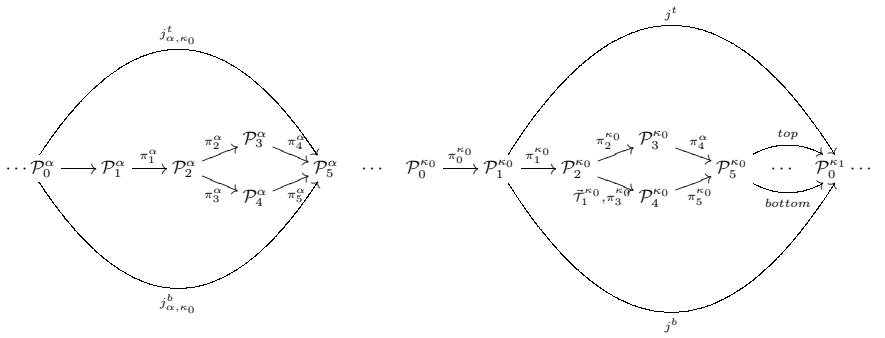


FIGURE 3.5.2. A bad sequence of length $\omega_1$

and if $\alpha < \kappa_0$

$$\pi(((\mathcal{P}^\alpha_i : i \leq 5), (\vec{\mathcal{T}}^\alpha_i : i \leq 4), (\pi^\alpha_i : i \leq 5), \xi^\alpha, \vec{\mathcal{U}}^\alpha)) = ((\mathcal{P}^\alpha_i : i \leq 5), (\vec{\mathcal{T}}^\alpha_i : i \leq 4), (\pi^\alpha_i : i \leq 5), \xi^\alpha, \vec{\mathcal{U}}^\alpha),\ k = 0, 1.$$

*Claim 1.* $j^t_{\kappa_0,\kappa_1} = \pi \restriction \mathcal{P}^{\kappa_0}_0 = j^b_{\kappa_0,\kappa_1}$ (see Figure 3.5.2).

The proof of the claim is just like the proof of the same claim in Lemma 2.49. Now, Let $\vec{\mathcal{T}} = \oplus_{\xi<\kappa_0}\vec{\mathcal{U}}^\xi$. We have that $\vec{\mathcal{T}}$ is according to $\Sigma$. The following claim produces the desired contradiction.

*Claim 2.* $B^{\kappa_0}$ is not a bad block for $(\mathcal{P}^{\kappa_0}_0, \Sigma_{\mathcal{P}^{\kappa_0}_0, \vec{\mathcal{T}}})$.

PROOF. Let $j^t : \mathcal{P}^{\kappa_0}_1 \to \mathcal{P}^{\kappa_1}_0$ be the embedding along the "top" and let $j^b : \mathcal{P}^{\kappa_0}_1 \to \mathcal{P}^{\kappa_1}_0$ be the embedding along the "bottom". Then we claim that $j^t = j^b$. To see this, let $x \in \mathcal{P}^{\kappa_0}_1$. There is then a function $f \in \mathcal{P}^{\kappa_0}_0$ and $a \in \delta(\vec{\mathcal{T}}^{\kappa_0}_0)^{<\omega}$ such that $x = \pi^{\kappa_0}_0(f)(a)$. Then

$$j^t(x) = j^t(\pi^{\kappa_0}_0(f))(j^t(a)) = j^t_{\kappa_0,\kappa_1}(f)(j^t(a)) = \pi(f)(j^t(a)).$$

Similarly, $j^b(x) = \pi(f)(j^b(a))$. Thus, it is enough to show that $j^t(a) = j^b(a)$. This follows from the fact that further iterations never disagree on how $a$ is moved (this is guaranteed by condition 3 and 8 of Definition 3.30).

It is now easy to get a contradiction. Let $j = j^t = j^b$. We have that $\vec{\mathcal{T}}^{\kappa_0}_1$ is a stack on $\mathcal{P}^{\kappa_0}_1(\xi^{\kappa_0})$ and $j \restriction \mathcal{P}^{\kappa_0}_1(\xi^{\kappa_0}) : \mathcal{P}^{\kappa_0}_1(\xi^{\kappa_0}) \to \mathcal{P}^{\kappa_1}_0(j(\xi^{\kappa_0}))$ is the iteration embedding according to $\Sigma_{\mathcal{P}^{\kappa_0}_1, (\oplus_{\xi<\kappa_0}\vec{\mathcal{U}}^\xi)^\frown \vec{\mathcal{T}}^{\kappa_0}_0}$. We have that $j = k \circ \pi^{\kappa_0}_2 \circ \pi^{\kappa_0}_1$. Because of 3 in Definition 3.30, we can apply the branch condensation of $\Sigma_{\mathcal{P}^{\kappa_0}_1(\xi^{\kappa_0}), \vec{\mathcal{T}}^\frown \vec{\mathcal{T}}^{\kappa_0}_0}$ to stacks $\vec{\mathcal{T}}^{\kappa_0}_1{}^\frown \vec{\mathcal{T}}^{\kappa_0}_2{}^\frown \mathcal{P}^{\kappa_0}_3$ and $\vec{\mathcal{T}}^{\kappa_0}_1{}^\frown \vec{\mathcal{T}}^{\kappa_0}_2{}^\frown \mathcal{P}^{\kappa_0}_4{}^\frown \vec{\mathcal{T}}^{\kappa_0}_5{}^\frown (\oplus_{\nu\in(\kappa_0,\kappa_1)}\vec{\mathcal{U}}^\nu)$. From here we get that $\vec{\mathcal{T}}^{\kappa_0}_1{}^\frown \vec{\mathcal{T}}^{\kappa_0}_2{}^\frown \mathcal{P}^{\kappa_0}_3$ is according to $\Sigma_{\mathcal{P}^{\kappa_0}_1(\xi^{\kappa_0}+1), \vec{\mathcal{T}}^\frown \vec{\mathcal{T}}^{\kappa_0}_0}$, contradiction! □

□

We now start proving the theorem by producing a bad sequence of length $\omega_1$. Fix, then, $(\mathcal{P}, \Sigma)$ as in Theorem 2.42. Suppose no tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$ is such that $\Lambda$ has branch condensation. We use this to build a bad sequence. Since branch condensation fails, there is $(\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{Q}, \mathcal{R}, \pi)$ such that

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(1) $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$,
(2) $\vec{\mathcal{U}}$ is a stack on $\mathcal{P}$ not according to $\Sigma$ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{U}}}$ exists,
(3) $\pi : \mathcal{Q} \to \mathcal{R}$ and $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi^{\vec{\mathcal{U}}}$.

Suppose $(\mathcal{M}_\alpha, \mathcal{M}^*_\alpha, \vec{\mathcal{U}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \eta)$ are the essential components of $\vec{\mathcal{U}}$. Let $\alpha$ be the least $\xi$ such that $\oplus_{\beta<\xi}\vec{\mathcal{U}}_\beta$ is according to $\Sigma$. Notice that $\alpha \neq 0$ because for every $\beta < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\beta)}$ has branch condensation. We say $\vec{\mathcal{U}}$ is a minimal counterexample to branch condensation of $\Sigma$ if

(1) $\eta = \alpha$,
(2) for all $\beta < \eta$ and for all $\gamma < \lambda^{\mathcal{M}_\beta}$, there is no stack $\vec{\mathcal{U}}^*$ on $\mathcal{M}_\beta(\gamma)$ with last model $\mathcal{Q}^*$ such that
   (a) $\pi^{\vec{\mathcal{U}}^*}$-exists,
   (b) the last normal component of $\vec{\mathcal{U}}^*$ has a successor length whose predecessor is limit,
   (c) $(\vec{\mathcal{U}}^*)^-$ is according to $\Sigma$ but $\vec{\mathcal{U}}^*$ isn't,
   (d) for some $\vec{\mathcal{S}}$ on $\mathcal{P}$ according to $\Sigma$ with last model $\mathcal{S}$ such that $\pi^{\vec{\mathcal{S}}}$-exists, there is $\sigma : \mathcal{Q}^* \to \mathcal{S}$ such that $\pi^{\vec{\mathcal{S}}} = \sigma \circ \pi^{\vec{\mathcal{U}}^*} \circ \pi^{\oplus_{\xi<\beta}\vec{\mathcal{U}}_\xi}$,
(3) $\vec{\mathcal{U}}_\eta$ has a last normal component which has a successor length whose predecessor is a limit and $\vec{\mathcal{U}}^-_\eta$ is according to $\Sigma_{\mathcal{M}^*_\eta, \oplus_{\beta<\eta}\vec{\mathcal{U}}_\beta}$.

It is not hard to see that there are minimal counterexamples to branch condensation. If $\vec{\mathcal{U}}$ is a minimal counterexample to branch condensation of $\Sigma$, we let $\xi(\vec{\mathcal{U}}) = \lambda^{\mathcal{M}^*_\eta}$. Notice that $\xi(\vec{\mathcal{U}})$ is a successor ordinal.

Now, we assume that $(\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{Q}, \mathcal{R}, \pi)$ is as above and $\vec{\mathcal{U}}$ is a minimal counterexample. Let $\Lambda = \Sigma^\pi_{\mathcal{R},\vec{\mathcal{T}}}$. Let $b$ be the branch of the last component of $\vec{\mathcal{U}}^-$ given by $\Sigma$. We would like to compare $(\mathcal{M}_b^{\vec{\mathcal{U}}^-}, \Sigma_{\mathcal{M}_b^{\vec{\mathcal{U}}^-}, \vec{\mathcal{U}}^- \frown \mathcal{M}_b^{\vec{\mathcal{U}}^-}}) = (\mathcal{S}, \Psi)$ with $(\mathcal{Q}, \Lambda)$. Because we do not know that $\Psi$ and $\Lambda$ have branch condensation, we cannot use Theorem 2.33 to do the comparison. Instead we would like to use Theorem 2.47, and for that we need to know that they are of the same kind in which case we can quote our comparison theorem. By our assumption, we have that for all $\alpha < \lambda^{\mathcal{Q}}$, $\Lambda_{\mathcal{Q}(\alpha)}$ has branch condensation. This means that $\mathcal{Q}$ is $\Gamma(\mathcal{Q}, \Lambda)$-fullness preserving iteration strategy (see Theorem 3.26). Notice, however, that $\Gamma(\mathcal{Q}, \Lambda) = \Gamma(\mathcal{P}, \Sigma)$ because $\Sigma = \Lambda^{\pi^{\vec{\mathcal{U}}}}$. Moreover, because $(\mathcal{S}, \Psi)$ is a tail of $(\mathcal{P}, \Sigma)$, $\Gamma(\mathcal{S}, \Psi) = \Gamma(\mathcal{P}, \Sigma)$ and hence, $\Psi$ is $\Gamma(\mathcal{P}, \Sigma)$-fullness preserving iteration strategy. This means that part a of the first clause and the second clause of Definition 2.46 hold for $\Psi$ and $\Lambda$. We then need to see that part b of the first clause also holds.

Fix, then, a tail $(\mathcal{S}^*, \Psi^*)$ of $(\mathcal{S}, \Psi)$ and $(\mathcal{Q}^*, \Lambda^*)$ of $(\mathcal{Q}, \Lambda)$ such that for some $\alpha < \min(\lambda^{\mathcal{S}^*}, \lambda^{\mathcal{Q}^*})$, $\mathcal{S}^*(\alpha+1) = \mathcal{Q}^*(\alpha+1)$ and $\Psi^*_{\mathcal{S}^*(\alpha)} = \Lambda^*_{\mathcal{Q}^*(\alpha)}$. We need to see that some tails of $(\mathcal{S}^*(\alpha+1), \Psi^*_{\mathcal{S}^*(\alpha+1)})$ and $(\mathcal{Q}^*(\alpha+1), \Lambda^*_{\mathcal{Q}^*(\alpha+1)})$ are close. We have that both strategies have branch condensation and both are in $\Gamma(\mathcal{P}, \Sigma)$. But then using Theorem 2.33, we can compare $(\mathcal{S}^*(\alpha+1), \Psi^*_{\mathcal{S}^*(\alpha+1)})$ and $(\mathcal{Q}^*(\alpha+1), \Lambda^*_{\mathcal{Q}^*(\alpha+1)})$, implying that the two pairs are indeed close. The discussion we just had will be useful later in the paper, and hence, we take a moment to summarize it in the following lemma.

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LEMMA 3.32. *Suppose* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$ *are such that* $\lambda^{\mathcal{P}}$ *and* $\lambda^{\mathcal{Q}}$ *are limit ordinals and for all* $(\vec{\mathcal{T}},\mathcal{R})\in B(\mathcal{P},\Sigma)$ *and* $(\vec{\mathcal{U}},\mathcal{S})\in B(\mathcal{Q},\Lambda)$, $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}$ *and* $\Lambda_{\mathcal{Q},\vec{\mathcal{U}}}$ *have branch condensation. Suppose further that* $\Gamma(\mathcal{P},\Sigma)=\Gamma(\mathcal{Q},\Lambda)$. *Then* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$ *are of the same kind and hence, can be compared.*

Let now $(\mathcal{M}_\alpha,\mathcal{M}^*_\alpha,\vec{\mathcal{U}}_\alpha,\pi_{\alpha,\beta}:\alpha<\beta\leq\eta)$ be the essential components of $\vec{\mathcal{U}}$. Let $(\vec{\mathcal{W}}_0,\mathcal{M})\in I(\mathcal{Q},\Lambda)$ and $(\vec{\mathcal{W}}_1,\mathcal{M})\in I(\mathcal{S},\Psi)$ be such that $\Lambda_{\mathcal{M},\vec{\mathcal{W}}_0}=\Psi_{\mathcal{M},\vec{\mathcal{W}}_1}$. We now let (by possibly redefining already used symbols)

(1) $\mathcal{P}_0=\mathcal{P}$, $\vec{\mathcal{T}}_0=\oplus_{\beta<\eta}\vec{\mathcal{U}}_\beta$ and $\pi_0=\pi^{\oplus_{\beta<\eta}\vec{\mathcal{U}}_\beta}$,
(2) $\mathcal{P}_1=\mathcal{M}_\eta$, $\xi=\lambda^{\mathcal{M}^*_\eta}$, $\pi_1=id$ and $\vec{\mathcal{T}}_1=\emptyset$,
(3) $\mathcal{P}_2=\mathcal{P}_1$, $\vec{\mathcal{T}}_2=\vec{\mathcal{U}}^-_\eta$, $\pi_2=\pi^{\vec{\mathcal{U}}}$ and $\pi_3=\pi^{\vec{\mathcal{U}}^-}_b$,
(4) $\mathcal{P}_3=\mathcal{Q}$ and $\mathcal{P}_4=\mathcal{M}^{\vec{\mathcal{U}}}_b$,
(5) $\vec{\mathcal{T}}_3=\vec{\mathcal{W}}_0$, $\vec{\mathcal{T}}_4=\vec{\mathcal{W}}_1$ and $\mathcal{P}_5=\mathcal{M}$,
(6) $\pi_4=\pi^{\vec{\mathcal{W}}_0}$ and $\pi_5=\pi^{\vec{\mathcal{W}}_1}$,
(7) $\vec{\mathcal{U}}=\vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{T}}_1{}^\frown\vec{\mathcal{T}}_2{}^\frown\mathcal{P}_4{}^\frown\vec{\mathcal{T}}_4$,
(8) $\Lambda=\Sigma^{\pi\circ\pi^{\vec{\mathcal{U}}_\eta}}_{\mathcal{R},\vec{\mathcal{T}}}$.

It is then not hard to see that $B_0=((\mathcal{P}_i:i\leq 5),(\vec{\mathcal{T}}_i:i\leq 4),\Lambda,(\pi_i:i\leq 5),\xi,\vec{\mathcal{U}})$ is a bad block for $(\mathcal{P},\Sigma)$. Because $(\mathcal{P}_5,\Sigma_{\mathcal{P}_5,\vec{\mathcal{U}}})$ is a tail of $(\mathcal{P},\Sigma)$ we can keep going and construct $B_1$, $B_2$ and etc. Notice that if $\alpha$ is limit and we have constructed $(B_\beta:\beta<\alpha)$ then the direct limit of $\mathcal{P}^\beta_5$'s under the iteration embeddings along the top and along the bottom are the same because of clause 3 and 8 of Definition 3.30. In this manner, we produce a sequence $(B_\alpha:\alpha<\omega_1)$ contradicting Lemma 3.31.

## 3.6. Generic comparisons

In this section, we introduce the method of generic comparisons. Such comparisons are useful when interpret the strategies of hod mice in the generic extensions of various models. Such comparison arguments were first used by Woodin in his work on $\mathrm{HOD}^{L[x][g]}$. In Section 3.7, we will use such a comparison argument to reorganize hod mice into a hierarchy for which the $S$-constructions can be performed (see Section 3.8). In general, we do not know how to do generic comparisons for hod pairs $(\mathcal{P},\Sigma)$. We seem to need a stronger fullness preservation condition.

DEFINITION 3.33 (Super fullness preservation). *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair.* $\Sigma$ *is super fullness preserving if it is fullness preserving and whenever* $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ *and* $\alpha<\lambda^{\mathcal{Q}}$, *the two sets*

$$U^\Sigma_{\mathcal{Q}(\alpha)}=\{(x,y)\in\mathbb{R}^2: x \text{ codes a transitive set } a\in HC \text{ and } y \text{ codes } \mathcal{M} \text{ such that } \mathcal{M}\trianglelefteq Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) \text{ and } \rho(\mathcal{M})=a\}$$

$$W^\Sigma_{\mathcal{Q}(\alpha)}=\{(x,y,z)\in\mathbb{R}^3:(x,y)\in U^\Sigma_{\mathcal{Q}(\alpha)} \text{ and if } \mathcal{M} \text{ is the } \Sigma_{\mathcal{Q}(\alpha)}\text{-mouse coded by } y \text{ then } z \text{ codes a tree according to the unique strategy of } \mathcal{M}\}.$$

*are term captured by* $(\mathcal{Q}[g],\Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ *whenever* $g\subseteq Coll(\omega,\mathcal{Q}(\alpha))$ *is* $\mathcal{Q}$*-generic. We let* $u^\Sigma_{\mathcal{Q}(\alpha)}$ *and* $w^\Sigma_{\mathcal{Q}(\alpha)}$ *be the term relations locally capturing* $U^\Sigma_{\mathcal{Q}(\alpha)}$ *and* $W^\Sigma_{\mathcal{Q}(\alpha)}$.

Notice that if $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ is super fullness preserving then whenever $\mathcal{Q}\in pI(\mathcal{P},\Sigma)$, $\alpha<\lambda^{\mathcal{Q}}$, $g\subseteq Coll(\omega,\mathcal{Q}(\alpha))$, $a\in HC$ and $x$ is a real coding $a$ such that $x$ is generic over $\mathcal{Q}[g]$ then

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$$Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) = \{\mathcal{M} : \text{there is } y \in \mathbb{R}^{\mathcal{Q}[x]} \text{ such that } y \text{ codes } \mathcal{M} \text{ and } (x,y) \in (u^{\Sigma}_{\mathcal{Q}(\alpha)})_{g*x})\}.$$

Thus, $Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) \in \mathcal{Q}[x]$. Moreover, if $h$ is $\mathcal{Q}[g]$-generic then the restriction of the function $b \to Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(b)$ to $\mathcal{Q}[g*h]$ is definable over $\mathcal{Q}[g*h]$. Also, continuing with the above setting, if $\mathcal{M}$ is a sound $\Sigma_{\mathcal{Q}(\alpha)}$-mouse over $a$ projecting to $a$ and $\Lambda$ is its unique iteration strategy then whenever $h$ is $\mathcal{Q}[g][x]$ generic and $\kappa$ is a cardinal of $\mathcal{Q}[g][x]$,

$$\Lambda \restriction H_{\kappa}^{\mathcal{Q}[g*x*h]} \in \mathcal{Q}[g*x*h].$$

This is because $\Lambda \restriction H_{\kappa}^{\mathcal{Q}[g*x*h]}$ can be defined over $\mathcal{Q}[g*x*h]$ using $w^{\Sigma}_{\mathcal{Q}(\alpha)}$.

Suppose $N$ is a model of $ZFC - Replacement$. We say $N$ *is* $\Sigma$*-closed* if $\Sigma \restriction N$ is definable over $N$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is super fullness preserving. Suppose $N$ is $\Sigma$-closed and suppose $g$ is $N$-generic. Below, using generic comparisons, we show that $N[g]$ is also $\Sigma$-closed.

LEMMA 3.34. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is super fullness preserving. Suppose $N$ is a $\Sigma$-closed model of $ZFC - Replacement$ and suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma) \cap N$. Let $\kappa$ be an $N$-cardinal such that $(\vec{\mathcal{T}}, \mathcal{Q}), \mathcal{P} \in H^N_{\kappa}$. Let $g \subseteq Coll(\omega, \kappa)$ be $N$-generic and suppose that for some $\beta < \lambda^{\mathcal{Q}}$, $\mathcal{N}[g]$ is $\Sigma_{\mathcal{Q}(\beta)}$-closed. Then if $\Lambda$ is the fragment of $\Sigma_{\mathcal{Q}}$ that acts on normal trees that are based on the window $[\delta^{\mathcal{Q}}_{\beta}, \delta^{\mathcal{Q}}_{\beta+1}]$ then $N[g]$ is closed under $\Lambda$.*

PROOF. Let $\Psi = \Sigma_{\mathcal{Q}(\beta)}$ and let $\mathcal{W} = \mathcal{Q}(\beta+1)$. Notice first that the operator $a \to Lp^{\Psi}(a)$ is definable over $N[g]$:

$b \in Lp^{\Psi}(a)$ iff letting $\sigma$ be such that $\sigma_g = (a, b)$ and $\eta > \kappa$ be an $N$-cardinal such that $\sigma \in H^N_{\eta}$, if $\mathcal{R} \in N$ is the $\Lambda$-iterate of $\mathcal{W}$ above $\delta^{\mathcal{W}}_{\beta}$ which makes $H^N_{\eta}$ generic for the extender algebra at $\delta^{\mathcal{R}}_{\beta+1}$, then whenever $k \subseteq Coll(\omega, \eta)$ is $N[g]$-generic, $h \in N[g][k]$ is $\mathcal{R}$-generic for $Coll(\omega, \mathcal{R}(\beta))$ and $(x, y) \in \mathbb{R}^2 \cap N[g][k]$ are such that $x$ codes $a$ and $y$ codes $b$ then there is $z \in \mathbb{R}^{\mathcal{R}[h][x,y]}$ such that $(x, z) \in (\pi^{\Lambda}_{\mathcal{W},\mathcal{R}}(u^{\Sigma}_{\mathcal{W}(\beta)}))_{h*x}$ such that $y$ is in the structure coded by $z$.

The same argument shows that if $a \in N[g]$ and $\mathcal{M} \trianglelefteq Lp^{\Psi}(a)$ is such that $\rho(\mathcal{M}) = a$ then letting $\Phi$ be the strategy of $\mathcal{M}$, $\Phi \restriction N[g]$ is definable over $N[g]$. Indeed, given a tree $\mathcal{U}$ on $\mathcal{M}$, to find its branch first fix $N$-cardinal $\eta$ such that $x, \mathcal{U} \in H^N_{\eta}[g]$, then iterate $\mathcal{W}$ according to $\Lambda$ above $\delta_{\beta}$ to get $\mathcal{S}$ such that $H^N_{\eta}$ is generic for the extender algebra of $\mathcal{S}$ at $\delta^{\mathcal{S}}_{\beta+1}$ and use the branch of $\mathcal{U}$ given by the strategy of $\mathcal{M}$ in $\mathcal{S}[x, \mathcal{U}]$.

Let now $\Lambda^s$ be the fragment of $\Lambda$ defined by $\Lambda^s(\mathcal{T}) = b$ if $\Lambda(\mathcal{T}) = b$ and $\mathcal{Q}(\mathcal{T}, b)$-exists. It is not hard to see that because of the above observation $\Lambda^s$ is definable over $N[g]$. Indeed, for $\mathcal{T} \in N[g]$, $\Lambda^s(\mathcal{T}) = b$ iff $\mathcal{T}$ is above $\delta^{\mathcal{W}}_{\beta}$ and either

(1) $\mathcal{T}$ has a fatal drop at $(\gamma, \nu)$ and $b$ is the unique branch according to the strategy of $\mathcal{O}^{\mathcal{M}^{\mathcal{T}}_{\gamma}}_{\nu}$, or
(2) $\mathcal{T}$ doesn't have a fatal drop and $b$ is the unique branch such that $\mathcal{Q}(\mathcal{T}, b)$-exists and $\mathcal{Q}(\mathcal{T}, b) \trianglelefteq Lp^{\Psi}(\mathcal{M}(\mathcal{T}))$.

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By Fact 3.6, the two cases above cover everything. We thus have that $\Lambda^s$ is definable over any generic extension of $N$. We now move to showing that in fact $\Lambda$ is definable over $N[g]$.

Fix an $N$-cardinal $\eta > \kappa$. In $N$, we can fix an enumeration of all names $(\dot{\mathcal{T}}_\alpha : \alpha < \nu)$ such that for each $\alpha$, $\dot{\mathcal{T}}_\alpha \in H^N_\eta$ is a name for a tree on $\mathcal{W}$ above $\delta^{\mathcal{W}}_\beta$ such that it is forced that all initial segments of $\dot{\mathcal{T}}$ are according to $\Lambda^s$ and $\dot{\mathcal{T}}$ isn't according to $\Lambda^s$. Suppose $p \in Coll(\omega, \kappa)$. We let $g_p$ be the set of conditions $q$ such that either $p \leq q$ or $q \leq p$ and for some $s \in Coll(\omega, \max(p)+1)$ such that $|s| = p$, $s^\frown(q \backslash p) \in g$. Then $g_p$ is $N$-generic and $N[g_p] = N[g]$. We let $\mathcal{Q}_{\alpha,p} = Lp^\Psi_\omega(\mathcal{M}((\dot{\mathcal{T}}_\alpha)_p))$.

What we would like to do is to compare all $\mathcal{Q}_{\alpha,p}$ in such a way that the final tree on $\mathcal{W}$ is in $N$. We do the comparison in $N[g]$. In this comparison, for each $(\alpha, p)$, we build a tree $\mathcal{U}_{\alpha,p}$ on $\mathcal{Q}_{\alpha,p}$. We allow "dummy" extensions, i.e., $\mathcal{U}_{\alpha,p}$ might be a padded tree. At some typical stage $\xi$ of the comparison, we have $(\mathcal{U}_{\alpha,p} \restriction \xi : \alpha < \nu \wedge p \in Coll(\omega, \kappa))$.

Suppose first $\xi$ is limit. If $\mathcal{T} \restriction \xi$ is the tree on $\mathcal{W}$ then we have two cases. Suppose first that $\Lambda^s(\mathcal{T} \restriction \xi)$ is undefined. Then we stop the construction. At this stage we must have that

$$Lp^\Psi(\mathcal{M}(\mathcal{U}_{\alpha,p} \restriction \xi)) \vDash \text{“}\delta(\mathcal{U}_{\alpha,p} \restriction \xi) \text{ is Woodin”}.$$

We let $\mathcal{U}_{\alpha,p}$ be the tree at stage $\xi$.

Suppose now that $\xi$ is limit and $\Lambda^s(\mathcal{T} \restriction \xi)$ is defined. Then we let $b = \Lambda^s(\mathcal{T} \restriction \xi)$ and we continue $\mathcal{T}$ by adding $b$ to it. We also have that for every $\alpha, p$, there is a branch $c_{\alpha,p}$ for $\mathcal{U}_{\alpha,p} \restriction \xi$ such that $\mathcal{Q}(c_{\alpha,p}, \mathcal{U}_{\alpha,p} \restriction \xi) = \mathcal{Q}(b, \mathcal{T} \restriction \xi)$. We then continue each $\mathcal{U}_{\alpha,p}$ by adding $c_{\alpha,p}$. Note that $c_{\alpha,p}$ only depends on $\mathcal{U}_{\alpha,p} \restriction \xi$.

Now suppose $\xi = \gamma + 1$. In this case, if there is no disagreement between $\mathcal{M}^{\mathcal{T}\restriction\xi}_\gamma$ and any $\mathcal{M}(\mathcal{U}_{\alpha,p} \restriction \xi)$ then we stop the construction. If there is a disagreement then we let $E \in \mathcal{M}^{\mathcal{T}\restriction\xi}_\gamma$ be the least extender which is part of some disagreement. For each $(\alpha, p)$ we let $E_{\alpha,p}$ be an extender on $\mathcal{M}^{\mathcal{U}_{\alpha,p}\restriction\xi}_\gamma$ which disagrees with $E$. Note that $E$ and $E_{\alpha,p}$ may be trivial but not simultaneously. We then expand $\mathcal{T} \restriction \xi$ and $\mathcal{U}_{\alpha,p}$ by applying $E$ and $E_{\alpha,p}$ to the appropriate models in $\mathcal{T} \restriction \xi$ and $\mathcal{U}_{\alpha,p}$ respectively. The usual proof of comparison lemma shows that our construction must stop before stage $(\eta^+)^N$ producing $\mathcal{T}$ on $\mathcal{W}$ and $\mathcal{U}_{\alpha,p}$ on $\mathcal{Q}_{\alpha,p}$. Notice that because of our construction $\mathcal{T} \in N$. Let $b = \Lambda(\mathcal{T})$. Then $b \in N$.

We can now define $\Lambda \cap H^N_\eta[g]$ by $\Lambda(\mathcal{S}) = c$ iff

(1) all initial segments of $\mathcal{S}$ are according to $\Lambda^s$,
(2) if $\Lambda^s(\mathcal{S})$ is defined then $c = \Lambda^s(\mathcal{S})$,
(3) if $\Lambda^s(\mathcal{S})$ is undefined then there is an embedding $\pi : \mathcal{M}^{\mathcal{S}}_c \to \mathcal{M}^{\mathcal{T}}_b$ such that $\pi^{\mathcal{T}}_b = \pi \circ \pi^{\mathcal{S}}_c$.

Because $\mathcal{S} = (\dot{\mathcal{T}}_\alpha)_p$ for some $\alpha < \nu$ and $p \in Coll(\omega, \kappa)$, we must have that if 1 holds, 2 fails and $\Lambda(\mathcal{S}) = c$ then 3 must hold: $\pi$ is just the embedding given by $b$ where $b$ is the branch of $\mathcal{U}_{\alpha,p}$ according to $\Sigma$. This completes the proof of Lemma 3.34. □

LEMMA 3.35. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is super fullness preserving. Suppose $N$ is a $\Sigma$-closed model of $ZFC -$ Replacement. Let $g$ be $N$-generic. Then $N[g]$ is $\Sigma$-closed.*

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PROOF. Using standard arguments from forcing theory, we can assume without loss of generality that $g$ is generic for some collapse. We then fix an $N$-cardinal $\kappa$ such that $\mathcal{P} \in H^N_\kappa$ and let $g \subseteq Coll(\omega, \kappa)$ be $N$-generic. We would like to define $\Sigma \restriction N[g]$ over $N[g]$. Fix an $N$-regular cardinal $\eta > \kappa$. We first define by induction a sequence of terms $(\dot{\Sigma}_{\mathcal{Q}} : \exists \vec{\mathcal{T}}(\vec{\mathcal{T}}, \mathcal{Q}) \in H^N_\eta \cap B(\mathcal{P}, \Sigma)\}$ such that $(\dot{\Sigma}_{\mathcal{Q}})_k = \Sigma_{\mathcal{Q}} \restriction N[k]$ for any $N$-generic $k \subseteq Coll(\omega, < \eta)$.

If $\mathcal{Q} = \mathcal{P}(0)$ then the above discussion on generic comparisons gives $\dot{\Sigma}_{\mathcal{Q}}$. Now suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma) \cap H^N_\eta$ is such that for any $(\vec{\mathcal{S}}, \mathcal{R}) \in H^N_\eta \cap B(\mathcal{Q}, \Sigma_{\mathcal{Q}})$, $\dot{\Sigma}_{\mathcal{R}}$ has been defined. We have two cases. Suppose first that $\lambda^{\mathcal{Q}} = \gamma + 1$. It means that we already have defined $\dot{\Sigma}_{\mathcal{Q}(\gamma)}$. Notice that for any $\dot{\vec{\mathcal{T}}} \in H^N_\eta$ which is forced to be a stack on $\mathcal{Q}(\gamma)$ according to $\dot{\Sigma}_{\mathcal{Q}(\gamma)}$ with a last model, the generic comparisons give us a term $\dot{\Lambda}_{\dot{\vec{\mathcal{T}}}}$ such that whenever $h \subseteq Coll(\omega, < \eta)$ is $N$-generic, letting $\mathcal{R}$ be such that $(\vec{\mathcal{T}}_h, \mathcal{R}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $\Lambda$ be the fragment of $\Sigma_{\mathcal{R}}$ which acts on stacks above $\delta_{\lambda^{\mathcal{R}}-1}$

$$(\dot{\Lambda}_{\dot{\vec{\mathcal{T}}}})_h = \Lambda \restriction H^N_\eta[h].$$

We then let

$$\dot{\Sigma}_{\mathcal{Q}} = \cup_{\dot{\vec{\mathcal{T}}} \in H^N_\eta} \dot{\Lambda}_{\dot{\vec{\mathcal{T}}}}.$$

Suppose now $\lambda^{\mathcal{Q}}$ is limit. If $\mathrm{cf}^{\mathcal{Q}}(\lambda^{\mathcal{Q}})$ isn't measurable in $\mathcal{Q}$ then let $\dot{\Sigma}_{\mathcal{Q}} = \cup_{\alpha < \lambda^{\mathcal{Q}}} \dot{\Sigma}_{\mathcal{Q}(\alpha)}$. Suppose then that $\mathrm{cf}^{\mathcal{Q}}(\lambda^{\mathcal{Q}})$ is measurable in $\mathcal{Q}$. If $\dot{\vec{\mathcal{T}}} \in H^N_\eta$ is forced to be a stack on $\mathcal{Q}$ whose last normal component is of limit length, we let $(\dot{\Sigma}_{\mathcal{Q}})_g(\dot{\vec{\mathcal{T}}}_g) = \dot{b}_g$ if letting $(\mathcal{M}_\alpha, \mathcal{M}^*_\alpha, \vec{\mathcal{T}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \nu)$ be the essential components of $\dot{\vec{\mathcal{T}}}_g$, there is $(\vec{\mathcal{S}}, \mathcal{R}) \in H^N_\eta \cap I(\mathcal{P}, \Sigma)$ and $\pi : \mathcal{M}_\eta \to \mathcal{R}$ such that

(1) $\pi^{\vec{\mathcal{S}}} = \pi \circ \pi_{0,\nu}$,
(2) if $\alpha$ is such that $\pi(\mathcal{M}^*_\nu) = \mathcal{R}(\alpha)$ then $\vec{\mathcal{T}}_\nu$ is according to $((\dot{\Sigma}_{\mathcal{R}(\alpha)})_g)^\pi$.

Fix $h \subseteq Coll(\omega, < \eta)$. We need to show that $(\dot{\Sigma}_{\mathcal{Q}})_h = \Sigma_{\mathcal{Q}} \restriction H^N_\eta[h]$. Let $\Psi = (\dot{\Sigma}_{\mathcal{Q}})_h$. By branch condensation, if $\vec{\mathcal{T}}$ is according to $\Psi$ then $\vec{\mathcal{T}}$ is according to $\Sigma_{\mathcal{Q}}$[4]. To complete the proof, it is enough to show that if $\vec{\mathcal{T}} \in H^N_\eta[h]$ is a stack on $\mathcal{Q}$ with essential components $(\mathcal{M}_\alpha, \mathcal{M}^*_\alpha, \vec{\mathcal{T}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \nu)$ such that $\vec{\mathcal{T}}_\nu$ is undefined and there is $(\vec{\mathcal{S}}, \mathcal{R}) \in H^N_\eta$ and $\pi$ such that $\pi : \mathcal{M}_\nu \to \mathcal{R}$ then whenever $\vec{\mathcal{U}}$ is a stack on $\mathcal{M}_\nu$ according to $\Sigma_{\mathcal{M}_\nu}$ which is based on $\mathcal{M}^*_\nu$ and has last model $\mathcal{W}$, there is $(\vec{\mathcal{S}}^*, \mathcal{R}^*) \in H^N_\eta[g] \cap I(\mathcal{R}, \Sigma_{\mathcal{R}})$ and there is $\sigma : \mathcal{W} \to \mathcal{R}^*$ with the property that $\pi^{\vec{\mathcal{S}}^*} \circ \pi = \sigma \circ \pi^{\vec{\mathcal{U}}}$.

Fix then such a stack $\vec{\mathcal{U}}$ on $\mathcal{M}_\nu$ which is based on $\mathcal{M}^*_\nu$. Let $\vec{\mathcal{U}}^* = \pi\vec{\mathcal{U}}$. Let $\mathcal{W}^*$ be the last model of $\vec{\mathcal{U}}^*$. Using the generic comparisons we can find $(\vec{\mathcal{S}}^*, \mathcal{R}^*) \in H^N_\eta[h] \cap I(\mathcal{R}, \Sigma_{\mathcal{R}})$ and $\sigma^* : \mathcal{W}^* \to \mathcal{R}^*$ with the property that $\pi^{\vec{\mathcal{S}}^*} = \sigma^* \circ \pi^{\vec{\mathcal{U}}^*}$. Let $\tau : \mathcal{W} \to \mathcal{W}^*$ come from copying construction. Then letting $\sigma = \sigma^* \circ \tau$ we have that $\pi^{\vec{\mathcal{S}}} \circ \pi = \sigma \circ \pi^{\vec{\mathcal{U}}}$. This then completes the description of $\dot{\Sigma}_{\mathcal{Q}}$. To get a term for $\Sigma$ we just repeat what we have just done. If $\lambda^{\mathcal{P}}$ is a successor then we do what

[4]We omit the part of the argument that deals with $\mathcal{Q}$-structures. It is just like the argument presented above while proving that $\Lambda^s$ is definable over $N[g]$

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we did in the case when $\lambda^{\mathcal{Q}}$ was a successor. If $\lambda^{\mathcal{P}}$ is limit then we do what we did in the case when $\lambda^{\mathcal{Q}}$ was limit. This finishes our proof of Lemma 3.35. □

## 3.7. Reorganizing hod mice

In order to do $S$-constructions we will need to change the particular way hod mice are defined. Readers familiar with $S$-constructions can recognize the problem. While carrying out $S$-constructions, we will need to translate hybrid mice over the ground model to hybrid mice over the generic extension. But the generic extension has more trees then the ground model and they don't come in a particular order. We solve the problem by changing the way the strategy is being fed into the hod mice. The new hod mice will only have trees that make initial segments of the model *generically generic*. It will then follow that when translating hybrid mice in this new hierarchy to hybrid mice over a generic extension, the trees that are on the sequence do not change[5].

DEFINITION 3.36. *Suppose $\Sigma$ is an iteration strategy and $a$ is a countable self-wellordered set. We let $\mathcal{M}_1^{\Sigma,\#}(a)$ be the minimal sound $\omega_1+1$-iterable $\Sigma$-mouse which has a unique Woodin cardinal and a last extender.*

Suppose $\Sigma$ is an iteration strategy and $N$ is a countable transitive set. Suppose $\mathcal{M}=\mathcal{M}_1^{\Sigma,\#}(a)$ where $a$ is some countable set $a$. We say $\mathcal{T}$ on $\mathcal{M}$ is the tree for making $N$ *generically generic* if the following holds:

(1) The first $o(N)+1$-models of $\mathcal{T}$ are obtained by iterating the first total measure of $\mathcal{M}$, $o(N)+1$ times.
(2) For $\alpha \geq o(N)+1$, $E^{\mathcal{T}}_\alpha$ is the least extender $F$ from the extender sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ such that there is some $p \in Coll(\omega, N)$ and a generic $g$ containing $p$ such that if $x=\{(n,m) : g(n) \in g(m)\}$ then $x$ doesn't satisfy some axioms from the extender algebra that involves $F$.

Below we only define the reorganized hybrid premice.

DEFINITION 3.37 (Reorganized hybrid strategy premouse). *A reorganized hybrid strategy premouse is a $\mathcal{J}$-structure $\mathcal{M}=\mathcal{J}^{\vec{E},f}_\alpha(\mathcal{N})$ with the following properties.*

(1) *$\mathcal{N}$ is an lhp.*
(2) *$\vec{E}$ is an extender sequence as in* [**20**] *and* [**33**].
(3) *For every $\beta \in \mathrm{dom}(\vec{E})$, if there are no Woodin cardinals in $\mathcal{M}|\beta$ then $\mathcal{M}|\beta$ is a hsp over $\mathcal{N}$ in the sense of Definition* 1.17.
(4) *Suppose there is $\beta \in dom(\vec{E})$ such that $\mathcal{M}|\beta \vDash$ "there is a Woodin cardinal". Let $\nu \in dom(\vec{E})$ be the least such that $\mathcal{M}|\nu \vDash$ "there is a Woodin cardinal" and let $\mathcal{Q}=\mathcal{M}|\nu$. Then $f \restriction (\mathcal{M}-\mathcal{M}|\nu)$ codes a fragment of an iteration strategy for $\mathcal{Q}$ in the following way. Let $\delta$ be the Woodin cardinal of $\mathcal{Q}$. Then for all $\beta \in (\nu,\alpha)$ such that $\beta = \sup f(a)$ for some $a \in dom(f)$,*

[5]We do not know if $S$-constructions are necessary to prove Theorem 6.19. We realized that $S$-constructions for hod mice aren't just a trivial generalization of $S$-constructions for mice very late in time when most of this work had already been written. This was pointed out by John Steel and Nam Trang. At this point it is easier for us to develop the theory in a way that it handles $S$-constructions rather than try to circumvent them. Moreover, both the new hierarchy and $S$-constructions have already found applications elsewhere. For example, the new hierarchy can be used to define hybrid mice over $\mathbb{R}$ or over any not selfwellorderd set.

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*letting* $\xi = \beta - o(a)$, $\mathcal{M}|\xi \vDash ZF - Replacement$ *and there is* $\gamma \in (\nu, \xi)$ *such that*

(a) $\mathcal{M}|\gamma \vDash ZF - Replacement$,
(b) $a = trc^{\mathcal{T}}$ *for some normal tree* $\mathcal{T}$ *on* $\mathcal{Q}$ *such that* $\mathcal{T}$ *is of limit length and is constructed according to the rules of* $\mathcal{M}|\gamma$*-generic genericty iteration of* $\mathcal{Q}$,
(c) *for every* $\zeta \in (\nu, \gamma)$ *such that* $\mathcal{M}|\zeta \vDash ZF - Replacement$, *there is a normal tree* $\mathcal{U} \in \mathcal{M}|\gamma$ *such that* $\mathcal{U}$ *is according to* $f^{\mathcal{M}|\gamma}$, $\mathcal{U}$ *is constructed according to the rules of* $\mathcal{M}|\zeta$*-generic genericity iterations,* $\mathcal{U}$ *has a last model* $\mathcal{P}$, *and* $\mathcal{M}|\zeta$ *is generically generic for the extender algebra of* $\mathcal{P}$ *at* $\pi^{\mathcal{U}}(\delta)$.

The definition of *reorganized layered hybrid premouse* and *reorganized hod premouse* are very similar to Definition 1.8 and Definition 1.7.1. In particular, if $\mathcal{P}$ is a reorganized hod premouse then $\mathcal{P}(\alpha+1)$ is a $\Sigma^{\mathcal{P}}_{\alpha}$–reorganized hybrid over $\mathcal{P}(\alpha)$. We can also define reorganized hod mouse and reorganized hod pair. The next lemma shows that the move from hod mice to reorganized hod mice is benign yet, as we will see later, quite useful.

LEMMA 3.38. *Suppose* $(\mathcal{P}, \Sigma)$ *is a reorganized hod pair such that* $\Sigma$ *is super fullness preserving. Then* $\mathcal{P}$ *is closed under* $\Sigma$ *and moreover, for any* $\mathcal{P}$*-generic* $g$, $\mathcal{P}[g]$ *is closed under* $\Sigma$.

PROOF. This is really a corollary to Lemma 3.35. To see that $\mathcal{P}$ is closed under $\Sigma$ let $\alpha < \lambda^{\mathcal{P}}$ and let $\vec{\mathcal{T}}$ be a stack on some $\mathcal{P}(\alpha)$. Let $\kappa$ be a regular cardinal of $\mathcal{P}$ such that $\vec{\mathcal{T}} \in \mathcal{P}|\kappa$ and let $\mathcal{U}$ be a normal tree on the sequence of $\mathcal{P}$ such that $\mathcal{U}$ is a tree on $\mathcal{M} = \mathcal{M}_1^{\Sigma_{\mathcal{P}(\alpha)},\#}$ and if $\mathcal{S}$ is the last model of $\mathcal{U}$ and $i : \mathcal{M} \to \mathcal{S}$ is the iteration embedding then $\mathcal{P}|\kappa$ is generic over $\mathcal{S}$ for the extender algebra at $i(\delta)$ where $\delta$ is the Woodin of $\mathcal{M}$. Then because $\mathcal{M}$ is a $\Sigma_{\mathcal{P}(\alpha)}$-mouse (in the old sense), it follows from Lemma 3.35 that if $b = \Sigma(\vec{\mathcal{T}})$ then $b \in \mathcal{S}[\vec{\mathcal{T}}]$. The rest of the argument is very similar. □

From now on, we will drop "reorganized" and will always assume that our hod premice are reorganized hod premice.

### 3.8. $S$-constructions

$S$-constructions were first fully introduced in [**30**] where they were called $P$-constructions. Such constructions are due to Steel and hence, we change the terminology and call them $S$-constructions. These constructions allow one to translate mice over some set $A$ to mice over some set $B$ provided $A$ and $B$ are somehow close. The complete proof of the following proposition is essentially the proof of Lemma 1.5 of [**30**].

PROPOSITION 3.39. *Suppose* $\mathcal{M}$ *is a sound mouse and* $\delta$ *is a strong cutpoint cardinal of* $\mathcal{M}$. *Suppose further that* $\mathcal{N} \in \mathcal{M}|\delta+1$ *is such that* $\delta \subseteq \mathcal{N} \subseteq H^{\mathcal{M}}_{\delta}$ *and there is a partial ordering* $\mathbb{P} \in L_{\omega}[\mathcal{N}]$ *such that whenever* $\mathcal{Q}$ *is a mouse over* $\mathcal{N}$ *such that* $H^{\mathcal{Q}}_{\delta} = \mathcal{N}$ *then* $\mathcal{M}|\delta$ *is* $\mathbb{P}$*-generic over* $\mathcal{Q}$. *Then there is a mouse* $\mathcal{S}$ *over* $\mathcal{N}$ *such that* $\mathcal{M}|\delta$ *is generic over* $\mathcal{S}$ *and* $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$.

It is clear what $\mathcal{S}$ must be. Because $\mathbb{P}$ is a small forcing with respect to the critical points of the extenders of $\mathcal{M}$ that have indices bigger than $\delta$, all such

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extenders can be put on a sequence of some mouse over $\mathcal{N}$. This is exactly what $S$-constructions do. An $S$-construction of $\mathcal{M}$ over $\mathcal{N}$ is a sequence of $\mathcal{N}$-mice $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha \leq \eta)$ such that

(1) $\mathcal{S}_0 = L_\omega[\mathcal{N}]$,
(2) if $\mathcal{M}|\delta$ is generic over $\bar{\mathcal{S}}_\alpha$ for a forcing in $L_\omega[\mathcal{M}|\delta]$ then $\bar{\mathcal{S}}_\alpha[\mathcal{N}] = \mathcal{M}|(\omega\times\alpha)$ and
   (a) if $\mathcal{M}||(\omega \times \alpha)$ is active then $\mathcal{S}_\alpha$ is the expansion of $\bar{\mathcal{S}}_\alpha$ by the last extender of $\mathcal{M}||(\omega \times \alpha)$ and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,
   (b) if $\mathcal{M}||(\omega \times \alpha)$ is passive then $\mathcal{S}_\alpha = \bar{\mathcal{S}}_\alpha$ and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,
(3) if $\lambda$ is limit then $\bar{S}_\lambda = \cup_{\alpha<\lambda}\mathcal{S}_\alpha$.

By the proof of Lemma 1.5 of [**30**], the $\mathcal{S}$-construction described in 1-3 cannot fail as long as the hypothesis of 2 holds. Thus, we always have a last model of $\mathcal{S}$-construction which might be some $\bar{\mathcal{S}}_\alpha$ instead of $\mathcal{S}_\alpha$. We let $\mathcal{S}$ be the last model of $\mathcal{S}$ construction. Then by the proof of Lemma 1.5 of [**30**], $\mathcal{S}[\mathcal{M}|\delta] \trianglelefteq \mathcal{M}$ and if the hypothesis of 2 never fails then in fact, $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$. Moreover, $\mathcal{S}$ inherits whatever iterability $\mathcal{M}$ has above $\delta$. $\mathcal{S}$-constructions are used in various places in inner model theory. A particularly important application for us is the following lemma.

LEMMA 3.40. *Suppose $\mathcal{M} \vDash ZFC - Powerset$ is a mouse and $\eta$ is a non-Woodin strong cutpoint cardinal of $\mathcal{M}$. Suppose $\gamma > \eta$ is a cardinal of $\mathcal{M}$ and $\mathcal{N} = L[\vec{E}]^{\mathcal{M}|\gamma}$. Suppose $L_\omega(\mathcal{N}|\eta) \vDash$ "$\eta$ is Woodin". Let $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha < \nu)$ be the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$. Then for some $\alpha < \nu$, $\mathcal{S}_\alpha \vDash$ "$\eta$ isn't Woodin".*

PROOF. Let $\mathcal{S}$ be the last model of the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$. Suppose $\eta$ is a Woodin cardinal of $\mathcal{S}$. Then $\mathcal{M}|\eta$ is generic for the $\eta$-generator version of the extender algebra of $L_\omega(\mathcal{N}|\eta)$. We also have that $\mathcal{M}|\eta$ is generic over $\mathcal{S}$ for the $\eta$-generator version of the extender algebra at $\eta$ and hence, $\mathcal{S}[\mathcal{M}|\eta] = \mathcal{M}|(\eta^+)^{\mathcal{M}}$. Thus, $\eta$ isn't Woodin in $\mathcal{S}[\mathcal{M}|\eta]$. Let $f : \eta \to \eta$ be the function in $\mathcal{M}$ witnessing that $\eta$ isn't Woodin. Then because the $\eta$-generator version of extender algebra is $\eta$-cc, there is $g \in \mathcal{S}$ which dominates $f$. Let $E$ be the extender that witnesses that $\eta$ is Woodin for $g$. Then if $E^*$ is the resurrection of $E$ then $E^*$ witnesses the Woodiness of $\eta$ for $f$ in $\mathcal{M}$, contradiction! □

We would like to prove the equivalent of Lemma 3.40 for hybrid mice. We do not know how to handle $S$-constructions in general. We restrict ourselves to hod mice that have super fullness preserving strategies with branch condensation. What makes $S$-constructions possible for such pairs is our reorganization of hybrid mice.

Let $(\mathcal{P}, \Sigma)$ be a hod pair such that $\Sigma$ has a branch condensation and is super fullness preserving. Suppose $\mathcal{M}$ is a sound $\Sigma$-mouse and $\delta$ is a cutpoint cardinal of $\mathcal{M}$. Suppose further that $\mathcal{N} \in \mathcal{M}|\delta + 1$ is such that $\delta \subseteq \mathcal{N} \subseteq H^{\mathcal{M}}_\delta$, $\mathcal{N}$ models a sufficiently strong fragment of $ZF - Replacement$, $\mathcal{N}$ is closed under $\Sigma$ and there is a partial ordering $\mathbb{P} \in L_\omega[\mathcal{N}]$ such that $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $L_\omega[\mathcal{N}]$. We would like to define $S$-construction of $\mathcal{M}$ over $\mathcal{N}$ relative to $\Sigma$.

DEFINITION 3.41. *An $S$-construction of $\mathcal{M}$ over $\mathcal{N}$ relative to $\Sigma$ is a sequence $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha \leq \eta)$ of $\Sigma$-mice over $\mathcal{N}$ such that*

(1) $\mathcal{S}_0 = L_\omega[\mathcal{N}]$,

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(2) *if* $\mathcal{M}|\delta$ *is generic over* $\bar{\mathcal{S}}_\alpha$ *for a forcing in* $L_\omega[\mathcal{N}]$ *then*
   (a) *if* $\mathcal{M}||(\omega\cdot\alpha)$ *is active and has a last branch* $b$ *then* $\mathcal{S}_\alpha$ *is the expansion of* $\bar{\mathcal{S}}_\alpha$ *by* $b$ *and* $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$.
   (b) *if* $\mathcal{M}||(\omega\cdot\alpha)$ *is active and has a last extender* $E$ *then* $\mathcal{S}_\alpha$ *is the expansion of* $\bar{\mathcal{S}}_\alpha$ *by* $E$ *and* $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,
   (c) *if* $\mathcal{M}||(\omega\times\alpha)$ *is passive then* $\mathcal{S}_\alpha = \bar{\mathcal{S}}_\alpha$ *and* $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,
(3) *if* $\lambda$ *is limit then* $\bar{\mathcal{S}}_\lambda = \cup_{\alpha<\lambda}\mathcal{S}_\alpha$.

We then get the following generalization of Proposition 3.39

LEMMA 3.42. *Suppose* $(\mathcal{P},\Sigma)$, $\mathcal{M},\mathcal{N}$ *are as above and* $\delta$ *is a strong cutpoint cardinal of* $\mathcal{M}$. *Suppose further that* $\mathcal{N}\in\mathcal{M}|\delta+1$ *is such that* $\delta\subseteq\mathcal{N}\subseteq H^{\mathcal{M}}_\delta$ *and there is a partial ordering* $\mathbb{P}\in L_\omega[\mathcal{N}]$ *such that whenever* $\mathcal{Q}$ *is a* $\Sigma$*-mouse over* $\mathcal{N}$ *such that* $H^{\mathcal{Q}}_\delta = \mathcal{N}$ *then* $\mathcal{M}|\delta$ *is* $\mathbb{P}$*-generic over* $\mathcal{Q}$. *Then there is a* $\Sigma$*-mouse* $\mathcal{S}$ *over* $\mathcal{N}$ *such that* $\mathcal{M}|\delta$ *is generic over* $\mathcal{S}$ *and* $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$.

We can also state the hybrid version of Lemma 3.40.

LEMMA 3.43. *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair such that* $\Sigma$ *is super fullness preserving and has branch condensation. Suppose* $\mathcal{M}\vDash ZFC-Replacement$ *is a* $\Sigma$*-mouse and* $\eta$ *is a strong cutpoint non-Woodin cardinal of* $\mathcal{M}$. *Suppose* $\gamma>\eta$ *is a cardinal of* $\mathcal{M}$ *and* $\mathcal{N} = L[\vec{E},\Sigma]^{\mathcal{M}|\gamma}$. *Suppose* $L_\omega(\mathcal{N}|\eta)\vDash$ "$\eta$ *is Woodin*". *Let* $(\mathcal{S}_\alpha,\bar{\mathcal{S}}_\alpha:\alpha<\nu)$ *be the* $\mathcal{S}$*-construction of* $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ *over* $\mathcal{N}|\eta$ *relative to* $\Sigma$. *Then for some* $\alpha<\nu$, $\mathcal{S}_\alpha\vDash$ "$\eta$ *isn't Woodin*".

PROOF. Suppose for all $\alpha<\nu$, $\mathcal{S}_\alpha\vDash$ "$\eta$ is Woodin". Let $\mathcal{S}$ be the final model of the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$ relative to $\Sigma$. Then $\mathcal{M}|\eta$ is generic over $\mathcal{S}$ for the extender algebra of $\mathcal{S}$ at $\eta$. By Lemma 3.42, $\mathcal{S}[\mathcal{M}|\eta]\vDash$ "$\eta$ isn't Woodin". But now we can finish as in Lemma 3.40. □

Notice that instead of super fullness preservation we could assume super $\Gamma$-fullness preservation. Below we re-state the results of this section for super $\Gamma$-fullness preserving strategies.

DEFINITION 3.44 (Super $\Gamma$-fullness preservation). *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair.* $\Sigma$ *is super* $\Gamma$*-fullness preserving if it is* $\Gamma$*-fullness preserving and whenever* $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ *and* $\alpha<\lambda^{\mathcal{Q}}$, *the two sets*

$$U^{\Gamma,\Sigma}_{\mathcal{Q}(\alpha)} = \{(x,y)\in\mathbb{R}^2 : x \text{ codes a transitive set } a\in HC \text{ and } y \text{ codes } \mathcal{M} \text{ such that } \mathcal{M}\trianglelefteq Lp^{\Gamma,\Sigma}_{\mathcal{Q}(\alpha)}(a) \text{ and } \rho(\mathcal{M}) = a\}$$

$$W^{\Gamma,\Sigma}_{\mathcal{Q}(\alpha)} = \{(x,y,z)\in\mathbb{R}^3 : (x,y)\in U^{\Sigma}_{\mathcal{Q}(\alpha)} \text{ and if } \mathcal{M} \text{ is the } \Sigma_{\mathcal{Q}(\alpha)}\text{-mouse coded by } y \text{ then } z \text{ codes a tree according to the unique strategy of } \mathcal{M}\}.$$

*are term captured by* $(\mathcal{Q}[g],\Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ *whenever* $g\subseteq Coll(\omega,\mathcal{Q}(\alpha))$ *is* $\mathcal{Q}$*-generic. We let* $u^{\Sigma}_{\mathcal{Q}(\alpha)}$ *and* $w^{\Sigma}_{\mathcal{Q}(\alpha)}$ *be the term relations locally capturing* $U^{\Sigma}_{\mathcal{Q}(\alpha)}$ *and* $W^{\Sigma}_{\mathcal{Q}(\alpha)}$.

LEMMA 3.45. *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair such that* $\Sigma$ *has branch condensation and is super* $\Gamma$*-fullness preserving for some pointclass* $\Gamma$ *closed under continuous images and preimages. Suppose* $N\vDash ZFC-Replacement$. *Suppose* $N$ *is* $\Sigma$*-closed and suppose* $g$ *is* $N$*-generic. Then* $N[g]$ *is* $\Sigma$*-closed.*

LEMMA 3.46. *Suppose* $(\mathcal{P},\Sigma)$, $\mathcal{M},\mathcal{N}$ *and* $(\mathcal{S}_\alpha,\bar{\mathcal{S}}_\alpha:\alpha\leq\eta)$ *are as in Definition* 3.41 *except that* $\Sigma$ *is super* $\Gamma$*-fullness preserving for some pointclass* $\Gamma$ *closed under continuous images and preimages. Suppose* $\delta$ *is a strong cutpoint cardinal of*

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*$\mathcal{M}$. Suppose further that $\mathcal{N} \in \mathcal{M}|\delta+1$ is such that $\delta \subseteq \mathcal{N} \subseteq H^{\mathcal{M}}_{\delta}$ and there is a partial ordering $\mathbb{P} \in L_\omega[\mathcal{N}]$ such that whenever $\mathcal{Q}$ is a $\Sigma$-mouse over $\mathcal{N}$ such that $H^{\mathcal{Q}}_{\delta} = \mathcal{N}$ then $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $\mathcal{Q}$. Then there is a $\Sigma$-mouse $\mathcal{S}$ over $\mathcal{N}$ such that $\mathcal{M}|\delta$ is generic over $\mathcal{S}$ and $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$.*

LEMMA 3.47. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ is $\Gamma$-super fullness preserving for some pointclass $\Gamma$ closed under continuous images and preimages and has branch condensation. Suppose $\mathcal{M} \vDash ZFC - \text{Replacement}$ is a $\Sigma$-mouse and $\eta$ is a strong cutpoint non-Woodin cardinal of $\mathcal{M}$. Suppose $\gamma > \eta$ is a cardinal of $\mathcal{M}$ and $\mathcal{N} = L[\vec{E}, \Sigma]^{\mathcal{M}|\gamma}$. Suppose $L_\omega(\mathcal{N}|\eta) \vDash$ "$\eta$ is Woodin". Let $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha < \nu)$ be the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$ relative to $\Sigma$. Then for some $\alpha < \nu$, $\mathcal{S}_\alpha \vDash$ "$\eta$ isn't Woodin".*

Using $S$-constructions we can show that the definition of $\Gamma$-hod pair constructions is equivalent to the following. Suppose $\Gamma$ is a pointclass closed under complements and under continuous preimages. We urge the reader to review the notation introduced before Definition 2.32.

DEFINITION 3.48 ($\Gamma$-full hod pair constructions revisited). *Suppose $\Gamma$ is a pointclass closed under continuous preimages and images and suppose that $A \subseteq \mathbb{R}$ is such that $w(A) = w(\Gamma)$. Suppose further $(M, \delta, \Sigma)$ is a self-capturing background triple such that $M$ locally Suslin, co-Suslin captures $A_\Gamma$. Then the* $\Gamma$-hod pair construction *of $M$ below $\delta$ is a sequence $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta, \gamma_\beta : \beta \leq \Omega)$ that satisfies the following properties.*

1. *$M \vDash$ " for all $\beta < \Omega, (\mathcal{P}_\beta, \Sigma_\beta)$ is a hod pair such that $\Sigma_\beta \in \Gamma$" [6].*
2. *$\delta_{\beta+1} < \gamma_{\beta+1} \leq \delta$, if they exist, are the first two cardinals of $M$ such that $Lp^{\Gamma,\Sigma_\beta}(V^M_{\delta_{\beta+1}}) \vDash$ "$\delta_{\beta+1}$ is Woodin" and $Lp^{\Gamma,\Sigma_\beta}(V^M_{\gamma_{\beta+1}}) \vDash$ "$\gamma_{\beta+1}$ is Woodin".*
3. *For $\beta \in [-1, \delta)$, letting $\Sigma_{-1} = \emptyset$, $\mathbb{C}_{\beta+1} = ((\mathcal{M}^{\beta+1}_\xi, \mathcal{N}^{\beta+1}_\xi : \xi \leq \gamma_\beta), (F^{\beta+1}_\xi : \xi < \gamma_\beta))$ is the output of $\mathcal{J}^{\vec{E},\Sigma_\beta}$-construction of $V^M_{\gamma_\beta}$.*
4. *For $\beta \in [-1, \delta)$, letting $\Sigma_{-1} = \emptyset$, if*
   (a) *$\delta_{\beta+1}$ exists,*
   (b) *$\mathcal{N}^{\beta+1}_{\gamma_{\beta+1}}$ doesn't have initial segments projecting across $\delta_\beta$,*
   (c) *if $\beta$ is a successor then $\mathcal{N}^{\beta+1}_{\gamma_{\beta+1}} \vDash$ "$\delta_\beta$ is Woodin" and*
   (d) *if $\beta$ is limit then $(\delta^+_\beta)^{\mathcal{P}_\beta} = (\delta^+_\beta)^{\mathcal{N}^{\beta+1}_{\gamma_{\beta+1}}}$,*

   *then $\mathcal{P}_{\beta+1} = \mathcal{N}^{\beta+1}_{\gamma_{\beta+1}}|(\delta^{+\omega}_{\beta+1})^{\mathcal{N}^{\beta+1}_{\gamma_{\beta+1}}}$ and $\Sigma_{\beta+1}$ is the strategy of $\mathcal{P}_{\beta+1}$ induced by $\Sigma$.*
5. *For limit ordinals $\beta$, letting $\mathcal{P}^*_\beta = \cup_{\gamma<\beta}\mathcal{P}_\beta$, $\Sigma^*_\beta = \oplus_{\gamma<\beta}\Sigma_\beta$ and $\delta_\beta = \sup_{\gamma<\beta}\delta_\gamma$, if $\delta_\beta$ is not measurable in $M$ and $\delta_\beta < \delta$ then $\gamma_\beta$, if it exists, is the least $M$-cardinal such that $Lp^{\Gamma,\Sigma^*_b}(V^M_{\gamma_\beta}) \vDash$ "$\gamma_\beta$ is Woodin". Then $\mathbb{C}_\beta = ((\mathcal{M}^\beta_\xi, \mathcal{N}^\beta_\xi : \xi \leq \gamma_\beta), (F^\beta_\xi : \xi < \gamma_\beta))$ is the output of $\mathcal{J}^{\vec{E},\Sigma^*_\beta}$-construction of $V^M_{\gamma_\beta}$. If $\mathcal{N}^\beta_{\gamma_\beta}$ doesn't project across $\delta_\beta$ then $\mathcal{P}_\beta = \mathcal{N}^\beta_{\gamma_\beta}|(\delta^{+\omega}_\beta)^{\mathcal{N}^\beta_{\gamma_\beta}}$.*

We leave it to the reader to show that the two constructions essentially produce the same model.

[6] We abuse our notation and let $\Sigma_\beta$ stand for both the induced strategy which acts on all trees in $V$ and for the strategy which acts on trees that are in $M$. We will do this throughout this paper.

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CHAPTER 4

# Analysis of HOD

Most of the modern terminology used in the analysis of HOD is due to Steel and Woodin who analyzed $(\mathrm{HOD}|\underset{\sim}{\delta}_1^2)^{L(\mathbb{R})}$ and $\mathrm{HOD}^{L[x][G]}$ respectively assuming respectively that $\mathcal{M}_\omega^\#$ is $(\omega_1,\omega_1)$-iterable in $V^{Coll(\omega,\mathbb{R})}$ and that $\mathcal{M}_1^\#$ is $(\omega_1,\omega_1)$ iterable (see [**43**] and [**44**]). In the later example, $x$ is a real coding $\mathcal{M}_1^\#$ and $G \subseteq Coll(\omega,<\kappa)$ is $L[x]$-generic where $\kappa$ is the least inaccessible of $L[x]$. In the analysis of $\mathrm{HOD}^{L[x][G]}$, the notion of $s$-iterability introduced by Woodin plays a crucial role. Here $s$ is a finite sequence of ordinals. This notion allows one to track the strategy of $\mathcal{M}_1$ inside $L[x][G]$. Woodin generalized the notion of $s$-iterability to $AD$ context and computed full HOD of $L(\mathbb{R})$. The details of this work will appear in [**44**]. We generalize the notion further to compute HOD of the minimal model of $AD_\mathbb{R}$ + "$\Theta$ is regular". As is customary, we start the exposition with *suitability*.

## 4.1. Suitability

Throughout this section, we fix a pointclass $\Gamma$ closed under Wadge reducibility.

Definition 4.1 (Suitable pair). *$(\mathcal{P},\Sigma)$ is a $(\Gamma,\Sigma)$-suitable pair if for some $k \in (1,\omega]$ the following holds:*

1. *There is a $\mathcal{P}$-cardinal $\delta$ such that $\delta$ is a Woodin cardinal of $\mathcal{P}$ and $\delta^{+k-1}$ is the largest cardinal of $\mathcal{P}$.*
2. *Letting $\mathcal{P}^+ = \mathcal{P}$ if $k=\omega$ and otherwise $\mathcal{P}^+ = Lp^{\Gamma,\Sigma}(\mathcal{P})$, $\mathcal{P}^+$ is a hod premouse and $\lambda^\mathcal{P}$ is a successor ordinal.*
3. *$((\mathcal{P}^+)^-,\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is $\Gamma$-fullness preserving.*
4. *$\mathcal{P}$ is a $\Sigma$-mouse above $\mathcal{P}_{\lambda^\mathcal{P}-1}$.*
5. *for any $\mathcal{P}$-cardinal $\eta > \delta^\mathcal{P}_{\lambda-1}$, if $\eta$ is a strong cutpoint then $\mathcal{P}|(\eta^+)^\mathcal{P} = Lp^{\Gamma,\Sigma}(\mathcal{P}|\eta)$.*

Suppose a $(\mathcal{P},\Sigma)$ is $(\Gamma,\Sigma)$-suitable pair. We then let $\delta^\mathcal{P}$ and $k^\mathcal{P}$ be the $\delta$ and $k$ of Definition 4.1. Also, we let $\mathcal{P}^- = (\mathcal{P}^+)^-$. Fix now a $(\Gamma,\Sigma)$-suitable pair $(\mathcal{P},\Sigma)$.

Definition 4.2 (Definition 2.1 of [**35**]). *Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{P}$ above $\mathcal{P}^-$; then $\mathcal{Q}(\mathcal{T})$ is the unique $\Sigma$-premouse extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1+1$-iterable above $\delta(\mathcal{T})$, and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$, if there is any such premouse.*

Given an iteration tree $\mathcal{T}$ on $\mathcal{P}$ above $\mathcal{P}^-$, we say $\mathcal{T}$ is *nice* if $\mathcal{T}$ has no fatal drops. Notice that $(\Gamma,\Sigma)$-suitable premice satisfy the hypothesis of Lemma 1.28. A nice tree $\mathcal{T}$ is $(\Gamma,\Sigma)$-*correctly guided* if for every limit $\alpha < lh(\mathcal{T})$, $\mathcal{Q}(\mathcal{T}\restriction\alpha)$ exists and



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$$\mathcal{Q}(\mathcal{T}\restriction\alpha)\trianglelefteq Lp^{\Gamma,\Sigma}(\mathcal{M}(\mathcal{T}\restriction\alpha)).$$

$\mathcal{T}$ is $(\Gamma,\Sigma)$-*short* if it is nice, $(\Gamma,\Sigma)$-correctly guided and $Lp^{\Gamma,\Sigma}(\mathcal{M}(\mathcal{T}))\vDash$ "$\delta(\mathcal{T})$ is not Woodin". $\mathcal{T}$ is $(\Gamma,\Sigma)$-*maximal* if it is nice, $(\Gamma,\Sigma)$-correctly guided yet not $(\Gamma,\Sigma)$-short. Notice that if $\mathcal{T}$ is a $(\Gamma,\Sigma)$-maximal tree and $b$ is a branch such that $\pi^{\mathcal{T}}_b(\delta^{\mathcal{P}})=\delta(\mathcal{T})$ then $\mathcal{T}$ doesn't have a nice normal continuation.

DEFINITION 4.3 ($(\Gamma,\Sigma)$-correctly guided finite stack). *We say $(\mathcal{T}_i,\mathcal{P}_i:i<m)$ is a $(\Gamma,\Sigma)$-correctly guided finite stack on $\mathcal{P}$ if*

1. *$\mathcal{P}_0=\mathcal{P}$,*
2. *for every $i<m$, $(\mathcal{P}_i,\Sigma)$ is $(\Gamma,\Sigma)$-suitable pair and $\mathcal{T}_i$ is a nice $(\Gamma,\Sigma)$-correctly guided tree on $\mathcal{P}_i$,*
3. *for every $i$ such that $i+1<m$ either $\mathcal{T}_i$ has a last model and $\pi^{\mathcal{T}}$-exists or $\mathcal{T}$ is maximal, and*
   (a) *if $\mathcal{T}_i$ has a last model then $\mathcal{P}_{i+1}$ is the last model of $\mathcal{T}_i$ and if $lh(\mathcal{T}_i)=\alpha+1$ where $\alpha$ is limit then if $\mathcal{T}_i^-$ is $\mathcal{T}_i$ without its last branch then $\mathcal{Q}(\mathcal{T}_i^-)$-exists and $\mathcal{Q}(\mathcal{T}_i^-)\trianglelefteq\mathcal{M}^{\mathcal{T}}_\alpha$,*
   (b) *if $\mathcal{T}_i$ is $(\Gamma,\Sigma)$-maximal then $\mathcal{P}_{i+1}=Lp^{\Gamma,\Sigma}_{k^{\mathcal{P}}}(\mathcal{M}(\mathcal{T}_i))$.*

Notice that if $(\mathcal{T}_i,\mathcal{P}_i:i\leq m)$ is a $(\Gamma,\Sigma)$-correctly guided finite stack on $\mathcal{P}$ then only $\mathcal{T}_m$ can have a dropping last branch.

DEFINITION 4.4 ($(\Gamma,\Sigma)$-correctly guided finite mixed stack). *We say*

$$(\mathcal{T}^k_i,\mathcal{P}^k_i,\vec{\mathcal{T}}^k:k\leq m\wedge i\leq n_k)$$

*is a $(\Gamma,\Sigma)$-correctly guided finite mixed stack on $\mathcal{P}$ if $\mathcal{P}_0=\mathcal{P}$ and*

1. *for every $k\leq m$, $(\mathcal{P}^k_0,\Sigma_{(\mathcal{P}^k_0)^-})$ is a $(\Gamma,\Sigma_{(\mathcal{P}^k_0)^-})$-suitable pair and $(\mathcal{T}^k_i,\mathcal{P}^k_i:i\leq n_k)$ is a $(\Gamma,\Sigma_{(\mathcal{P}^k_0)^-})$-correctly guided finite stack on $\mathcal{P}^k_0$,*
2. *for every $k$ such that $k<m$, $(\vec{\mathcal{T}}^k,(\mathcal{P}^{k+1}_0)^-)\in I((\mathcal{P}^k_{m_k})^-,\Sigma_{(\mathcal{P}^k_{m_k})^-})$ and $\mathcal{P}^{k+1}_0$ is the last model of $\vec{\mathcal{T}}^k$ when this stack is regarded as a stack on $\mathcal{P}^k_{m_k}$.*
3. *$\vec{\mathcal{T}}^m$ is a stack on $(\mathcal{P}^m_{n_m})^-$ according to $\Sigma_{(\mathcal{P}^m_{n_m})^-}$.*

DEFINITION 4.5 (The last model of a $(\Gamma,\Sigma)$-correctly guided finite mixed stack). *Suppose $\vec{\mathcal{T}}=(\mathcal{T}^k_i,\mathcal{P}^k_i,\vec{\mathcal{T}}^k:k\leq m\wedge i\leq n_k)$ is a $(\Gamma,\Sigma)$-correctly guided finite mixed stack on $\mathcal{P}$. We say $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}$ if one of the following holds:*

1. *$\vec{\mathcal{T}}^m$ is defined and has a last model, and $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}^m$ when it is regarded as a stack on $\mathcal{P}^m_{n_m}$,*
2. *$\vec{\mathcal{T}}^m$ is undefined and $\mathcal{T}^m_{n_m}$ is defined and has a last model, and $\mathcal{R}$ is the last model of $\mathcal{T}^m_{n_m}$,*
3. *$\vec{\mathcal{T}}^m$ is undefined and $\mathcal{T}^m_{n_m}$ is of limit length, $\mathcal{T}^m_{n_m}$ is $(\Gamma,\Sigma_{(\mathcal{P}^m_0)^-})$-short and there is a cofinal well-founded branch $b$ of $\mathcal{T}^m_{n_m}$ such that $\mathcal{Q}(b,\mathcal{T}^m_{n_m})$ exists,*
$$\mathcal{Q}(b,\mathcal{T})\trianglelefteq Lp^{\Gamma,\Sigma}(\mathcal{M}(\mathcal{T}^m_{n_m}))$$
*and $\mathcal{R}=\mathcal{M}^{T^m_{n_m}}_b$,*
4. *$\vec{\mathcal{T}}^m$ is undefined and $\mathcal{T}^m_{n_m}$ is of limit length, $\mathcal{T}^m_{n_m}$ is $(\Gamma,\Sigma_{(\mathcal{P}^m_0)^-})$-maximal, $\mathcal{R}$ is $(\Gamma,\Sigma_{(\mathcal{P}^m_0)^-})$-suitable and*
$$\mathcal{R}=Lp^{\Gamma,\Sigma_{(\mathcal{P}^m_0)^-}}_{k^{\mathcal{P}}}(\mathcal{M}(\mathcal{T}^m_{n_m})).$$

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*We say $\mathcal{R}$ is a $(\Gamma,\Sigma)$-correct iterate of $\mathcal{P}$ if there is a $(\Gamma,\Sigma)$-correctly guided finite mixed stack on $\mathcal{P}$ with last model $\mathcal{R}$. We also say $(\mathcal{R},\Lambda)$ is a $(\Gamma,\Sigma)$-correct tail of $(\mathcal{P},\Sigma)$ if $\mathcal{R}$ is a $(\Gamma,\Sigma)$-correct iterate of $\mathcal{P}$ and $\Lambda=\Sigma_{\mathcal{R}^-}$.*

DEFINITION 4.6 ($S(\Gamma,\Sigma)$ and $F(\Gamma,\Sigma)$). *We let $S(\Gamma,\Sigma)=\{\mathcal{Q}:\mathcal{Q}^-\in pI(\mathcal{P},\Sigma)\wedge(\mathcal{Q},\Sigma_{\mathcal{Q}^-})$ is $(\Gamma,\Sigma_{\mathcal{Q}^-})$-suitable$\}$. Also, we let $F(\Gamma,\Sigma)$ be the set of functions $f$ such that $dom(f)=S(\Gamma,\Sigma)$ and for each $\mathcal{Q}\in S(\Gamma,\Sigma)$, $f(\mathcal{Q})\subseteq\mathcal{Q}$ and $f(\mathcal{Q})$ is amenable to $\mathcal{Q}$, i.e., for every $X\in\mathcal{Q}$, $X\cap f(\mathcal{Q})\in\mathcal{Q}$.*

Given $\mathcal{Q}\in S(\Gamma,\Sigma)$ and $f\in F(\Gamma,\Sigma)$ we let for $n<k^{\mathcal{P}}$, $f^n(\mathcal{Q})=f(\mathcal{Q})\cap\mathcal{Q}|((\delta^{\mathcal{Q}})^{+n})^{\mathcal{P}}$. Then $f(\mathcal{Q})=\cup_{n<k^{\mathcal{P}}}f^n(\mathcal{Q})$. We also let

$$\gamma_f^{\mathcal{Q}}=\delta^{\mathcal{Q}}\cap\mathcal{H}_1^{\mathcal{Q}}(\mathcal{Q}^-\cup\{f^n(\mathcal{Q}):n<k^{\mathcal{P}}\}).$$

Notice that

$$\gamma_f^{\mathcal{Q}}=\delta^{\mathcal{Q}}\cap\mathcal{H}_1^{\mathcal{Q}}(\gamma_f^{\mathcal{Q}}\cup\{f^n(\mathcal{Q}):n<k^{\mathcal{P}}\}).$$

We then let

$$H_f^{\mathcal{Q}}=\mathcal{H}_1^{\mathcal{Q}}(\gamma_f^{\mathcal{Q}}\cup\{f^n(\mathcal{Q}):n<k^{\mathcal{P}}\}).$$

If $\mathcal{Q}\in S(\Gamma,\Sigma)$, $f\in F(\Gamma,\Sigma)$ and $i:\mathcal{Q}\to\mathcal{R}$ is an embedding then we let $i(f(\mathcal{Q}))=\cup_{n<k^{\mathcal{P}}}i(f^n(\mathcal{Q}))$.

DEFINITION 4.7 ($f$-iterability). *Suppose $\mathcal{Q}\in S(\Gamma,\Sigma)$ and $f\in F(\Gamma,\Sigma)$. We say $\mathcal{Q}$ is $f$-iterable if whenever $(\vec{\mathcal{T}}^k,\mathcal{T}_i^k,\mathcal{Q}_i^k:k\leq m\wedge i\leq n_k)$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite stack on $\mathcal{Q}$ with last model $\mathcal{R}$ then there is a sequence $(b_i^k:k\leq m\wedge i\leq n_k)$ such that the following holds.*

(1) *For $k\leq m-1$ and $i\leq n_k$ and for $k=m$ and $i\leq n_m-1$,*

$$b_i^k=\begin{cases}\emptyset & :\mathcal{T}_i^k \text{ has a successor length}\\ \text{cofinal well-founded branch} & \\ \text{such that } \mathcal{M}_{b_i^k}^{\mathcal{T}_i^k}=\mathcal{Q}_i^\kappa & :\mathcal{T}_i^k \text{ is } (\Gamma,\Sigma_{(\mathcal{P}_0^k)^-})\text{-maximal}\end{cases}$$

(2) *The following three cases hold.*
   (a) *If $\mathcal{T}_{n_m}^m$ has a successor length then $b_{n_m}^m=\emptyset$.*
   (b) *If $\mathcal{T}_{n_m}^m$ is $(\Gamma,\Sigma_{(\mathcal{P}_0^m)^-})$-short then there is a cofinal well-founded branch $b$ such that $\mathcal{Q}(b,\mathcal{T}_{n_m}^m)$ exists, $\mathcal{Q}(b,\mathcal{T}_{n_m}^m)\trianglelefteq Lp^{\Gamma,\Sigma}(\mathcal{M}(\mathcal{T}_{n_m}^m))$ and $b_{n_m}^m$ is the unique such branch.*
   (c) *If $\mathcal{T}_{n_m}^m$ is $(\Gamma,\Sigma_{(\mathcal{P}_0^m)^-})$-maximal then $b_{n_m}^m$ is a cofinal well-founded branch.*

(3) *Letting for $k\leq m$ and $i\leq n_k$*

$$\pi_i^k=\begin{cases}\pi^{\mathcal{T}_i^k} & :\mathcal{T}_i^k \text{ has a successor length}\\ \pi_{b_i^k}^{\mathcal{T}_i^k} & :\mathcal{T}_i^k \text{ is } (\Gamma,\Sigma_{(\mathcal{P}_0^k)^-})\text{-maximal}\end{cases}$$

*$\pi_k=\pi^{\vec{\mathcal{T}}^k}\circ(\circ_{i\leq n_k}\pi_i^k)$ and $\pi=\pi_{m-1}\circ\pi_{m-2}\circ\cdots\pi_0$ then*

$$\pi(f(\mathcal{Q}))=f(\mathcal{R}).$$

Suppose again that $\mathcal{Q}\in S(\Gamma,\Sigma)$ and $f\in F(\Gamma,\Sigma)$. Suppose $\vec{\mathcal{T}}=(\vec{\mathcal{T}}^k,\mathcal{T}_i^k,\mathcal{Q}_i^k:k\leq m\wedge i\leq n_k)$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite stack on $\mathcal{Q}$ with last model $\mathcal{R}$. We say $\vec{b}=(b_i^k:k\leq m\wedge i\leq n_m)$ *witness $f$-iterability for $\vec{\mathcal{T}}$* if clause 3 above is satisfied. We then let $\pi_{\vec{\mathcal{T}},\vec{b}}$ be the embedding $\pi$ define above.

Continuing with the notation of the previous paragraph, let $\vec{b}$ and $\vec{c}$ be two $f$-iterability branches for $\vec{\mathcal{T}}$. It then follows from Theorem 1.13 that

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$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{Q}}_f = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H^{\mathcal{Q}}_f.$$

LEMMA 4.8 (Uniqueness of $f$-iterability embeddings). *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$, $f \in F(\Gamma,\Sigma)$ and $\vec{\mathcal{T}}$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite mixed stack on $\mathcal{Q}$. Suppose $\vec{b}$ and $\vec{c}$ are two $f$-iterability branches for $\vec{\mathcal{T}}$. Then*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{Q}}_f = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H^{\mathcal{Q}}_f.$$

*Moreover, if $\vec{\mathcal{T}}$ consists of just one normal tree $\mathcal{T}$, $\mathcal{Q}$ is the last model of $\mathcal{T}$ and $b$ and $c$ witness $f$-iterability for $\mathcal{T}$ then if $\xi \in b$ is the least such that $crit(E^{\mathcal{T}}_\xi) > \gamma^{\mathcal{Q}}_f$ then $b \cap \xi = c \cap \xi$.*

DEFINITION 4.9. *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$ and $f$-iterable. Given a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided maximal $\mathcal{T}$ on $\mathcal{Q}$ with last model $\mathcal{Q}$, we let $b_{\mathcal{T},f} = b \cap \xi$ where $b$ witnesses $f$-iterability of $\mathcal{Q}$ for $\mathcal{T}$ and $\xi \in b$ is the least such that $crit(E^{\mathcal{T}}_\xi) > \gamma^{\mathcal{Q}}_f$.*

Notice that, if $\mathcal{Q}$ is $f$-iterable, $\vec{\mathcal{T}}$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite mixed stack on $\mathcal{Q}$, and $\vec{b}$ witnesses $f$-iterability of $\mathcal{Q}$ for $\vec{\mathcal{T}}$, then even though $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{Q}}_f$ is independent of $\vec{b}$ it may very well depend on $\vec{\mathcal{T}}$. This observation motivates the following definition.

DEFINITION 4.10 (Strong $f$-iterability). *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$. We say $\mathcal{Q}$ is* strongly $f$-iterable *if $\mathcal{Q}$ is $f$-iterable and whenever $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are two $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite mixed stacks on $\mathcal{Q}$ with common last model $\mathcal{R}$, $\vec{b}$ witnesses $f$-iterability for $\vec{\mathcal{T}}$ and $\vec{c}$ witnesses $f$-iterability for $\vec{\mathcal{U}}$ then $\pi_{\vec{\mathcal{T}},\vec{b}}$ is defined iff $\pi_{\vec{\mathcal{U}},\vec{c}}$ is defined and*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{Q}}_f = \pi_{\vec{\mathcal{U}},\vec{c}} \restriction H^{\mathcal{Q}}_f.$$

If $\mathcal{Q}$ is strongly $f$-iterable and $\vec{\mathcal{T}}$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correctly guided finite stack on $\mathcal{Q}$ with last model $\mathcal{R}$ then we let

$$\pi_{\mathcal{Q},\mathcal{R},f} : H^{\mathcal{Q}}_f \to H^{\mathcal{R}}_f$$

be the embedding given by any $\vec{b}$ which witnesses the $f$-iterability of $\vec{\mathcal{T}}$, i.e., fixing $\vec{b}$ which witnesses $f$-iterability for $\vec{\mathcal{T}}$,

$$\pi_{\mathcal{Q},\mathcal{R},f} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{Q}}_f.$$

Clearly, $\pi_{\mathcal{Q},\mathcal{R},f}$ is independent of $\vec{\mathcal{T}}$ and $\vec{b}$.

Given a finite sequence of functions $\vec{f} = (f_i : i < n) \in F(\Gamma,\Sigma)$, we let $\oplus_{i<n} f_i \in F(\Gamma,\Sigma)$ be the function given by $(\oplus_{i<n} f_i)(\mathcal{Q}) = (f_i(\mathcal{Q}) : i < n)$. We set $\oplus\vec{f} = \oplus_{i<n} f_i$. Let then

$$\mathcal{I}_{\Gamma,\Sigma,F} = \{(\mathcal{Q},\vec{f}) : \mathcal{Q} \in S(\Gamma,\Sigma),\ \vec{f} \in (F(\Gamma,\Sigma))^{<\omega} \text{ and } \mathcal{Q} \text{ is strongly } \oplus\vec{f}\text{-iterable}\}.$$

DEFINITION 4.11. *Given $F \subseteq F(\Gamma,\Sigma)$, we say $F$ is closed if for any $\vec{f} \subseteq F^{<\omega}$ there is $\mathcal{Q}$ such that $(\mathcal{Q},\oplus\vec{f}) \in \mathcal{I}_{\Gamma,\Sigma,F}$ and for any $\vec{g} \subseteq F^{<\omega}$, there is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correct iterate $\mathcal{R}$ of $\mathcal{Q}$ such that $(\mathcal{R},\vec{f}\cup\vec{g}) \in \mathcal{I}_{\Gamma,\Sigma,F}$.*

Fix now a closed $F \subseteq F(\Gamma,\Sigma)$. Let

$$\mathcal{F}_{\Gamma,\Sigma,F} = \{H^{\mathcal{Q}}_f : (\mathcal{Q},f) \in \mathcal{I}_{\Gamma,\Sigma,F}\}.$$

We then define $\preceq_{\Gamma,F}$ on $\mathcal{I}_{\Gamma,\Sigma,F}$ by letting $(\mathcal{Q},\vec{f}) \preceq_{\Gamma,\Sigma,F} (\mathcal{R},\vec{g})$ iff $\mathcal{R}$ is a $(\Gamma,\Sigma_{\mathcal{Q}^-})$-correct iterate of $\mathcal{Q}$ and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{Q},\vec{f}) \preceq_{\Gamma,\Sigma,F} (\mathcal{R},\vec{g})$, we have that

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$$\pi_{\mathcal{Q},\mathcal{R},\vec{f}} : H^{\mathcal{Q}}_{\oplus\vec{f}} \to H^{\mathcal{Q}}_{\oplus\vec{f}}.$$

Notice that if $F$ is closed then $\preceq_{\Gamma,\Sigma,F}$ is directed. Let then

$$\mathcal{M}_{\infty,\Gamma,\Sigma,F}$$

be the direct limit of $(\mathcal{F}_{\Gamma,F}, \preceq_{\Gamma,\Sigma,F})$ under $\pi_{\mathcal{Q},\mathcal{R},\vec{f}}$. Given $(\mathcal{Q},\vec{f}) \in \mathcal{I}_{\Gamma,\Sigma,F}$, we let $\pi_{\mathcal{Q},\vec{f},\infty} : H^{\mathcal{Q}}_{\oplus\vec{f}} \to \mathcal{M}_{\infty,\Gamma,\Sigma,F}$ be the direct limit embedding.

LEMMA 4.12. *$\mathcal{M}_{\infty,\Gamma,\Sigma,F}$ is wellfounded.*

PROOF. If not then we can fix $((\mathcal{Q}_i, f_i) : i < \omega) \subseteq F$ and $(\alpha_i : i < \omega)$ such that $\alpha_i \in H^{\mathcal{Q}_i}_{f_i}$, $(\mathcal{Q}_i, \oplus_{j\leq i} f_j) \in \mathcal{I}_{\Gamma,F}$ and $\pi_{\mathcal{Q}_{i+1},f_{i+1},\infty}(\alpha_{i+1}) < \pi_{\mathcal{Q}_i,f_i,\infty}(\alpha_i)$. By simultaneously comparing $\mathcal{Q}_i$'s we get a common $(\Gamma, \Sigma_{\mathcal{Q}^-})$-correct iterate $\mathcal{Q}$ such that if $\beta_i = \pi_{\mathcal{Q}_i,\mathcal{Q},f_i}(\alpha_i)$ then $\beta_{i+1} < \beta_i$, contradiction! □

It turns out that $V^{\mathrm{HOD}}_{\Theta}$ of many models of determinacy can be obtained as $\mathcal{M}_{\infty,\Gamma,\Sigma,F}$ for some $\Gamma$ and $F$. We will give the details in the next few sections. We end this section by introducing two useful notions.

DEFINITION 4.13. *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$. Suppose $\Lambda$ is an iteration strategy for $\mathcal{Q}$ such that $\Lambda_{\mathcal{Q}^-} = \Sigma_{\mathcal{Q}^-}$ and $(\mathcal{Q},\Lambda)$ is a hod pair such that $\Lambda$ has branch condensation and is $\Gamma$-fullness preserving. We say $\Lambda$ respects $f$ if whenever $i : \mathcal{P} \to \mathcal{Q}$ and $j : \mathcal{Q} \to \mathcal{R}$ are iterations produced via $\Lambda$ then $j(f(\mathcal{Q})) = f(\mathcal{R})$. We say $\Lambda$ strongly respects $f$ if whenever $i, j, \mathcal{P}, \mathcal{Q}, \mathcal{R}$ are as above and $\mathcal{S}$ is such that there are $\sigma : \mathcal{Q} \to \mathcal{S}$ and $\tau : \mathcal{S} \to \mathcal{R}$ such that $j = \tau \circ \sigma$ then $\mathcal{S} \in S(\Gamma,\Sigma)$ and $\sigma(f(\mathcal{Q})) = f(\mathcal{S})$.*

If $\Sigma$ strongly respects many functions then it is possible to show that $\Sigma$ has branch condensation. Fix some $F \subseteq F(\Gamma)$.

DEFINITION 4.14. *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$. Suppose $\Lambda$ is an iteration strategy for $\mathcal{Q}$ such that $\Lambda_{\mathcal{Q}^-} = \Sigma_{\mathcal{Q}^-}$ and $(\mathcal{Q},\Lambda)$ is a hod pair such that $\Lambda$ is $\Gamma$-fullness preserving. Suppose $F \in F(\Gamma,\Sigma)^\omega$. We say $\Lambda$ is guided by $F$ if $F$ is closed, for every $f \in F$, $\Lambda$ respects $f$ and whenever $(\vec{\mathcal{T}},\mathcal{R}) \in I(\mathcal{Q},\Lambda)$, $\sup_{f\in F} \gamma^{\mathcal{R}}_f = \delta^{\mathcal{R}}$. We say $\Lambda$ is strongly guided by $F$ if for every $f \in F$, $\Lambda$ strongly respects $f$.*

Strategies strongly guided by $F$ as above have branch condensation.

LEMMA 4.15. *Suppose $\mathcal{Q} \in S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$. Suppose $\Lambda$ is an iteration strategy for $\mathcal{Q}$ such that $\Lambda_{\mathcal{Q}^-} = \Sigma_{\mathcal{Q}^-}$ and $(\mathcal{Q},\Lambda)$ is a hod pair such that $\Lambda$ is $\Gamma$-fullness preserving. Suppose $F \in F(\Gamma,\Sigma)^\omega$ strongly guides $\Lambda$. Then $\Lambda$ has branch condensation.*

PROOF. Let $(\mathcal{R},\vec{\mathcal{T}},\mathcal{S},\vec{\mathcal{U}},c,\pi)$ be such that $(\vec{\mathcal{T}},\mathcal{R}) \in I(\mathcal{Q},\Lambda)$, $\vec{\mathcal{U}}$ is according to $\Lambda$, $c$ is a branch of $\vec{\mathcal{U}}$, $\mathcal{S} = \mathcal{M}^{\vec{\mathcal{U}}}_c$, $\pi : \mathcal{S} \to \mathcal{R}$ and $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi^{\vec{\mathcal{U}}}_c$. We need to see that $c = \Lambda(\vec{\mathcal{U}})$. Let $(\mathcal{U}_\alpha, \mathcal{W}_\alpha : \alpha \leq \eta)$ be the normal components of $\vec{\mathcal{U}}$. If $\mathcal{U}_\eta$ is a tree on $(\mathcal{W}_\eta)^-$ then the claim follows from the branch condensation of $\Sigma$. Suppose then $\mathcal{U}_\eta$ isn't a tree on $(\mathcal{W}_\eta)^-$. There is then $\alpha < lh(\mathcal{U}_\eta)$ such that $\pi^{\mathcal{U}_\eta}_{0,\alpha}$ exists and letting $\mathcal{U}$ be $\mathcal{U}_\eta$ after stage $\alpha$, $\mathcal{U}$ is a tree on $\mathcal{W} =_{def} \mathcal{M}^{\mathcal{U}_\eta}_\alpha$ above $\mathcal{W}^-$. Let $b$ be the branch of $\vec{\mathcal{U}}$ according to $\Lambda$. The case that either $\mathcal{Q}(\mathcal{U},c)$ or $\mathcal{Q}(\mathcal{U},b)$ exists is easy and we leave it to the reader. We then assume that both $\mathcal{Q}(\mathcal{U},c)$and $\mathcal{Q}(\mathcal{U},b)$ don't exist. Notice that we have that for every $f \in F$, $\pi^{\mathcal{U}}_c(f(\mathcal{W})) = f(\mathcal{S})$ and

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$\pi^{\mathcal{U}}_b(f(\mathcal{W})) = f(\mathcal{S})$. This implies that $\{\gamma^{\mathcal{S}}_f : f \in F\} \subseteq rng(\pi^{\mathcal{U}}_b) \cap rng(\pi^{\mathcal{U}}_c)$. Because $\sup_{f\in F} \gamma^{\mathcal{S}}_f = \delta^{\mathcal{S}}$ and $\delta(\mathcal{U}) = \delta^{\mathcal{S}}$, it then follows from Lemma 1.13 that $b = c$. □

Finally, if $\Gamma = \mathcal{P}(\mathbb{R})$ then we drop $\Gamma$ from our terminology.

## 4.2. $B$-iterability

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has branch condensation. In this section, we introduce a large collection of functions $f \in F(\Sigma)$. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are such that $\Sigma$ and $\Lambda$ have branch condensation and are fullness preserving, we let $(\mathcal{P}, \Sigma) \leq_{DJ} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$. Here $DJ$ stands for Dodd-Jensen. Notice that $\leq_{DJ}$ is a well-founded relation. We then let $\alpha(\mathcal{P}, \Sigma) = |(\mathcal{P}, \Sigma)|_{\leq_{DJ}}$, and we let $[\mathcal{P}, \Sigma]$ be the $=_{DJ}$ equivalence class of $(\mathcal{P}, \Sigma)$, i.e.,

$(\mathcal{Q}, \Lambda) \in [\mathcal{P}, \Sigma]$ iff $(\mathcal{Q}, \Lambda)$ is a hod pair such that $\Lambda$ has branch condensation and is fullness preserving, and $\alpha(\mathcal{Q}, \Lambda) = \alpha(\mathcal{P}, \Sigma)$.

Notice that $[\mathcal{P}, \Sigma]$ is independent of $(\mathcal{P}, \Sigma)$. We let

$$\mathbb{B}(\mathcal{P}, \Sigma) = \{B \subseteq [\mathcal{P}, \Sigma] \times \mathbb{R} : B \text{ is } OD\}.$$

Note that $\mathbb{B}(\mathcal{P}, \Sigma)$ is defined for hod pairs not suitable pairs. The following standard lemma features prominently in our computations of HOD. Compare it with Lemma 4.5 of [**32**].

LEMMA 4.16. *Assume $SMC$ and suppose $(\mathcal{P}, \Sigma)$ is a suitable pair such that $\Sigma$ has branch condensation and is fullness preserving. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ and $\kappa > \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is a $\mathcal{P}$-cardinal. Then there is $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ such that $(\mathcal{P}, \tau)$ locally term captures $B_{(\mathcal{P},\Sigma)}$ at $\kappa$ for comeager set of $\mathcal{P}$-generics.*

PROOF. The claim follows from the fact that whenever $g \subseteq Coll(\omega, \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1})$ is $\mathcal{P}$-generic and $x \in \mathcal{P}[g]$ is a real coding $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$ then by $SMC$ and clause 4 of Definition 4.1, $C_{\Sigma^2_1(Code(\Sigma))}(x) \in \mathcal{P}[g]$. In particular, $B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g] \in \mathcal{P}[g]$. Below we give more details.

Let $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ be the set of all $(p, \sigma)$ such that $\sigma$ is a standard term for a pair of reals such that for comeager many $\mathcal{P}$-generics $g \subseteq Coll(\omega, \kappa)$, either $p \notin g$ or $\tau_g \in B_{(\mathcal{P}^-,\Sigma)}$. By $SMC$, $\tau \in \mathcal{P}$. We claim that for comeager many $g$, $\tau_g = B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g]$.

To show this, first let

$$C_{p,\sigma} = \{g \subseteq Coll(\omega, \kappa)\text{: } g \text{ isn't } \mathcal{P}\text{-generic, or } g \text{ is } \mathcal{P} \text{ generic but } p \notin g, \text{ or } g \text{ is } \mathcal{P}\text{-generic, } p \in g \text{ and } \sigma_g \in B_{(\mathcal{P}^-,\Sigma)}\},$$

$$C'_{p,\sigma} = \{g \subseteq Coll(\omega, \kappa)\text{: } g \text{ isn't } \mathcal{P}\text{-generic, or } g \text{ is } \mathcal{P} \text{ generic but } p \notin g, \text{ or } g \text{ is } \mathcal{P}\text{-generic, } p \in g \text{ and } \sigma_g \notin B_{(\mathcal{P}^-,\Sigma)}\}.$$

Then we have that $\tau = \{(p, \sigma) : C_{p,\sigma} \text{ is comeager }\}$. We claim that the set $D_\sigma = \{p : C_{p,\sigma} \text{ or } C'_{p,\sigma} \text{ is comeager }\}$ is dense in $Coll(\omega, \kappa)$. To see this, fix a condition $q$. Suppose $C_{q,\sigma}$ isn't comeager. Then there is $r \leq q$ such that for comeager many $g$, either $r \notin g$ or $r \in g$ but $g \notin C_{q,\sigma}$. Let $C$ be this comeager set. Then $C \subseteq C'_{r,\sigma}$ and hence, $C'_{r,\sigma}$ is comeager.

Let now $C^*_{p,\sigma}$ be the comeager one from the pair $(C_{p,\sigma}, C'_{p,\sigma})$. Let $C = \cap C^*_{p,\sigma}$. Then $C$ is comeager because there are only countably many $C^*_{p,\sigma}$'s. We claim that whenever $g \in C$ is $\mathcal{P}$-generic, $\tau_g = B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g]$. To see this, let $g \in C$. Fix

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$(p,\sigma) \in \tau$ such that $p \in g$. We want to see that $\sigma_g \in B_{(\mathcal{P}^-,\Sigma)}$. Let $r \leq p$ be such that either $C_{r,\sigma}$ or $C'_{r,\sigma}$ is comeager and $r \in g$. Notice that $g \notin C'_{r,\sigma}$ because then $C'_{r,\sigma}$ is comeager implying that $(p,\sigma) \notin \tau$. Hence, $g \in C_{r,\sigma}$, implying that $\sigma_g \in B_{(\mathcal{P}^-,\Sigma)}$. This shows that $\tau_g \subseteq B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g]$. To establish the reverse direction, let $(x,y) \in B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g]$. Let $\sigma$ be a standard term for a pair of reals such that $\sigma_g = (x,y)$. Let $r \in g$ be such that either $C_{r,\sigma}$ or $C'_{r,\sigma}$ is comeager. Notice that $C'_{r,\sigma}$ cannot be comeager because then $g \in C'_{r,\sigma}$ implying that $\sigma_g \notin B_{(\mathcal{P}^-,\Sigma)}$. Hence, $g \in C_{r,\sigma}$ implying that $(r,\sigma) \in \tau$. Hence, $(x,y) \in \tau_g$. □

If $B$ is locally term captured for comeager many set generics over a suitable pair $(\mathcal{P},\Sigma)$ then we let $\tau^{\mathcal{P},\Sigma}_{B,\kappa}$ be the invariant term in $\mathcal{P}$ locally term capturing $B$ at $\kappa$ for comeager many set generics. We then let $f_B \in F(\Sigma)$ be given by $f_B(\mathcal{Q}) = \oplus_{\kappa<o(\mathcal{Q})}\tau^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}_{B,\kappa}$. We then say $\mathcal{P}$ is $B$-iterable if it is $f_B$-iterable. Similarly, all the notions defined in Section 4.1 carry over. Notice, however, that if $B \in [\mathcal{P},\Sigma]$ and $\mathcal{Q} \in S(\Sigma)$ then $\tau^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}_{B,\kappa}$ depends on $\Sigma$. Because of this we let $\gamma_B^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}$ stand for $\gamma^{\mathcal{Q}}_{f_B}$. Similarly for $H^{\mathcal{Q}}_{f_B}$.

## 4.3. The direct limit of iterates of hod mice

Suppose $\Gamma$ is a pointclass closed under Wadge reducibility and $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is $\Gamma$-fullness preserving. By Corollary 2.44, $\Sigma$ is commuting and therefore, we can form the direct limit of all iterates of $\mathcal{P}$ by $\Sigma$. We let

$$\mathcal{F}(\mathcal{P},\Sigma) = \{(\mathcal{Q},\Sigma_{\mathcal{Q}}) : \mathcal{Q} \in pB(\mathcal{P},\Sigma)\}$$
$$\mathcal{F}^+(\mathcal{P},\Sigma) = \{(\mathcal{Q},\Sigma_{\mathcal{Q}}) : \mathcal{Q} \in pI(\mathcal{P},\Sigma)\}.$$

Given $\mathcal{Q},\mathcal{R} \in pI(\mathcal{P},\Sigma) \cup pB(\mathcal{P},\Sigma)$, we let $\mathcal{Q} \leq^{\mathcal{P},\Sigma} \mathcal{R}$ if either

(1) $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ and $\mathcal{R} \in pI(\mathcal{Q},\Sigma_{\mathcal{Q}})$

or

(2) $\mathcal{Q} \in pB(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Sigma_{\mathcal{Q}}) \leq_{DJ} (\mathcal{R},\Sigma_{\mathcal{R}})$.

The following is an easy consequence of our comparison theorem, Theorem 2.28.

LEMMA 4.17. *$\leq^{\mathcal{P},\Sigma}$ is directed.*

Given then $\mathcal{Q},\mathcal{R} \in pI(\mathcal{P},\Sigma) \cup pB(\mathcal{P},\Sigma)$ such that for some $\beta \leq \lambda^{\mathcal{R}}$, $\mathcal{R}(\beta) \in pI(\mathcal{Q},\Sigma_{\mathcal{Q}})$, we let

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}(\beta)$$

be the iteration embedding given by $\Sigma_{\mathcal{Q}}$. We can then form the direct limit of $\mathcal{F}(\mathcal{P},\Sigma)$ and $\mathcal{F}^+(\mathcal{P},\Sigma)$ under the maps $\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}$. We let

$$\mathcal{M}_\infty(\mathcal{P},\Sigma) = dirlim(\mathcal{F}(\mathcal{P},\Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q},\mathcal{R} \in pB(\mathcal{P},\Sigma) \wedge \exists \beta \leq \lambda^{\mathcal{R}}(\mathcal{R}(\beta) \in pI(\mathcal{Q},\Sigma_{\mathcal{Q}}))$$
$$\mathcal{M}^+_\infty(\mathcal{P},\Sigma) = dirlim(\mathcal{F}^+(\mathcal{P},\Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q},\mathcal{R} \in pI(\mathcal{P},\Sigma) \wedge \mathcal{Q} \leq^{\mathcal{P},\Sigma} \mathcal{R}).$$

Note that if $\lambda^{\mathcal{P}}$ is limit then the height of $\mathcal{M}_\infty(\mathcal{P},\Sigma)$ is the sup of its Woodins. If $\lambda^{\mathcal{P}}$ is a successor ordinal then $\mathcal{M}_\infty(\mathcal{P},\Sigma)$ is a hod premouse. We let $\delta_\infty(\mathcal{P},\Sigma)$ be the sup of the Woodins of $\mathcal{M}_\infty(\mathcal{P},\Sigma)$. We let $\delta^+_\infty(\mathcal{P},\Sigma)$ be the sup of the Woodins of $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)$ and $\lambda_\infty(\mathcal{P},\Sigma) = \lambda^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}$. For $\mathcal{Q} \in pB(\mathcal{P},\Sigma)$ and $\mathcal{R} \in pI(\mathcal{P},\Sigma)$, we let

$$\pi^{\Sigma}_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P},\Sigma)$$
$$\sigma^{\Sigma}_{\mathcal{R},\infty} : \mathcal{R} \to \mathcal{M}^+_\infty(\mathcal{P},\Sigma)$$

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be the iteration maps. The next two lemmas are quite standard and are easy consequences of comparison. We prove the first and skip the proof of the second.

LEMMA 4.18 (The equivalence of direct limits). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and $\Sigma$ has branch condensation and is $\Gamma$-fullness preserving. Then*

(1) $\delta_\infty(\mathcal{P},\Sigma)=\delta^+_\infty(\mathcal{P},\Sigma)$;
(2) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\delta^+_\infty(\mathcal{P},\Sigma)=\mathcal{M}_\infty(\mathcal{P},\Sigma)$.

PROOF. The proof of clause 1 is straightforward and we leave it to the reader. We start by defining a map

$$\pi:\mathcal{M}_\infty(\mathcal{P},\Sigma)|\delta_\infty(\mathcal{P},\Sigma)\to_{\Sigma_1}\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\delta^+_\infty(\mathcal{P},\Sigma)$$

and show that $\pi$ is onto. It then follows that $\pi=id$. Suppose $x\in\mathcal{M}_\infty(\mathcal{P},\Sigma)$. Let $\mathcal{M}\in pB(\mathcal{P},\Sigma)$ be such that there is $\bar{x}\in\mathcal{M}$ such that $\pi^\Sigma_{\mathcal{M},\infty}(\bar{x})=x$. Then let $\mathcal{R}$ be such that $\mathcal{R}\in I(\mathcal{P},\Sigma)$ and $\mathcal{M}\trianglelefteq_{hod}\mathcal{R}$. Set $\pi(x)=\sigma^\Sigma_{\mathcal{R},\infty}(\bar{x})$.

*Claim 1.* $\pi$ is well-defined.

PROOF. First we show that $\pi$ is independent of the choice of $\mathcal{M}$. Suppose $\mathcal{N}\in pB(\mathcal{P},\Sigma)$ is such that for some $y\in\mathcal{N}$, $\pi^\Sigma_{\mathcal{N},\infty}(y)=x$. Let $\mathcal{Q}$ and $\mathcal{R}$ be such that $\mathcal{R},\mathcal{Q}\in pI(\mathcal{P},\Sigma)$ and $\mathcal{M}\trianglelefteq_{hod}\mathcal{R}$ and $\mathcal{N}\trianglelefteq_{hod}\mathcal{Q}$. Let $\mathcal{W}\in pI(\mathcal{P},\Sigma)$ be such that $(\mathcal{W},\Sigma_{\mathcal{W}})$ is a common tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ and $(\mathcal{R},\Sigma_{\mathcal{R}})$. Then we must have that $\pi^\Sigma_{\mathcal{R},\mathcal{W}}(\bar{x})=\pi^\Sigma_{\mathcal{Q},\mathcal{W}}(y)$. Therefore,

$$\sigma^\Sigma_{\mathcal{R},\infty}(\bar{x})=\sigma^\Sigma_{\mathcal{W},\infty}(\pi^\Sigma_{\mathcal{R},\mathcal{W}}(\bar{x}))=\sigma^\Sigma_{\mathcal{W},\infty}(\pi^\Sigma_{\mathcal{Q},\mathcal{W}}(y))=\sigma^\Sigma_{\mathcal{Q},\infty}(y).$$

□

*Claim 2.* $\pi$ is onto.

PROOF. Suppose $y\in\mathcal{M}^+_\infty(\mathcal{P},\Sigma)$. Let $\mathcal{R}\in pI(\mathcal{P},\Sigma)$ be such that there is $\bar{y}\in\mathcal{R}$ such that $\sigma^\Sigma_{\mathcal{R},\infty}(\bar{y})=y$. Let $\mathcal{Q}\trianglelefteq_{hod}\mathcal{R}$ be such that $\bar{y}\in\mathcal{Q}$. Then $\pi(i^\Sigma_{\mathcal{Q},\infty}(\bar{y}))=y$. □

Thus, $\pi=id$, and clause 2 follows. □

We will show in the next section that for $\alpha\leq\lambda_\infty$, $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}=\theta_\alpha$. The next lemma shows that $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)\subseteq$ HOD. It follows easily from Lemma 4.19 and from the observation that if $\mathcal{Q}\in pI(\mathcal{P},\Sigma)$ then $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)=\mathcal{M}^+_\infty(\mathcal{Q},\Lambda)$

LEMMA 4.19. *Suppose $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are two hod pairs such that both $\Sigma$ and $\Lambda$ have branch condensation and are $\Gamma$-fullness preserving. Suppose there is a good pointclass $\Gamma$ such that $Code(\Sigma)\in\Delta_{\underset{\sim}{\Gamma}}$. Then one of the following holds:*

(1) $(\mathcal{Q},\Lambda)\leq_{DJ}(\mathcal{P},\Sigma)$ *and for some* $\mathcal{R}\in pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma)$, $\mathcal{M}^+_\infty(\mathcal{R},\Sigma_{\mathcal{R}})=\mathcal{M}^+_\infty(\mathcal{Q},\Lambda)$,
(2) $(\mathcal{P},\Sigma)\leq_{DJ}(\mathcal{Q},\Lambda)$ *and for some* $\mathcal{R}\in pI(\mathcal{Q},\Lambda)\cup pB(\mathcal{Q},\Lambda)$, $\mathcal{M}^+_\infty(\mathcal{R},\Lambda_{\mathcal{R}})=\mathcal{M}^+_\infty(\mathcal{P},\Sigma)$.

COROLLARY 4.20. *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is $\Gamma$-fullness preserving. Suppose there is a good pointclass $\Gamma$ such that $Code(\Sigma)\in\Delta_{\underset{\sim}{\Gamma}}$. Then $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)\subseteq$ HOD.*

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PROOF. We have that $x \in \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$ iff there is a hod pair $(\mathcal{Q}, \Lambda)$ such that $\Lambda$ has branch condensation and is $\Gamma$-fullness preserving and $x \in \mathcal{M}_\infty^+(\mathcal{Q}, \Lambda)$. □

## 4.4. The computation of Hod

Assume $AD^+ + SMC$ and let $(\theta_\alpha : \alpha \leq \Omega)$ be the Solovay sequence. In this section, our goal is to compute $V_{\theta_\alpha}^{\mathrm{HOD}}$ for $\alpha \leq \Omega$ under some additional hypothesis. In the next chapter, we will show that our additional hypothesis almost follows from $AD^+ + SMC$. The computation of HOD can be carried out from just $AD^+ + SMC$. However, in this paper, we always have a little bit more room which can be used to simplify some of the arguments. Below we will give two computations, one done from the assumption that $\phi$ holds (see below) and the other from the assumption that $\psi$ holds (see below). In the next chapter, we proof that $\phi$ follows from $AD^+ + SMC$. However, we only prove that $\psi$ follows from a little bit more (for instance, from the assumption that our current model is completely mouse full in some bigger model of $AD^+$). The proof that $\psi$ follows from $AD^+ + SMC$ uses the theory of scales of $Lp^\Sigma(\mathbb{R})$. This theory will appear in [**28**] and the computation will appear in [**25**].

Recall that $\alpha(\mathcal{P}, \Sigma) = |(\mathcal{P}, \Sigma)|_{\leq_{DJ}}$. We let $\phi$ be the sentence: for every $\alpha < \Omega$, there is $(\mathcal{P}, \Sigma)$ such that

(1) $\alpha(\mathcal{P}, \Sigma) = \alpha$,
(2) $\Sigma$ is fullness preserving and has branch condensation,
(3) for any $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$, if $\lambda^{\mathcal{Q}}$ is a successor ordinal then
   (a) there is a sequence $(B_i : i < \omega) \subseteq \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ which guides $\Sigma_{\mathcal{Q}}$ and
   (b) for any $B \in \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ there is $\mathcal{R} \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ such that $\Sigma_{\mathcal{R}}$ respects $B$.

Recall that if $\mathcal{M}$ is a hod premouse then $f^{\mathcal{M}}$ is the shift of the amenable function coding a fragment of a strategy of $\mathcal{M}$. We will use the following notation in the proof of the next lemma. Suppose $\mathcal{Q}$ is a hod premouse such that $\lambda^{\mathcal{Q}}$ is a successor ordinal. Let then for $m < \omega$, $\eta_m^{\mathcal{Q}} = ((\delta^{\mathcal{Q}})^{+m})^{\mathcal{Q}}$ and let $\mathcal{Q}_m = \mathcal{Q}|\eta_m^{\mathcal{Q}}$.

LEMMA 4.21. *Suppose $\phi$ holds and $\alpha < \Omega$. Let $(\mathcal{P}, \Sigma)$ be such that $\Sigma$ has branch condensation and is fullness preserving and $\alpha(\mathcal{P}, \Sigma) = \alpha$. Let $\mathcal{M} = \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$, $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda$ be the strategy coded by $f^{\mathcal{M}}$. Then $\mathcal{M} \subseteq \mathrm{HOD}$ and*

$$\delta_\infty(\mathcal{P}, \Sigma) = \theta_\alpha \text{ and } \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)|\theta_\alpha = (V_{\theta_\alpha}^{\mathrm{HOD}}, \vec{E} \restriction \theta_\alpha, \Lambda \restriction V_{\theta_\alpha}^{\mathrm{HOD}}, \in).$$

PROOF. We already have that $\mathcal{M} \subseteq \mathrm{HOD}$ (see Theorem 4.18). By induction, we show that for every $\beta \leq \lambda^{\mathcal{M}}$, $\delta_\beta^{\mathcal{M}} = \theta_\beta$ and

$$\mathcal{M}|\theta_\beta = (V_{\theta_\beta}^{\mathrm{HOD}}, \vec{E} \restriction \theta_\beta, \Lambda \restriction V_{\theta_\beta}^{\mathrm{HOD}}, \in) \qquad (*)_\beta.$$

We have that for limit $\beta \leq \lambda^{\mathcal{M}}$, $\delta_\beta^{\mathcal{M}} = \sup_{\xi<\beta} \delta_\xi^{\mathcal{M}}$ and $\mathcal{M}(\beta)|\theta_\beta = \cup_{\xi<\beta} \mathcal{M}(\xi)$. It then follows that if $\beta \leq \lambda^{\mathcal{M}}$ is limit and for all $\xi < \beta$ we have that $\delta_\xi^{\mathcal{M}} = \theta_\xi$ and $(*)_\xi$ then we also have that $\delta_\beta^{\mathcal{M}} = \theta_\beta$ and $(*)_\beta$. Thus, it is enough to show that if $\beta + 1 \leq \lambda^{\mathcal{M}}$ is such that $\delta_\beta^{\mathcal{M}} = \theta_\beta$ and $(*)_\beta$ then $\delta_{\beta+1}^{\mathcal{M}} = \theta_{\beta+1}$ and $(*)_{\beta+1}$.

Fix then $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{Q}}$ is a successor ordinal and $\mathcal{M}_\infty^+(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-}) = \mathcal{M}(\beta)$. Note that $\alpha(\mathcal{Q}, \Sigma_{\mathcal{Q}}) = \beta + 1$. Without loss of generality, we assume, by comparing $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ with a pair witnessing $\phi$ for $\beta + 1$ if necessary, that $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$

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witnesses $\phi$ for $\beta+1$. Fix then $\vec{B} = (B_i : i < \omega)$ that guides $\Sigma_{\mathcal{Q}}$. We then have that

$$\delta^{\mathcal{M}}_{\beta+1} = \sup_{i<\omega} \pi^{\Sigma_{\mathcal{Q}}}_{\mathcal{Q},\infty}(\gamma^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}_{B_i}).$$

*Claim.* Suppose $A \subseteq \delta^{\mathcal{M}}_{\beta+1}$ is a bounded $OD$ subset of $\delta^{\mathcal{M}}_{\beta+1}$. Then $A \in \mathcal{M}(\beta+1)$.

Proof. Let $A \subseteq \alpha < \delta^{\mathcal{M}}_{\beta+1}$. Fix $i < \omega$ and $m \in (1,\omega)$ such that $\alpha < \pi^{\Sigma_{\mathcal{Q}_m}}_{\mathcal{Q}_m,\infty}(\gamma^{\mathcal{Q}_m,\Sigma_{\mathcal{Q}^-_m}}_{B_i})$. Define $C \in \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ by $((\mathcal{S},\Lambda), y) \in C$ if

(1) $\alpha(\mathcal{S},\Lambda) = \beta$,
(2) $y$ codes a pair $(\mathcal{N},\gamma)$ such that $(\mathcal{N},\Lambda)$ is suitable, $k^{\mathcal{N}} = m$, $\lambda^{\mathcal{N}}$ is a successor ordinal, $\mathcal{S} = \mathcal{N}^-$, $\gamma < \gamma^{\mathcal{N}_m,\Lambda}_{B_i}$ and there is a strategy $\Psi$ extending $\Lambda$ such that $(Lp^{\Lambda}_{\omega}(\mathcal{N}),\Psi)$ is a hod pair, $\Psi$ has branch condensation and is fullness preserving, $\Psi$ respects $B$ and $\pi^{\Psi}_{\mathcal{N},\infty}(\gamma) \in A$.

Then $C \in \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$. We assume without loss of generality that $\Sigma_{\mathcal{Q}}$ respects $C$ (otherwise we will just work with an iterate of $\mathcal{Q}$). We let $\tau = \pi^{\mathcal{Q},\Sigma_{\mathcal{Q}}}_{\mathcal{Q},\infty}(\tau^{\mathcal{Q},\Sigma_{\mathcal{Q}}}_{C})$. We can now define $A$ over $\mathcal{M}$ by

$$(*) \qquad \xi \in A \leftrightarrow \mathcal{M} \vDash\Vdash_{Coll(\omega,\delta^{+m}_{\beta+1})} \text{“if } y \text{ is a code of } (\mathcal{M}_m,\xi) \text{ then } y \in \tau\text{”}.$$

To see that this defines $A$, first suppose $\xi \in A$. Fix $\mathcal{R} \in pI(\mathcal{Q},\Sigma_{\mathcal{Q}})$ such that for some $\gamma < \gamma^{\mathcal{R}_m,\Sigma_{\mathcal{R}^-}}_{B_i}$, $\xi = \pi^{\Sigma}_{\mathcal{R},\infty}(\gamma)$. Then letting $\delta = \delta^{\mathcal{R}}$ we have that

$$\mathcal{R} \vDash\Vdash_{Coll(\omega,\delta^{+m})} \text{“ if } y \text{ is a code of } (\mathcal{R}_m,\gamma) \text{ then } y \in \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}(\tau^{\mathcal{Q}_m,\Sigma_{\mathcal{Q}^-_m}}_{C})\text{”}.$$

By applying $\pi^{\Sigma}_{\mathcal{R},\infty}$ we get the right hand side of (*). On the other hand, if the right hand side of (*) holds then we can get $\mathcal{R} \in pI(\mathcal{Q},\Sigma_{\mathcal{Q}})$ such that for some $\gamma < \gamma^{\mathcal{R}_m,\Sigma_{\mathcal{R}^-_m}}_{B_i}$, $\xi = \pi^{\Sigma}_{\mathcal{R},\infty}(\gamma)$ and letting $\delta = \delta^{\mathcal{R}}$ we have that

$$\mathcal{R} \vDash\Vdash_{Coll(\omega,\delta^{+m})} \text{“ if } y \text{ is a code of } (\mathcal{R}_m,\gamma) \text{ then } y \in \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}(\tau^{\mathcal{Q}_m,\Sigma_{\mathcal{Q}^-_m}}_{C})\text{”}.$$

This means that there is a strategy $\Psi$ for $\mathcal{R}$ such that $(Lp^{\Sigma_{\mathcal{R}^-}}_{\omega}(\mathcal{R}),\Psi)$ is a hod pair, $\Psi$ has branch condensation and is fullness preserving, $\Sigma_{\mathcal{R}^-} = \Psi_{\mathcal{R}^-}$ and $\pi^{\Psi}_{\mathcal{R},\infty}(\gamma) \in A$. It is then enough to show that $\pi^{\Psi}_{\mathcal{R},\infty}(\gamma) = \xi$. By Theorem 2.28, there is a normal tree $\mathcal{T}$ on $\mathcal{R}$ without a last branch such that if $b = \Sigma_{\mathcal{R}}(\mathcal{T})$ and $c = \Psi(\mathcal{T})$ then $\mathcal{M}^{\mathcal{T}}_b = \mathcal{M}^{\mathcal{T}}_c =_{def} \mathcal{S}$ and $\Sigma_{\mathcal{S}} = \Psi_{\mathcal{S}}$. Moreover, since both $\Psi$ and $\Sigma_{\mathcal{R}}$ respect $B$, we have that (by the proof of Lemma 1.13)

$$\pi^{\Psi}_{\mathcal{R},\mathcal{S}}(\gamma) = \pi^{\Sigma_{\mathcal{R}}}_{\mathcal{R},\mathcal{S}}(\gamma).$$

This then easily gives that $\pi^{\Psi}_{\mathcal{R},\infty}(\gamma) = \pi^{\Sigma_{\mathcal{R}}}_{\mathcal{R},\infty}(\gamma) = \xi$. □

It follows from the claim that $\delta^{\mathcal{M}}_{\beta+1}$ is a cardinal in HOD. Moreover, because there are no Woodin cardinals in $\mathcal{M}$ in the interval $(\delta^{\mathcal{M}}_{\beta}, \delta^{\mathcal{M}}_{\beta+1})$, we get that $\delta^{\mathcal{M}}_{\beta+1} \leq \theta_{\beta+1}$. It also follows from the claim that

$$\mathcal{M}|\delta^{\mathcal{M}}_{\beta+1} = (V^{\mathrm{HOD}}_{\delta^{\mathcal{M}}_{\beta+1}}, \vec{E} \restriction \delta^{M}_{\beta+1}, \Lambda \restriction V^{\mathrm{HOD}}_{\delta^{\mathcal{M}}_{\beta+1}}, \in).$$

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To finish the proof, it is enough to show that $\delta^{\mathcal{M}}_{\beta+1} = \theta_{\beta+1}$. To prove this, it is enough to show that $\delta^{\mathcal{M}}_{\beta+1} < \theta_{\beta+1}$ is impossible. Suppose towards a contradiction that $\delta^{\mathcal{M}}_{\beta+1} < \theta_{\beta+1}$. Let $\pi = \pi^{\Sigma}_{\mathcal{Q},\infty}$. Then by coding lemma,

$$\pi, \mathcal{M}(\beta+1) \in L(\{A \subseteq \mathbb{R} : w(A) < \theta_{\beta+1}\}, \mathbb{R}).$$

This then implies that $\Sigma_{\mathcal{Q}} \in L(\{A \subseteq \mathbb{R} : w(A) < \theta_{\beta+1}\}, \mathbb{R})$ as $\Sigma_{\mathcal{Q}}$ can be defined in $L(\{A \subseteq \mathbb{R} : w(A) < \theta_{\beta+1}\}, \mathbb{R})$ from $\Sigma_{\mathcal{Q}^-}$ and $\pi$. We give the definition for normal trees and leave the general case to the reader. We have that $\Sigma_{\mathcal{Q}}(\mathcal{T}) = b$ iff

(1) the $\mathcal{T}$ which is based on $\mathcal{Q}^-$ is according to $\Sigma_{\mathcal{Q}^-}$,
(2) if $\pi^{\mathcal{T}}_b$ exists then there is $\sigma : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{M}(\beta+1)$ such that $\pi = \sigma \circ \pi^{\mathcal{T}}_b$,
(3) if $\mathcal{T}$ isn't entirely based on $\mathcal{Q}^-$ and $\pi^{\mathcal{T}}_b$ doesn't exist then letting $\mathcal{R}$ be the node in $\mathcal{T}$ such that the part of $\mathcal{T}$ from $\mathcal{Q}$-to-$\mathcal{R}$ is based on $\mathcal{Q}^-$ and the rest of $\mathcal{T}$ is a tree on $\mathcal{R}$ above $\mathcal{R}^-$ then $\mathcal{Q}(\mathcal{T}, b)$-exists and is a $\Sigma_{\mathcal{R}^-}$-mouse.

Using the fact that $\Sigma_{\mathcal{Q}}$ has branch condensation, it is easy to see that the definition given above indeed defines $\Sigma_{\mathcal{Q}}$ in $L(\{A \subseteq \mathbb{R} : w(A) < \theta_{\beta+1}\}, \mathbb{R})$.

We now have that $\Sigma_{\mathcal{Q}}$ is $OD$ from $\Sigma_{\mathcal{Q}^-}$ and a real $x$. Let $\gamma = w(Code(\Sigma_{\mathcal{Q}}))$. For each hod pair $(\mathcal{S}, \Lambda)$ such that $\Lambda$ has branch condensation and is fullness preserving and $\alpha(\mathcal{S}, \Lambda) = \beta$, let $A_{\mathcal{S}} \subseteq \mathbb{R}$ be the least $OD(\Lambda_{\mathcal{S}})$ set such that $w(A_{\mathcal{S}}) > \gamma$. Let $B \in \mathbb{B}[\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-}]$ be given by

$$((\mathcal{S}, \Lambda), y) \in B \leftrightarrow \alpha(\mathcal{S}, \Lambda) = \beta \text{ and } y \in A_{\mathcal{S}}.$$

Let $\mathcal{R} \in pI(\mathcal{Q}, \Sigma)$ be such that $\Sigma_{\mathcal{R}}$ respects $B$. Then it is easy to see that whenever $\mathcal{S} \in pI(\mathcal{R}^-, \Sigma_{\mathcal{R}})$, then $w(A_{\mathcal{S}}) \leq w(Code(\Sigma_{\mathcal{R}}))$. This then implies that $w(Code(\Sigma_{\mathcal{R}})) > \gamma = w(Code(\Sigma_{\mathcal{Q}}))$. However, $w(Code(\Sigma_{\mathcal{R}})) = w(Code(\Sigma_{\mathcal{Q}}))$, contradiction! □

Lemma 4.21 computes $\mathrm{HOD}|\theta_{\alpha+1}$ in all cases except when $\alpha+1 = \Omega$. Let then $\alpha+1 = \Omega$. Below we describe the hypothesis that we need to compute $V^{\mathrm{HOD}}_{\theta_{\alpha+1}}$. We let $\mathcal{I} = \{(\mathcal{Q}, \Lambda, \vec{B}) :$

(1) $(\mathcal{Q}, \Lambda)$ is suitable and $\alpha(\mathcal{Q}^-, \Lambda) = \alpha$,
(2) $\Lambda$ has branch condensation and is fullness preserving,
(3) $\vec{B} \subseteq (\mathbb{B}(\mathcal{Q}^-, \Lambda))^{<\omega}$,
(4) $(\mathcal{Q}, \Lambda)$ is strongly $\vec{B}$-iterable$\}$.

Define $\leq^*$ on $\mathcal{I}$ by

$$(\mathcal{P}, \Sigma, \vec{B}) \leq^* (\mathcal{Q}, \Lambda, \vec{C}) \leftrightarrow \vec{B} \subseteq \vec{C} \text{ and } (\mathcal{Q}, \Lambda) \text{ is a } \Sigma\text{-correct tail of } (\mathcal{P}, \Sigma).$$

Suppose now $(\mathcal{R}, \Psi, \vec{B}) \leq^* (\mathcal{Q}, \Lambda, \vec{C})$. There is then a canonical map $i : H^{\mathcal{R},\Psi}_{\vec{B}} \to H^{\mathcal{Q},\Lambda}_{\vec{B}}$ that is independent of $\vec{B}$-iterable branches. We let $\pi_{(\mathcal{R},\Psi,\vec{B}),(\mathcal{Q},\Lambda,\vec{B})}$ be this map. It then follows that $(\mathcal{I}, \leq^*)$ is a directed system.

Next, we let

$$\mathcal{F} = \{H^{\mathcal{Q},\Lambda}_{\vec{B}} : (\mathcal{Q}, \Lambda, \vec{B}) \in \mathcal{I}\}.$$

and we let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ under the iteration maps $\pi_{(\mathcal{R},\Psi,\vec{B}),(\mathcal{Q},\Lambda,\vec{B})}$. Let $\delta_\infty = \delta^{\mathcal{M}_\infty}$. For $(\mathcal{Q}, \Lambda, B) \in \mathcal{I}$, we let $\pi_{(\mathcal{Q},\Lambda,B),\infty} : H^{\mathcal{Q},\Lambda}_{B} \to \mathcal{M}_\infty$.

Let then $\psi$ be the conjunction of the following sentences:

(1) $\phi$.
(2) There is a suitable $(\mathcal{P}, \Sigma)$ such that

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(a) $(\mathcal{P}^-, \Sigma)$ is a hod pair such that $\alpha(\mathcal{P}^-, \Sigma) = \alpha$ and $\Sigma$ is fullness preserving and has branch condensation,
(b) for any $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$, there is a $\Sigma$-correct tail $(\mathcal{Q}, \Phi)$ of $(\mathcal{P}, \Sigma)$ such that $(\mathcal{Q}, \Phi)$ is strongly $B$-iterable.

(3) $\mathcal{M}_\infty$ is well-founded and $\delta_\infty = \Theta = \theta_{\alpha+1}$.

The proof of the claim from the proof of Lemma 4.21 can be used to show that indeed $\psi$ gives a computation of HOD.

LEMMA 4.22. *Assume $\psi$ holds. Let $\mathcal{M} = \mathcal{M}_\infty$, $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda$ be the strategy coded by $f^{\mathcal{M}}$. Then $\mathcal{M} \subseteq \mathrm{HOD}$ and*

$$\delta^{\mathcal{M}} = \Theta \text{ and } \mathcal{M}|\Theta = (V_\Theta^{\mathrm{HOD}}, \vec{E} \restriction \Theta, \Lambda \restriction V_\Theta^{\mathrm{HOD}}, \in).$$

PROOF. Let $\mathcal{M} = \mathcal{M}_\infty$. It is enough to show that whenever $A \subseteq \Theta$ is a bounded $OD$ subset of $\Theta$, $A \in \mathcal{M}$. Fix such an $A$ and let $\beta$ be such that $A \subseteq \beta < \Theta$. Fix $(\mathcal{P}, \Sigma)$ as in $\psi$. Notice that it follows from clause 3 of $\psi$ that

$$\Theta = \sup_{(\mathcal{Q},\Lambda,B)\in\mathcal{I}} \pi_{(\mathcal{Q},\Lambda,B),\infty}(\gamma_B^{\mathcal{Q},\Lambda}).$$

For $B \in \mathbb{B}(P^-, \Sigma)$ let $\gamma_{B,\infty} = \pi_{(\mathcal{Q},\Lambda,B),\infty}(\gamma_B^{\mathcal{Q},\Lambda})$ where $(\mathcal{Q}, \Lambda, B) \in \mathcal{I}$ (it is not hard to see that $\gamma_{B,\infty}$ is independent of the choice of $(\mathcal{Q}, \Lambda, B)$).

Fix then $B \in \mathbb{B}(P^-, \Sigma)$ such that $\beta < \gamma_{B,\infty}$. Define $C \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ by $((\mathcal{S}, \Lambda), y) \in C$ if

(1) $\alpha(\mathcal{S}, \Lambda) = \alpha$,
(2) $y$ codes a pair $(\mathcal{N}, \gamma)$ such that $(\mathcal{N}, \Lambda)$ is suitable, $k^{\mathcal{N}} = k^{\mathcal{P}}$, $(\mathcal{N}, \Lambda, B) \in \mathcal{I}$, $\gamma < \gamma_B^{\mathcal{N},\Lambda}$ and $\pi_{(\mathcal{N},\Lambda,B),\infty}(\gamma) \in A$.

Then $C \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. We assume without loss of generality that $(\mathcal{P}, \Sigma, B \oplus C) \in \mathcal{I}$ (otherwise we will just work with an iterate of $\mathcal{P}$). We let $\tau = \pi_{(\mathcal{P},\Sigma,B\oplus C),\infty}(\tau_C^{\mathcal{P},\Sigma})$ and $m = k^{\mathcal{P}}$. We can now define $A$ over $\mathcal{M}$ by

$$(*) \qquad \xi \in A \leftrightarrow \mathcal{M} \vDash\Vdash_{Coll(\omega,\delta_{\alpha+1}^{+m})} \text{“if } y \text{ is a code of } (\mathcal{M}|\delta_\beta^{+m}, \xi) \text{ then } y \in \tau\text{”}.$$

To see that this defines $A$, first suppose $\xi \in A$. Let $(\mathcal{R}, \Lambda, B \oplus C) \in \mathcal{I}$ be such that for some $\gamma < \gamma_B^{\mathcal{R},\Lambda}$, $\xi = \pi_{(\mathcal{R},\Lambda,B),\infty}(\gamma)$. We can assume that $(\mathcal{R}, \Lambda)$ is a $\Sigma$-correct tail of $(\mathcal{P}, \Sigma)$. Then letting $\delta = \delta^{\mathcal{R}}$ we have that

$$\mathcal{R} \vDash\Vdash_{Coll(\omega,\delta^{+m})} \text{“if } y \text{ is a code of } (\mathcal{R}|\delta^{+m}, \gamma) \text{ then } y \in \pi_{(\mathcal{P},\Sigma,C),(\mathcal{R},\Lambda,C)}(\tau_C^{\mathcal{P},\Sigma})\text{”}.$$

By applying $\pi_{(\mathcal{R},\Lambda,B),\infty}$ we get the right hand side of (*).

On the other hand, if the right hand side of (*) holds then we can get $(\mathcal{R}, \Lambda, B \oplus C) \in \mathcal{I}$ such that for some $\gamma < \gamma_B^{\mathcal{R},\Lambda}$, $\xi = \pi_{(\mathcal{R},\Lambda,B),\infty}(\gamma)$. Again, we can assume that $(\mathcal{R}, \Lambda)$ is a $\Sigma$-correct tail of $(\mathcal{P}, \Sigma)$. Letting $\delta = \delta^{\mathcal{R}}$ we have that

$$\mathcal{R} \vDash\Vdash_{Coll(\omega,\delta^{+m})} \text{“ if } y \text{ is a code of } (\mathcal{R}|\delta^{+m}, \gamma) \text{ then } y \in \pi_{(\mathcal{P},\Sigma,C),(\mathcal{R},\Lambda,C)}(\tau_C^{\mathcal{P},\Sigma})\text{”}.$$

This, by the definition of $C$, gives that $\pi_{(\mathcal{R},\Lambda,B),\infty}(\gamma) \in A$ and because

$$\pi_{(\mathcal{R},\Lambda,B),\infty}(\gamma) = \xi,$$

we get that $\xi \in A$. □

We will use the following lemma to establish $\psi$.

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LEMMA 4.23. *Suppose* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $L(\Gamma,\mathbb{R}) \vDash AD^+ + SMC + \Omega = \alpha+1$ *and* $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma,\mathbb{R})$. *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \vDash AD^+$ *and there is a hod a pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

1. $\Sigma$ *has branch condensation and is* $\Gamma$*-fullness preserving,*
2. $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma,\mathbb{R}) \vDash$ "$(\mathcal{P},\Sigma_{\mathcal{P}^-})$ *is a suitable pair such that* $\alpha(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}) = \alpha$",
3. *there is a sequence* $(B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *which guides* $\Sigma$,
4. *for any* $B \in (\mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *there is* $\mathcal{R} \in pI(\mathcal{P},\Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* $B$.

*Then* $L(\Gamma,\mathcal{R}) \vDash \psi$ *and* $\mathcal{M}_\infty = \mathcal{M}_\infty^+(\mathcal{P},\Sigma)$.

PROOF. We already have that $(\mathcal{P},\Sigma_{\mathcal{P}^-})$ witness clause 1 and 2 of $\psi$. Thus, we only need to prove clause 3 of $\psi$. We prove clause 3 by defining $\pi : (\mathcal{M}_\infty)^{L(\Gamma,\mathbb{R})} \to \mathcal{M}_\infty^+(\mathcal{P},\Sigma)$ as follows: given $x \in \mathcal{M}_\infty$ let $(\mathcal{Q},\Lambda,B) \in \mathcal{I}^{L(\Gamma,\mathbb{R})}$ be such that for some $\pi_{(\mathcal{Q},\Lambda,B),\infty}(z) = x$ where $z \in H_B^{\mathcal{Q},\Lambda_{\mathcal{Q}^-}}$. Let $(\mathcal{R},\Psi)$ be a tail of $(\mathcal{P},\Sigma)$ such that $\Psi$ respects $B$. We can assume that $\mathcal{R}$ is an iterate of $\mathcal{Q}$. Let then

$$\pi(x) = \pi^{\Sigma}_{\mathcal{R},\infty}(\pi_{(\mathcal{Q},\Lambda,B),(\mathcal{R},\Psi_{\mathcal{R}^-},B)}(z)).$$

It follows from comparison that $\pi$ is independent of the choice of $(\mathcal{Q},\Lambda,B)$ and $(\mathcal{R},\Psi)$. Thus, $\pi$ is $\Sigma_1$-elementary. This then shows that $\mathcal{M}_\infty$ is well-founded. It follows from the proof of Lemma 4.21 that $\delta_\infty = \theta^\Gamma$ as otherwise, as at the end of the proof of Lemma 4.21, $Code(\Sigma) \in \Gamma$ which is a contradiction.

It remains to show that $\mathcal{M}_\infty = \mathcal{M}_\infty^+(\mathcal{P},\Sigma)$. For this, it is enough to show that $\pi$ is the identity. To see this, let $x \in \mathcal{M}_\infty^+(\mathcal{P},\Sigma)|\delta^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)}$. There is then $\mathcal{R} \in pI(\mathcal{P},\Sigma)$ such that for some $z \in \mathcal{R}|\delta^{\mathcal{R}}$, $\pi^{\Sigma}_{\mathcal{R},\infty}(z) = x$. We can then fix $i$ such that $z \in H_{B_i}^{\mathcal{R},\Sigma_{\mathcal{R}^-}}$. Then $(\mathcal{R},\Sigma_{\mathcal{R}^-},B_i) \in \mathcal{I}$ and by clause 3 above, we get that

$$\pi(\pi_{(\mathcal{R},\Sigma_{\mathcal{R}^-},B_i),\infty}(z)) = \pi^{\Sigma}_{\mathcal{R},\infty}(z) = x.$$

Hence, $\mathcal{M}_\infty^+(\mathcal{P},\Sigma)|\delta^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} \subseteq rng(\pi)$. This implies that

$$\delta^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} = \delta_\infty = \theta^\Gamma \text{ and } \mathcal{M}_\infty|\theta^\Gamma = \mathcal{M}_\infty^+(\mathcal{P},\Sigma)|\theta^\Gamma.$$

But then it follows that $crit(\pi) > \theta^\Gamma$ and hence, $\pi = id$. □

Let us summarize what we have shown in this section.

THEOREM 4.24 (Computation of HOD). *Assume* $AD^+$. *Suppose* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma,\mathbb{R})$. *Let* $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. *Then the following holds:*

1. *If* $L(\Gamma,\mathbb{R}) \vDash \phi$ *then whenever* $(\mathcal{P},\Sigma) \in \Gamma$ *is such that* $\alpha(\mathcal{P},\Sigma) < \Omega^\Gamma$, *then letting* $\mathcal{M} = \mathcal{M}_\infty^+(\mathcal{P},\Sigma)$, $\vec{E} = \vec{E}^{\mathcal{M}}$ *and* $\Lambda$ *be the strategy coded by* $f^{\mathcal{M}}$, *for every* $\alpha \leq \alpha(\mathcal{P},\Sigma)$
$$\delta^{\mathcal{M}}_\alpha = \theta^\Gamma_\alpha \text{ and } \mathcal{M}|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \vec{E} \restriction \theta^\Gamma_\alpha, \Lambda \restriction V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \in).$$
2. *If* $L(\Gamma,\mathbb{R}) \vDash \psi$ *then if* $\mathcal{M}_\infty$ *is as above, then letting* $\mathcal{M} = \mathcal{M}_\infty$ $\vec{E} = \vec{E}^{\mathcal{M}}$ *and* $\Lambda$ *be the strategy coded by* $f^{\mathcal{M}}$, *for every* $\alpha \leq \Omega^\Gamma$
$$\delta^{\mathcal{M}}_\alpha = \theta^\Gamma_\alpha \text{ and } \mathcal{M}|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \vec{E} \restriction \theta^\Gamma_\alpha, \Lambda \restriction V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \in).$$
3. *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \vDash AD^+$ *and there is a hod a pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

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(a) $\Sigma$ *has branch condensation and is* $\Gamma$*-fullness preserving,*
(b) $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ *is a suitable pair such that* $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$",
(c) *there is a sequence* $(B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *which guides* $\Sigma$,
(d) *for any* $B \in (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *there is* $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* $B$.

*Then* $L(\Gamma, \mathcal{R}) \vDash \psi$ *and* $\mathcal{M}_\infty = \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$.

In the next chapter, we will need a somewhat weaker form of Theorem 4.24. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor ordinal and $\Sigma$ has branch condensation and is fullness preserving. Then we let $\mathcal{N}_\infty(\mathcal{P}, \Sigma)$ be the direct limit of all $\Sigma$-iterates of $\mathcal{P}$ that are above $\mathcal{P}^-$. What we would like to show is that $\mathcal{N}^+_\infty$ agrees with $\mathrm{HOD}_{\Sigma_{\mathcal{P}^-}}$ up to $\theta_{\alpha+1}$ where $\alpha = \alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$. Suppose $A \subseteq \mathbb{R}$ is a set of reals which is $OD$ from $\Sigma_{\mathcal{P}^-}$. Let then $\phi$ and $s \in Ord^{<\omega}$ be such that

$$x \in A \leftrightarrow \phi[\Sigma_{\mathcal{P}^-}, x, s].$$

We then let $A^* \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ be such that

$$((\mathcal{Q}, \Lambda), y) \in A^* \leftrightarrow \Lambda \text{ is fullness preserving and has branch condensation,}$$
$$\alpha(\mathcal{Q}, \Lambda) = \alpha, \text{ and } \phi[\Lambda, y, s].$$

We say $\Sigma$ respects $A$ if $\Sigma$-respect $A^*$. Given a sequence $\vec{A} = (A_i : i < \omega) \subseteq OD(\Sigma_{\mathcal{P}^-})$, we say $\Sigma$ is guided by $\vec{A}$ if $\Sigma$ is guided by $(A^*_i : i < \omega)$. Given $A \in OD(\Sigma)$ we let $\gamma^{\mathcal{P},\Sigma}_A = \gamma^{\mathcal{P},\Sigma}_{A^*}$ and $H^{\mathcal{P},\Sigma}_A = H^{\mathcal{P},\Sigma}_{A^*}$. The weaker form of Theorem 4.24 that we will need is the following.

THEOREM 4.25 (Computation of HOD relative to $\Sigma$). *Assume* $AD^+$. *Suppose* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. *Suppose* $(\mathcal{P}, \Sigma)$ *is such that*

(1) $\lambda^{\mathcal{P}}$ *is a successor,*
(2) $\Sigma$ *is fullness preserving and has branch condensation,*
(3) $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$,
(4) *there is a sequence* $(A_i : i < \omega) \subseteq (OD(\Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *which guides* $\Sigma$,
(5) *for any* $A \in (OD(\Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ *there is* $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ *such that* $\mathcal{R}^- = \mathcal{P}^-$ *and* $\Sigma_{\mathcal{R}}$ *respects* $A$.

*Let* $\alpha = \alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ *and let* $\mathcal{N} = \mathcal{N}_\infty(\mathcal{P}, \Sigma)$, $\vec{E} = \vec{E}^{\mathcal{N}}$ *and* $\Lambda$ *be the strategy coded by* $f^{\mathcal{N}}$. *Let* $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}_{\Sigma_{\mathcal{P}^-}}$. *Then*

$$\mathcal{N}|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \vec{E} \restriction \theta^\Gamma_\alpha, \Lambda \restriction V^{\mathcal{H}}_{\theta^\Gamma_\alpha}, \in).$$

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CHAPTER 5

# Hod pair constructions

In this chapter we analyze hod pair constructions assuming either the existence of a hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation and is fullness preserving or assuming that the construction converges. In the next chapter, using Theorem 5.22, we will show that hod pair constructions done inside sufficiently strong background triples converge (see Theorem 6.1). We start the chapter by introducing the stack over a model produced by a fully backgrounded construction and showing that the resulting stack has a covering property (see Theorem 5.2). We will use Theorem 5.2 to show that if a hod pair construction converges then the induced strategy has branch condensation (see Theorem 5.11). In Section 5.2, we show that clause 4 of Definition 1.7.1 cannot be an obstacle to the convergence of hod pair constructions (see Lemma 5.5). In Section 5.3, we show that provided hod pair construction converges, the induced strategy is fullness preserving (see Lemma 5.7). In Section 5.6, we define $\Gamma(\mathcal{P}, \Sigma)$ when $\lambda^{\mathcal{P}}$ is a successor. The rest of the chapter is devoted to showing that hod pair constructions produce pairs that are $B$-iterable.

## 5.1. Stacking mice

We would like to adopt Jensen's theory of stacks to the context of fully backgrounded constructions and use it to prove branch condensation of strategies of hod pairs constructed via hod pair constructions (see Lemma 5.11). See [**7**] for some background information on stacks.

Recall from [**7**] that if $\mathcal{W}$ is a mouse then

$$S(\mathcal{W}) = \cup\{\mathcal{M} : \mathcal{M} \text{ is a sound countably iterable mouse, } \mathcal{W} \trianglelefteq \mathcal{M} \text{ and } \rho_\omega(\mathcal{M}) = o(\mathcal{W})\}$$

In general, $S(\mathcal{W})$ doesn't have to be anything reasonable as we are not assuming that $\mathcal{W}$ is a cutpoint of mice that are being stacked. However, if $o(\mathcal{W}) = \kappa$ is a regular uncountable cardinal then in fact whenever $\mathcal{M}$ and $\mathcal{N}$ are sound mice extending $\mathcal{W}$ and projecting to $\kappa$, either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. Jensen showed that if $\mathcal{W} = K^c||\kappa$ where $\kappa \geq \aleph_3$ is an $\omega$-closed regular cardinal which is a limit cardinal in $K^c$ then $\mathrm{cf}(o(S(\mathcal{W}))) \geq \kappa$ provided there is no superstrong cardinal in $K^c$ (see [**7**]). Steel, in an unpublished work, showed that in many cases the same holds when $\mathcal{W}$ comes from fully backgrounded constructions. We adopt Steel's result to hybrid mice. Suppose $\Sigma$ is a strategy and $\mathcal{W}$ is a $\Sigma$-mouse. Then

$$S^\Sigma(\mathcal{W}) = \cup\{\mathcal{M} : \mathcal{M} \text{ is a sound countably iterable } \Sigma\text{-mouse, } \mathcal{W} \trianglelefteq \mathcal{M} \text{ and } \rho_\omega(\mathcal{M}) = o(\mathcal{W})\}.$$

The proof of the following lemma is just like the proof of Lemma 3.1 and Lemma 3.3 of [**7**]. We only sketch the proof. Below $M_\Sigma$ is the structure that $\Sigma$ iterates. In what follows we let $\phi(\delta, \Sigma, W, \mathcal{N})$ be the conjunction of the following statements:



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(1) $\delta$ is a Woodin cardinal ,
(2) $\Sigma$ is a $\delta^+$-iteration strategy with hull condensation such that $M_\Sigma \in V_\delta$,
(3) $W \vDash ZFC$, $W \subseteq H_{\delta^+}$, and $L_1(\Sigma \restriction V_\delta, V_\delta) \subseteq W$,
(4) $V_\delta$ is $\delta+1$-iterable for trees that are in $W$,
(5) letting $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \delta), (F_\gamma : \gamma < \delta))$ be the output of $\mathcal{J}^{\vec{E},\Sigma}$-construction of $V_\delta$, $\mathcal{N} = \mathcal{N}_\delta$.

LEMMA 5.1. *Suppose $\phi[\delta, \Sigma, W, \mathcal{N}]$ holds. Then the following holds.*

(1) *If $\mathcal{M}_0$ and $\mathcal{M}_1$ are such that for $i \in \{0,1\}$, $\mathcal{M}_i$ is sound countably iterable $\Sigma$-mouse, $\mathcal{N} \trianglelefteq \mathcal{M}_i$ and $\rho_\omega(\mathcal{M}_i) = \delta$ then either $\mathcal{M}_0 \trianglelefteq \mathcal{M}_1$ or $\mathcal{M}_1 \trianglelefteq \mathcal{M}_0$.*
(2) *For all sound countably iterable $\Sigma$-mice $\mathcal{M}$ such that $\mathcal{N} \trianglelefteq \mathcal{M}$ with $\rho_\omega(\mathcal{M}) = \delta$ there is a sound countably iterable $\Sigma$-mouse $\mathcal{W}$ such that $\mathcal{M} \triangleleft \mathcal{W}$ and $\rho_\omega(\mathcal{W}) = \delta$.*

*In particular, if $\mathcal{S} = S(\mathcal{N})$ then $\mathcal{S} \vDash ZFC - Powerset$ and $\delta$ is the largest cardinal of $\mathcal{S}$.*

PROOF. The proof of clause 2 is easy. Let $\mathcal{W}$ be the least level of $\mathcal{J}[\mathcal{M}]$ projecting to $\delta$. We sketch the proof of clause 1 as we would like to use such arguments to prove more general facts. Suppose that $\mathcal{M}_0$ and $\mathcal{M}_1$ aren't compatible. Let then $\pi : H \to H_{\delta^{++}}$ be an elementary such that each $\mathcal{M}_i \in rng(\pi)$, $\Sigma \in rng(\pi)$, $|H| < \delta$ and $rng(\pi) \cap \delta = crit(\pi)$. Let $\bar{\mathcal{N}}$, $\bar{\mathcal{M}}_i$ and $\bar{\Sigma}$ be the collapses of $\mathcal{N}$, $\mathcal{M}_i$ and $\Sigma$. Notice that because $\Sigma$ has hull condensation, $\Sigma \restriction H = \bar{\Sigma}$. Also, $\rho_\omega(\bar{\mathcal{M}}_i) = crit(\pi)$. This means that by condensation[1], each $\bar{\mathcal{M}}_i \trianglelefteq \mathcal{N}$ and therefore, the $\bar{\mathcal{M}}_i$ are compatible, contradiction! □

LEMMA 5.2 (Covering for fully backgrounded constructions, Steel). *Suppose $\phi[\delta, \Sigma, W, \mathcal{N}]$ holds. Let $\mathcal{S} = S(\mathcal{N})$. Then $\mathrm{cf}(o(\mathcal{S})) \geq \delta$.*

PROOF. This is the equivalent of Theorem 3.4 of [**7**]. Here the proof is somewhat different and uses the ideas from the proof of Lemma 2.13. Suppose not. Let $\kappa = \mathrm{cf}(o(\mathcal{S}))$. Let $f : \kappa \to o(\mathcal{S})$ be a cofinal function. Let $\theta = \delta^{+5}$ and let $\pi : H \to H_\theta$ be such that

(1) $H^\omega \subseteq H$ and $|H| = crit(\pi)$,
(2) $rng(\pi) \cap \delta = crit(\pi)$,
(3) $\kappa < crit(\pi)$,
(4) $\{f, \mathcal{N}, \Sigma, \mathcal{S}\} \in rng(\pi)$,

We say $(\pi, H)$ is a *good* submodel of $H_\theta$. Let $\vec{X} = (X_\alpha : \alpha < \delta)$ be continuous sequence of submodels of $H_\theta$. We say $\vec{X}$ is *good* if letting for each $\alpha < \delta$, $\pi_\alpha : H_\alpha \to X_\alpha \prec H_\theta$ be the inverse of transitive collapse, $(\pi_{\alpha+1}, H_{\alpha+1})$ is a good submodel of $H_\theta$. As a consequence of Theorem 3.4 of [**7**] we get that whenever $(X_\alpha : \alpha < \delta)$ is a good continuous sequence of submodels of $H_\theta$, letting for each $\alpha < \delta$, $\pi_\alpha : H_\alpha \to X_\alpha \prec H_\theta$ be the inverse of transitive collapse, there is a stationary set $S \subseteq \delta$ such that whenever $\alpha \in S$, letting $\mathcal{S}_\alpha = \pi_\alpha^{-1}(\mathcal{S})$ and $\eta_\alpha = crit(\pi_\alpha)$, we have that $\mathcal{S}_\alpha = \mathcal{O}^{\mathcal{N}}_{\eta_\alpha, \eta_\alpha}$.

Fix now a continuous chain $(X_\alpha : \alpha < \delta)$ of submodels of $H_\theta$ and let for each $\alpha < \delta$, $\pi_\alpha : H_\alpha \to X_\alpha \prec H_\theta$ be the inverse of transitive collapse. Define $h : \delta \to \delta$ as follows:

[1] "Condensation" here refers to the well-known condensation theorem for mice, see Theorem 5.1 of [**43**].

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$$h = \begin{cases} 0 & : \text{for all } \alpha < \delta,\ \xi \neq \eta_\alpha \\ \sup\{g(\xi) : g \in X \wedge g : \delta \to \delta\} & :\ \xi = \eta_\alpha. \end{cases}$$

Using the Woodiness of $\delta$ we can now find an extender $E$ such that

(1) $lh(E)$ is inaccessible, $V_{lh(E)} \subseteq Ult(V, E)$ and $crit(E) = \eta_\alpha$,
(2) $E$ witnesses Woodiness property for $h$, i.e., $j_E(h)(\eta_\alpha) < \nu_E$,
(3) $\mathcal{N}|lh(E) = i_E(\mathcal{N})|lh(E)$,
(4) $(\eta_\xi : \xi < lh(E)) = i_E((\eta_\xi : \xi < \delta)) \restriction lh(E)$

Let $E^* = E \cap \mathcal{N}$. Then $E^* \in \vec{E}^{\mathcal{N}}$ (for details see the proof of Theorem 11.3 of [**20**]). It then follows that

(1.) for any $g : \eta_\alpha \to \eta_\alpha$ such that $g \in \mathcal{S}_\alpha$, $j_{E^*}(g)(\eta_\alpha) < \nu_{E^*}$.

To see (1.), fix such a $g$. Then $g$ is in $\mathcal{S}_\alpha$ and hence, $\pi_\alpha(g)(\eta_\alpha) < h(\eta_\alpha)$. Notice that the set $A = \{\eta_\xi : \xi < \alpha,\ g \restriction \eta_\xi \in \mathcal{S}_\xi \text{ and } \pi_\xi(g) = \pi_\alpha(g)\} \in E_{\eta_\alpha}$ and for any $\eta_\xi \in A$, $\pi_\xi(g)(\eta_\xi) < h(\eta_\xi)$. This means that

$$\pi_E(\pi_\alpha(g))(\eta_\alpha) < \pi_E(h)(\eta_\alpha).$$

But because $g = \pi_\alpha(g) \restriction \eta_\alpha$, we have that

$$\pi_E(g) = \pi_E(\pi_\alpha(g)) \restriction \pi_E(\eta_\alpha).$$

Therefore,

$$\pi_E(g)(\eta_\alpha) = \pi_E(\pi_\alpha(g))(\eta_\alpha) < \pi_E(h)(\eta_\alpha) < \nu_E = \nu_{E^*}.$$

It now follows that letting $\nu = \sup\{\pi_{E^*}(g)(\eta_\alpha) : g \in \mathcal{S}_\alpha\}$, $E^* \restriction \nu$ witness that $\eta_\alpha$ is a superstrong cardinal in $\mathcal{N}$. □

Next we would like to prove facts about thick hulls of the stack over a universal model. In particular, a fact that we will need later is that a thick hull of the stack over a universal model is itself universal.

LEMMA 5.3. *Suppose $\phi[\delta, \Sigma, W, \mathcal{N}]$ holds. Suppose $\Phi$ is the iteration strategy of $V_\delta$ that acts on trees in $W$. Let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ and $\Psi$ be the strategy of $\mathcal{N}$ induced by $\Phi$. Suppose $\mathcal{T} \in L_1(\Sigma \restriction V_\delta, V_\delta)$ is a tree on $\mathcal{N}$ of length $\delta + 1$ which is according to $\Psi$ with last model $\mathcal{M}$ such that $\pi^{\mathcal{T}}$ exists and $\pi^{\mathcal{T}}(\delta) = \delta$. Let $i = \pi^{\mathcal{T}}$. Then $i$ lifts to $i^* : \mathcal{S} \to \mathcal{S}(\mathcal{M})$.*

PROOF. Let $\kappa = crit(i)$ and let $E$ be the $(\kappa, \delta)$-extender derived from $i$. We first show that $Ult(\mathcal{S}, E)$ is well-founded. Let $\mathcal{W} = \mathcal{S}(\mathcal{M})$. If not then we can find a countably closed submodel $\pi : H \to H_{\delta^{+5}}$ such that $\mathcal{S}, E, \mathcal{T}, i, \mathcal{W} \in rng(\pi)$, $crit(\pi) < \delta$, $crit(\pi)$ is an inaccessible cardinal and letting $(\mathcal{S}^*, E^*, \mathcal{T}^*, i^*, \mathcal{W}^*) = \pi^{-1}(\mathcal{S}, E, \mathcal{T}, i, \mathcal{W})$, $\mathcal{S}^* \trianglelefteq \mathcal{N}|(\kappa^+)^{\mathcal{N}}$ (this follows from condensation). Let $\lambda = crit(\pi)$. Regarding $\mathcal{T}^*$ as a tree on $\mathcal{N}$, let $\mathcal{N}^*$ be the last model of $\mathcal{T}^*$. Let $k = \pi^{\mathcal{T}^*} : \mathcal{N} \to \mathcal{N}^*$. Then there is $\sigma : Ult(\mathcal{S}^*, E^*) \to k(\mathcal{S}^*)$ given by

$$\sigma(\pi_{E^*}(f)(a)) = k(f)(i^*(a))$$

where $f \in \mathcal{S}^*$ and $a \in \lambda^{<\omega}$. Thus, $Ult(\mathcal{S}^*, E^*)$ is well-founded and therefore, by elementarity of $\pi$, $Ult(\mathcal{S}, E)$ is well-founded.

Let then $\mathcal{W}^* = Ult(\mathcal{S}, E)$. We claim that $\mathcal{W}^* = \mathcal{W}$. We have that $\mathcal{W}^*|\delta = \mathcal{W}|\delta = \mathcal{M}$. It is easy to see, using Lemma 5.1, that $\mathcal{W}^* \trianglelefteq \mathcal{W}$. Indeed, fix some

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$\mathcal{M} \trianglelefteq \mathcal{W}^{**} \triangleleft \mathcal{W}^*$ such that $\rho(\mathcal{W}^{**}) = \delta$. It is enough to show that $\mathcal{W}^{**}$ is countably iterable. Let then $\sigma : \mathcal{K} \to \mathcal{W}^{**}$ be such a countable hull. Let $\pi : H \to V_{\delta^+}$ be such that $\mathcal{S}, E, \mathcal{W}, \mathcal{W}^*, \mathcal{W}^{**}, \sigma[\mathcal{K}], \mathcal{T} \in rng(\pi)$, $H^\omega \subseteq H$, $crit(\pi) = \kappa$ is a regular cardinal, and $\pi(\kappa) = \delta$. Then $\pi^{-1}(\mathcal{W}^{**}) \trianglelefteq \pi^{-1}(\mathcal{W}^*)$ and the later is an iterate of $\pi^{-1}(\mathcal{S})$ via $\Psi$. This implies that $\pi^{-1}(\mathcal{W}^{**})$ is countably iterable. Hence, as $\pi^{-1}(\sigma) : \mathcal{K} \to \pi^{-1}(\mathcal{W}^{**})$, $\mathcal{K}$ is countably iterable.

It remains to show that $\mathcal{W} \trianglelefteq \mathcal{W}^*$. Suppose not and let $\mathcal{W}^{**} \trianglelefteq \mathcal{W}$ be the least level of $\mathcal{W}$ such that $\mathcal{W}^* \trianglelefteq \mathcal{W}^{**}$ and $\rho(\mathcal{W}^{**}) = \delta$. Let again $\pi : H \to V_{\delta^+}$ be such that $\mathcal{S}, E, \mathcal{W}, \mathcal{W}^*, \mathcal{W}^{**}, \mathcal{T}, i \in rng(\pi)$, $H^\omega \subseteq H$, $crit(\pi) = \kappa$ is a regular cardinal, and $\pi(\kappa) = \delta$. Then let $\mathcal{S}^*, E^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_1^{**}, \mathcal{T}^*, i^*$ be the collapses of $\mathcal{S}, E, \mathcal{W}, \mathcal{W}^*, \mathcal{W}^{**}, \mathcal{T}, i$. Then by condensation, $\mathcal{W}_1^{**} \trianglelefteq \mathcal{M}$.

Notice that we can regard $\mathcal{T}^*$ as a tree on $\mathcal{N}$ and that $\mathcal{T}^*$, when regarded as a tree on $\mathcal{N}$, is a subtree of $\mathcal{T}$. Let now $\mathcal{N}^*$ be the last model of $\mathcal{T}^*$ when we regard $\mathcal{T}^*$ as a tree on $\mathcal{N}$. Then $\mathcal{N}^*$ is a model on $\mathcal{T}$ and in fact, the iteration embedding $\sigma : \mathcal{N}^* \to \mathcal{M}$ exists. Moreover, on the branch leading from $\mathcal{N}^*$ to $\mathcal{M}$ no extender of length $< \kappa$ is used. This means that $\mathcal{M}|(\kappa^+)^{\mathcal{M}} \trianglelefteq \mathcal{N}^*|(\kappa^+)^{\mathcal{N}^*}$. Because $\mathcal{W}_1^{**} \trianglelefteq \mathcal{M}$, we have that $\mathcal{W}_1^{**} \trianglelefteq \mathcal{N}^*$.

Let then $k = \pi^{\mathcal{T}^*} : \mathcal{N} \to \mathcal{N}^*$. Suppose first that $\mathcal{S}^* \triangleleft \mathcal{N}|(\kappa^+)^{\mathcal{N}}$. Let then $\mathcal{K} \trianglelefteq \mathcal{N}$ be the least such that $\mathcal{S}^* \triangleleft \mathcal{K}$ and $\rho(\mathcal{K}) = \kappa$. We then have that $k(\mathcal{K}) = \mathcal{W}_1^{**}$. Because $k \restriction \mathcal{S}^* \in H$ and $\mathcal{W}_1^{**} \in H$, we have that $\mathcal{K}^{**} \in H$. Because $H$ is countably closed, we have that in $H$, $\mathcal{K}$ is countably iterable. This then implies that $\mathcal{K} \trianglelefteq \mathcal{S}^*$, contradiction!.

Assume then that $\mathcal{S}^* = \mathcal{N}|(\kappa^+)^{\mathcal{N}}$. Notice that there is $\sigma : \mathcal{W}_1^* \to k(\mathcal{S}^*) = \mathcal{N}^*|(\kappa^+)^{\mathcal{N}^*}$ such that $k = \sigma \circ i^*$. Since $k[(\kappa^+)^{\mathcal{N}}]$ is cofinal, we have that $i^*$ must be the identity. Hence, $\mathcal{W}_1^{**} \trianglelefteq \mathcal{W}_1^*$. □

Assume that now $\phi[\delta, \Sigma, W, \mathcal{N}]$ holds and let $\mathcal{S} = \mathcal{S}(\mathcal{N})$. We say that $\Gamma \subseteq [\delta, o(\mathcal{S})]$ is a thick set if $\Gamma$ contains an $\omega$-club which is unbounded in $o(\mathcal{S})$. Let $\mathcal{S}_\Gamma = Hull^{\mathcal{S}}(\Gamma)$ and let $\pi_\Gamma : \mathcal{S}_\Gamma \to \mathcal{S}$.

LEMMA 5.4 (Universality of thick hulls). *Suppose that $\phi[\delta, \Sigma, W, \mathcal{N}]$ holds and let $\mathcal{S} = \mathcal{S}(\mathcal{N})$. Suppose further that for some $\kappa < \delta$ and a generic $g \subseteq Coll(\omega, \kappa)$ there is $(\mathcal{M}, \Lambda) \in V[g]$ such that $\mathcal{M} \in V_\delta[g]$ is a $\Sigma$-mouse and $\Lambda \in W[g]$ is a $\delta+1$-iteration strategy for $\mathcal{M}$. Suppose $\Gamma$ is a thick set and suppose that no initial segment of $\mathcal{N}$ satisfies "there is a superstrong cardinal". Then $\mathcal{S}_\Gamma$ wins the coiteration with $\mathcal{M}$. Moreover, if $\mathcal{N}^*$ is such that $\pi_\Gamma(\mathcal{N}^*) = \mathcal{N}$ then $\mathcal{S}_\Gamma = \mathcal{S}(\mathcal{N}^*)$.*

PROOF. Let $\mathcal{W} = \mathcal{S}_\Gamma$ and $\pi_\Gamma = \pi$. Assume for a moment that $\mathcal{W} = \mathcal{S}(\mathcal{N}^*)$ and if $\mathcal{T}$ is a tree on $\mathcal{W}$ with last model $\mathcal{W}^*$ such that $\pi^{\mathcal{T}}$ exists and $\pi^{\mathcal{T}}(\delta) = \delta$ then $\mathcal{W}^* = \mathcal{S}(\mathcal{W}^*|\delta)$. Suppose towards a contradiction that $\mathcal{M}$ wins the coiteration with $\mathcal{W}$. This process then produces a tree $\mathcal{T}$ on $\mathcal{M}$ and a tree $\mathcal{U}$ on $\mathcal{W}$ with last models $\mathcal{M}^*$ and $\mathcal{W}^*$ such that $\mathcal{W}^* \trianglelefteq \mathcal{M}^*$. We get that $\pi^{\mathcal{U}}$ exists and $\pi^{\mathcal{U}}(\delta) = \delta$. Then, letting $\lambda = o(\mathcal{W}^*)$, it follows from Lemma 5.2 that $\mathrm{cf}(\lambda) \geq \delta$. Let $\mathcal{M}_\alpha^{\mathcal{T}}$ be such that $\pi_\alpha^{\mathcal{T}}$ exists and for some $\lambda^* < \delta^*$, $\pi_\alpha^{\mathcal{T}}(\delta^*) = \delta$ and $\pi_\alpha^{\mathcal{T}}(\lambda^*) = \lambda$. Then $\pi_\alpha^{\mathcal{T}}[\lambda^*]$ cannot be cofinal in $\lambda$ (as $\lambda^* < \delta$). Notice that it follows that $\lambda^*$ cannot be a cardinal of $\mathcal{M}_\alpha^{\mathcal{T}}$ as otherwise $\pi_\alpha^{\mathcal{T}}[\lambda^*]$ will be cofinal in $\lambda$. We then let $\mathcal{Q} \trianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$ be the largest initial segment of $\mathcal{M}_\alpha^{\mathcal{T}}$ such that $\mathcal{Q} \vDash$ "$\lambda^*$ is a cardinal". It follows that $\rho(\mathcal{Q}) = \delta^*$. Then letting $\mathcal{Q}^* = \pi_\alpha^{\mathcal{T}}(\mathcal{Q})$, we have that $\rho(\mathcal{Q}^*) = \delta$. It follows that $\mathcal{Q}^* \not\trianglelefteq \mathcal{W}^*$. However, $\mathcal{Q}^* \trianglelefteq \mathcal{S}(\mathcal{W}^*|\delta)$ implying that $\mathcal{Q}^* \trianglelefteq \mathcal{W}^*$, contradiction!

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To finish the proof, it is enough to show then $\mathcal{W} = \mathcal{S}(\mathcal{N}^*)$ and that if $\mathcal{T}$ is a tree on $\mathcal{W}$ with last model $\mathcal{W}^*$ such that $\pi^{\mathcal{T}}$ exists and $\pi^{\mathcal{T}}(\delta) = \delta$ then $\mathcal{W}^* = \mathcal{S}(\mathcal{W}^*|\delta)$. We only prove that $\mathcal{W} = \mathcal{S}(\mathcal{N}^*)$ as the second half is similar. We already have that $\mathcal{W} \trianglelefteq \mathcal{S}(\mathcal{N}^*)$. Towards a contradiction, suppose there is $\mathcal{K} \trianglelefteq \mathcal{S}(\mathcal{N}^*)$ such that $\rho(\mathcal{K}) = \delta$ and $\mathcal{K} \not\trianglelefteq \mathcal{W}$ (so that $\mathcal{W} \triangleleft \mathcal{K}$). We assume that $\mathcal{K}$ is minimal with these properties. Let $\pi = \pi_\Gamma$.

Let $\sigma : H \to V_{\delta^+}$ be such that $\mathcal{S}, \mathcal{W}, \mathcal{K}, \mathcal{S}(\mathcal{N}^*), \pi \in rng(\sigma)$, $H^\omega \subseteq H$, and if $\kappa = crit(\sigma)$ then $\sigma(\kappa) = \delta$ and $\kappa$ is a regular cardinal. Let $\mathcal{Q} = \mathcal{S}(\mathcal{N}^*)$. Then let $\mathcal{S}^*, \mathcal{W}^*, \mathcal{K}^*, \mathcal{Q}^*, \pi^*, \mathcal{R}$ be the collapse of $\mathcal{S}, \mathcal{W}, \mathcal{K}, \mathcal{Q}, \pi, \mathcal{N}$. Let $E$ be the $(\kappa, \kappa)$-extender derived from $\pi^*$. We claim that $Ult(\mathcal{W}, E)$ is well-founded. This is because $\pi^*(\kappa) = \kappa$ and all the measures of $E$ concentrate on ordinals $< \kappa$. Hence, if $F$ is the $(\delta, \delta)$-extender derived from $\pi$ then $E$ is a subextender of $F$.

Because $\mathcal{K}^* \trianglelefteq \mathcal{Q}$, we have that $\mathcal{K}^* \trianglelefteq \mathcal{W}$, and because $Ult(\mathcal{W}, E)$ is well-founded, we get that $Ult(\mathcal{K}^*, E)$ is well-founded. Because $Ult(\mathcal{K}^*, E) \in H$ and $H \vDash$ "$\mathcal{S}^* = \mathcal{S}(\mathcal{R})$", we have that $Ult(\mathcal{K}^*, E) \trianglelefteq \mathcal{S}^*$ (as $Ult(\mathcal{K}^*, E)$ is countably iterable which can be established by an easy hull argument similar to the one used in the proof of Lemma 5.3). By elementarity, $Ult(\mathcal{K}, F) \trianglelefteq \mathcal{S}$, which is a contradiction as $\pi[o(\mathcal{W})]$ is cofinal in $o(\mathcal{S})$. □

## 5.2. Clause 4

In this section, we show that hod pair constructions do not fail because of clause 4 of Definition 1.7.1. We assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

LEMMA 5.5 (Clause 4). *Let $\Gamma$ be a good pointclass and let $F$ be as in Theorem 2.25 for $\Gamma$. Suppose $x$ is such that $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined. Suppose for some $\beta$, the hod mouse construction of $\mathcal{N}^*_x|\delta_x$ reaches $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta)$[2]. Suppose*

1. *$\Sigma_\beta$ is super fullness preserving and has branch condensation,*
2. *$\mathcal{N}_{\beta+1}$ doesn't project across $\delta_\beta$,*
3. *$\mathcal{N}_{\beta+1}$ has at least two Woodin cardinals $> \delta_\beta$,*
4. *if $\beta = 0$ or is a successor ordinal then $\mathcal{N}_{\beta+1} \vDash$ "$\delta_\beta$ is Woodin" and*
5. *if $\beta$ is limit then $(\delta^+_\beta)^{\mathcal{P}_\beta} = (\delta^+_\beta)^{\mathcal{N}_{\beta+1}}$.*

*Then $\mathcal{P}_{\beta+1}$ exists and is a hod premouse.*

PROOF. We only need to show that $\mathcal{P}_{\beta+1}$ satisfies clause 4 of Definition 1.7.1 and we only need to do this for the last block of $\mathcal{P}$ as it follows from our hypothesis that $\mathcal{P}_\beta$ is a hod premouse. Let $\mathcal{P} = \mathcal{P}_{\beta+1}$. Fix then a $\mathcal{P}$-cardinal $\nu \in (\delta_\beta, \delta)$. Let $\Lambda$ be the fragment of $\Sigma_{\beta+1}$ that acts on non-dropping stacks which are based on $\mathcal{P}|\nu$ . We need to show that $\Lambda \restriction \mathcal{P}$ is definable over $\mathcal{P}$. Let then $\delta < \eta$ be the first two Woodin cardinals of $\mathcal{N}_{\beta+1}$ above $\delta_\beta$. Thus $\delta_{\beta+1} = \delta$.

We claim that $\mathcal{T}$ is according to $\Lambda$ iff for every limit $\alpha < lh(\mathcal{T})$ the branch of $\mathcal{T} \restriction \alpha$ is the unique well-founded branch $b$ such that $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$-exists and is $< o(\mathcal{P})$-iterable in $\mathcal{P}$. To see that the equivalence holds, it is enough to show that if $x \in \mathcal{P}$ then

$$Lp^{\Sigma_\beta}(x) \trianglelefteq (Lp^{\Sigma_\beta}(x))^{\mathcal{P}}.$$

[2]We allow $\beta = -1$ in which case $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta) = \emptyset$.

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Fix then $x \in \mathcal{P}$ and let $\mathcal{Q}$ be a sound $\Sigma_\beta$-mouse over $x$ such that $\rho(\mathcal{Q}) = x$. Then by universality of $\mathcal{N}_{\beta+1}$, $\mathcal{Q} \trianglelefteq \mathcal{N}_{\beta+1}$. We then just need to show that $\mathcal{Q}$ is $< o(\mathcal{P})$-iterable in $\mathcal{P}$. Let $((\mathcal{M}_\xi, \mathcal{N}_\xi : \xi \leq \eta), (F_\xi : \xi < \eta))$ be the output of $\mathcal{J}^{\vec{E}, \Sigma_\beta}(x)$-construction of $\mathcal{N}_{\beta+1}|\eta$ in which the extenders used have critical points $> \delta$. Let $\mathcal{N} = \mathcal{N}_\eta$.

*Claim.* $\mathcal{Q} \trianglelefteq \mathcal{N}$.

PROOF. Suppose $\mathcal{Q}$ isn't an initial segment of $\mathcal{N}$. It then follows that $\mathcal{Q}$ wins the comparison with $\mathcal{N}$. It also follows from Lemma 2.12 that in the comparison of $\mathcal{Q}$ with $\mathcal{N}$ only $\mathcal{Q}$ moves. Notice that it follows from Lemma 3.43 that $\mathcal{N}$ has no Woodin cardinals. It then follows from Lemma 3.6 that the comparison of $\mathcal{Q}$ and $\mathcal{N}$ can be carried out in $\mathcal{P}$. This is because no $\mathcal{Q}$-structures used on the $\mathcal{Q}$-side may have overlaps, and hence, any $\mathcal{Q}$-structure used on the $\mathcal{Q}$-side of the iteration is an initial segment of $\mathcal{N}$. It then follows from universality of $\mathcal{N}_{\beta+1}$ that if $\mathcal{U}$ is the tree on $\mathcal{Q}$ coming from the comparison process then $\mathcal{U} \in \mathcal{N}_{\beta+1}$ (without its last branch). Let then $\mathcal{R} = \mathcal{M}_b^{\mathcal{U}}$ where $b$ is the branch of $\mathcal{U}$ given by the strategy of $\mathcal{Q}$. We have that $\mathcal{N} \trianglelefteq \mathcal{R}$. Moreover, either $\mathcal{R} \vDash$ "$\eta$ isn't Woodin" or definable over $\mathcal{R}$ there is a counterexample to the Woodiness of $\eta$. Either way, using $S$-constructions, we get that $\mathcal{N}_{\beta+1} \vDash$ "$\eta$ isn't Woodin". This contradiction implies that in fact $\mathcal{Q} \trianglelefteq \mathcal{N}$. □

It follows from the claim that there is some $\xi$ such that $\mathcal{C}(\mathcal{N}_\xi) = \mathcal{Q}$. To show that $\mathcal{Q}$ is $< o(\mathcal{P})$-iterable in $\mathcal{P}$, it is enough to show that $\mathcal{N}_\xi$ is $< \eta$-iterable in $\mathcal{N}_{\beta+1}|\eta$. To show this, it is enough to show that for every $\mathcal{N}_{\beta+1}$-cardinal $\gamma \in (\delta, \eta)$, $\mathcal{N}_{\beta+1} \vDash$ "$\mathcal{N}_{\beta+1}|\gamma$ is $< \eta$-iterable for non-dropping trees".

This can be done by repeating the argument once again. Let $\Psi$ be the fragment of the strategy of $\mathcal{N}_{\beta+1}$ that acts on non-dropping trees which are based on $\mathcal{N}_{\beta+1}|\gamma$ and are above $\delta$. Let $\mathcal{W} = \mathcal{N}_{\beta+1}|\gamma$. Given a non-dropping tree $\mathcal{T} \in \mathcal{N}_{\beta+1}|\eta$ which is based on $\mathcal{W}$ and is above $\delta$, we have that $\mathcal{T}$ is according to $\Psi$ iff for every limit $\alpha < lh(\mathcal{T})$ if $b$ is the branch of $\mathcal{T} \restriction \alpha$ then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists and letting $((\mathcal{K}_\zeta, \mathcal{S}_\zeta : \zeta \leq \eta), (G_\zeta : \zeta < \eta))$ be the output of $\mathcal{J}^{\vec{E}, \Sigma_\beta}(\mathcal{M}(\mathcal{T}))$-construction of $\mathcal{N}_{\beta+1}|\eta$ in which the extenders used in the construction have critical points $> \gamma$, $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{N}_\eta$. It follows from claim applied to $x = \mathcal{M}(\mathcal{T})$ that the equivalence is indeed true. This finishes our proof of Lemma 5.5. □

The following is a version of Lemma 5.5 for $\Gamma$-full hod pair constructions.

LEMMA 5.6. *Suppose $\Gamma$ is a mouse full pointclass. Suppose $A \subseteq \mathbb{R}$ is such that $w(A) \geq w(\Gamma)$ and suppose $\Gamma^*$ is a good pointclass such that $A_\Gamma \in \Delta_{\Gamma^*}$. Let $F$ be as in Theorem 2.25 for $\Gamma^*$. Suppose $x$ is such that $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined and $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $(A_\Gamma, A)$. Suppose for some $\beta$, the $\Gamma$-hod pair construction of $\mathcal{N}_x^*|\delta_x$ reaches $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta)$[3]. Let $\mathbb{C}_{\beta+1} = ((\mathcal{M}_\xi^{\beta+1}, \mathcal{N}_\xi^{\beta+1} : \xi \leq \delta_x), (F_\xi^{\beta+1} : \xi < \delta_x))$. Suppose*

(1) *$\Sigma_\beta$ is super $\Gamma$-fullness preserving and has branch condensation,*
(2) *$\mathcal{N}_{\delta_{\beta+1}}^{\beta+1}$ doesn't project across $\delta_\beta$,*
(3) *$\mathcal{N}_{\delta_{\beta+1}}^{\beta+1}$ has at least two Woodin cardinals $> \delta_\beta$,*
(4) *if $\beta = 0$ or is a successor then $\mathcal{N}_{\delta_{\beta+1}}^{\beta+1} \vDash$ "$\delta_\beta$ is Woodin" and*

[3]We allow $\beta = -1$ in which case $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta) = \emptyset$.

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(5) *if $\beta$ is limit then $(\delta^+_\beta)^{\mathcal{P}_\beta} = (\delta^+_\beta)^{\mathcal{N}^{\beta+1}_{\delta_{\beta+1}}}$.*

*Then $\mathcal{P}_{\beta+1}$ exists and is a hod premouse.*

The proof is just like the proof of Lemma 5.5.

## 5.3. Fullness preservation

The next lemma establishes fullness preservation of the induced strategy.

LEMMA 5.7 (Fullness preservation). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose for some $\gamma$ such that $\theta_\gamma < \theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\gamma\}$. Let $\Gamma^*$ be a good pointclass such that $\Gamma \subseteq \Delta_{\Gamma^*}$ and let $F$ be as in Theorem 2.25 for $\Gamma^*$. Suppose $x$ is such that $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \Sigma_x, \delta_x)$ is defined and $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $(A_\Gamma, A)$ where $A \subseteq \mathbb{R}$ is such that $w(A) \geq \theta_\gamma$. Suppose for some $\beta$, the hod mouse construction of $\mathcal{N}^*_x|\delta_x$ reaches $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta)$[4] and $w(Code(\Sigma_\beta)) < \theta_{\gamma+1}$. Suppose for every $\alpha < \beta$, $\Sigma_\alpha$ is super fullness preserving. Then $\Sigma_\beta$ is fullness preserving. Moreover, if $\mathcal{N}_\beta$ has a Woodin cardinal above $\delta_\beta$ then $\Sigma_\beta$ is super fullness preserving.*

PROOF. We only verify the first part of the fullness preservation as the second is a trivial generalization of the proof. Suppose $\Sigma_\beta$ isn't fullness preserving. Then there is $(\vec{\mathcal{T}}, \mathcal{R}, \alpha, \mathcal{M}, \eta)$ such that

(1) $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}_\beta, \Sigma_\beta)$,
(2) $\alpha + 1 \leq \lambda^{\mathcal{R}}$ and $\eta \in (\delta^{\mathcal{R}}_\alpha, \delta^{\mathcal{R}}_{\alpha+1}]$ is a strong cutpoint of $\mathcal{R}$,
(3) $\mathcal{M} \trianglelefteq Lp^{\Sigma_{\mathcal{R}(\alpha),\vec{\mathcal{T}}}}(\mathcal{R}|\eta)$ and $\mathcal{M} \not\trianglelefteq \mathcal{R}$.

We have that $\mathcal{M}$ has a unique strategy which is in $\Gamma$. Because of clause 8 of Theorem 2.25, the statement above is absolute between $V$ and $\mathcal{N}^*_x[g]$ where $g \subseteq Coll(\omega, \mathcal{P}_\beta)$ is an $\mathcal{N}^*_x$-generic. Assume then that $(\vec{\mathcal{T}}, \mathcal{R}, \alpha, \mathcal{M}, \eta) \in \mathcal{N}^*_x[g]$ such that the above conditions hold.

Notice that we can apply $\vec{\mathcal{T}}$ to $\mathcal{N}_\beta$. Let $\mathcal{N}^*$ be the last model of $\vec{\mathcal{T}}$ when applied to $\mathcal{N}_\beta$. Let $\mathcal{N}$ be the last model of $\mathcal{J}^{\vec{E},\Sigma_{\mathcal{R}(\alpha)}}(\mathcal{R}|\eta)$-construction of $\mathcal{N}^*$. It follows from Lemma 2.13 that $\mathcal{M}$ must lose the comparison with $\mathcal{N}$. However, because of clause 3 above, $\mathcal{M}$ must win the comparison, contradiction!

Assume now that $\mathcal{N}_\beta$ has a Woodin cardinal above $\delta_\beta$. To finish, we need to show that $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}_\beta, \Sigma_\beta)$ and $\alpha < \lambda^{\mathcal{Q}}$, $U^\Sigma_{\mathcal{Q}(\alpha)}$ and $W^\Sigma_{\mathcal{Q}(\alpha)}$ are term captured by $(\mathcal{Q}[g], \Sigma_{\mathcal{Q}})$ whenever $g \subseteq Coll(\omega, \mathcal{Q}(\alpha))$ is $\mathcal{Q}$-generic. We only show this for $U^\Sigma_{\mathcal{Q}(\alpha)}$ and leave the other case to the reader. Notice that it is enough to show that whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}_\beta, \Sigma_\beta)$ and $\mathcal{N}^*$ is the last model of $\vec{\mathcal{T}}$ when we regard it as a stack on $\mathcal{N}_\beta$ then $u^\Sigma_{\mathcal{Q}(\alpha)}$ is definable over $\mathcal{N}^*$ uniformly in $(\mathcal{Q}, \alpha)$.

Fix then such a triple $(\mathcal{N}^*, \mathcal{Q}, \alpha)$. Let $\phi(p, \sigma, \kappa)$ be the formula expressing the conjunction of the following statements:

(1) $\kappa > \delta^{\mathcal{Q}}_\alpha$ is a $\mathcal{Q}$-cardinal and $p \in Coll(\omega, \kappa)$.
(2) $\sigma \in \mathcal{Q}^{Coll(\omega,\kappa)}$ is a name for a real.
(3) In $\mathcal{N}^*$, $p$ forces that $\sigma = (\dot{x}, \dot{\mathcal{M}})$ such that $\dot{\mathcal{M}}$ is a sound $\Sigma_\alpha$-mouse over $\dot{x}$ such that $\rho(\dot{\mathcal{M}}) = \dot{x}$ and if $\dot{\mathcal{K}}$ is the last model $\mathcal{J}^{\vec{E},\Sigma_\alpha}(\dot{x})$-construction of $\mathcal{N}^*$ then $\dot{\mathcal{M}} \trianglelefteq \dot{\mathcal{K}}$.

[4]We allow $\beta = -1$ in which case $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta) = \emptyset$.

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Let $\tau_\kappa^{\mathcal{Q}} = \{(p,\sigma) : \mathcal{N}^* \vDash \phi[p,\sigma,\kappa]\}$. Then the proof of the fact that $\Sigma_\beta$ is fullness preserving (and also the proof of Lemma 5.5) shows that in fact $\oplus_{\kappa<o(\mathcal{Q})}\tau_\kappa^{\mathcal{Q}} = u_{\mathcal{Q}}^{\Sigma_\beta}$. The claim then follows.

□

The next theorem is a version of Theorem 5.7 for $\Gamma$-hod pair constructions provided there is already a hod pair $(\mathcal{Q},\Lambda)$ such that $\Lambda$ has branch condensation and is $\Gamma$-fullness preserving.

LEMMA 5.8 (Super $\Gamma$-fullness preservation on a tail). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $\Gamma$ is a mouse full pointclass and $(\mathcal{Q},\Lambda)$ is a hod pair such that $\Lambda$ has branch condensation and is $\Gamma$-fullness preserving. Suppose there is an $A \subseteq \mathbb{R}$ such that $w(A) = w(\Gamma)$ and there is a good pointclass $\Gamma^*$ such that $A, Code(\Lambda) \in \Delta_{\widetilde{\Gamma^*}}$. Let $F$ be as in Theorem 2.25 for $\Gamma^*$. Suppose $x$ is such that $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined and $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $(A_\Gamma, A)$ and $Code(\Lambda)$. Then for some $\beta$, the $\Gamma$-hod mouse construction of $\mathcal{N}_x^*|\delta_x$ as done in Definition 3.48 reaches $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta, \gamma_\beta)$[5] such that $(\mathcal{P}_\beta, \Sigma_\beta)$ is a tail of $(\mathcal{Q},\Lambda)$ and $\Sigma_\beta$ is super $\Gamma$-fullness preserving.*

PROOF. The proof is very much like the proof of Lemma 5.7. Notice that $\gamma_\beta$ always exists as $\delta_x$ can be taken to be $\gamma_\beta$. That there is a $\beta$ such that $(\mathcal{P}_\beta, \Sigma_\beta)$ is a tail of $(\mathcal{Q},\Lambda)$ follows from the proof of our comparison theorem, Theorem 2.33. We just need to show that $\Sigma_\beta$ is super $\Gamma$-fullness preserving. If $\beta = \alpha + 1$ then we can do it as in the proof of Theorem 5.7. In the limit case the proof doesn't apply, but we can use our theorem on derived models. Assume then that $\beta$ is limit and let $(\mathcal{P},\Sigma) = (\mathcal{P}_\beta, \Sigma_\beta)$. Then, without loss of generality, we can assume that for every $\alpha < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}$ is super $\Gamma$-fullness preserving, and hence, super $\Gamma(\mathcal{P},\Sigma)$-fullness preserving. Then it follows from Theorem 3.19, that whenever $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ and $\alpha < \lambda^{\mathcal{Q}}$ then $\Sigma_{\mathcal{Q}(\alpha)}$ has super $\Gamma(\mathcal{P},\Sigma)$-fullness preservation. But because $\Sigma$ is both $\Gamma(\mathcal{P},\Sigma)$ and $\Gamma$ fullness preserving, it follows that $\Gamma(\mathcal{P},\Sigma) \trianglelefteq_{mouse} \Gamma$. Hence, $\Sigma$ is super $\Gamma$-fullness preserving. □

### 5.4. The comparison argument revisited

In this subsection, we will prove Theorem 2.28 without the additional hypothesis on the existence of good pointclasses.

THEOREM 5.9. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ has branch condensation and is fullness preserving. Then $Code(\Sigma)$ is Suslin, co-Suslin in $L(\Sigma,\mathbb{R})$.*

PROOF. Suppose not. We work in $L(\Sigma,\mathbb{R})$. Because $L(\Sigma,\mathbb{R}) \vDash AD^+ + V = L(\mathcal{P}(\mathbb{R}))$, using Theorem A.10, we get some $\alpha$ and $(\mathcal{Q},\Lambda)$ such that

(1) $(\mathcal{Q},\Lambda)$ is a hod pair such that $\Lambda$ has branch condensation and $L_\alpha(\Lambda,\mathbb{R}) \vDash$ "$\Lambda$ is fullness preserving",
(2) $L_\alpha(\Lambda,\mathbb{R}) \vDash ZF - Replacement +$ "$Code(\Lambda)$ isn't Suslin, co-Suslin".
(3) $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\Lambda,\mathbb{R}) \subsetneq \underset{\sim}{\Delta}^2_1$.

Let $\Gamma = \mathcal{P}(\mathbb{R}) \cap L_\alpha(\Lambda,\mathbb{R})$. Notice that $\Lambda$ is $\Gamma$-fullness preserving, and because of clause 3 above, there is a good pointclass $\Gamma^*$ such that $\Gamma \subseteq \Delta_{\widetilde{\Gamma^*}}$. By Lemma 5.8, there is $(\mathcal{R},\Psi)$ which is a tail of $(\mathcal{Q},\Lambda)$ such that $\Psi$ is super $\Gamma$-fullness preserving.

[5]We allow $\beta = -1$ in which case $(\mathbb{C}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta, \gamma_\beta) = \emptyset$.

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But because $Code(\Psi) \in \Gamma^*$, it follows that for every $x$, $\mathcal{M}_1^{\#,\Psi}(x)$-exists[6]. Let $\Phi$ be the iteration strategy of $\mathcal{M}_1^{\#,\Psi}$. Notice that $\Phi$ has branch condensation, is commuting and positional. Let $\mathcal{M}$ be the direct limit of all iterates of $\mathcal{M}_1^{\#,\Psi}$ via $\Phi$. Notice that $Code(\Phi) \in \Gamma$ and if $\pi : \mathcal{M}_1^{\#,\Psi} \to \mathcal{M}$ is the iteration map then $\pi \in L_\alpha(\Gamma, \mathbb{R})$. Let $\delta$ be the Woodin cardinal of $\mathcal{M}$ and let $\eta$ be the Woodin cardinal of $\mathcal{M}_1^{\#,\Psi}$. We claim that $Code(\Psi)$ is $\delta$-Suslin.

To see this, consider the tree of attempts to construct a pair $(x, y, \sigma)$ such that $x, y \in \mathbb{R}$, $y$ codes a non-dropping tree $\mathcal{T}$ on $\mathcal{M}_1^{\#,\Psi}$ with last model $\mathcal{R}$ such that if $i = \pi^{\mathcal{T}}$ then

(1) $x$ is generic for the extender algebra of $\mathcal{R}$ at $i(\eta)$,
(2) $\sigma : \mathcal{R} \to \mathcal{M}$ such that $\pi = \sigma \circ i$,
(3) $\mathcal{R}[x] \vDash$ "$x$ codes a member of $\Psi$".

Notice that it follows from clause 2 that $\mathcal{R}$ is a $\Psi$-mouse. Notice also that it follows from Lemma 3.45 that we can interpret $\Psi$ onto the generic extensions of $\mathcal{R}$ and hence, clause 3 above is meaningful. Let $T$ be the tree described above. Clearly $T \in L_\alpha(\Gamma, \mathbb{R})$ and $p[T] = Code(\Psi)$. By the same argument, there is also a tree $S \in L_\alpha(\Gamma, \mathbb{R})$ such that $p[S]$ is the complement of $Code(\Psi)$. Thus, $L_\alpha(\Gamma, \mathbb{R}) \vDash$ "$Code(\Psi)$ is Suslin, co-Suslin". But because $w(Code(\Lambda)) = w(Code(\Psi))$, we get that $L_\alpha(\Gamma, \mathbb{R}) \vDash$ "$Code(\Lambda)$ is Suslin, co-Suslin", contradiction! □

Theorem 5.9 allows us to restate comparison theorem without any extra hypothesis. One can also state the $\Gamma$-fullness preserving version of it but we leave that for the reader.

THEOREM 5.10 (Comparison of hod pairs revisited). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that both $\Sigma$ and $\Lambda$ have branch condensation and are fullness preserving. Then there are $(\mathcal{T}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\mathcal{U}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ such that either*

(1) *$\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}$*
*or*
(2) *$\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Lambda_{\mathcal{S},\mathcal{U}} = \Sigma_{\mathcal{S},\mathcal{T}}$.*

PROOF. Apply Lemma 5.9 and Theorem 2.28 in $L(\Sigma, \Lambda, \mathbb{R})$. □

## 5.5. Branch condensation

In this section we show that hod pair constructed via hod pair constructions inherit strategies with branch condensation.

LEMMA 5.11 (Branch condensation). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose for some $\alpha$ such that $\theta_\alpha < \theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Let $\Gamma^*$ be a good pointclass such that $\Gamma \subseteq \Delta_{\widetilde{\Gamma^*}}$ and let $F$ be as in Theorem 2.25 for $\Gamma^*$. Suppose $x$ is such that $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined and $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $(A_\Gamma, A)$ where $A \subseteq \mathbb{R}$ is such that $w(A) \geq \theta_\alpha$. Suppose for some $\beta$, the hod mouse construction of $\mathcal{N}_x^*|\delta_x$ reaches $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta)$[7] such that $w(Code(\Sigma_\beta)) < \theta_{\alpha+1}$. Suppose for every $\alpha < \beta$, $\Sigma_\alpha$ has branch condensation. Then $\Sigma_\beta$ has branch condensation.*

[6]Fix $F$ as in Theorem 2.25 for $\Gamma^*$ and let $z$ be such that $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Code(\Psi)$. Then if $(\mathcal{J}^{\vec{E},\Psi})^{\mathcal{N}_z^*|\delta_z}$ reaches $\mathcal{M}_1^{\#,\Psi}(x)$.

[7]We allow $\beta = -1$ in which case $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta) = \emptyset$.

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PROOF. Towards a contradiction, assume that $\Sigma_\beta$ doesn't have branch condensation. Like in the proof of Lemma 5.7, this is absolute between $V$ and $\mathcal{N}_x^*[g]$ where $g \subseteq Coll(\omega, \mathcal{P}_\beta)$ is an $\mathcal{N}_x^*$-generic. Thus, we can fix $(\vec{\mathcal{T}}, \mathcal{R}, \vec{\mathcal{U}}, \mathcal{Q}, \pi) \in \mathcal{N}_x^*[g]$ such that

(1) $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}_\beta, \Sigma_\beta)$,
(2) $\vec{\mathcal{U}}$ is a stack on $\mathcal{P}_\beta$ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{U}}}$ exists and $\vec{\mathcal{U}}$ isn't according to $\Sigma_\beta$,
(3) $\pi : \mathcal{Q} \to \mathcal{R}$ such that $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi^{\vec{\mathcal{U}}}$.

We assume $\beta$ is a limit ordinal as the other case is easier and is contained in the proof that follows. Let then $(\mathcal{M}_\alpha, \mathcal{M}_\alpha^*, \vec{\mathcal{U}}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \eta)$ be the essential components of $\vec{\mathcal{U}}$. Without losing generality, we can impose the following minimality conditions on $\vec{\mathcal{U}}$:

(1) For every $\alpha < \eta$, $\vec{\mathcal{U}} \restriction \alpha$ is according to $\Sigma$.
(2) For every $\alpha < \eta$ and $\gamma < \lambda^{\mathcal{M}_\alpha}$, there is no stack on $\mathcal{M}_\alpha(\gamma)$ which witnesses that branch condensation fails for $\Sigma$.
(3) $\mathcal{M}_\eta^*$ is the least hod initial segment $\mathcal{W}$ of $\mathcal{M}_\eta$ such that there is a stack on $\mathcal{W}$ witnessing the failure of branch condensation for $\Sigma$.
(4) Only the last normal component of $\vec{\mathcal{U}}_\eta$ isn't according to $\Sigma$ and this last component has length $\zeta + 1$ for some limit ordinal $\zeta$.

Notice that $\vec{\mathcal{T}}$ can be applied to $\mathcal{N}_\beta$. Let $\mathcal{N}$ be the result of applying $\vec{\mathcal{T}}$ to $\mathcal{N}_\beta$. We can apply $\vec{\mathcal{U}}$ to $\mathcal{N}_\beta$ in the following way. Let $E$ be the extender from $\pi^{\vec{\mathcal{U}}} : \mathcal{P}_\beta \to \mathcal{Q}$. Then we let $\mathcal{M} = Ult(\mathcal{N}_\beta, E)$. Notice that $\mathcal{M}$ is well founded because $\pi$ can be extended to $\pi^* : \mathcal{M} \to \mathcal{N}$ by $\pi^*([a, f]) = \pi^{\vec{\mathcal{T}}}(f)(\pi(a))$. Let $\mathcal{P} = \mathcal{P}_\beta$, $\Sigma = \Sigma_\beta$ and let $\mathcal{K}$ be the result of applying $\vec{\mathcal{U}} \restriction \eta$ to $\mathcal{N}_\beta$.

Let $b = \Sigma(\vec{\mathcal{U}} \restriction \eta^\frown \vec{\mathcal{U}}_\eta^*)$ where $\vec{\mathcal{U}}_\eta^*$ is the same as $\vec{\mathcal{U}}_\eta$ without its last branch (notice that it follows from Lemma 2.13 that $\mathcal{M}$ is universal and hence, if $\mathcal{U}$ is the last normal component of $\vec{\mathcal{U}}_\eta$ then $\mathcal{Q}(\mathcal{M}(\mathcal{U}^-))$ doesn't exists). Let $\xi + 1 = \lambda^{\mathcal{M}_\eta^*}$. Notice that the proof of Lemma 5.7 implies that $\mathcal{M}_\eta^*$ is full with respect to $\Sigma_{\mathcal{M}_\eta^*(\xi), \vec{\mathcal{U}} \restriction \eta}$-mice and $\mathcal{Q}(\pi^{\vec{\mathcal{U}}_\eta}(\xi + 1))$ is full with respect to $\Sigma_{\mathcal{Q}(\pi^{\vec{\mathcal{U}}_\eta}(\xi)), \vec{\mathcal{U}}^{**}}$-mice where $\vec{\mathcal{U}}^{**} = (\vec{\mathcal{U}}^*)^\frown \mathcal{M}_b^{\vec{\mathcal{U}}^*}$. This means that $\mathcal{Q}(\pi^{\vec{\mathcal{U}}_\eta}(\xi + 1)) = \mathcal{M}_b^{\vec{\mathcal{U}}_\eta^*}$. Let $i = \pi^{\vec{\mathcal{U}}_\eta}$ and $j = \pi_b^{\vec{\mathcal{U}} \restriction \eta^\frown \vec{\mathcal{U}}_\eta^*}$. Let $\mathcal{W} = \mathcal{M}_b^{\vec{\mathcal{U}} \restriction \eta^\frown \vec{\mathcal{U}}_\eta^*}$. Then $i : \mathcal{M}_\eta \to \mathcal{Q}$ and $j : \mathcal{M}_\eta \to \mathcal{W}$. We also have that $i(\xi + 1) = j(\xi + 1) =_{def} \gamma$ and $\mathcal{Q}(\gamma) = \mathcal{W}(\gamma)$.

Let $\mathcal{K}^*$ be the last model of $\mathcal{J}^{\vec{E}, \Sigma_{\mathcal{M}_\eta^*(\xi), \vec{\mathcal{U}} \restriction \eta}}(\mathcal{M}_\eta^*)$-construction of $\mathcal{K}$ and let $\mathcal{K}^{**} = S(\mathcal{K}^*)$ be the stack over $\mathcal{K}^*$ (see Section 5.1). Notice that the extenders derived from $i$ and $j$ can be applied to $\mathcal{K}^{**}$. Let then $\mathcal{Q}^* = Ult(\mathcal{K}^{**}, E_i)$ and $\mathcal{W}^* = Ult(\mathcal{K}^{**}, E_j)$ where $E_i$ is $(\delta^{\mathcal{M}_\eta}, \delta^{\mathcal{Q}})$-extender derived from $i$ and $E_j$ is $(\delta^{\mathcal{M}_\eta}, \delta^{\mathcal{W}})$-extender derived from $j$. Then it follows from Lemma 5.3 that

$$\mathcal{Q}^* = S(\mathcal{Q}^* | \delta_x) \text{ and } \mathcal{W}^* = S(\mathcal{W}^* | \delta_x)$$

We can now compare $\mathcal{Q}^*$ and $\mathcal{W}^*$. This produces $\mathcal{S}$ and iteration embeddings $m : \mathcal{Q}^* \to \mathcal{S}$ and $n : \mathcal{W}^* \to \mathcal{S}$. We abuse our notation and let $i$ and $j$ act on $\mathcal{K}^{**}$. Notice that $i \circ m(\delta_x) = j \circ n(\delta_x)$. It follows from Lemma 5.2 that the set

$$\{\beta < o(\mathcal{K}^{**}) : m(i(\beta)) = n(j(\beta))\}$$

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contains an $\omega$-club. Let $C$ be such a club. Let $\delta = \delta^{\mathcal{M}^*_\eta}$. Then we have that

$$s =_{def} \delta \cap Hull_1^{\mathcal{K}^{**}}(\mathcal{M}^*_\eta(\xi) \cup C)$$

is cofinal in $\delta$, and because $m \restriction i(\delta) = n \restriction j(\delta)$,

$$m \circ i[s] = n \circ j[s].$$

This means that

$$i[s] = j[s].$$

Hence, $rng(i) \cap rng(j) \cap i(\delta)$ is cofinal in $i(\delta)$ and $rng(i) \cap rng(j) \cap j(\delta)$ is cofinal in $j(\delta)$. But because $i(\delta) = j(\delta)$, using Lemma 1.13, we get that $b$ is the last branch of $\vec{\mathcal{U}}_\eta$, contradictions! □

The following is an easy corollary of Theorem 2.42 and Lemma 5.11.

COROLLARY 5.12. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose for some $\alpha$ such that $\theta_\alpha < \theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Let $\Gamma^*$ be a good pointclass such that $\Gamma \subseteq \Delta_{\widetilde{\Gamma^*}}$ and let $F$ be as in Theorem 2.25 for $\Gamma^*$. Suppose $x$ is such that $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined and $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $(A_\Gamma, A)$ where $A \subseteq \mathbb{R}$ is such that $w(A) \geq \theta_\alpha$. Suppose for some $\beta$, the hod mouse construction of $\mathcal{N}^*_x|\delta_x$ reaches $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta)$[8] such that $w(Code(\Sigma_\beta)) < \theta_{\alpha+1}$. Suppose for every $\alpha < \beta$, $\Sigma_\alpha$ has branch condensation. Then $\Sigma_\beta$ is positional and commuting.*

We then have that $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are two hod pairs that are constructed via hod pair construction of some $\mathcal{N}^*_x$ then $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ can be compared. Our next block of lemmas go towards showing that the hod pair constructions produce strategies that are strongly guided by some $\vec{B}$. We start by showing that there is always a "nice" pointclass associated to hod pairs $(\mathcal{P},\Sigma)$ such that $\Sigma$ is fullness preserving and has branch condensation. If $\lambda^{\mathcal{P}}$ is limit then the "nice" pointclass we have in mind is $\Gamma(\mathcal{P},\Sigma)$. In the next subsection our main goal is to describe this 'nice" pointclass in the case $\lambda^{\mathcal{P}}$ is a successor.

## 5.6. $\Gamma(\mathcal{P},\Sigma)$ when $\lambda^{\mathcal{P}}$ is successor

We assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ throughout this section. Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor ordinal and $\Sigma$ has branch condensation and is fullness preserving. Because hod pair constructions produce super fullness preserving strategies, we assume that $\Sigma$ is in fact super fullness preserving. It then follows that for any tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$, $Code(Mice_{\Lambda_{\mathcal{Q}^-}}) \leq_w Code(\Sigma)$. By Lemma 5.9, $Code(\Sigma)$ is Suslin, co-Suslin and, therefore, there is a scaled pointclass closed under continuous images and preimages and $\exists^{\mathbb{R}}$ which contains $Mice_{\Sigma_{\mathcal{P}^-}}$. We let $\Gamma^*_\Sigma$ be the least such pointclass. The next lemma shows that $\Gamma^*_\Sigma$ is the boldface version of some good pointclass and more.

LEMMA 5.13. *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor and $\Sigma$ has branch condensation and is super fullness preserving. There is then a tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$ such that*

$$\Gamma^*_\Lambda = (\underset{\sim}{\Sigma}^2_1(Code(\Lambda_{\mathcal{Q}^-})))^{L(Mice_{\Lambda_{\mathcal{Q}^-}},\mathbb{R})},$$

[8]We allow $\beta = -1$ in which case $(\mathcal{N}_\beta, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta) = \emptyset$.

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*and the complement of* $Code(Mice_{\Lambda_{\mathcal{Q}^-}})$ *is not in* $\Gamma^*_\Lambda$.

PROOF. Towards a contradiction, suppose our claim is false. Notice that because $\Sigma$ is super fullness preserving, $Code(Mice_{\Sigma_{\mathcal{P}^-}}) < w(Code(\Sigma)$. Thus, our claim fails in $L(\Sigma,\mathbb{R})$ implying that whenever $(\mathcal{Q},\Lambda)$ is a tail of $(\mathcal{P},\Sigma)$, we must have that $\Gamma^*_\Lambda \subseteq (\underset{\sim}{\Sigma}^2_1(Code(\Lambda_{\mathcal{Q}^-})))^{L(Mice_{\Lambda_{\mathcal{Q}^-}},\mathbb{R})}$. There is then some set of reals $A \in L(\Sigma,\mathbb{R})$ such that $Code(\Sigma) <_w A$, $A$ is Suslin, co-Suslin in $L(\Sigma,\mathbb{R})$ and our claim is false in $L_\alpha(A,\mathbb{R})$ where $\alpha$ is the least $A$-admissible ordinal, i.e., whenever $(\mathcal{Q},\Lambda)$ is a tail of $(\mathcal{P},\Sigma)$, we must have that $\Gamma^*_\Lambda \subseteq (\underset{\sim}{\Sigma}^2_1(Code(\Lambda_{\mathcal{Q}^-})))^{L_\alpha(Mice_{\Lambda_{\mathcal{Q}^-}},\mathbb{R})}$. It follows that whenever $(\mathcal{R},\Psi)$ is a tail of $(\mathcal{P},\Sigma)$, $Mice_{\Psi_{\mathcal{R}^-}} <_w A$. We can then let $B \subseteq \mathbb{R}$ be a set codding the structure

$$(L_\alpha(A,\mathbb{R}),(\mathcal{P},\Sigma),\in)$$

along with its first order theory.

Next we fix a good pointclass $\Gamma$ such that $Code(\Sigma), B \in \Delta_{\underset{\sim}{\Gamma}}$ and $\Gamma^*_\Sigma \subseteq \Delta_{\underset{\sim}{\Gamma}}$. Let $F$ be as in Theorem 2.25 and let $x$ be such that $(\mathcal{N}^*_x,\delta_x,\Sigma_x)$ Suslin, co-Suslin captures $(C_\Gamma, C)$, $B$ and $Code(\Sigma)$ where $C$ is a set of reals such that $w(C) = w(\Gamma)$. Let $(\mathcal{N}_\beta,\mathcal{P}_\beta,\Sigma_\beta,\delta_\beta : \beta < \varsigma)$ be the output of the hod pair construction of $\mathcal{N}^*_x|\delta_x$. By the proof of Theorem 2.28, there is $\beta$ such that $(\mathcal{P}_\beta,\Sigma_\beta)$ is a tail of $(\mathcal{P},\Sigma)$.

Notice that if $\gamma \in (\delta_\beta,\delta_x)$ and if $\mathcal{W}$ is the last model of $\mathcal{J}^{\vec{E},\Sigma_{\beta-1}}$-construction of $\mathcal{N}_\beta$ where the extenders used in the construction have critical points $> \gamma$ then $\mathcal{W}$ has a Woodin cardinal. This is because if $\mathcal{R} \in pI(\mathcal{P},\Sigma) \cap \mathcal{N}^*_x|(\delta^+_{\beta-1})^{\mathcal{N}^*_x}$ is such that $\mathcal{R}(\beta-1) = \mathcal{P}_{\beta-1}$ then $\mathcal{R}$ iterates to some initial segment of $\mathcal{W}$ and the iteration is above $\delta_{\beta-1}$. Using then $\mathcal{S}$-constructions, namely Theorem 3.43, we get

$$\mathcal{N}_\beta \vDash \text{“the least strong cardinal is a limit of Woodin cardinals”}.$$

(This is because for a club of $\eta < \delta_x$, $Lp^{\Sigma_{\beta-1}}(\mathcal{N}^*_x|\eta) \vDash$ “$\eta$ is Woodin”.) Moreover, it follows from Lemma 2.13 that if $\eta$ is a strong cutpoint of $\mathcal{N}_\beta$ then $Lp^{\Sigma_\beta}(\mathcal{N}_\beta|\eta) \trianglelefteq \mathcal{N}_\beta$.

We let $(\mathcal{Q},\Lambda) = (\mathcal{P}_\beta,\Sigma_\beta)$ and $\mathcal{N} = \mathcal{N}_\beta$. Let $\kappa$ be the least strong cardinal of $\mathcal{N}$. Let $g \subseteq Coll(\omega,<\kappa)$ be $\mathcal{N}^*_x$-generic. Let $\bar{\mathbb{R}} = \mathbb{R} \cap \mathcal{N}^*_x[g]$, $\bar{B} = \mathcal{N}^*_x[g] \cap B$, $\bar{\Sigma} = \Sigma \restriction HC^{\mathcal{N}^*_x[g]}$, $\bar{\Lambda} = \Lambda \restriction HC^{\mathcal{N}^*_x[g]}$, $\bar{A} = A \cap \mathcal{N}^*_x[g]$ and $\bar{\alpha}$ be such that $(L_{\bar{\alpha}}(\bar{A},\bar{\mathbb{R}}),(\mathcal{P},\bar{\Sigma}),\bar{\Gamma}_{\bar{\Sigma}},\in)$ is the structure coded by $\bar{B}$. Then

$$(L_{\bar{\alpha}}(\bar{A},\bar{\mathbb{R}}),(\mathcal{P},\bar{\Sigma}),\in) \prec_1 (L_\alpha(A,\mathbb{R}),(\mathcal{P},\Sigma),\in).$$

Let $M$ be the derived model of $\mathcal{N}$ at $\kappa$ in $\mathcal{N}^*_x[g]$. Let $C = Mice_{\Lambda_{\mathcal{Q}^-}}$ and let $\bar{C} = C \cap HC^{\mathcal{N}^*_x[g]}$. Then we claim that $\bar{C} \in M$. The proof is like the arguments used in the proof of Lemma 5.5 and Lemma 5.7.

*Claim.* $\bar{C} \in M$. In fact, $M \vDash \bar{C} = Mice_{\bar{\Lambda}_{\mathcal{Q}^-}}$.

PROOF. First fix $(y,\mathcal{M}) \in \bar{C}$. We need to show that $M \vDash$ “$\mathcal{M}$ is a sound $\bar{\Lambda}_{\mathcal{Q}^-}$-mouse over $y$ projecting to $y$”. For this we need to describe a strategy for $\mathcal{M}$ in $M$. Let then $\eta$ be a Woodin cardinal of $\mathcal{N}$ such that $y \in \mathcal{N}[g \cap Coll(\omega,\eta)]$. Let $\nu$ be the least Woodin of $\mathcal{N}$ above $\eta$. Let $\mathcal{N}^*$ be the last model of $\mathcal{J}^{\vec{E},\Lambda_{\mathcal{Q}^-}}(y)$-construction of $\mathcal{N}|\nu[g \cap Coll(\omega,\eta)]$. Then as in the proof of Lemma 5.5 and Lemma 5.7, $\mathcal{M} \trianglelefteq \mathcal{N}^*$.

Let $((\mathcal{M}_\xi,\mathcal{N}_\xi : \xi \leq \nu),(F_\xi : \xi < \nu))$ be the output of $\mathcal{J}^{\vec{E},\Lambda_{\mathcal{Q}^-}}(y)$-construction of $\mathcal{N}|\nu$. We have just shown that there is $\xi$ such that $\mathcal{C}(\mathcal{N}_\xi) = \mathcal{M}$. It follows then

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that to show that $\mathcal{M}$ is iterable in $M$, it is enough to show that $\mathcal{N}_\xi$ is $\omega_1^M+1$-iterable in $M$ for non-dropping trees. For this, it is enough to show that if $\gamma\in[\eta,\nu]$ is a successor cardinal of $\mathcal{N}_x^*$ and $\Psi\in\mathcal{N}_x^*[g]$ is the fragment of the strategy of $\mathcal{N}_x^*|\gamma$ that acts on stacks which never drop and are above $\eta$ then $\Psi\restriction HC^M\in M$. This follows from clause 8 of Theorem 2.25. This completes our proof that if $(y,\mathcal{M})\in\bar{C}$ then $\mathcal{M}$ has an iteration strategy in $M$.

On the other hand, if $(y,\mathcal{M})$ is such that $M\vDash$ "$\mathcal{M}$ is a sound $\bar{\Lambda}_{\mathcal{Q}^-}$-mouse over $y$ projecting to $y$" then in fact, $\mathcal{M}$ is iterable. This can be seen by repeating the above proof. If $\eta$ is a Woodin cardinal of $\mathcal{N}$ such that $\mathcal{M}\in\mathcal{N}[g\cap Coll(\omega,\eta)]$ and $\nu$ is the least Woodin cardinal of $\mathcal{N}_\beta$ above $\eta$, then $\mathcal{M}$ can be reached by the $\mathcal{J}^{\vec{E},\Lambda_{\mathcal{Q}^-}}$-construction of $\mathcal{N}|\nu[g\cap Coll(\omega,\eta)]$. We then get that $\bar{C}\in M$. □

Notice now that $M\vDash$ "$Code(\bar{C})\in\Sigma^2_1(Code(\bar{\Lambda}_{\mathcal{Q}^-}))$" and hence,

$$(\Gamma^*_\Lambda)^{L_{\bar{\alpha}}(\bar{A},\mathbb{R})}\subseteq(\underset{\sim}{\Sigma}^2_1(Code(\bar{\Lambda}_{\mathcal{Q}^-})))^M.$$

To finish, we just need to show that $M\vDash$ "$\bar{C}^c$ isn't Suslin" as it will imply that $M\vDash$ "$Code(\bar{C})$ is a universal $\Sigma^2_1(Code(\bar{\Lambda}_{\mathcal{Q}^-}))$ set".

To see this, suppose $\bar{C}$ is Suslin co-Suslin in $M$. Then $\bar{C}\in(\underset{\sim}{\Delta}^2_1(Code(\bar{\Lambda}_{\mathcal{Q}^-})))^M$ implying that there is a sjs $\vec{A}=(A_i:i<\omega)\in M$ such that $A_0=\bar{C}$ and $A_1=\bar{C}^c$. Let then $\eta$ be a Woodin cardinal of $\mathcal{N}$ such that there are $\kappa$-complementing trees $T,S\in\mathcal{N}[g\cap Coll(\omega,\eta)]$ such that $(p[T])^{\mathcal{N}[g]}=\vec{A}$ and $(p[S])^{\mathcal{N}[g]}=(\vec{A})^c$. Let $\nu$ be the least Woodin cardinal of $\mathcal{N}$ above $\eta$. Let $T^*,S^*\in\mathcal{N}|(\nu^+)^{\mathcal{N}}[g\cap Coll(\omega,\eta)]$ be the $\nu$-complementing subtrees of $T$ and $S$ respectively. Then let $X\prec\mathcal{N}|(\nu^+)^{\mathcal{N}}[g\cap Coll(\omega,\eta)]$ be such that $X\in\mathcal{N}[g\cap Coll(\omega,\eta)]$, $\eta+1\subseteq X$, $X\cap\nu\in\nu$ and $T^*,S^*\in X$. Let $\pi:\mathcal{H}\to\mathcal{N}|(\nu^+)^{\mathcal{N}}[g\cap Coll(\omega,\eta)]$ be the collpase of $X$. Let $\nu^*=\pi^{-1}(\nu)$. Then $\mathcal{H}\vDash$ "$\nu^*$ is Woodin". Moreover, by Lemma 2.21, $Lp^{\Lambda_{\mathcal{Q}^-}}(\mathcal{H}|\nu^*)\in\mathcal{H}$. Hence, by Lemma 3.43, $\mathcal{N}\vDash$ "$\nu^*$ is Woodin". However, $\nu^*\in(\eta,\nu)$ contradicting the fact that $\nu$ is the least Woodin of $\mathcal{N}$ above $\eta$. This contradiction completes our proof. □

Suppose now $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ is super fullness preserving and has branch condensation and

$$\Gamma^*_\Sigma=(\Sigma^2_1(Code(\Sigma_{\mathcal{P}^-})))^{L(Mice_{\Sigma_{\mathcal{P}^-}},\mathbb{R})}.$$

We then let $\Gamma_\Sigma=\Gamma^*_\Sigma$. Notice that $\Gamma_\Sigma$ is a lightface good pointclass. Also $Mice_{\Sigma_{\mathcal{P}^-}}$ belongs to $\Gamma_\Sigma$ and is a universal $\Gamma_\Sigma$ set. We let

$$\Gamma(\mathcal{P},\Sigma)=\{A:\text{for cone of }x\in\mathbb{R},\ A\cap C_{\Gamma_\Sigma}(x)\in C_{\Gamma_\Sigma}(x)\}=Env(\Gamma_\Sigma).$$

Notice that if $(\mathcal{Q},\Lambda)$ is a tail of $(\mathcal{P},\Sigma)$ then $\Gamma(\mathcal{Q},\Lambda)=\Gamma(\mathcal{P},\Sigma)$.

LEMMA 5.14. *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ is super fullness preserving and has branch condensation and $\Gamma_\Sigma$ is defined. Then for any real $x$ coding $\mathcal{P}^-$,*

$$C_{\Gamma_\Sigma}(x)=Lp^{\Sigma_{\mathcal{P}^-}}(x).$$

PROOF. First we show that the claim is true for a cone of $x$. We clearly have that for every $x$, $Lp^{\Sigma_{\mathcal{P}^-}}(x)\subseteq C_{\Gamma_\Sigma}(x)$. Let $M=Code(Mice_{\Sigma_{\mathcal{P}^-}})$ and $N=\{(x,y):y\in C_{\Gamma_\Sigma}(x)\}$. Towards a contradiction, suppose that for a cone of $x$, $C_{\Gamma_\Sigma}(x)\not\subseteq Lp^{\Sigma_{\mathcal{P}^-}}(x)$. We claim that this implies that $M<_w N$. To see this, suppose that $N\leq_w M$. Let $y$ be a base of this cone and fix a real $z$ coding a Wadge reduction of $N$ to $M$. Let $(\mathcal{T},\mathcal{Q})\in I(\mathcal{P},\Sigma)$ be such that $\mathcal{T}$ is above $\mathcal{P}^-$ and $(y,z)$ is generic for the extender algebra of $\mathcal{Q}$ at $\delta^{\mathcal{Q}}$. We have that for comeager

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many generic reals $w$ coding $(\mathcal{Q}|\delta^{\mathcal{Q}}, z, y)$, $C_{\Gamma_\Sigma}(w) = C_{\Gamma_\Sigma}(\mathcal{Q}|\delta^{\mathcal{Q}}, z, y)[w]$. Fix one such $w$.

Then by super fullness preservation of $\Sigma$, we get that

(1) $M \cap HC^{\mathcal{Q}[w]} \in \mathcal{Q}[w]$.

This also means that $N \cap \mathcal{Q}[w] \in \mathcal{Q}[w]$ and in particular, $C_{\Gamma_\Sigma}(w) \in \mathcal{Q}[w]$. But we have that

$$C_{\Gamma_\Sigma}(w) \not\subseteq Lp^{\Sigma_{\mathcal{P}^-}}(w).$$

To get a contradiction, we have to show that

$$\mathbb{R}^{\mathcal{Q}[w]} \subseteq Lp^{\Sigma_{\mathcal{P}^-}}(w).$$

This follows from Lemma 3.42: $\mathcal{Q}[w]$ can be reorganized as $\Sigma_{\mathcal{Q}^-}$-mouse over $w$ giving what we want. This contradiction shows that for a cone of $x$, $C_{\Gamma_\Sigma}(x) \subseteq Lp^{\Sigma_{\mathcal{P}^-}}(x)$, and hence,

$$C_{\Gamma_\Sigma}(x) = Lp^{\Sigma_{\mathcal{P}^-}}(x).$$

We now need to show that $C_{\Gamma_\Sigma}(x) = Lp^{\Sigma_{\mathcal{P}^-}}(x)$ for all $x$ coding $\mathcal{P}^-$. Fix, then, such a real $x$. Let $y$ be a real such that whenever $y \leq_T z$, $C_{\Gamma_\Sigma}(z) = Lp^{\Sigma_{\mathcal{P}^-}}(z)$. Fix a good pointclass $\Gamma$ such that $Code(\Sigma) \in \Delta_{\widetilde{\Gamma}}$. Let $F$ be as in Theorem 2.25 and let $z$ be such that $F(z) = (\mathcal{N}_z^*, \mathcal{M}_z, \Sigma_z, \delta_z)$ is define and $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Code(\Sigma)$ and $z$ codes $x, y$. Let $\mathcal{N}$ be the last model of $\mathcal{J}^{\vec{E},\Sigma_{\mathcal{P}^-}}(\mathcal{P}^-)$ construction of $\mathcal{N}_z^*|\delta_z$. Let $\delta < \nu$ be the first two Woodin cardinals of $\mathcal{N}$ above $\mathcal{P}^-$. (That there are such Woodin cardinals follows from the proof of Lemma 5.13 especially from the argument that the least strong of $\mathcal{N}_\beta$ is a limit of Woodin cardinals). Then $(x, y)$ is generic over $\mathcal{N}$ for the extender algebra at $\delta$ and at $\nu$. Let $\mathcal{M}$ be the last model of $\mathcal{J}^{\vec{E},\Sigma_{\mathcal{P}^-}}(x)$-construction of $\mathcal{N}[x]$. Then $\nu$ is Woodin in $\mathcal{M}$ and $y$ is generic over $\mathcal{M}$ for the extender algebra at $\nu$. Because of the choice of $y$, Lemma 3.42 and universality of $\mathcal{M}$, we have that $C_{\Gamma_\Sigma}(\mathcal{M}|\nu) \subseteq \mathcal{M}$. But $C_{\Gamma_\Sigma}(x) \subseteq C_{\Gamma_\Sigma}(\mathcal{M}|\nu)$, and hence, $C_{\Gamma_\Sigma}(x) \subseteq \mathcal{M}|\nu$. This implies that $C_{\Gamma_\Sigma}(x) \subseteq Lp^{\Sigma_{\mathcal{P}^-}}(x)$, and hence, $C_{\Gamma_\Sigma}(x) = Lp^{\Sigma_{\mathcal{P}^-}}(x)$. □

It follows that the complement of $Code(Mice_{\Sigma_{\mathcal{P}^-}})$ doesn't carry a scale in $\Gamma(\mathcal{P}, \Sigma)$ and also that the complement of $Code(Mice_{\Sigma_{\mathcal{P}^-}})$ carries a scale all of whose norms are in $\Gamma(\mathcal{P}, \Sigma)$. The following is a consequence of the aforementioned fact.

LEMMA 5.15. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ is super fullness preserving and has branch condensation and $\Gamma_\Sigma$ is defined. Then $Code(\Sigma) \notin \Gamma(\mathcal{P}, \Sigma)$.*

PROOF. Suppose that $Code(\Sigma) \in \Gamma(\mathcal{P}, \Sigma)$. Then $Code(\Sigma) \cap C_{\Gamma_\Sigma}(x) \in C_{\Gamma_\Sigma}(x)$ for a cone of $x$. Let $y$ be a base of this cone and let $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ be such that $\mathcal{T}$ is above $\mathcal{P}^-$ and $(y, \mathcal{P})$ is generic over $\mathcal{Q}$ for the extender algebra at $\delta^{\mathcal{Q}}$. Then we must have that $Code(\Sigma) \cap \mathcal{Q}[y] \in \mathcal{Q}[y]$. This then means that $\mathcal{Q}[y]$ can compute $\mathcal{T}$ and thus, $\delta^{\mathcal{Q}}$ is countable in $\mathcal{Q}[y]$, contradiction! □

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LEMMA 5.16. *Suppose now $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ is super fullness preserving and has branch condensation and $\Gamma_\Sigma$ is defined. Then there is a tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$ such that*

$$\mathcal{P}(\mathbb{R})\cap L(\Gamma(\mathcal{Q},\Lambda),\mathbb{R})=\Gamma(\mathcal{Q},\Lambda).$$

PROOF. Towards a contradiction suppose that there is no such tail. Notice that for every tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$, $Code(\Sigma)\notin\Gamma(\mathcal{Q},\Lambda)$ as otherwise $Code(\Lambda)\in\Gamma(\mathcal{Q},\Lambda)$. This implies that if $A_{\mathcal{Q},\Lambda}\in L(\Gamma(\mathcal{Q},\Lambda),\mathbb{R})$ is of least Wadge rank such that $A_{\mathcal{Q},\Lambda}\notin\Gamma(\mathcal{Q},\Lambda)$ then $A_{\mathcal{Q},\Lambda}\leq_w Code(\Sigma)$. Let $\alpha_{\mathcal{Q},\Lambda}$ be the least ordinal $\gamma$ such that $A_{\mathcal{Q},\Lambda}$ is definable over $L_\gamma(\Gamma(\mathcal{Q},\Lambda),\mathbb{R})$. Then the structure

$$(L_{\alpha_{\mathcal{Q},\Lambda}}(\Gamma(\mathcal{Q},\Lambda),\mathbb{R}),\in)$$

can be coded by a set, say $B_{\mathcal{Q},\Lambda}\subseteq\mathbb{R}$, such that $B_{\mathcal{Q},\Lambda}$ is projective in $Code(\Sigma)$. Let $\gamma=\sup_{(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)}\alpha_{\mathcal{Q},\Sigma_{\mathcal{Q}}}$. Then $\gamma<\delta^2_1(Code(\Sigma))$. Let $\beta>\gamma$ be the least ordinal $\xi>\gamma$ such that

$$L_\xi(Code(\Sigma),\mathbb{R})\vDash\text{“}ZF-Replacement\text{”}.$$

We can then fix $A\subseteq\mathbb{R}$ which codes $L_\beta(Code(\Sigma),\mathbb{R})$. Notice that the statement that “for any tail $(\mathcal{Q},\Lambda)$ of $(\mathcal{P},\Sigma)$, $\Gamma(\mathcal{Q},\Lambda)\cap L(\Gamma(\mathcal{Q},\Lambda),\mathbb{R})\neq\Gamma(\mathcal{Q},\Lambda)$” is a projective fact about $A$ and $Code(\Sigma)$. Let then $B\subseteq\mathbb{R}$ code the theory of $(HC,A,\Sigma,\in)$ with real parameters. We can then fix a good pointclass $\Gamma$ such that $\Delta_{\underset{\sim}{\Gamma}}$ contains all sets projective in $Code(\Sigma)$ and $B\in\Delta_{\underset{\sim}{\Gamma}}$.

Let now $F$ be as in Theorem 2.25 and let $x$ be such that $F(x)$ is defined and $(\mathcal{N}^*_x,\delta_x,\Sigma_x)$ Suslin, co-Suslin captures $B$. Notice that if $\eta<\delta_x$, $g\subseteq Coll(\omega,\eta)$ is $\mathcal{N}^*_x$-generic, $A^*=A\cap\mathcal{N}^*_x[g]$, $\Sigma^*=\Sigma\restriction HC^{\mathcal{N}^*_x[g]}$, and $B^*=B\cap\mathcal{N}^*_x[g]$ then

$$(HC^{\mathcal{N}^*_x[g]},A^*,\Sigma^*,\in)\prec(HC,A,\Sigma,\in).$$

Next we have that there is some $\gamma$ such that the $\gamma$th model of hod pair construction of $\mathcal{N}^*_x$ is a tail of $(\mathcal{P},\Sigma)$. We let $(\mathcal{N},\mathcal{Q},\Lambda)=(\mathcal{N}_\gamma,\mathcal{P}_\gamma,\Sigma_\gamma)$. Then, as in Lemma 5.13, the least strong cardinal of $\mathcal{N}$ is a limit of Woodin cardinals. Let $\kappa$ be the least strong cardinal of $\mathcal{N}$ and let $g\subseteq Coll(\omega,<\kappa)$ be $\mathcal{N}^*_x$-generic. Let $A^*=A\cap\mathcal{N}^*_x[g]$, $\Sigma^*=\Sigma\restriction HC^{\mathcal{N}^*_x[g]}$, $\Lambda^*=\Lambda\restriction HC^{\mathcal{N}^*_x[g]}$, $B^*=B\cap\mathcal{N}^*_x[g]$, and $\mathbb{R}^*=\mathbb{R}\cap\mathcal{N}^*_x[g]$. Let

$$\Gamma^*=\{E\subseteq\mathbb{R}^*:L(A^*,\mathbb{R}^*)\vDash\text{“ for cone of }y,\ E\cap C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(y)\in C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(y)\text{”}\}.$$

Then we have that there is some $\alpha$ such that there is a subset of $\mathbb{R}^*$ definable over $L_\alpha(\Gamma^*,\mathbb{R}^*)$ which is not in $\Gamma^*$. Let $C$ be this set. We let $M$ be the derived model of $\mathcal{N}$ at $\kappa$ in $\mathcal{N}^*_x[g]$. We have that by the proof of Lemma 5.13 and the choice of $A$, for any $y\in\mathbb{R}^*$

$$(Lp^{\Lambda^*_{\mathcal{Q}^-}}(y))^{L(A^*,\mathbb{R}^*)}=(Lp^{\Lambda^*_{\mathcal{Q}^-}}(y))^{\mathcal{N}^*_x[g]}=(Lp^{\Lambda^*_{\mathcal{Q}^-}}(y))^M.$$

It then follows from Lemma 5.14, that for any $y\in\mathbb{R}^*$

$$(C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(y))^{L(A^*,\mathbb{R}^*)}=(C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(y))^M$$

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We claim that $\Gamma^* = (\mathcal{P}(\mathbb{R}^*))^M$. Suppose $E \in \Gamma^*$. Then there is a real $z \in \mathcal{N}_x^*[g]$ such that $z$ codes a Wadge reduction of $E$ to $A^*$. Let now $G$ be the set of reals which is Wadge reducible to $A$ via $z$. Then, $E = G \cap \mathcal{N}_x^*[g]$. Now, if $G \notin \Gamma(\mathcal{Q}, \Lambda)$ then this statement is a projective sentence about $(HC, A, \Sigma, z)$ and therefore, that sentence holds in

$$(HC^{\mathcal{N}_x^*[g]}, A^*, \Sigma^*, z)$$

implying that $E \notin \Gamma^*$. This contradiction shows that $G \in \Gamma(\mathcal{Q}, \Lambda)$. It then follows that $G \cap \mathcal{N}(\mathbb{R}^*) \in \mathcal{N}(\mathbb{R}^*)$. (Whenever $h \subseteq Coll(\omega, \mathbb{R}^*)$ is an $\mathcal{N}(\mathbb{R}^*)$-generic, $G \cap \mathcal{N}(\mathbb{R}^*)[h] \in \mathcal{N}(\mathbb{R}^*)[h]$. Therefore, $G \cap \mathcal{N}(\mathbb{R}^*) \in \mathcal{N}(\mathbb{R}^*)$.) But $E = G \cap \mathcal{N}(\mathbb{R}^*)$ and hence, $E \in \mathcal{N}(\mathbb{R}^*)$. Because $L(E, \mathbb{R}^*) \vDash AD^+$, we have that $E \in M$. This shows that $\Gamma^* \subseteq M$.

Let now $E = Code(Mice_{\Lambda_{\mathcal{Q}^-}}) \cap \mathcal{N}_x^*[g]$. Then we have already shown that $E \in M$ (see the proof of Lemma 5.13) and moreover, the complement of $E$ doesn't have a scale in $M$. For this reason for any set $F \in M$, we have that in $M$, for a cone of $x$, $F \cap C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(x) \in C_{\Gamma_{\Lambda^*_{\mathcal{Q}^-}}}(x)$. Hence, $F \in \Gamma^*$.

We then immediately obtain a contradiction because what we have shown is that $M = L(\Gamma^*, \mathbb{R}^*)$ and $(\mathcal{P}(\mathbb{R}^*))^M = \Gamma^*$, while because of our assumption, we have that $\Gamma^* \neq \mathcal{P}(\mathbb{R}^*) \cap L(\Gamma^*, \mathbb{R}^*)$. □

Finally, as a corollary to Lemma 5.14 and Lemma 5.16, we obtain that $MC$ holds in $L(\Gamma(\mathcal{P}, \Sigma), \mathbb{R})$.

COROLLARY 5.17. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ is fullness preserving and has branch condensation such that $\Gamma_\Sigma$ is defined. Then whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\alpha < \lambda^{\mathcal{Q}}$,*

$$L(\Gamma(\mathcal{P}, \Sigma), \mathbb{R}) \vDash \text{“}MC \text{ with respect to } \Sigma_{\mathcal{Q}(\alpha)}\text{”}.$$

## 5.7. $B$-iterability

In this section we show that hod pair constructions produce pairs whose strategy is guided via some $\vec{B}$.

LEMMA 5.18 (Strong $B$-condensation). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ is such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ has branch condensation and is super fullness preserving and $\Gamma(\mathcal{P}, \Sigma)$ is defined. Suppose $B \in (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma(\mathcal{P}, \Sigma), \mathbb{R})}$ and suppose $B^* \subseteq \mathbb{R}$ codes $B$. Let $\Gamma$ be a good pointclass such that $Mice_{\Sigma_{\mathcal{P}^-}} \in \Delta_{\underset{\sim}{\Gamma}}$ and there is sjs $\vec{A} \in \Delta_{\underset{\sim}{\Gamma}}$ such that $A_0 = Mice_{\Sigma_{\mathcal{P}^-}}$. Let $F$ be as in Theorem 2.25 for $\Gamma$. Then for cone of $x$ such that $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \Sigma_x, \delta_x, )$ is defined and $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Code(\Sigma)$, $\vec{A}$ and $B^*$ if $\beta$ is such that letting $(\mathcal{N}_{\beta+1}, \mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$ be the $\beta+1$st triple constructed via the hod pair construction of $\mathcal{N}_x^*|\delta_x$, $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$ is a tail of $(\mathcal{P}, \Sigma)$, then $\Sigma_{\beta+1}$ strongly respect $B$.*

PROOF. Let $\Gamma^* = \Gamma(\mathcal{P}, \Sigma)$. Because $w(\Gamma^*) = \theta^{\Gamma^*}$ is a successor Suslin cardinal, we must have that $\theta^{\Gamma^*}$ has cofinality $\omega$. It follows then that we can fix $(B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma^*, \mathbb{R})}$ such that

(1) $B_i \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$,
(2) $B_0 = B$,
(3) for any tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$, $((B_i)_{(\mathcal{Q}^-, \Lambda_{\mathcal{Q}^-})} : i < \omega)$ is cofinal in $\theta^{\Gamma^*}$.

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Let $\eta > \theta^{\Gamma^*}$ be the least such that

$$L_\eta(\Gamma^*, \mathbb{R}) \vDash KP$$

and let $A \subseteq \mathbb{R}$ be a set of reals coding $L_\eta(\Gamma^*, \mathbb{R})$. Let $C$ be a set of reals coding the the first order theory of

$$(HC, A, (\mathcal{P}, \Sigma), (B_i : i < \omega), \in).$$

Let then $x$ be such that $(C_\Gamma, C)$ is Suslin, co-Suslin captured by $(\mathcal{N}^*_x, \Sigma_x)$[9].

As in the proof of Lemma 5.7, we have that the first strong cardinal of $\mathcal{N}_{\beta+1}$ is a limit of Woodin cardinals. Let $\nu$ be the least strong cardinal of $\mathcal{N}_{\beta+1}$ and let $\mathcal{Q} = Lp^{\Sigma_\beta}_\omega(\mathcal{N}_{\beta+1}|\nu)$. Let $g \subseteq Coll(\omega, < \nu)$ be $\mathcal{N}^*_x$-generic. Let $\Gamma^{**} = (\Gamma(\mathcal{P}, \Sigma))^{\mathcal{N}^*_x[g]}$. Then it follows from the proof of Lemma 5.16, that if $M$ is the derived model of $\mathcal{Q}$ at $\nu$ computed in $\mathcal{N}^*_x[g]$, then

$$M = L(\Gamma^{**}, \mathbb{R}).$$

Let $\mathbb{R}^* = \mathbb{R} \cap \mathcal{N}^*_x[g]$ and $B^*_i = B_i \cap \mathcal{N}^*_x[g]$. We have that if $\eta^* > \theta^M$ is the least such that $L_{\eta^*}(\Gamma^{**}, \mathbb{R}^*) \vDash KP$ then

$$L_{\eta^*}(\Gamma^{**}, B^*_i, \mathbb{R}^*) \prec L_\eta(\Gamma^*, B_i, \mathbb{R}).$$

Because each $B_i$ is $OD$ in $L_\eta(\Gamma^*, B_i, \mathbb{R})$, we must have that each $B^*_i$ is $OD$ in $L_{\eta^*}(\Gamma^{**}, \mathbb{R}^*)$. Let $s$ be a sequence of ordinals defining $B^*_0$ in $L_{\eta^*}(\Gamma^{**}, \mathbb{R}^*)$.

Suppose now towards a contradiction that no tail of $\Sigma_{\beta+1}$ strongly respects $B$. Let $\Lambda$ be the induced strategy of $\mathcal{Q}$. Then it follows from the proof of Lemma 5.7 that $\Lambda$ is fullness preserving. Let $\delta = \delta_{\beta+1}$ and let $h = g \cap Coll(\omega, (\delta^{+\omega})^{\mathcal{N}^*_x[g]})$. We then get $(\vec{\mathcal{T}}_i, \vec{\mathcal{S}}_i, \mathcal{Q}_i, \mathcal{R}_i, \pi_i, \sigma_i, j_i : i < \omega) \in \mathcal{N}^*_x[g]$ such that

(1) $(\vec{\mathcal{T}}_i, \vec{\mathcal{S}}_i : i < \omega) \in \mathcal{N}^*_x[h]$,
(2) $\mathcal{Q}_0 = \mathcal{Q}$, $\vec{\mathcal{T}}_0$ is a stack on $\mathcal{Q}_0|\delta$ according to $\Lambda$ with last model $\mathcal{Q}_1$, $\pi_1 = \pi^{\vec{\mathcal{T}}_0}$, $\vec{\mathcal{S}}_0$ is a stack on $\mathcal{Q}_0|\delta$ with last model $\mathcal{R}_0$, $\sigma_0 = \pi^{\vec{\mathcal{S}}_0}$ and $j_0 : \mathcal{R}_0 \to \mathcal{Q}_1$,
(3) $\vec{\mathcal{T}}_k$ is a stack on $\mathcal{Q}_k|(\pi_{k-1} \circ ... \circ \pi_0(\delta))$ according to $\Lambda$ with last model $\mathcal{Q}_{k+1}$, $\pi_k = \pi^{\vec{\mathcal{T}}_k}$, $\vec{\mathcal{S}}_k$ is a stack on $\mathcal{Q}_k|(\pi_{k-1} \circ ... \circ \pi_0(\delta))$ with last model $\mathcal{R}_k$, $\sigma_k = \pi^{\vec{\mathcal{S}}_k}$ and $j_k : \mathcal{R}_k \to \mathcal{Q}_{k+1}$,
(4) for all $k < \omega$, $\pi_k = j_k \circ \sigma_k$,
(5) $j_k(\tau_B^{\mathcal{R}_k, \Psi_k}) \neq \tau_B^{\mathcal{Q}_{k+1}, \Lambda_{\kappa+1}}$.

Let $\mathcal{Q}_\omega$ be the direct limit of $\mathcal{Q}_i$'s under $\pi_i$. Let for $i \leq \omega$, $\Lambda_i$ be the corresponding tail of $\Lambda$ and let $\Psi_i = \Lambda^{j_i}_{i+1}$. We then, working in $\mathcal{N}^*_x[g]$, simultaneously iterate $\mathcal{Q}_i$ and $\mathcal{R}_i$ to make $\mathbb{R}^*$-generic in the following way.

Let $(x_i : i < \omega)$ be a generic over $\mathcal{N}^*_x[g]$-enumeration of $\mathbb{R}^*$. We do our genericity iteration in a way that it produces $(\mathcal{Q}^i_k, \mathcal{R}^i_k, \pi^i_k, \sigma^i_k, j^i_k, : i, k < \omega)$ such that

(1) $(\mathcal{Q}^i_0, \mathcal{R}^i_0, \pi^i_0, \sigma^i_0, j^i_0) = (\mathcal{Q}_0, \mathcal{R}_0, \pi_0, \sigma_0, j_0)$,
(2) $\mathcal{Q}^{i+1}_k$ is an iterate of $\mathcal{Q}^i_k$ to make $x_k$ generic at the $k$th Woodin of $\mathcal{Q}^i_k$,
(3) $\mathcal{R}^{i+1}_k$ is an iterate of $\mathcal{R}^i_k$ to make $x_k$ generic at the $k$th Woodin of $\mathcal{R}^i_k$,
(4) $\pi^i_k : \mathcal{Q}^i_k \to \mathcal{Q}^i_{k+1}$ is the iteration map,
(5) $\sigma^i_k : \mathcal{Q}^i_k \to \mathcal{R}^i_\kappa$ and $j^i_k : \mathcal{R}^i_k \to \mathcal{R}^i_{k+1}$ are such that $\pi^i_k = j^i_k \circ \sigma^i_k$,
(6) all the iteration embeddings fix $\nu$.

[9]One can show that $C \in \Delta_{\underset{\sim}{\Gamma}}$, we leave it to the reader.

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We can obtain such an iteration by lifting genericity trees from the earlier models to the current model and then starting a genericicty iteration of the current model. We then get $(\mathcal{Q}^\omega_k, \mathcal{R}^\omega_k, \pi^\omega_k, \sigma^\omega_k, j^\omega_k, : k < \omega)$ such that

(1) $\mathcal{Q}^\omega_k$ and $\mathcal{R}^\omega_k$ are the direct limits of respectively $\mathcal{Q}^i_k$ and $\mathcal{R}^i_k$ under the corresponding iteration embeddings,
(2) $\pi^\omega_k : \mathcal{Q}^\omega_k \to \mathcal{Q}^\omega_{k+1}$, $\sigma^\omega_k : \mathcal{Q}^\omega_k \to \mathcal{R}^\omega_\kappa$ and $j^\omega_k : \mathcal{R}^\omega_k \to \mathcal{R}^\omega_{k+1}$ are such that $\pi^\omega_k = j^\omega_k \circ \sigma^\omega_k$.

Notice that the direct limit of $\mathcal{Q}^\omega_k$ is well-founded as the entire matrix can be lifted to $\mathcal{Q}_\omega$.

We must then have that for some $k$, for all $n \geq k$, $\pi^\omega_n(s) = s$. Let $\Lambda^\omega_k$ be the corresponding tail of $\Lambda_k$ and let $\Psi^\omega_k$ be the corresponding tail of $\Psi_k$. Using the proof of Lemma 5.16, it is not hard to show that the derived model of $\mathcal{Q}^\omega_k$ at $\nu$ as computed in $\mathcal{N}^*_x[g]$ is $L(\Gamma^{**}, \mathbb{R}^*)$. This implies that for all $n \geq k$,

$$\pi^\omega_n(\tau_B^{\mathcal{Q}^\omega_n, \Lambda^\omega_n}) = \tau_B^{\mathcal{Q}^\omega_{n+1}, \Lambda^\omega_{n+1}}.$$

Also, for all $n \geq k$, $\sigma^\omega_n(s) = s$ and $j^\omega_n(s) = s$. Therefore,

$$\sigma^\omega_n(\tau_B^{\mathcal{Q}^\omega_n, \Lambda^\omega_n}) = \tau_B^{\mathcal{R}^\omega_n, \Psi^\omega_n};$$
$$j^\omega_n(\tau_B^{\mathcal{R}^\omega_n, \Psi^\omega_n}) = \tau_B^{\mathcal{Q}^\omega_{n+1}, \Lambda^\omega_{n+1}}.$$

This contradiction completes our proof. □

## 5.8. Strongly $\vec{B}$-guided strategies

Here we show that a tail of a hod pair is strongly $\vec{B}$-guided.

LEMMA 5.19 (Strongly $\vec{B}$-guided strategies). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ is such that $\lambda^{\mathcal{P}}$ is a successor, $\Sigma$ has branch condensation and is super fullness preserving and $\Gamma(\mathcal{P}, \Sigma)$ is defined. Then there is $\vec{B} = (B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma(\mathcal{P},\Sigma),\mathbb{R})}$ such that some tail of $(\mathcal{P}, \Sigma)$ is strongly guided by $\vec{B}$.*

PROOF. Let $\Gamma = \Gamma(\mathcal{P}, \Sigma)$. Because all successor Suslin cardinals have cofinality $\omega$ and because it follows from the definition of $\Gamma(\mathcal{P}, \Sigma)$ that $\theta^\Gamma$ is a successor Suslin cardinal, we have that $\mathrm{cf}(\theta^\Gamma) = \omega$. Let $(\alpha_i : i < \omega)$ be an increasing sequence cofinal in $\theta^\Gamma$. It follows then that we can fix $(B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ such that

(1) $B_i \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$,
(2) for any tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$, $((B_i)_{(\mathcal{Q}^-, \Lambda_{\mathcal{Q}^-})} : i < \omega)$ is cofinal in $\theta^\Gamma$.

Using Lemma 5.18, we can assume without loss of generality, that $\Sigma$ strongly respects $B_i$ for each $i$.

Let now $C = Mice_{\Sigma_{\mathcal{P}^-}}$. Then there is a sequence $\vec{A} = (A_i : i < \omega) \subseteq (OD(\Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ such that $\vec{A}$ is a semi-scale on $C^c$. Suppose $\phi_i$ and $s_i \in Ord^{<\omega}$ are such that

$$x \in A_i \leftrightarrow L(\Gamma, \mathbb{R}) \vDash \phi_i[\Sigma_{\mathcal{P}^-}, s_i, x].$$

Let then $A^*_i \in (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma^*,\mathbb{R})}$ be such that

$$((\mathcal{Q}, \Lambda), y) \in A^*_i \leftrightarrow L(\Gamma, \mathbb{R}) \vDash \alpha(\mathcal{Q}, \Lambda) = \alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) \wedge \phi[\Lambda, s_i, y].$$

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Using the proof of Lemma 5.18[10], we can find a tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$ such that $\mathcal{Q}^- = \mathcal{P}^-$ and $\Lambda$ respects $A_i^*$ for stacks that are above $\delta^{\mathcal{Q}}_{\lambda^{\mathcal{Q}}-1}$. It then follows from Lemma 2.21 that

$$\sup_{i<\omega} \gamma^{\mathcal{Q},\Lambda}_{A_i^*} = \delta^{\mathcal{Q}}.$$

*Claim.* $\sup_{i<\omega} \gamma^{\mathcal{Q},\Lambda}_{B_i} = \delta^{\mathcal{Q}}$.

Proof. To see this, suppose not. Let $\delta^{\mathcal{Q}} = \delta$ and let $\gamma = \sup_{i<\omega} \gamma^{\mathcal{Q},\Lambda}_{B_i}$. Let

$$X = Hull_1^{\mathcal{Q}}(\mathcal{Q}^- \cup \{\tau^{\mathcal{Q},\Lambda}_{B_i} : i < \omega\}).$$

We claim that $X \cap (\delta^+) < \delta^+$. First fix $i$ such that $\gamma^{\mathcal{Q}}_{A_i^*} > \gamma$. Let then $\mathcal{U} \in \mathcal{Q}$ be the tree on $\mathcal{Q}$above $\mathcal{Q}^-$ that makes $\mathcal{Q}|\delta$-generic. Then $\delta(\mathcal{U}) = (\delta^+)^{\mathcal{Q}}$. Letting $b \in \mathcal{Q}$ be any branch of $\mathcal{U}$ that moves $A_i^*$-correctly, we have that $X \cap (\delta^+)^{\mathcal{Q}} \subseteq \pi^{\mathcal{U}}_b(\gamma)$. On the other hand, notice that if

$$Y = Hull_1^{\mathcal{Q}}(\mathcal{Q}^- \cup \{\tau^{\mathcal{Q},\Lambda}_{A_i^*} : i < \omega\})$$

then $Y \cap (\delta^+)^{\mathcal{Q}} = \delta^+$ and hence, there is some $i$ such that $X \in H^{\mathcal{Q},\Lambda}_{A_i^*}$.

It then follows that for every $k$, $(B_k)_{(\mathcal{Q}^-,\Lambda_{\mathcal{Q}^-})}$ can be computed from $A_i$, $Mice_{\Lambda_{\mathcal{Q}^-}}$ and $\mathcal{Q}$ in $L(\Gamma, \mathbb{R})$: $y \in (B_k)_{(\mathcal{Q}^-,\Lambda_{\mathcal{Q}^-})}$ if letting $\mathcal{R}$ be a $\Lambda_{\mathcal{Q}^-}$-correct iterate of $\mathcal{Q}$ via a non-dropping tree which is above $\mathcal{Q}^-$ such that $y$ is generic for the extender algebra of $\mathcal{R}$ at $\delta^{\mathcal{R}}$, $\mathcal{R}[y] \Vdash \Vdash_{Coll(\omega,\delta^{\mathcal{R}})}$ "$\check{y} \in \pi_{(\mathcal{Q},A_i^*),(\mathcal{R},A_i^*)}(\tau^{\mathcal{Q},\Lambda_{\mathcal{Q}^-}}_{B_k})$". The equivalence follows from the fact that $\Lambda$ respects $A_i^*$ and $(B_k : k < \omega)$. This is then a contradiction as it implies that $\sup_{k<\omega}(B_k)_{\mathcal{Q}^-,\Lambda_{\mathcal{Q}^-}} < w((A_i^*)_{\mathcal{Q}^-,\Lambda_{\mathcal{Q}^-}} < \theta^{\Gamma}$ while $\sup_{k<\omega}(B_k)_{\mathcal{Q}^-,\Lambda_{\mathcal{Q}^-}} = \theta^{\Gamma}$. □

It is now easy to check, using the fact that $\Sigma$ respects $B_i$ and that the iteration embeddings are continuous at $\delta^{\mathcal{Q}}$, that whenever $\mathcal{R} \in pI(\mathcal{Q}, \Lambda)$,

$$\sup_{i<\omega} \gamma^{\mathcal{R},\Lambda_{\mathcal{R}^-}}_{B_i} = \delta^{\mathcal{R}}.$$

It then follows that $\Lambda$ is indeed strongly guided by $(B_i : i < \omega)$. □

## 5.9. Summary

The following theorem summarizes most of what we have proved in this chapter.

Theorem 5.20. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma$ has a branch condensation and is fullness preserving and $\Gamma(\mathcal{P}, \Sigma)$ is defined. There is then $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ such that whenever $\mathcal{R} \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$, $\alpha < \lambda^{\mathcal{R}}$ and $B \in (\mathbb{B}(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1),\Sigma_{\mathcal{R}(\alpha+1)}))}$*

(1) *$\Sigma_{\mathcal{R}(\alpha+1)}$ is super fullness preserving and there is*

$$\vec{B} = (B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1),\Sigma_{\mathcal{R}(\alpha+1)}))}$$

*such that $\vec{B}$ strongly guides $\Sigma_{\mathcal{R}(\alpha+1)}$,*

(2) *there is $\mathcal{S} \in pI(\mathcal{R}(\alpha+1), \Sigma_{\mathcal{R}(\alpha+1)})$ such that $\Sigma_{\mathcal{S}}$ respects $B$.*

[10] We can essentially apply the lemma to $\Sigma_{\mathcal{P}^-}$-hod mice.

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PROOF. The proof is by induction. Suppose $\lambda^{\mathcal{P}} = 0$. Then what we want to prove has already been established in Lemma 5.18 and Lemma 5.19. Suppose then $\lambda^{\mathcal{P}} = \alpha + 1$ and our theorem is true for $(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$. Let then $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ be such that the conclusion of our theorem is true for $(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$. In this case, we can again finish by just citing Lemma 5.18 and Lemma 5.19, as all we have to do now is to prove the theorem for $\lambda^{\mathcal{Q}}$. If $\lambda^{\mathcal{P}}$ is limit but of non-measurable cofinality then by induction hypothesis we can assume that for every $\alpha < \lambda^{\mathcal{P}}$, the conclusion of the theorem is true for $(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$ and therefore, it is true for $(\mathcal{P}, \Sigma)$. The only hard case, then, is when $\lambda^{\mathcal{P}}$ has a measurable cofinality in $\mathcal{P}$. Without loss of generality, by changing $\mathcal{P}$ if necessary, we can assume that for every $\alpha < \lambda^{\mathcal{P}}$, the conclusion of our theorem is true for $(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$. To show that the conclusion of our theorem is true for $(\mathcal{P}, \Sigma)$, fix some $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$. Let $\alpha < \lambda^{\mathcal{R}}$ and let $i : \mathcal{P} \to \mathcal{R}$ be the iteration embedding. Then because for every $\gamma < \lambda^{\mathcal{P}}$ the conclusion of out theorem for $(\mathcal{P}(\gamma), \Sigma_{\mathcal{P}(\gamma)})$ is true in the derived model of $\mathcal{P}$, we have that the conclusion of our theorem for $(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)})$ is true in the derived model of $\mathcal{R}$. But this means that the conclusion of our theorem is true for $(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)})$. □

As a corollary to Theorem 4.24 and Theorem 5.20, we obtain the following theorem on HOD.

COROLLARY 5.21 (Computation of HOD revisited). *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Suppose* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ *and* $L(\Gamma, \mathbb{R}) \vDash SMC$. *Suppose further that there is a hod pair* $(\mathcal{P}, \Sigma)$ *such that* $\Sigma$ *has branch condensation and either*

(1) $\lambda^{\mathcal{P}}$ *is a successor,* $\Sigma$ *is fullness preserving and* $\Gamma(\mathcal{P}, \Sigma) = \Gamma$
*or*
(2) $\lambda^{\mathcal{P}}$ *is limit and* $\Gamma(\mathcal{P}, \Sigma) = \Gamma$.

*Then* $L(\Gamma, \mathbb{R}) \vDash \phi$ *and if* $\mathcal{M} = \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$, $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$, $\vec{E} = \vec{E}^{\mathcal{M}}$ *and* $\Lambda$ *is the strategy coded by* $f^{\mathcal{M}}$ *then for any* $\alpha \leq \Omega^\Gamma$, $\theta_\alpha^\Gamma = \delta_\alpha^{\mathcal{M}}$ *and*

$$\mathcal{M}|\theta_\alpha^\Gamma = (V_{\theta_\alpha^\Gamma}^{\mathcal{H}}, \vec{E} \restriction \theta_\alpha^\Gamma, \Lambda \restriction V_{\theta_\alpha^\Gamma}^{\mathcal{H}}, \in).$$

Thus, working in a model of $AD^+$, if $\alpha < \Omega$ then to compute $\mathrm{HOD}|\theta_\alpha$ we only need to produce a hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. We will show that this is true in any model of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ provided that there is no proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular". We finish this section by showing that the existence of a certain hod pair implies that there is an inner model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" containing the reals.

THEOREM 5.22. *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ *and suppose that* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\lambda^{\mathcal{P}}$ *is limit,* $\delta^{\mathcal{P}}$ *is a regular cardinal of* $\mathcal{P}$ *and* $\Sigma$ *is fullness preserving and has branch condensation. Then there is a proper class inner model containing the reals and satisfying* $AD_{\mathbb{R}}$ + "$\Theta$ *is regular*".

PROOF. Let $\Gamma = \Gamma(\mathcal{P}, \Sigma)$. We claim that $L(\Gamma, \mathbb{R}) \vDash AD_{\mathbb{R}}$ + "$\Theta$ is regular". Because of Theorem 3.19, we have that $\mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma$. Then by Corollary 5.21, if $\mathcal{M} = \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$, $\vec{E} = \vec{E}^{\mathcal{M}}$, $\Lambda$ is the strategy coded by $f^{\mathcal{M}}$ and $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ then

$$\mathcal{M}|\theta^\Gamma = (V_{\theta^\Gamma}^{\mathcal{H}}, \vec{E} \restriction \theta^\Gamma, \Lambda \restriction V_{\theta^\Gamma}^{\mathcal{H}}, \in).$$

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But because $\Omega^\Gamma$ is limit,

$$\mathcal{H} \vDash V = L(V_{\theta^\Gamma}).$$

Because $\theta^\Gamma = \delta^{\mathcal{M}}$ and $\mathcal{M} \vDash$ "$\delta$ is a weakly inaccessible", we have that $\mathcal{H} \vDash$ "$\theta^\Gamma$ is weakly inaccessible". But this implies that $L(\Gamma, \mathbb{R}) \vDash$ "$\Theta$ is regular". Because $\Omega^\Gamma$ is limit, we have that $L(\Gamma, \mathbb{R}) \vDash AD_{\mathbb{R}} +$ "$\Theta$ is regular". □

In the next chapter, we will use Theorem 5.22 to prove that hod pair constructions converge.

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CHAPTER 6

# A proof of the mouse set conjecture

The main goal of this chapter is to give a proof of the mouse set conjecture assuming that there is no proper class inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular". This will be done in Section 6.4. We will start by proving that assuming there is no proper class inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}}$ + "$\Theta$ is regular", every mouse full pointclass can be represented as $\Gamma(\mathcal{P}, \Sigma)$ for some perhaps anomalous hod pair $(\mathcal{P}, \Sigma)$ (see Theorem 6.1). Next we will show that, under the aforementioned hypothesis, given a hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation and is fullness preserving, there is a mouse which "captures" a tail of $\Sigma$ (see Theorem 6.5). In Section 6.4, we will show that the two aforementioned results imply SMC. The last section of this chapter is devoted to showing that $AD_{\mathbb{R}}$ + "$\Theta$ is regular" is a weaker theory then $ZFC$+"there is an inner model with a superstring cardinal".

## 6.1. The generation of the mouse full pointclasses

In this section, our goal is to show that if SMC holds and $\Gamma$ is a mouse full pointclass then there is $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma$.

THEOREM 6.1 (The generation of the mouse full pointclasses). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and suppose there is no $M$ such that $\mathcal{P}(\mathbb{R})^M \neq \mathcal{P}(\mathbb{R})$, $\mathbb{R} \in M$, $Ord \subseteq M$ and $M \vDash$ "$AD_{\mathbb{R}} + \Theta$ is regular". Suppose $\Gamma \neq \mathcal{P}(\mathbb{R})$ is a mouse full pointclass such that $\Gamma \vDash SMC$. Then*

1. *If $\Gamma$ is completely mouse full then letting $A \subseteq \mathbb{R}$ witness it, there is $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ such that $L(A, \mathbb{R}) \vDash$ "$\Sigma$ has branch condensation and is fullness preserving and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$".*
2. *If $\Gamma$ is mouse full but not completely mouse full then there is a hod pair or an anomalous hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$.*

PROOF. The proof is by induction. Suppose $\Gamma$ is a mouse full pointclass such that whenever $\Gamma^*$ is properly contained in $\Gamma$ and is a mouse full pointclass then there is a hod pair $(\mathcal{P}, \Sigma)$ as in 1 or 2. We first claim that $\theta^\Gamma$ isn't the largest Suslin cardinal. Towards a contradiction, suppose it is. Then because largest Suslin cardinal is always regular and because all successor Suslin cardinals are singular, we must have that $\theta^\Gamma$ is regular and that $\Omega^\Gamma$ is a limit ordinal. But then $L(\Gamma, \mathbb{R}) \vDash$ "$AD_{\mathbb{R}} + \Theta$ is regular", contradiction! We then get that there are good pointclasses beyond $\Gamma$.

Let then $A \subseteq \mathbb{R}$ be such that $w(A) = \Gamma$ and let $B = A_\Gamma$. For each hod pair $(\mathcal{P}, \Sigma) \in \Gamma$, there is a sjs $(A_i : i < \omega)$ such that for every $i < \omega$, $A_i \in \Gamma$ and $Mice^\Gamma_\Sigma = A_0$. We then let $C$ be the set of reals $\sigma$ coding a continuous function such that $\sigma^{-1}[A]$ codes a hod pair $(\mathcal{P}, \Sigma)$ such that $Code(\Sigma) \in \Gamma$ and a sjs $(A_i :$



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$i < \omega) \subseteq \Gamma$ such that $Mice^{\Gamma}_{\Sigma} = A_0$. Let $\Gamma^*$ be a good pointclass such that $A, B, C \in \Delta_{\Gamma^*}$ and let $F$ be as in Theorem 2.25 for $\Gamma^*$. Let $x \in dom(F)$ be such that $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $A, B, C$. We claim that some model of $\Gamma$-hod pair construction of $\mathcal{N}^*_x$ is as desired. To prove this, we just show that otherwise $\Gamma$-hod pair constructions never break down and hence, they produce a hod pair $(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{P}}$ is limit and $\delta^{\mathcal{P}}$ is regular inside $\mathcal{P}$. This then implies via Theorem 5.22 that there is a proper class model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular" containing all the reals and ordinals, contradiction!

To show that the $\Gamma$-hod pair construction never fail we complete the following steps.

(1) $(\mathcal{P}_0, \Sigma_0)$ always exists,
(2) if $(\mathcal{P}_\beta, \Sigma_\beta)$ exists then $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$-exists,
(3) if $(\mathcal{P}_\beta, \Sigma_\beta)$ exists for all $\beta < \alpha$ where $\alpha$ is limit then $(\mathcal{P}_\alpha, \Sigma_\alpha)$-exists.

We start by showing that $(\mathcal{P}_0, \Sigma_0)$ exists. For this it is enough to show that there is $\eta < \delta_x$ such that $Lp^{\Gamma}(\mathcal{N}^*_x|\eta) \vDash$ "$\eta$ is Woodin". The existence of such an $\eta$ follows from the fact that because of our choice of $x$, there is $\sigma \in \mathcal{N}^*_x$ such that $\sigma^{-1}[A]$ codes a sjs $(A_i : i < \omega) \subseteq \Gamma$ such that $A_0 = Mice^{\Gamma}$. It follows that $Lp^{\Gamma}(\mathcal{N}^*_x|\delta_x) \vDash$ "$\delta_x$ is Woodin". Next an easy Skolem hull argument using Lemma 2.21 shows that there is an $\eta < \delta_x$ such that $Lp^{\Gamma}(\mathcal{N}^*_x|\eta) \vDash$ "$\eta$ is Woodin". The argument presented here can be used to show that $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$ exists as long as

(1) if $\beta = \gamma + 1$ then $\mathcal{N}_{\beta+1} \vDash$ "$\delta_\beta$ is Woodin",
(2) if $\beta$ is limit then
    (a) no level of $\mathcal{N}_\beta$ projects across $\delta_\beta$,
    (b) no level of $\mathcal{N}_{\beta+1}$ projects across $\delta_\beta$,
    (c) $(\delta^+_\beta)^{\mathcal{N}_{\beta+1}} = (\delta^+_\beta)^{\mathcal{P}_\beta}$.

For limit $\beta$, to show that $(\mathcal{P}_\beta, \Sigma_\beta)$ exists provided for all $\gamma < \beta$, $(\mathcal{P}_\gamma, \Sigma_\gamma)$ exists, it is enough to show that $Lp^{\Gamma, \oplus_{\gamma<\beta}\Sigma_\gamma}(\cup_{\gamma<\beta}\mathcal{P}_\gamma)$ doesn't have a level projecting across $\delta_\beta$.

Towards a contradiction assume that there is $\beta$ such that $\mathcal{P}_\beta$ is undefined. Let $\beta$ be the least such that $\mathcal{P}_\beta$ doesn't exist. We now derive a contradiction provided that $\beta$ is limit or is a successor and clause 1, 2a or 2b fail. If it is clause 1 that fails, we let $\mathcal{M} \trianglelefteq \mathcal{N}_{\beta+1}$ be the largest initial segment of $\mathcal{N}_{\beta+1}$ such that $\mathcal{M} \vDash$ "$\delta_\beta$ is Woodin". If it is clause 2a that fails, we let $\mathcal{M}$ be the first initial segment of $\mathcal{N}_\beta$ such that $\rho(\mathcal{M}) < \delta_\beta$. Finally if is clause 2b that fails, we let $\mathcal{M}$ be the first initial segment of $\mathcal{N}_{\beta+1}$ such that $\rho(\mathcal{M}) < \delta_\beta$. Let $\Sigma$ be the strategy of $\mathcal{M}$. Notice that $(\mathcal{M}, \Sigma)$ is an anomalous hod pair: it is of type I if $\beta$ is a successor and clause 1 fails, it is of type II if $\beta$ is limit and clause 2a fails and finally, it is of type III if $\beta$ is limit and clause 2b fails.

By our assumption, in both cases we have that $Code(\Sigma) \in \Gamma$. Also if $\mathcal{M}$ is defined because clause 1 fails then $\Sigma$ is $\Gamma$-fullness preserving and if $\mathcal{M}$ is defined because either clause 2a or 2b fails then for all $\gamma < \beta$, $\Sigma_\gamma$ is $\Gamma$-fullness preserving. Let $\Psi = \Gamma(\mathcal{P}_\beta, \Sigma_\beta)$ if $\mathcal{M}$ is defined because clause 1 fails and let $\Psi = \Gamma(\mathcal{M}, \Sigma)$ if $\mathcal{M}$ is defined because either 2a or 2b fails (notice that because of our inductive hypothesis and because of Theorem 2.7, the hypothesis of Theorem 3.27 holds, and hence, $\Gamma(\mathcal{M}, \Sigma)$ is defined). We let $M = L(\Sigma, \mathbb{R})$. Notice that by the above

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discussion, for all $\gamma$ such that $\gamma+1 \leq \beta$, $\Sigma_{\gamma+1}$ is fullness preserving in $L(\Sigma, \mathbb{R})$. This also implies that for any $\gamma < \beta$, in $M$, SMC holds relative to $\Sigma_\gamma$.

Using Theorem 5.20, we can get a tail $(\mathcal{N}, \Lambda)$ of $(\mathcal{M}, \Sigma)$ such that if $i : \mathcal{M} \to \mathcal{N}$ is the iteration embedding then for every $\gamma < \beta$, $\mathcal{R} \in pI(\mathcal{N}(i(\gamma)), \Lambda_{\mathcal{N}(i(\gamma))})$, $\alpha < \lambda^{\mathcal{R}}$ and $B \in (\mathbb{B}(\mathcal{R}(\alpha), \Lambda_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1), \Lambda_{\mathcal{R}(\alpha+1)}))}$

(1) $\Lambda_{\mathcal{R}(\alpha+1)}$ is super fullness preserving and there is
$$\vec{B} = (B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{R}(\alpha), \Lambda_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1), \Lambda_{\mathcal{R}(\alpha+1)}))}$$
such that $\vec{B}$ strongly guides $\Lambda_{\mathcal{R}(\alpha+1)}$,
(2) there is a $\mathcal{S} \in pI(\mathcal{R}(\alpha+1), \Lambda_{\mathcal{R}(\alpha+1)})$ such that $\Lambda_{\mathcal{S}}$ respects $B$.

Without loss of generality we assume that $\mathcal{N} = \mathcal{M}$ and $\Lambda = \Sigma$.

We now work in $M$. Let $\alpha$ be such that $\theta_\alpha = \sup_{\gamma<\beta} \theta_{Code(\Sigma_\gamma)}$. Then letting $\mathcal{F}$ be the set of triples $(\mathcal{Q}, \Lambda, \mathcal{N})$ such that

(1) if $\beta$ is a successor then
  (a) $(\mathcal{N}, \Lambda)$ is an anomalous hod pair of type I as witnessed by $\mathcal{Q} \trianglelefteq \mathcal{N}$,
  (b) $\mathcal{Q} \trianglelefteq \mathcal{N}$ and $(\mathcal{Q}, \Lambda_{\mathcal{Q}})$ is a hod pair such that $\Lambda_{\mathcal{Q}}$ is fullness preserving and has branch condensation,
  (c) $\mathcal{N}$ is a sound $\Lambda_{\mathcal{Q}}$-mouse over $\mathcal{Q}$ such that either $\rho(\mathcal{Q}) < \delta^{\mathcal{Q}}$ or there is a function witnessing non-Woodiness of $\delta^{\mathcal{Q}}$ via extenders in $\vec{E}^{\mathcal{Q}}$ definable over $\mathcal{N}$,
  (d) $\Gamma(\mathcal{Q}, \Lambda) = \Psi$,
(2) if $\beta$ is limit then
  (a) $(\mathcal{N}, \Lambda)$ is an anomalous hod pair of type II or III,
  (b) the hypothesis of Theorem 3.27 holds for $(\mathcal{N}, \Lambda)$ where $\Gamma = \Psi$
  (c) $\Gamma(\mathcal{N}, \Lambda) = \Psi$, $\mathcal{Q} = \mathcal{N}^-$ and $\rho(\mathcal{N}) < \delta^{\mathcal{N}}$,
  (d) for every $\xi < \lambda^{\mathcal{N}}$, $\Lambda_{\mathcal{N}(\xi)}$ is fullness preserving and has branch condensation.

We define $\leq^*$ on $\mathcal{F}$ by $(\mathcal{Q}, \mathcal{N}, \Lambda) \leq^* (\mathcal{R}, \mathcal{S}, \Phi)$ if $\mathcal{S}$ is a $\Lambda$-iterate of $\mathcal{N}$ and $\Phi = \Lambda_{\mathcal{S}}$. Using Theorem 2.28 and Theorem 2.33, we can form a direct limit of $(\mathcal{F}, \leq^*)$ under the iteration maps. Let $\mathcal{W}$ be this direct limit. Let $\vec{E} = \vec{E}^{\mathcal{W}}$ and $\Lambda$ be the strategy coded by $f^{\mathcal{W}}$. Then by our assumption on $(\mathcal{M}, \Sigma)$, using clause 3 of Theorem 4.24, we have that,

$$\mathcal{W}|\theta_\alpha = (V_{\theta_\alpha}^{\mathrm{HOD}}, \vec{E} \restriction \theta_\alpha, \Lambda \restriction V_{\theta_\alpha}^{\mathrm{HOD}}, \in).$$

Moreover, either $\rho(\mathcal{W}) < \theta_\alpha$ and $\mathcal{W}$ is $\theta_\alpha$-sound or $\alpha$ is a successor ordinal and $\theta_\alpha$ is not Woodin in $L_1[\mathcal{W}]$. Because $\mathcal{W} \in \mathrm{HOD}$, we have that either there is a bounded $OD$ subset of $\theta_\alpha$ which is not in $V_{\theta_\alpha}^{\mathrm{HOD}}$ or $\alpha$ is a successor ordinal and $\theta_\alpha$ isn't Woodin in HOD. Either way we get a contradiction.

It remains to show that clause 2c cannot fail. Assume then that it fails and let $\mathcal{M} \trianglelefteq \mathcal{N}_{\beta+1}$ be such that $\rho(\mathcal{M}) = \delta_\beta$ and $\mathcal{P}_\beta \trianglelefteq \mathcal{M}$. Let $\kappa = (\mathrm{cf}(\lambda^{\mathcal{P}_\beta}))^{\mathcal{P}_\beta}$ and let $f : \kappa \to \lambda^{\mathcal{P}_\beta} \in \mathcal{P}_\beta$ be an increasing and continuous function unbounded in $\lambda^{\mathcal{P}_\beta}$. Let $\mathcal{Q} = \mathcal{P}_\beta$ and $\delta = \delta_\beta$. (Notice that clause 2c can only fail when $\kappa$ is a measurable cardinal of $\mathcal{P}_\beta$.) Regarding $\mathcal{M}$ as a subset of $\delta$ not in $\mathcal{Q}$, we let $(\mathcal{M}_\alpha : \alpha < \kappa)$ be such that $\mathcal{M}_\alpha = \mathcal{M} \cap \delta^{\mathcal{Q}}_{f(\alpha)}$. We have that $\mathcal{M}_\alpha \in \mathcal{Q}$ for all $\alpha < \kappa$. Let $\mathcal{M}^* = Ult_\omega(\mathcal{M}, E)$ and $\mathcal{Q}^* = Ult(\mathcal{Q}, E)$ where $E \in \vec{E}^{\mathcal{Q}}$ is the Mitchell order 0 extender with $crit(E) = \kappa$. Then we have that

$$(\mathcal{P}(\delta))^{\mathcal{Q}^*} = (\mathcal{P}(\delta))^{\mathcal{M}^*}.$$

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Let $(\mathcal{M}^*_\alpha : \alpha < \pi_E(\kappa))$ be such that $\mathcal{M}^*_\alpha = \mathcal{M}^* \cap \delta^{\mathcal{Q}^*}_{\pi_E(f)(\alpha)}$. We have that $\mathcal{M}^*_\alpha = \pi_E(\mathcal{M}_\alpha)$ for $\alpha < \kappa$. Because $\mathcal{Q}$ is $\Gamma$-full, we have that $\mathcal{P}(\delta)^{\mathcal{Q}^*} \subseteq \mathcal{Q}$ and since $(\mathcal{M}^*_\alpha : \alpha < \kappa) \in \mathcal{Q}^*|(\delta^+)^{\mathcal{Q}^*}$, we get that $(\mathcal{M}^*_\alpha : \alpha < \kappa) \in \mathcal{Q}$. Therefore, $(\mathcal{M}_\alpha : \alpha < \kappa) \in \mathcal{Q}$ as it can be easily computed from $(\mathcal{M}^*_\alpha : \alpha < \kappa)$. But then $\mathcal{M} = \cup(\mathcal{M}_\alpha : \alpha < \kappa) \in \mathcal{Q}$ producing a contradiction. This completes the proof of Theorem 6.1. □

Using Theorem 5.20, we get the following corollary.

COROLLARY 6.2. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + SMC$ and suppose there is no $M$ such that $\mathcal{P}(\mathbb{R})^M \neq \mathcal{P}(\mathbb{R})$, $\mathbb{R} \in M$, $Ord \subseteq M$ and $M \vDash$ "$AD_{\mathbb{R}} + \Theta$ is regular". Suppose $\Gamma \neq \mathcal{P}(\mathbb{R})$ is a mouse full pointclass such that $\Gamma \vDash SMC$*[1]*. Then*

(1) *If $\Gamma$ is completely mouse full then letting $A \subseteq \mathbb{R}$ witness it, there is $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ such that $L(A, \mathbb{R}) \vDash$ "$\Sigma$ has branch condensation and is fullness preserving and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$". Moreover, $(\mathcal{P}, \Sigma)$ satisfies the conditions of Theorem* 5.20*.*
(2) *If $\Gamma$ is mouse full but not completely mouse full then there is a hod pair or an anomalous hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has branch condensation and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$. Moreover, $(\mathcal{P}, \Sigma)$ satisfies the conditions of Theorem* 5.20*.*

As corollary to Theorem 4.24 and Corollary 6.2, we obtain the following revised form of Theorem 4.24.

THEOREM 6.3 (Revised form of computation of HOD). *Assume $AD^+ + SMC$. Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a completely mouse full pointclass such that $\Gamma \vDash SMC$*[2]*. Then $L(\Gamma, \mathbb{R}) \vDash \phi$ and whenever $(\mathcal{P}, \Sigma) \in \Gamma$ is such that $\alpha(\mathcal{P}, \Sigma) < \Omega^\Gamma$, letting $\mathcal{M} = \mathcal{M}_\infty(\mathcal{P}, \Sigma)^+$, $\vec{E} = \vec{E}^{\mathcal{M}}$, $\Lambda$ be the strategy coded by $f^{\mathcal{M}}$ and $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$, $\mathcal{M} \subseteq \mathcal{H}$ and for every $\beta \leq \alpha(\mathcal{P}, \Sigma)$*

$$\delta^{\mathcal{M}}_\beta = \theta^\Gamma_\beta \text{ and } \mathcal{M}|\theta^\Gamma_\beta = (V^{\mathcal{H}}_{\theta^\Gamma_\beta}, \vec{E} \restriction \theta_\beta, \Lambda \restriction V^{\mathcal{H}}_{\theta^\Gamma_\beta}, \in).$$

*If $\Gamma \neq \mathcal{P}(\mathbb{R})$ then $L(\Gamma, \mathbb{R}) \vDash \psi$ and if $\mathcal{M}_\infty$ is as in Theorem* 4.24*, letting $\mathcal{M} = \mathcal{M}_\infty$, $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda$ be the strategy coded by $f^{\mathcal{M}}$, $\mathcal{M} \subseteq \mathcal{H}$ and for every $\beta \leq \Omega^\Gamma$,*

$$\delta^{\mathcal{M}}_\beta = \theta^\Gamma_\beta \text{ and } \mathcal{M}|\theta^\Gamma_\beta = (V^{\mathcal{H}}_{\theta^\Gamma_\beta}, \vec{E} \restriction \theta_\beta, \Lambda \restriction V^{\mathcal{H}}_{\theta^\Gamma_\beta}, \in).$$

### 6.2. An analysis of stacks

In this section, we present an alternative way of analyzing stacks on hod premice. Up to now we used essential components of stacks to analyze iterations of hod premice. Here we will use *terminal nodes*. We will use the terminology developed in this section in the proof of Theorem 6.5 and in particular, in the case $\lambda^{\mathcal{P}}$ is a limit ordinal.

Suppose $M$ is a transitive structure and $\mathcal{T}$ is an iteration tree on $M$[3]. Let $\mathcal{S}$ be a node in $\mathcal{T}$. Then we write $\mathcal{T}_{\geq \mathcal{S}}$ for the component of $\mathcal{T}$ that comes after stage $\mathcal{S}$ and $\mathcal{T}_{\leq \mathcal{S}}$ for the component of $\mathcal{T}$ up to stage $\mathcal{S}$. We say $\mathcal{T}$ is *reducible* if there is a node $\mathcal{S}$ in $\mathcal{T}$ such that $\mathcal{T}_{\geq \mathcal{S}}$ is a tree on $\mathcal{S}$. Otherwise we say $\mathcal{T}$ is *irreducible*. We say $\mathcal{T}$ has a *last irreducible component* if there is a node $\mathcal{S}$ in $\mathcal{T}$ such that $\mathcal{T}_{\geq \mathcal{S}}$ is an irreducible tree on $\mathcal{S}$.

[1] Again, this requirement can be removed by generalizing Theorem 16.1 of [**42**] to hod mice.
[2] Again, this requirement can be removed by generalizing Theorem 16.1 of [**42**] to hod mice.
[3] Recall that all trees are normal.

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Suppose now that $\mathcal{P}$ is a hod premouse and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$. Recall that the definition of a stack on a hod premouse $\mathcal{P}$ is such that it guarantees that for every $\alpha < \eta$, $\pi^{\vec{\mathcal{T}}}_{0,\alpha} : \mathcal{M}_0 \to \mathcal{M}_\alpha$ exists.

DEFINITION 6.4. *We say $\mathcal{R}$ is a terminal node in $\vec{\mathcal{T}}$ if for some $\alpha < \eta$ and $\beta < lh(\mathcal{T}_\alpha)$, $\mathcal{R} = \mathcal{M}^{\mathcal{T}_\alpha}_\beta$ and $\pi^{\mathcal{T}_\alpha}_{0,\beta}$ exists. We say $\mathcal{R}$ is a non-trivial terminal node if letting $(\alpha, \beta)$ be as in the previous sentence, $E^{\mathcal{T}_\alpha}_\beta$ is applied to $\mathcal{R}$.*

If $\mathcal{R}$ is non-trivial terminal node then $\xi^{\vec{\mathcal{T}},\mathcal{R}}$ is the least $\xi$ such that $E^{\mathcal{T}_\alpha}_\beta \in \mathcal{R}(\xi+1)$. We also let $\vec{\mathcal{T}}_\mathcal{R}$ be the largest initial segment of $\vec{\mathcal{T}}$ that can be regarded as a tree on $\mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}}+1)$. Also let $\pi^{\vec{\mathcal{T}}}_\mathcal{R}$ be the iteration embedding from $\mathcal{P}$-to-$\mathcal{R}$ and set

$$\mathrm{tn}(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a terminal node in } \vec{\mathcal{T}}\}$$
$$\mathrm{ntn}(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a non-trivial terminal node in } \vec{\mathcal{T}}\}.$$

Notice that if $\mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$ then player $I$ can legitimately start a new round on $\mathcal{R}$. Next, given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ if, in $\vec{\mathcal{T}}$, $\mathcal{Q}$-to-$\mathcal{R}$ iteration embedding exists. If $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ then we let $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the iteration embedding given by $\vec{\mathcal{T}}$. Again given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}},s} \mathcal{R}$ if[4] $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ and if $\vec{\mathcal{U}}$ is the part of $\vec{\mathcal{T}}$ between $\mathcal{Q}$ and $\mathcal{R}$ then $\vec{\mathcal{U}}$ is an iteration of $\mathcal{Q}$. We then let $\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{R}}$ stand for the part of $\vec{\mathcal{T}}$ between $\mathcal{Q}$ and $\mathcal{R}$.

Suppose now that $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ is a stack on $\mathcal{P}$ and $C \subseteq tn(\vec{\mathcal{T}})$. We say $C$ is linear if it is linearly ordered by $\preceq^{\vec{\mathcal{T}}}$. We say $C \subseteq tn(\vec{\mathcal{T}})$ is strongly linear if $C$ is linearly ordered by $\preceq^{\vec{\mathcal{T}},s}$. Suppose $C$ is strongly linear and $(\mathcal{R}_\alpha : \alpha < \eta)$ is a $\preceq^{\vec{\mathcal{T}},s}$-increasing enumeration of $C$. We let $lh(C) = \eta$. Suppose further that $\eta$ is a limit ordinal. Then we let $\mathcal{R}^{\vec{\mathcal{T}}}_C$ be the direct limit of the $\mathcal{R}_\alpha$ under the iteration embeddings $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}_\alpha,\mathcal{R}_\beta}$. We then say $C \subseteq tn(\vec{\mathcal{T}})$ is closed if it is strongly linear and for every limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha} \in C$. Notice that strong linearity implies that for each limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha}$ is a node in $\vec{\mathcal{T}}$. We say $C$ is cofinal if for every node $\mathcal{S}$ of $\vec{\mathcal{T}}$ either $\mathcal{S} \in C$ or there are $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q} \in C$ such that $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$. Notice that if $C$ is closed and cofinal and $\mathcal{S} \notin C$ then there is a $\preceq^{\vec{\mathcal{T}},s}$-largest $\mathcal{R} \in C$ such that for any $\mathcal{Q} \in C$ such that $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q}$, $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$.

Notice also that if $\vec{\mathcal{T}}$ doesn't have a last model but there is a strongly linear closed and unbounded $C \subseteq tn(\vec{\mathcal{T}})$ then $C$ uniquely identifies the branch of $\vec{\mathcal{T}}$. Indeed, let $D = \{\mathcal{S} \in tn(\vec{\mathcal{T}}) : \exists \mathcal{R}, \mathcal{Q} \in C(\mathcal{R} \preceq^{\vec{\mathcal{T}}} \mathcal{S} \preceq^{\vec{\mathcal{T}}} \mathcal{Q})\}$. Let $\mathcal{R} \in D$ be the $\preceq^{\vec{\mathcal{T}}}$-minimal member of $D$ and let $b$ be the set of indices of the nodes of $\vec{\mathcal{T}}$ between $\mathcal{P}$ and $\mathcal{R}$. Then the union of $b$ with the indices of the nodes of $D$ constitute a branch $b_C$ of $\vec{\mathcal{T}}$. It is not hard to see that we have $\mathcal{M}^{\vec{\mathcal{T}}}_{b_C} = \mathcal{R}^{\vec{\mathcal{T}}}_C$.

Suppose now that $\vec{\mathcal{T}}$ doesn't have a last model and there is no strongly linear closed and cofinal $C \subseteq tn(\vec{\mathcal{T}})$. It follows that $\eta$ must be a successor ordinal. Let $\alpha = \eta - 1$ and $\mathcal{T} = \mathcal{T}_\alpha$. It then follows that there is $\mathcal{S} \in tn(\mathcal{T})$ such that $\mathcal{T}_{\geq\mathcal{S}}$ is an irreducible tree on $\mathcal{S}$ or there is $\mathcal{W} \notin tn(\mathcal{T})$, $\mathcal{T}_{\geq\mathcal{W}}$ is a tree on $\mathcal{W}$. Let then $D = \{\mathcal{S} \in tn(\mathcal{T}) : \mathcal{T}_{\geq\mathcal{S}} \text{ is a tree on } \mathcal{S}\}$. It follows from our discussion that $D$ has a

[4] "s" stands for "strongly".

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$\preceq^{\vec{\mathcal{T}},s}$-largest element. We then let $\mathcal{S}_{\vec{\mathcal{T}}}$ be this largest element. Such an analysis of stacks is very useful because we have that $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is a normal tree such that for some $\alpha < \lambda^{\mathcal{S}_{\vec{\mathcal{T}}}}$, $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is based on $\mathcal{S}_{\vec{\mathcal{T}}}(\alpha+1)$ and is above $\delta_\alpha^{\mathcal{S}_{\vec{\mathcal{T}}}}$.

### 6.3. Capturing of hod pairs

The proof of MSC we will present in the next section uses the fact that hod pairs can be "captured" in a certain way by ordinary mice. More precisely, given a hod pair $(\mathcal{P},\Sigma)$ such that $\Sigma$ is fullness preserving and has branch condensation, we can find an ordinary mouse $\mathcal{M}$ such that there is $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ such that $\mathcal{Q} \in \mathcal{M}$ and $\Sigma_{\mathcal{Q}} \restriction \mathcal{M}$ is definable over $\mathcal{M}$. Moreover, we will also have that $Mice_{\Sigma_{\mathcal{Q}}} \cap \mathcal{M}$ is also definable over $\mathcal{M}$. To prove MSC using such a capturing, we first show that if the mouse capturing is false and $(x,y) \in \mathbb{R}^2$ is such that $x \in OD(y)$ but $x$ is not in a $y$-mouse, then there is a hod pair $(\mathcal{P},\Sigma)$ such that $\Sigma$ has branch condensation, is fullness preserving, and $x$ is in a $\Sigma$-mouse over $y$. Then we capture $\Sigma$ in some $y$-mouse $\mathcal{M}$ as above. Letting $\mathcal{Q}$ be as above, since $Mice_{\Sigma_{\mathcal{Q}}} \cap \mathcal{M}$ is definable over $\mathcal{M}$, we have that $x \in \mathcal{M}$. In this section we present the aforementioned capturing result.

THEOREM 6.5 (Capturing of hod pairs). *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ is super fullness preserving and has branch condensation. Moreover, suppose that whenever $\mathcal{Q} \in pI(\mathcal{P},\Sigma) \cup pB(\mathcal{P},\Sigma)$ is such that $\lambda^{\mathcal{Q}}$ is a successor, there is a sequence $\vec{B} = (B_i : i < \omega) \subseteq (\mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-}))^{L(\Gamma(\mathcal{Q},\Sigma_{\mathcal{Q}}),\mathbb{R})}$ such that $\vec{B}$ strongly guides $\Sigma_{\mathcal{Q}}$. Suppose $\Gamma$ is a good pointclass such that $Code(\Sigma) \in \Delta_\Gamma$. Let $F$ be as in Theorem 2.25 and let $x$ be such that if $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ then $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin captures $Code(\Sigma)$. Let $\mathcal{N}^*$ be the last model of $\mathcal{J}^{\vec{E}}$-construction of $\mathcal{N}_x^* | \delta_x$ and let $\mathcal{N} = L[\mathcal{N}^*]$. Then there is $\mathcal{Q} \in pI(\mathcal{P},\Sigma) \cap \mathcal{N}^*$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N}^* \in \mathcal{N}$.*

We devote this entire section to the proof of Theorem 6.5. We start by relativizing the notion of fullness preservation to $ZFC$ context. More precisely, we would like to have a notion of fullness preservation that $\mathcal{N}$ of Theorem 6.5 can identify. We start by fixing $(\mathcal{P},\Sigma)$, $\Gamma$, $F$ and $x$ as in Theorem 6.5.

Suppose for a moment that we are working in some model of $ZFC$. Suppose $\kappa$ is an inaccessible cardinal. We say that $(\mathcal{Q},\Lambda)$ is a *hod pair at $\kappa$*, if

(1) $(\mathcal{Q},\Lambda)$ is a hod pair,
(2) $\mathcal{Q} \in HC$,
(3) $\Lambda$ is $(\kappa,\kappa)$-iteration strategy,
(4) $\Lambda$ is $\kappa$-universally Baire.

Suppose $(\mathcal{Q},\Lambda)$ is a hod pair at $\kappa$. Then we let

$$Lp^{\Lambda,\kappa}(a) = \cup\{\mathcal{M} : \mathcal{M} \text{ is a } \Lambda\text{-mouse over } a \text{ such that } \rho_\omega(\mathcal{M}) = a \text{ and } \mathcal{M} \trianglelefteq (\mathcal{J}^{\vec{E},\Lambda}(a))^{V_\kappa}\}.$$

DEFINITION 6.6 (Fullness preservation in models of $ZFC$). *Suppose now that $(\mathcal{Q},\Lambda)$ is a hod pair at $\kappa$. We then say $\Lambda$ is $\kappa$-fullness preserving if whenever $(\vec{\mathcal{T}},\mathcal{R}) \in I(\mathcal{Q},\Lambda) \cap V_\kappa$, $\alpha+1 \leq \lambda^{\mathcal{R}}$ and $\eta \geq \delta_\alpha$ is a cutpoint of $\mathcal{R}(\alpha)$ then*

$$\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)} = Lp^{\Lambda_{\mathcal{Q}(\alpha),\vec{\mathcal{T}},\kappa}}(\mathcal{Q}|\eta).$$

*and*

$$\mathcal{Q}|(\delta_\alpha^+)^{\mathcal{Q}} = Lp^{\oplus_{\beta<\alpha}\Lambda_{\mathcal{Q}(\beta+1),\vec{\mathcal{T}},\kappa}}(\mathcal{Q}(\alpha)).$$

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Continuing our work inside some model of $ZFC$, suppose $(\mathcal{Q},\Lambda)$ is a hod pair at $\kappa$ such that $\Lambda$ has branch condensation and is $\kappa$-fullness preserving. Suppose $\lambda<\kappa$ is an inaccessible cardinal. Then we say

DEFINITION 6.7 (Universal tail). *$(\mathcal{Q}^*,\Lambda^*)$ is a $\lambda$-universal tail of $(\mathcal{Q},\Lambda)$ if there is a stack $\vec{\mathcal{T}}$ according to $\Lambda$ on $\mathcal{Q}$ with last model $\mathcal{Q}^*$ such that*

1. *$lh(\vec{\mathcal{T}})=\lambda$ and for all $\beta<lh(\vec{\mathcal{T}})$, $\vec{\mathcal{T}}\restriction\beta\in V_\lambda$;*
2. *for any $(\vec{\mathcal{S}},\mathcal{R})\in I(\mathcal{Q},\Lambda)\cap V_\lambda$ there is a stack $\vec{\mathcal{U}}$ on $\mathcal{R}$ according to $\Lambda_{\mathcal{R}}$ with its last model on the main branch of $\vec{\mathcal{T}}$.*

*If $\vec{\mathcal{T}}$ is as above then we say $\vec{\mathcal{T}}$ is a $\lambda$-universal stack on $\mathcal{Q}$ according to $\Lambda$.*

We now resume the proof of Theorem 6.5. and start working in $\mathcal{N}_x^*$. Before we move on, we observe that because of our assumption on $(\mathcal{P},\Sigma)$, whenever $\mathcal{Q},\mathcal{R}\in pI(\mathcal{P},\Sigma)$, $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ and $(\mathcal{R},\Sigma_{\mathcal{R}})$ are of the same kind in the sense of Theorem 2.47. Then using either Theorem 2.47 or absoluteness and Theorem 2.28, we get that if $\kappa$ is an $\mathcal{N}_x^*$-cardinal and $\mathcal{Q},\mathcal{R}\in pI(\mathcal{P},\Sigma)\cap\mathcal{N}_x^*|\kappa$ then there is a $\mathcal{W}\in pI(\mathcal{P},\Sigma)$ such that $(\mathcal{W},\Sigma_{\mathcal{W}})$ is a common tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ and $(\mathcal{R},\Sigma_{\mathcal{R}})$. This means that whenever $\kappa<\delta_x$ is a cardinal of $\mathcal{N}_x^*$ and $\mathcal{Q}\in(pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma))\cap\mathcal{N}_x^*|\kappa$, we can form the direct limit of all $\Sigma_{\mathcal{Q}}$ iterates of $\mathcal{Q}$ that are in $\mathcal{N}_x^*|\kappa$. Let $\mathcal{R}_\kappa^{\mathcal{Q},\Sigma_{\mathcal{Q}}}$ be this direct limit. the next lemma shows that the universal tails are unique.

LEMMA 6.8 (Uniqueness of universal tails). *Suppose $\mathcal{Q}\in pI(\mathcal{P},\Sigma)\cap\mathcal{N}_x^*|\delta_x$. Then for each inaccessible $\kappa<\delta_x$ such that $\mathcal{Q}\in\mathcal{N}_x^*|\kappa$ and $\alpha\leq\kappa^{\mathcal{Q}}$, there is a unique $\kappa$-universal tail of $(\mathcal{Q}(\alpha),\Sigma_{\mathcal{Q}(\alpha)})$. In fact, letting $\mathcal{R}=\mathcal{R}_\kappa^{\mathcal{Q}(\alpha),\Sigma_{\mathcal{Q}(\alpha)}}$, $(\mathcal{R},\Sigma_{\mathcal{R}})$ is the unique $\kappa$-universal tail of $(\mathcal{Q}(\alpha),\Sigma_{\mathcal{Q}(\alpha)})$*

PROOF. The proof is like the proof of Theorem 4.18. Given two $\kappa$-universal tails $(\mathcal{Q}_1,\Lambda_1)$ and $(\mathcal{Q}_2,\Lambda_2)$ of $(\mathcal{Q}(\alpha),\Sigma_{\mathcal{Q}(\alpha)})$, using the proof of Theorem 4.18, we can get a surjective embedding $\pi:\mathcal{Q}_1\to_{\Sigma_1}\mathcal{Q}_2$. This then implies that $\mathcal{Q}_1=\mathcal{Q}_2$. By branch condensation, $\Lambda_1=\Lambda_2$. □

Suppose $\mathcal{Q}\in(pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma))\cap\mathcal{N}_x^*|\delta_x$ and $\kappa$ is an $\mathcal{N}$ cardinal such that $\mathcal{Q}\in\mathcal{N}_x^*|\kappa$.

DEFINITION 6.9. *Then we say $\mathcal{N}$ captures a tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ below $\kappa$ if there is a hod pair $(\mathcal{R},\Lambda)\in\mathcal{N}$ such that $\Lambda$ is a $(\kappa,\kappa)$-iteration strategy and there is a term relation $\tau\in\mathcal{N}^{Coll(\omega,<\kappa)}$ such that whenever $g\subseteq Coll(\omega,|\mathcal{R}|^+)$ is $\mathcal{N}$-generic,*

1. *$\mathcal{N}[g]\vDash$ "$(\mathcal{R},\tau_g)$ is a hod pair at $\kappa$ such that $\tau_g$ is $\kappa$-fullness preserving" and $\tau_g\restriction\mathcal{N}=\Lambda$,*
2. *for some $\lambda<\kappa$, $\mathcal{R}=\mathcal{R}_\lambda^{\mathcal{Q}}$ and letting $T,U\in\mathcal{N}[g]$ witness that $\tau_g$ is $\kappa$-uB, whenever $h\subseteq Coll(\omega,<\kappa)$ is $\mathcal{N}[g]$-generic, $(p[T])^{\mathcal{N}[g][h]}=Code(\Sigma_{\mathcal{R}})\cap\mathcal{N}[g][h]$.*

*We say $\mathcal{N}$ captures $B(\mathcal{Q},\Sigma_{\mathcal{Q}})$ below $\kappa$ if whenever $\mathcal{R}\in pB(\mathcal{Q},\Sigma_{\mathcal{Q}})\cap\mathcal{N}_x^*|\kappa$, $\mathcal{N}$ captures $(\mathcal{R},\Sigma_{\mathcal{R}})$ below $\kappa$.*

Now, towards a contradiction suppose $\mathcal{N}$ doesn't capture $(\mathcal{P},\Sigma)$ below $\delta_x$. We let $\delta=\delta_x$. We can then assume, without loss of generality, that $\mathcal{N}$ captures $B(\mathcal{P},\Sigma)$ below $\delta_x$. For each $\mathcal{Q}\in pB(\mathcal{P},\Sigma)$ we let $\lambda_{\mathcal{Q}}$ be the least inaccessible cardinal $\nu$ such that $\mathcal{N}$ captures the $\nu$-universal tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$. We let $(\mathcal{R}^{\mathcal{Q}},\Psi^{\mathcal{Q}})$ be the $\lambda_{\mathcal{Q}}$-universal tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$. For each inaccessible cardinal $\nu$ such that $\mathcal{Q}\in\mathcal{N}|\nu$, we let $(\mathcal{R}_\nu^{\mathcal{Q}},\Psi_\nu^{\mathcal{Q}})$ be the $\nu$-universal tail of $(\mathcal{Q},\Sigma_{\mathcal{Q}})$. We then have two cases: either

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$\lambda^{\mathcal{P}}$ is a successor or $\lambda^{\mathcal{P}}$ is limit. We first consider the case when $\lambda^{\mathcal{P}}$ is a successor.

**$\lambda^{\mathcal{P}}$ is a successor.**

Using our hypothesis, we can fix $\vec{B} = (B_i : i \in \omega) \subseteq (\mathbb{B}(\mathcal{P}, \Sigma))^{L(\Gamma(\mathcal{P},\Sigma),\mathbb{R})}$ such that $\vec{B}$ strongly guides $\Sigma$. We can further assume that $\vec{B}$ is captured by $\mathcal{N}_x^*$ as by absoluteness one such sequence is captured by $\mathcal{N}_x^*$. We let $(\mathcal{R}, \Phi) = (\mathcal{R}^{\mathcal{P}^-}, \Psi^{\mathcal{P}^-})$. Let $\vec{\mathcal{T}}$ on $\mathcal{P}^-$ be a $\lambda_{\mathcal{P}^-}$-universal stack according to $\Sigma_{\mathcal{P}^-}$. Then we have that $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}$. Let $\kappa$ be the least $< \delta$-strong above $\lambda_{\mathcal{P}^-}$. Let $\mathcal{N}_1^* = (\mathcal{J}^{\vec{E},\Phi\restriction\mathcal{N}^*})^{\mathcal{N}\restriction\delta}$ where the extenders used all have critical points $> \lambda_{\mathcal{P}^-}$, and $\mathcal{N}_1 = L[\mathcal{N}_1^*]$. As in the proof of Lemma 5.13, $\mathcal{N}_1 \vDash$ "$\kappa$ is a limit of Woodins".

Let $g \subseteq Coll(\omega, < \kappa)$ be $\mathcal{N}_x^*$-generic. Let $M$ be the derived model of $\mathcal{N}_1$ at $\kappa$. Then using the proof of Theorem 5.16, we get that letting $\Gamma = (\Gamma(\mathcal{P}, \Sigma))^{\mathcal{N}_x^*[g]}$ and $\mathbb{R}^* = \mathbb{R} \cap \mathcal{N}_x^*[g]$,

$$M = L(\Gamma, \mathbb{R}^*)$$

and moreover, for each $i$, if $B_i^* = B_i \cap \mathcal{N}_x^*[g]$ and $\Sigma^* = \Sigma \restriction (HC)^{\mathcal{N}_x^*[g]}$ then

$$M \vDash \text{"}B_i \in B(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}^*)\text{"}.$$

By the proof of Theorem 5.16, we also get that for any $\mathcal{Q} \in B(\mathcal{P}, \Sigma) \cap (HC)^{\mathcal{N}_x^*[g]}$

$$\sup_{i<\omega} w((B_i^*)_{\Sigma_{\mathcal{Q}}^*})^M = \Theta^M.$$

Using now 3 of Theorem 4.24, we get that

$$\mathcal{R}_\kappa^{\mathcal{P}} | \Theta^M = (V_{\Theta^M})^{\mathrm{HOD}^M}.$$

Let $\mathcal{M} = \mathcal{R}_\kappa^{\mathcal{P}}$. It follows from Lemma 2.13 that $\mathcal{M} \in \mathcal{N}_1$ and hence, $\mathcal{M} \in \mathcal{N}$. (Indeed, notice that by universality of $\mathcal{N}$, if $\lambda = \Theta^M$ and $\mathcal{W} = (\mathcal{J}^{\vec{E},\Phi\restriction\mathcal{N}^*}(\mathcal{M}|\lambda))^{\mathcal{N}|\delta}$ then $\mathcal{R}_\kappa^{\mathcal{P}} = \mathcal{W}|(\lambda^{+\omega})^{\mathcal{W}}$.)

Notice that $\Sigma_{\mathcal{P}^-}^* \in M$. Working in $M$ let, for $n < \omega$,

$$\mathcal{F}_n = \{(\mathcal{S}, \Upsilon, (B_i^* : i \leq n)) : M \vDash \text{"}(\mathcal{S}, \Upsilon) \text{ is suitable and } (\mathcal{S}, \Upsilon) \text{ is } (B_i^* : i \leq n)\text{-iterable "}\}.$$

Notice that for every $n < \omega$, $(\mathcal{P}, \Sigma_{\mathcal{P}^-}^*, (B_i^* : i \leq n)) \in \mathcal{F}_n$ and moreover, by the proof of Theorem 4.24, $\mathcal{R}_\kappa^{\mathcal{P}}$ is the direct limit of

$$(\cup_{n<\omega} \mathcal{F}_n, (\leq^*)^M \restriction \cup_{n<\omega} \mathcal{F}_n).$$

under the appropriate partial iteration maps (here $\leq^*$ is the relation defined on $\mathcal{F}$ while computing HOD). Let now $\Psi = \Psi_\kappa^{\mathcal{P}}$. We would like to show that $\Psi \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. The following claim establishes that.

*Claim:* $\Psi \restriction (\mathcal{N}|\delta) \in \mathcal{N}$.

Proof. We show that if $\Psi^n$ is the fragment of $\Psi$ that acts on normal trees then $\Psi^n \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. The general case is only notationally more complicated and we leave it to the reader. Let $\mathcal{Q} = \mathcal{R}_\kappa^{\mathcal{P}}$. We define a $\delta$-iteration strategy $\Lambda \in \mathcal{N}$ and show that $\Lambda = \Psi^n \restriction (\mathcal{N}|\delta)$. We let $\mathcal{T} \in dom(\Lambda)$ if the following holds:

(1) The fragment of $\mathcal{T}$ that is based on $\mathcal{Q}^-$ is according to $\Phi_{\mathcal{Q}^-} \restriction \mathcal{N}|\delta$.
(2) Suppose $\mathcal{T}$ isn't entirely based on $\mathcal{Q}^-$. Let $\mathcal{S}$ be the node on $\mathcal{T}$ such that the fragment of $\mathcal{T}$ from $\mathcal{Q}$-to-$\mathcal{S}$ is based on $\mathcal{Q}^-$ and the rest of $\mathcal{T}$ is above $\mathcal{S}^-$. Let $\mathcal{U}$ be the fragment of $\mathcal{T}$ after $\mathcal{S}$. Then one of the following holds:

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(a) $\mathcal{U}$ has a fatal drop at $(\alpha,\gamma)$ and $\mathcal{U}$ after stage $\alpha$ is according to the strategy of $\mathcal{O}_\gamma^{\mathcal{M}_\alpha^{\mathcal{U}}}$.
(b) $\mathcal{U}$ doesn't have fatal drops and for every limit $\alpha < lh(\mathcal{U})$, $\mathcal{Q}(\mathcal{U}\restriction\alpha)$ exists and $\mathcal{Q}(\mathcal{U}\restriction\alpha)\trianglelefteq(\mathcal{J}^{\vec{E},\Phi_{\mathcal{S}^-}})^{\mathcal{N}|\delta}$.

Notice that it follows from Lemma 2.13 and from our proof of Lemma 5.5 that for $\mathcal{T}\in\mathcal{N}|\delta$, $\mathcal{T}\in dom(\Psi^n)\leftrightarrow\mathcal{T}\in dom(\Lambda)$. Given now $\mathcal{T}\in dom(\Lambda)$ we let $\Lambda(\mathcal{T})=b$ if one of the following holds:

(1) $\mathcal{T}$ is based on $\mathcal{Q}^-$ and $b=\Psi_{\mathcal{Q}^-}(\mathcal{T})$.
(2) $\mathcal{T}$ has is not entirely based on $\mathcal{Q}^-$ and if $(\mathcal{S},\mathcal{U})$ are as in clause 2 above then one of the following holds:
   (a) $\mathcal{U}$ has a fatal drop at $(\alpha,\gamma)$ and letting $\mathcal{W}$ be the part of $\mathcal{U}$ after stage $\alpha$, $\mathcal{W}^\frown\{\mathcal{M}_b^{\mathcal{U}}\}$ is according to the strategy of $\mathcal{O}_\gamma^{\mathcal{M}_\alpha^{\mathcal{U}}}$.
   (b) $\mathcal{U}$ doesn't have a fatal drop, $\mathcal{Q}(\mathcal{U},b)$ exists and $\mathcal{Q}(\mathcal{U},b)\trianglelefteq(\mathcal{J}^{\vec{E},\Psi_{\mathcal{S}^-}})^{\mathcal{N}|\delta}$.
   (c) None of the above holds and there is an extender $E\in\vec{E}^{\mathcal{N}}$ with $crit(E)=\kappa$ and such that $\mathcal{T}\in\mathcal{N}|\nu(E)$ and there is $\sigma:\mathcal{M}_b^{\mathcal{T}}\to\pi_E(\mathcal{Q})$ with the property that $\pi_E\restriction\mathcal{Q}=\sigma\circ\pi_b^{\mathcal{T}}$.

We claim that $\Psi\restriction\mathcal{N}|\delta=\Lambda$. To see this fix $\mathcal{T}\in dom(\Psi)\cap dom(\Lambda)$. Let $b=\Lambda(\mathcal{T})$. Suppose then $b$ is defined by clause 1, clause 2a or clause 2b. Then it follows from universality of $\mathcal{N}$ (see Lemma 2.13) and from our proof of Lemma 5.5 that in fact $\Psi(\mathcal{T})=b$. Suppose then $b$ is defined by clause 2c. Let $E$ be as in clause 2c and let $E^*$ be the resurrection of $E$. Let $\sigma:\mathcal{M}_b^{\mathcal{T}}\to\pi_E(\mathcal{Q})$ be such that $\pi_E\restriction\mathcal{Q}=\sigma\circ\pi_b^{\mathcal{T}}$. Let $i:Ult(\mathcal{N},E)\to\pi_{E^*}(\mathcal{N})$ be such that $\pi_{E^*}\restriction\mathcal{N}=i\circ\pi_E$.

Notice now that we have that $\pi_{E^*}\restriction\mathcal{Q}$ is the iteration embedding according to $\Psi$. Also, notice that $\pi_{E^*}\restriction\mathcal{Q}=i\circ\sigma\circ\pi_b^{\mathcal{T}}$. Because $\Psi$ has branch condensation, we get that $b=\Psi(\mathcal{T})$.

It remains to show that whenever $\mathcal{T}\in dom(\Lambda)$ then $\Lambda(\mathcal{T})$ is defined. Fix then such a $\mathcal{T}\in dom(\Lambda)$. We have that $\Psi(\mathcal{T})$ is defined. Let $b=\Psi(\mathcal{T})$. We claim that $\Lambda(\mathcal{T})=b$. It is easy to see (again using universality and the proof of Lemma 5.5) that if $\mathcal{Q}(\mathcal{T},b)$ exists or that $\mathcal{T}$ is based on $\mathcal{Q}^-$ then indeed $b\in\mathcal{N}$ and $\Lambda(\mathcal{T})=b$.

Suppose then $\mathcal{Q}(\mathcal{T},b)$ does't exist and that $\mathcal{T}$ is not entirely based on $\mathcal{Q}^-$. Let $\mathcal{S}=\mathcal{M}_b^{\mathcal{T}}$. Notice that since $\mathcal{S}\trianglelefteq(\mathcal{J}^{\vec{E},\Psi_{\mathcal{S}^-}}(\mathcal{M}(\mathcal{T})))^{\mathcal{N}|\delta}$, $\mathcal{S}\in\mathcal{N}$. We then need to find $E,\sigma$ as in clause 2c above. Let $E\in\vec{E}^{\mathcal{N}}$ be any extender such that $\mathcal{T}\in\mathcal{N}|\nu(E)$. Let $h\subseteq Coll(\omega,\pi_E(\kappa))$ be $\mathcal{N}$-generic such that $g=h\cap Coll(\omega,\kappa)$. Then $\pi_E$ can be extended to $\pi_E^+:\mathcal{N}[g]\to Ult(\mathcal{N},E)[g][h]$.

Notice that in $\pi_E^+(M)$, $(\mathcal{S},\Psi_{\mathcal{S}^-})$ is a suitable pair that is $\vec{D}_k=(\pi_E^+(B_i^*):i<k)$-iterable for every $k<\omega$. Let then $\sigma_k=(\pi^{\Psi_{\mathcal{S}^-}}_{\mathcal{S},\vec{D}_k,\infty})^{\pi^+(M)}$. Because we have that $\sup_{k<\omega}\gamma^{\mathcal{S},\Psi_{\mathcal{S}^-}}_{\vec{D}_k}=\delta^{\mathcal{S}}$ we have that if $\sigma=\cup_{k<\omega}\sigma_k$ then $\sigma:\mathcal{S}\to\pi_E(\mathcal{Q})$. We claim that $\pi_E\restriction\mathcal{Q}=\sigma\circ\pi_b^{\mathcal{T}}$.

First notice that $\mathcal{Q}=\cup_{k<\omega}H^{\mathcal{Q},\Psi_{\mathcal{Q}^-}}_{\vec{D}_k}$. Let then for $k<\omega$, $\pi_k=\pi_E\restriction H^{\mathcal{Q},\Psi_{\mathcal{Q}^-}}_{\vec{D}_k}$. It is then enough to show that for every $k$, $\pi_k=\sigma_k\circ\pi^{\mathcal{T}}_{\mathcal{Q},\mathcal{S},\vec{D}_k}$. Fix then $k<\omega$ and let $E^*$ be the resurrection of $E$. Let $i:Ult(N,E)\to\pi_{E^*}(\mathcal{N})$ be such that $\pi_{E^*}\restriction\mathcal{N}=i\circ\pi_E$. Notice that because $\pi_{E^*}\restriction\mathcal{Q}$ is the iteration embedding via $\Psi$ and $\pi_k\in Ult(\mathcal{N},E)$, it follows that $i(\pi_k)=\pi_{E^*}\restriction H^{\mathcal{Q},\Psi_{\mathcal{Q}^-}}_{\vec{D}_k}=\pi^{\Psi_{\mathcal{Q}^-}}_{\mathcal{Q},\pi_{E^*}(\mathcal{Q}),(B_i:i<k)}$. Also, $i(\sigma_k)=\pi^{\Psi_{\mathcal{S}^-}}_{\mathcal{S},\pi_{E^*}(\mathcal{Q}),(B_i:i<k)}$. Therefore, we have that $i(\pi_\kappa)=i(\sigma_k)\circ\pi^{\mathcal{T}}_{\mathcal{Q},\mathcal{S},(B_i:i<k)}$.

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Because $i(\pi^{\mathcal{T}}_{\mathcal{Q},\mathcal{S},(B_i:i<k)}) = \pi^{\mathcal{T}}_{\mathcal{Q},\mathcal{S},(B_i:i<k)}$, we get that $\pi_\kappa = \sigma_k \circ \pi^{\mathcal{T}}_{\mathcal{Q},\mathcal{S},(B_i:i<k)}$. This finishes the proof of the claim and also the proof of Theorem 6.5 in the case $\lambda^{\mathcal{P}}$ is a successor. □

**$\lambda^{\mathcal{P}}$ is limit.**

We first assume that $\lambda^{\mathcal{P}}$ has a measurable cofinality in $\mathcal{P}$. The next lemma is very useful for our further computations. It essentially says that if for some $\mathcal{Q} \in pB(\mathcal{P},\Sigma)$, $\mathcal{N}$ captures a tail of some $(\mathcal{Q},\Sigma_{\mathcal{Q}})$ below some strong cardinal $\lambda$ then $\mathcal{N}$ captures $(\mathcal{Q},\Sigma_{\mathcal{Q}})$.

LEMMA 6.10 (Capturing up to a strong cardinal). *Suppose $\nu$ is an inaccessible cardinal, $\lambda > \nu$ is a strong cardinal, $(\vec{\mathcal{S}},\mathcal{Q}) \in B(\mathcal{P},\Sigma) \cap \mathcal{N}^*_x|\nu$, $\mathcal{R}^{\mathcal{Q}}_\nu \in \mathcal{N}$, and $\Psi^{\mathcal{Q}}_\nu \restriction (\mathcal{N}|\lambda) \in \mathcal{N}$. Then, $\Psi^{\mathcal{Q}}_\nu \restriction (\mathcal{N}|\delta) \in \mathcal{N}$.*

PROOF. Let $(\mathcal{R},\Psi) = (\mathcal{R}^{\mathcal{Q}}_\nu, \Psi^{\mathcal{Q}}_\nu)$. We have that $\mathcal{R}$ is the last model of some $\nu$-universal stack $\vec{\mathcal{T}}$ and $\Psi = \Sigma_{\mathcal{R}}$. We need to show that $\Psi \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. Let $\Phi = \Psi \restriction (\mathcal{N} \restriction \lambda)$. Then $\Phi \in \mathcal{N}$. Working in $\mathcal{N}$, we define $\Phi^*$, an extension of $\Phi$ to all trees in $\mathcal{N}|\delta$. Given $\vec{\mathcal{U}}$ on $\mathcal{R}$ we set $\Phi^*(\vec{\mathcal{U}}) = b$ if there is an extender $E \in \vec{E}^{\mathcal{N}}$ with critical point $\lambda$ such that $\vec{\mathcal{U}} \in \mathcal{N}|\nu(E)$ and $b = \pi_E(\Phi)(\vec{\mathcal{U}})$. We need to see that $\Phi^*$ is well defined and in fact, $\Psi \restriction \mathcal{N} = \Phi^*$.

Fix $\vec{\mathcal{U}}$ on $\mathcal{R}$. Suppose $E$ is as above and $b = \pi_E(\Phi)(\vec{\mathcal{U}})$. Let $E^*$ be the resurrection of $E$ and let $\sigma : Ult(\mathcal{N},E) \to \pi_{E^*}(\mathcal{N})$ be the factor embedding. Then

$$\pi_{E^*}(\Phi) = \pi_{E^*}(\Psi) \restriction (\pi_{E^*}(\mathcal{N}) \restriction \pi_{E^*}(\lambda)).$$

But

$$\pi_{E^*}(\Psi) = \pi_{E^*}(\Sigma_{\mathcal{R}} \restriction \mathcal{N}^*_x) = \Sigma_{\mathcal{R}} \restriction Ult(\mathcal{N}^*_x, E^*) = \Psi \restriction Ult(\mathcal{N}^*_x, E^*).$$

The middle equality holds because $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $\Sigma$. Thus, because $\sigma \restriction (Ult(\mathcal{N},E)|\lambda) = id$, we have that

$$\begin{aligned}
\Phi^*(\vec{\mathcal{U}}) = b &\leftrightarrow \pi_E(\Phi)(\vec{\mathcal{U}}) = b \\
&\leftrightarrow \sigma(\pi_E(\Phi))(\vec{\mathcal{U}}) = b \\
&\leftrightarrow \pi_{E^*}(\Phi)(\vec{\mathcal{U}}) = b \\
&\leftrightarrow \pi_{E^*}(\Psi)(\vec{\mathcal{U}}) = b \\
&\leftrightarrow \Psi(\vec{\mathcal{U}}) = b.
\end{aligned}$$

This completes the proof of the lemma. □

Let $\kappa$ be the least cardinal of $\mathcal{N}$ such that $\mathcal{N} \vDash$ "$\kappa$ reflects the set of $< \delta$-strong cardinals". Let $\mathcal{E} = \{E \in \vec{E}^{\mathcal{N}}$ : for all $\eta \in (\kappa,\nu(E))$, $\mathcal{N} \vDash$ "$\eta$ is strong" if and only if $Ult(\mathcal{N},E) \vDash$ "$\eta$ is strong"$\}$.

LEMMA 6.11 (Capturing below $\kappa$). *Suppose $\mathcal{Q} \in pB(\mathcal{P},\Sigma) \cap \mathcal{N}^*_x|\kappa$. Then $\lambda_{\mathcal{Q}} < \kappa$. Thus, $\mathcal{R}^{\mathcal{Q}} \in \mathcal{N}|\kappa$.*

PROOF. Let $\lambda = \lambda_{\mathcal{Q}}$ and $(\mathcal{R},\Psi) = (\mathcal{R}^{\mathcal{Q}},\Psi^{\mathcal{Q}})$. We have that $\Psi \restriction (\mathcal{N}|\delta) \in \mathcal{N}$ and $\Sigma_{\mathcal{R}} \restriction \mathcal{N}^*_x = \Psi$. Let $\nu > \max(o(\mathcal{R}),\kappa)$ be a $< \delta$-strong cardinal. Let $E \in \mathcal{E}$ be an extender such that $crit(E) = \kappa$ and $lh(E) > (\nu^+)^{\mathcal{N}^*_x}$. We then have a factor map $\sigma : Ult(\mathcal{N},E) \to \pi_{E^*}(\mathcal{N})$ such that $\sigma \restriction (Ult(\mathcal{N},E) \restriction \nu(E)) = id$. It follows that $\nu$ is a strong cardinal in $\pi_{E^*}(\mathcal{N})$ because it is a strong cardinal in $Ult(\mathcal{N},E)$.

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Notice that because $\Sigma$ is Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$, we have that $\Psi \restriction Ult(\mathcal{N}_x^*, E^*) \in Ult(\mathcal{N}_x^*, E^*)$. We claim that $\Psi \restriction (\pi_{E^*}(\mathcal{N}|\delta)) \in \pi_{E^*}(\mathcal{N})$. It follows from Lemma 6.10 that it is enough to show that $\Psi \restriction (\pi_{E^*}(\mathcal{N})|\nu) \in \pi_{E^*}(\mathcal{N})$. We have that $\Psi \restriction (\mathcal{N}|\nu) = \Psi \restriction (Ult(\mathcal{N}, E)|\nu)$ and therefore, $\Psi \restriction (Ult(\mathcal{N}, E)|\nu) \in Ult(\mathcal{N}, E)$. But $Ult(\mathcal{N}, E)|(\nu^+)^{Ult(\mathcal{N},E)} = \pi_{E^*}(\mathcal{N})|(\nu^+)^{Ult(\mathcal{N},E)}$, which implies that $\Psi \restriction (\pi_{E^*}(\mathcal{N})|\nu) \in \pi_{E^*}(\mathcal{N})$.

We now have that in $Ult(\mathcal{N}_x^*, E^*)$, $(\mathcal{R}, \Psi \restriction Ult(\mathcal{N}_x^*, E^*))$ is a universal tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $\Psi \restriction (\pi_{E^*}(\mathcal{N}|\delta)) \in \pi_{E^*}(\mathcal{N})$. Because $\lambda < \pi_{E^*}(\kappa)$, by elementarity of $\pi_{E^*}$, we get $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ is captured by $\mathcal{N}$ below $\kappa$ implying that $\lambda < \kappa$. □

LEMMA 6.12. *Suppose $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, $\lambda > \kappa$ is a strong cardinal such that $\lambda_{\mathcal{Q}} < \lambda$, and $E \in \mathcal{E}$ is an extender with critical point $\kappa$ such that $\nu(E) > (\lambda^+)^{\mathcal{N}_x^*}$. Then $\Psi^{\mathcal{Q}} \restriction (Ult(\mathcal{N}, E)|\delta) \in Ult(\mathcal{N}, E)$.*

PROOF. Let $(\mathcal{R}, \Psi) = (\mathcal{R}^{\mathcal{Q}}, \Psi^{\mathcal{Q}})$. Because $\mathcal{N} \restriction (\lambda^+)^{\mathcal{N}} = Ult(\mathcal{N}, E) \restriction (\lambda^+)^{\mathcal{N}}$, we have that $\Psi \restriction (Ult(\mathcal{N}, E) \restriction \lambda) \in Ult(\mathcal{N}, E)$. Let $E^*$ be the resurrection of $E$. We have a factor embedding $\sigma : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ such that $\pi_{E^*} \restriction \mathcal{N} = \sigma \circ \pi_E$ and $\sigma \restriction (\mathcal{N}|\nu(E)) = id$.

Because $\Sigma$ is Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$, we have that

$$\Psi \restriction Ult(\mathcal{N}_x^*, E^*) \in Ult(\mathcal{N}_x^*, E^*).$$

Let $\Phi = \Psi \restriction Ult(\mathcal{N}_x^*, E^*) \in Ult(\mathcal{N}_x^*, E^*)$. Because $\sigma \restriction (\mathcal{N}|(\lambda^+)^{\mathcal{N}}) = id$, we have that $\pi_{E^*}(\mathcal{N})|(\lambda^+)^{\mathcal{N}} = \mathcal{N}|(\lambda^+)^{\mathcal{N}}$. This then implies that $\Phi \restriction (\pi_{E^*}(\mathcal{N})|\lambda) \in \pi_{E^*}(\mathcal{N})$ because $\Phi \restriction (\pi_{E^*}(\mathcal{N})|\lambda) = \Psi \restriction (\mathcal{N}|\lambda)$. Applying Lemma 6.10 in $Ult(\mathcal{N}_x^*, E^*)$, we see that because $\lambda$ is strong in $\pi_{E^*}(\mathcal{N})$, $\Phi \restriction (\pi_{E^*}(\mathcal{N})|\delta) \in \pi_{E^*}(\mathcal{N})$.

Let $\Psi^* = \Psi \restriction (\mathcal{N}|\lambda)$. Then we have that $\Psi^* \in Ult(\mathcal{N}, E)$ and $\sigma(\Psi^*) = \Psi^* = \Phi \restriction (\pi_{E^*}(\mathcal{N})|\lambda)$. By the proof of Lemma 6.10, we have that $\Phi \restriction (\pi_{E^*}(\mathcal{N})|\delta)$ is definable in $\pi_{E^*}(\mathcal{N})$ from $\Psi^*$, $\lambda$ and extenders on $\pi_{E^*}(\mathcal{N})$ that have critical point $\lambda$. This means that the same definition defines a $< \delta$-strategy $\Phi^* \in Ult(\mathcal{N}, E)$. We must have that $\sigma(\Phi^*) = \Phi \restriction (\pi_{E^*}(\mathcal{N})|\delta)$. But using the hull-condensation of $\Sigma$, we see that $\Phi^* = \Psi \restriction (Ult(\mathcal{N}, E)|\delta)$. □

Now, working in $\mathcal{N}$, let

$$\mathcal{F} = \{(\mathcal{Q}, \Lambda) : \mathcal{Q} \in \mathcal{N}|\delta \wedge \mathcal{N} \vDash \text{“}(\mathcal{Q}, \Lambda) \text{ is a hod pair at } \delta \text{ and } \Lambda \text{ has branch condensation and is } \delta\text{-fullness preserving”}\}.$$

Because of Theorem 2.47, we have that $\mathcal{F}$ is a directed system under $\leq^{\mathcal{P},\Sigma} \restriction \mathcal{N}$. Let for $\lambda \leq \delta$, $\mathcal{F} \restriction \lambda = \{(\mathcal{Q}, \Lambda) \in \mathcal{F} : \mathcal{Q} \in \mathcal{N}|\lambda\}$. We let $\mathcal{R}^*$ be the direct limit of $(\mathcal{F} \restriction \kappa, \leq^{\mathcal{P},\Sigma} \restriction \mathcal{N}|\kappa)$ under the iteration maps. Let $\mathcal{R} = \mathcal{R}_\kappa^{\mathcal{P}}$.

LEMMA 6.13 (Capturing $\mathcal{R}$). *Either $\mathcal{R} \trianglelefteq_{hod} \mathcal{R}^*$ or $\mathcal{R}|\delta^{\mathcal{R}} = \mathcal{R}^*$. Moreover, $\mathcal{R} \in \mathcal{N}$.*

PROOF. The first claim is an easy consequence of Lemma 6.11 and Theorem 4.18, and we omit it. We now want to show that $\mathcal{R} \in \mathcal{N}$. Let $\Lambda = \oplus_{\alpha < \lambda^{\mathcal{R}}}(\Sigma_{\mathcal{R}(\alpha)})$. Then we claim that $\Lambda \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. (To see that this is enough, note that $\mathcal{R} = Lp_\omega^\Lambda(\mathcal{R}^-)$ and that $\mathcal{N}$, because of universality, can find $Lp_\omega^\Lambda(\mathcal{R}^-)$).

To see this, it is enough to show that $(\Sigma_{\mathcal{R}_\alpha} \restriction (\mathcal{N}|\delta) : \alpha < \lambda^{\mathcal{R}}) \in \mathcal{N}$. We can define $\Sigma_{\mathcal{R}(\alpha)} \restriction \mathcal{N}$ inside $\mathcal{N}$ as follows. Fix some $(\mathcal{Q}, \Phi) \in \mathcal{F}$ such that $\mathcal{R}(\alpha)$ is a $\Phi$-iterate of $\mathcal{Q}$. Then working in $\mathcal{N}$ we can find a $\kappa$-universal stack $\vec{\mathcal{S}} \in \mathcal{N}$ according to $\Phi$ on $\mathcal{Q}$ with last model $\mathcal{R}(\alpha)$. Using Theorem 2.47 and Lemma 6.11, we get that $\Sigma_{\mathcal{R}(\alpha)} = \Phi_{\mathcal{R}(\alpha)}$. □

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To finish the proof of Theorem 6.5 we need to show that

$$\Sigma_{\mathcal{R}} \restriction (\mathcal{N}|\delta) \in \mathcal{N}.$$

Let $\Sigma^* = \oplus_{\alpha<\lambda^{\mathcal{R}}} \Sigma_{\mathcal{R}(\alpha)}$. We have already shown that $\Sigma^* \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. Working in $\mathcal{N}$, we define an iteration strategy $\Lambda$ for $\mathcal{R}$ and show that it must be $\Sigma_{\mathcal{R}} \restriction (\mathcal{N}|\delta)$.

DEFINITION 6.14 ($\pi_E$-realizable iterations). *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{R}$ and $E \in \mathcal{E}$. We say $\vec{\mathcal{T}}$ is $\pi_E$-realizable if there is a strong cardinal $\lambda < \nu(E)$ such that $\vec{\mathcal{T}} \in \mathcal{N}|\lambda$, a sequence $(\sigma_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ and a sequence $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) \in \mathcal{F} \restriction \lambda : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ such that the following holds:*

1. *$\sigma_{\mathcal{R}} = \pi_E \restriction \mathcal{R}$, for all terminal nodes $\mathcal{Q}$ of $\vec{\mathcal{T}}$, $\sigma_{\mathcal{Q}} : \mathcal{Q} \to \pi_E(\mathcal{R})$ and whenever $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{S}$, $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{S}} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{S}}$.*
2. *For every non-trivial terminal node $\mathcal{Q}$ of $\vec{\mathcal{T}}$, letting $\mathcal{S}_{\mathcal{Q}} \trianglelefteq \pi_E(\mathcal{R})$ be such that $\mathcal{S}_{\mathcal{Q}}$ is a $\Lambda_{\mathcal{Q}}$-iterate of $\mathcal{W}_{\mathcal{Q}}$, we have that $\mathcal{S}_{\mathcal{Q}} = \sigma_{\mathcal{Q}}(\mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1))$ and $\sigma_{\mathcal{Q}}[\mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1)] \subseteq rng(\pi^{\Lambda_{\mathcal{Q}}}_{\mathcal{W}_{\mathcal{Q}},\mathcal{S}_{\mathcal{Q}}})$.*
3. *For every non-trivial terminal node $\mathcal{Q}$, letting $k_{\mathcal{Q}} : \mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1) \to \mathcal{W}_{\mathcal{Q}}$ be given by $k_{\mathcal{Q}}(x) = y$ if and only if $\sigma_{\mathcal{Q}}(x) = \pi^{\Lambda_{\mathcal{Q}}}_{\mathcal{W}_{\mathcal{Q}},\mathcal{S}_{\mathcal{Q}}}(y)$, $k_{\mathcal{Q}}\vec{\mathcal{T}}_{\mathcal{Q}}$ is according to $\Lambda_{\mathcal{Q}}$.*
4. *Suppose $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$. Then $(\mathcal{Q}(\xi^{\vec{\mathcal{T}}}+1), \Lambda^{k_{\mathcal{Q}}}_{\mathcal{Q}}) \in \pi_E(\mathcal{F})$ and $\sigma_{\mathcal{Q}} \restriction \mathcal{Q}(\xi^{\vec{\mathcal{T}}}+1)$ is the embedding given by $\Lambda^{k_{\mathcal{Q}}}_{\mathcal{Q}}$.*
5. *For every $\mathcal{Q}, \mathcal{K} \in ntn(\vec{\mathcal{T}})$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{K}$,*
$$(\Lambda^{k_{\mathcal{K}}}_{\mathcal{K}})_{\mathcal{K}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{K}}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1))} = (\Lambda^{k_{\mathcal{Q}}}_{\mathcal{Q}})_{\mathcal{K}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{K}}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1))}.$$
6. *Suppose $\mathcal{Q}$ is a trivial terminal node of $\vec{\mathcal{T}}$. Then for every $\xi < \lambda^{\mathcal{Q}}$, there is $(\mathcal{W}, \Lambda) \in \mathcal{F}|\lambda$ such that letting $\mathcal{S} \trianglelefteq \pi_E(\mathcal{R})$ be the iterate of $\mathcal{W}$ via $\Lambda$, $\sigma_{\mathcal{Q}}[\mathcal{Q}(\xi+1)] \subseteq rng(\pi^{\Lambda}_{\mathcal{W},\mathcal{S}})$.*

*We say that $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ are the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ are the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$.*

DEFINITION 6.15 (The definition of $dom(\Lambda)$). *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{R}$ such that either there is a strongly linear closed and cofinal set $C \subseteq tn(\vec{\mathcal{T}})$ or $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is of limit length. We let $\vec{\mathcal{T}} \in dom(\Lambda)$ if for some $\xi$ whenever $E \in \mathcal{E}$ is such that $\nu(E) > \xi$, $\vec{\mathcal{T}}$ is $\pi_E$-realizable. We let $\Lambda(\vec{\mathcal{T}}) = b$ if there $\xi$ such that whenever $E \in \mathcal{E}$ is such that $\nu(E) > \xi$, $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}$ is $\pi_E$-realizable.*

First we show that $\Lambda$ is well-defined everywhere on its domain. The following is a useful lemma.

LEMMA 6.16. *Suppose $\vec{\mathcal{T}} \in dom(\Lambda)$ and $\xi$ witnesses this. Let $E \in \mathcal{E}$ be such that for every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$ and for every $\zeta < \lambda^{\mathcal{Q}}$, $\nu_E > \lambda_{\mathcal{Q}(\zeta)}$ and that there is a strong cardinal $\lambda \in (\sup_{\mathcal{Q}\in tnt(\vec{\mathcal{T}}),\zeta<\lambda^{\mathcal{Q}}} \lambda_{\mathcal{Q}(\zeta)}, \nu(E))$. Let $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$. Let $E^*$ be the resurrection of $E$ and let $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor embedding. Then for every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$, $i \circ \sigma_{\mathcal{Q}}$ is the iteration embedding according to $\Sigma_{\mathcal{Q}}$.*

PROOF. We already have that $\pi_{E^*} = i \circ \sigma_{\mathcal{R}}$ is the iteration embedding according to $\Sigma_{\mathcal{R}}$. Suppose now that $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$ is such that for every $\mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$, $i \circ \sigma_{\mathcal{Q}^*}$ is

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according to $\Sigma_{\mathcal{Q}^*}$. Suppose first that there is no largest $\mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$. It then follows that $\mathcal{Q}$ is the direct limit of $\{\mathcal{Q}^* : \mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}\}$ under $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}^{**}}$ and $\sigma_{\mathcal{Q}}$ is given by

$$\sigma_{\mathcal{Q}}(x) = \sigma_{\mathcal{Q}^*}(y)$$

where $y \in \mathcal{Q}^*$ is such that $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(y) = x$. It then follows that $i \circ \sigma_{\mathcal{Q}}$ is according to $\Sigma_{\mathcal{Q}}$.

Next suppose that there is a largest $\mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$. Let $\zeta = \xi^{\vec{\mathcal{T}},\mathcal{Q}^*}$. We then have that $\vec{\mathcal{T}}_{\mathcal{Q}^*,\mathcal{Q}}$ is based on $\mathcal{Q}^*(\zeta+1)$ and it follows from the branch condensation of $\Sigma_{\mathcal{Q}^*}$ that $\vec{\mathcal{T}}_{\mathcal{Q}^*,\mathcal{Q}}$ is according to $\Sigma_{\mathcal{Q}^*}$. It then follows that

$$\sigma_{\mathcal{Q}}(x) = \sigma_{\mathcal{Q}^*}(f)(\sigma_{\mathcal{Q}}(a))$$

where $f \in \mathcal{Q}^*$ and $a \in [\mathcal{Q}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(\zeta+1))]^{<\omega}$ are such that $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(f)(a) = x$. It is then enough to show that $\sigma_{\mathcal{Q}} \restriction \mathcal{Q}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(\zeta+1))$ is according to $\Sigma_{\mathcal{Q}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(\zeta+1))}$. It follows from clause 5 of Definition 6.14 and our inductive hypothesis that this would follow provided we show that $(\Lambda^{k^{\mathcal{Q}^*}}_{\mathcal{Q}^*})_{\mathcal{Q}^*(\zeta+1)} = \Sigma_{\mathcal{Q}^*(\zeta+1)} \restriction (\mathcal{N}|\delta)$.

Let then $\Phi = (\Lambda^{k^{\mathcal{Q}^*}}_{\mathcal{Q}^*})_{\mathcal{Q}^*(\zeta+1)}$ and let $\nu < \nu(E)$ be a strong cardinal such that $\mathcal{Q}^* \in \mathcal{N}|\nu$. It follows from Lemma 6.10 that it is enough to show that $\Phi \restriction (\mathcal{N}|\nu) = \Sigma_{\mathcal{Q}^*(\zeta+1)} \restriction (\mathcal{N}|\nu)$. Notice that

$$Ult(\mathcal{N}, E) \vDash \Phi \restriction (\mathcal{N}|\nu) = ((\Lambda_{\mathcal{Q}^*})_{\mathcal{S}_{\mathcal{Q}^*}(\sigma_{\mathcal{Q}^*}(\zeta+1))})^{\sigma_{\mathcal{Q}^*}} \restriction (\mathcal{N}|\nu).$$

Notice however that because of our choice of $E$,

$$i((\Lambda_{\mathcal{Q}^*})_{\mathcal{S}_{\mathcal{Q}^*}(\sigma_{\mathcal{Q}^*}(\zeta+1))}) = \Sigma_{\mathcal{S}_{\mathcal{Q}^*}(\sigma_{\mathcal{Q}^*}(\zeta+1))} \restriction (\mathcal{N}|\delta) \qquad (*).$$

Indeed, this follows from the fact that we can find, in $Ult(\mathcal{N}, E)$, a common tail $(\mathcal{Q}^{**}, \Lambda^*)$ of $(\mathcal{Q}^*, \Lambda_{\mathcal{Q}^*})$ and $(\mathcal{R}^{\mathcal{Q}^*}, \Psi^{\mathcal{Q}^*} \restriction (Ult(\mathcal{N}, E)))$ such that $\mathcal{Q}^{**} \in \mathcal{N}|\lambda$. Because $i \circ \sigma_{\mathcal{Q}^*}$ is the iteration embedding according to $\Sigma_{\mathcal{Q}^*}$ and because $i(\Phi)$ is $i \circ \sigma_{\mathcal{Q}^*}$-pullback of $i((\Lambda_{\mathcal{Q}^*})_{\mathcal{S}_{\mathcal{Q}^*}(\sigma_{\mathcal{Q}^*}(\zeta+1))})$, it follows from $(*)$ that $i(\Phi) = \Sigma_{\mathcal{Q}^*(\zeta+1)} \restriction (\pi_{E^*}(\mathcal{N}))$. Since $i \restriction \lambda = id$, it follows that $\Phi \restriction (\mathcal{N}|\nu) = \Sigma_{\mathcal{Q}^*(\zeta+1)} \restriction (\mathcal{N}|\nu)$. This finishes the proof of the claim. □

If $\vec{\mathcal{T}}$ and $E$ are as in the hypothesis of Lemma 6.16 then we say $E$ is *above* $\vec{\mathcal{T}}$.

LEMMA 6.17. *Suppose $\vec{\mathcal{T}}$ is $\pi_E$-realized for some $E$ that is above $\vec{\mathcal{T}}$. Then $\vec{\mathcal{T}} \in dom(\Sigma_{\mathcal{R}})$.*

PROOF. Let $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$. We need to show that for any $\mathcal{Q} \in tn(\vec{\mathcal{T}})$, $\mathcal{R}$-to-$\mathcal{Q}$ part of $\vec{\mathcal{T}}$ is according to $\Sigma_{\mathcal{R}}$ and if $\mathcal{S}_{\vec{\mathcal{T}}}$ exists then $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is according to $\Sigma_{\mathcal{S}_{\vec{\mathcal{T}}}}$. The first claim follows easily from branch condensation of $\Sigma_{\mathcal{R}}$. Let $E^*$ be the resurrection of $E$ and let $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor embedding. We then have that $\pi_{E^*} \restriction \mathcal{R} = i \circ \sigma_{\mathcal{Q}} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{R},\mathcal{Q}}$. Because $\pi_{E^*} \restriction \mathcal{R}$ is just the iteration embedding according to $\Sigma_{\mathcal{R}}$, it follows from the branch condensation of $\Sigma_{\mathcal{R}}$, that $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$ is according to $\Sigma_{\mathcal{R}}$.

Next, suppose that $\mathcal{S}_{\vec{\mathcal{T}}}$ exists. Let $\mathcal{K} = \mathcal{S}_{\vec{\mathcal{T}}}$ and let $\mathcal{T} = \vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$. We need to see that $\mathcal{T}$ is according to $\Sigma_{\mathcal{K}(\eta+1)}$ where $\eta = \xi^{\vec{\mathcal{T}},\mathcal{W}}$. We have that $\mathcal{T}$ is according to $\Lambda^{k^*_{\mathcal{K}}}_{\mathcal{K}}$. Let $E \in \mathcal{E}$ be above $\vec{\mathcal{T}}$. We then have that $\Psi^{\mathcal{W}(\eta)} \restriction (\mathcal{N}|\delta) \in \mathcal{N}$. Let again

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$(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$. Let $E^*$ be the resurrection of $E$ and let $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor embedding.

Notice that we have that in $Ult(\mathcal{N}, E)$, $(\Lambda_{\mathcal{K}})_{\mathcal{S}_{\mathcal{K}}} = \Sigma_{\mathcal{S}_{\mathcal{W}}} \restriction (Ult(\mathcal{N}, E)|\delta)$. This is simply because we can find a common tail $(\mathcal{W}^*, \Lambda^*)$ of $(\mathcal{W}_{\mathcal{K}}, \Lambda_{\mathcal{K}})$ and $(\mathcal{R}^{\mathcal{K}(\eta+1)}, \Psi^{\mathcal{K}(\eta+1)} \restriction (Ult(\mathcal{N}, E)|\delta))$ such that $\mathcal{W}^* \in \mathcal{N}|\lambda$ where $\lambda < \nu_E$ is some strong cardinal such that $\vec{\mathcal{T}} \in \mathcal{N}|\lambda$. We also have that $\mathcal{T}$ is according to $\sigma_{\mathcal{K}}$-pullback of $(\Lambda_{\mathcal{K}})_{\mathcal{S}_{\mathcal{K}}}$. This then implies that $\mathcal{T}$ is according to $i \circ \sigma_{\mathcal{K}}$-pullback of $i((\Lambda_{\mathcal{K}})_{\mathcal{S}_{\mathcal{K}}})$. But

$$i((\Lambda_{\mathcal{K}})_{\mathcal{S}_\kappa}) = i((\Psi^{\mathcal{K}(\eta+1)})_{\mathcal{S}_{\mathcal{K}}}) = \Sigma_{i(\mathcal{S}_{\mathcal{K}})} \restriction (\pi_{E^*}(\mathcal{N}|\delta)).$$

But because $i \circ \sigma_{\mathcal{K}}$ is the iteration embedding according to $\Sigma_{\mathcal{K}(\eta+1)}$ (see Lemma 6.16), we have that $\mathcal{T}$ is is according to $\Sigma_{\mathcal{K}(\eta+1)}$. □

It follows from Lemma 6.17 that to show that $\Sigma_{\mathcal{R}} \restriction (\mathcal{N}|\delta) = \Lambda$ it is enough to show that $\Lambda$ is defined everywhere on its domain. The following lemma then finishes the proof of Theorem 6.5.

LEMMA 6.18. *Suppose $\vec{\mathcal{T}} \in dom(\Lambda)$. Then $\Lambda(\vec{\mathcal{T}})$ is defined.*

PROOF. Let $\xi$ witnesses the fact that $\vec{\mathcal{T}} \in dom(\Lambda)$. We have that $\vec{\mathcal{T}} \in dom(\Sigma_{\mathcal{R}})$. Suppose for a moment that $E$ is above $\vec{\mathcal{T}}$. Let $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$. Suppose first $\mathcal{S}_{\vec{\mathcal{T}}}$ is defined. Then letting $\mathcal{Q} = \mathcal{S}_{\vec{\mathcal{T}}}$ and $\mathcal{T} = \vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$, let $b = \Lambda_{\mathcal{Q}}(\kappa_{\mathcal{Q}}\mathcal{T})$. It follows from the proof of Lemma 6.17 that $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}$ is according to $\Sigma_{\mathcal{R}}$. Because $E$ was arbitrary, it follows that $b$ is independent of the choice of $E$.

Let $\xi^*$ be such that every $E \in \mathcal{E}$ with the property that $\nu(E) > \xi^*$ is above $\vec{\mathcal{T}}$. If $\pi^{\vec{\mathcal{T}}}_b$ doesn't exist then follows that $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}$ is according to $\Lambda$ as witnessed by $\xi^*$. Suppose next that $\pi^{\vec{\mathcal{T}}}_b$ exists. Notice that $\xi^*$ witnesses all the clauses of Definition 6.14 except possibly clause 6. Let then $\mathcal{S} = \mathcal{M}^{\vec{\mathcal{T}}}_b$ and let $\lambda^* = \sup_{\mathcal{K} \in tnt(\vec{\mathcal{T}}^\frown\{\mathcal{S}\}), \zeta < \lambda^{\mathcal{K}}} \lambda_{\mathcal{K}(\zeta)}$. Let $\lambda > \max(\lambda^*, \xi^*)$ be a strong cardinal and let $E \in \mathcal{E}$ be such that $\nu(E) > \lambda$. Let $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable embeddings of $\vec{\mathcal{T}}$ and $((\mathcal{W}_{\mathcal{Q}}, \Lambda_{\mathcal{Q}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ be the $\pi_E$-realizable pairs of $\vec{\mathcal{T}}$. Let $\sigma_{\mathcal{S}} : \mathcal{S} \to \pi_E(\mathcal{R})$ be given by $\sigma_{\mathcal{S}}(x) = \sigma_{\mathcal{Q}}(f)(\tau(a))$ where $f \in \mathcal{Q}$ and $a \in (\delta(\mathcal{T}))^{<\omega}$ are such that $\pi^{\mathcal{T}}_b(f)(a) = x$ and letting $\zeta = \xi^{\vec{\mathcal{T}}}$, $\tau = \pi^{\Lambda^{k_{\mathcal{Q}}}_{\mathcal{Q}}}_{\mathcal{S}(\pi^{\vec{\mathcal{T}}}_b(\zeta+1)), \mathcal{S}_{\mathcal{Q}}}$.

It follows from Lemma 6.16 that if $E^*$ is the resurrection of $E$ and $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ is the factor map then $i \circ \sigma_{\mathcal{S}} = \pi^{\Sigma_{\mathcal{R}}}_{\mathcal{S}, \mathcal{S}_{\mathcal{Q}}}$. Therefore, letting $j = i \circ \sigma_{\mathcal{S}}$, for every $\phi < \lambda^{\mathcal{S}}$,

$$j[\mathcal{S}(\phi)] \subseteq \pi^{\Psi^{\mathcal{S}(\phi)}}_{\mathcal{S}(\phi), \pi_{E^*}(\mathcal{S}_{\mathcal{Q}})(\sigma_{\mathcal{S}}(\phi))}.$$

It then follows that $(\mathcal{R}^{\mathcal{S}(\phi)}, \Psi^{\mathcal{S}(\phi)} : \phi < \lambda^{\mathcal{S}})$ witnesses clause 6. Therefore, as $E$ was arbitrary, $\lambda$ witnesses that $\vec{\mathcal{T}}^\frown\{\mathcal{S}\}$ is according to $\Lambda$.

Next suppose $\mathcal{S}_{\vec{\mathcal{T}}}$ is undefined. In this case, we have a strongly linear closed and unbounded $C \subseteq ntn(\vec{\mathcal{T}})$. Let then $b = b_C$ be the branch of $\vec{\mathcal{T}}$ given by $C$. Recall that $b$ consists of downward closure of $C$ in $\vec{\mathcal{T}}$. Let again $\lambda^* =$

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$\sup_{\mathcal{K}\in tnt(\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}),\zeta<\lambda^{\mathcal{K}}}\lambda_{\mathcal{K}(\zeta)}$. It follows from the proof given above that if $\lambda > \max(\lambda^*,\xi)$ then $\lambda$ witnesses that $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is according to $\Lambda$. □

## 6.4. The mouse set conjecture

In this section we prove that the Mouse Capturing holds in the minimal model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular".

THEOREM 6.19 (The Mouse Set Conjecture). *Assume $AD^+$ and that there is no $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$ such that $L(\Gamma,\mathbb{R}) \vDash$ "$AD_{\mathbb{R}}+\Theta$ is regular". Then Strong Mouse Capturing holds.*

We will need the following theorem which is essentially due to Steel and Woodin. We adopted it for hod mice. Readers familiar with their proof will have no difficulties proving our version.

THEOREM 6.20 (Steel and Woodin, [**37**]). *Assume $AD^+$ and suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $L(\Sigma,\mathbb{R}) \vDash$ "$\Sigma$ has branch condensation and is $\Gamma$-super fullness preserving for some $\Gamma$". Then $L(\Sigma,\mathbb{R}) \vDash$ "MC relative to $\Sigma$".*

We devote this entire section to the proof of Theorem 6.19. Without loss of generality, let us assume that $MC$ fails. In the general case, we will have to repeat the arguments that follow except relativized to some hod pair $(\mathcal{P},\Sigma)$ where $\Sigma$ is fullness preserving and has branch condensation. Fix then $x,y\in\mathbb{R}$ such that $y$ is $OD(x)$ yet there is no $x$-mouse containing $y$. There is then some $\Gamma\subsetneq\mathcal{P}(\mathbb{R})$ and some $\alpha$ such that $L_{\alpha+1}(\Gamma,\mathbb{R}) \vDash$ "$y$ is $OD(x)$", $L_\alpha(\Gamma,\mathbb{R}) \vDash$ "$y$ isn't $OD(x)$", and

$$\mathcal{P}(\mathbb{R})\cap L_\alpha(\Gamma,\mathbb{R}) = \Gamma.$$

LEMMA 6.21. *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\Sigma$ is super fullness preserving and has branch condensation. Suppose also that $Code(\Sigma)\in\Gamma$. Moreover, suppose that whenever $\mathcal{Q}\in pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma)$ is such that $\lambda^{\mathcal{Q}}$ is a successor, there is a sequence $\vec{B}=(B_i : i<\omega)\subseteq(\mathbb{B}(\mathcal{Q}^-,\Sigma_{\mathcal{Q}^-}))^{L(\Gamma(\mathcal{Q},\Sigma_{\mathcal{Q}}),\mathbb{R})}$ such that $\vec{B}$ strongly guides $\Sigma$. Then for cone of $z$, every sound $\Sigma$-mouse $\mathcal{M}$ over $z$ such that $\rho(\mathcal{M})=z$ has a strategy $\Phi$ such that $Code(\Phi)\in\Gamma$.*

PROOF. Suppose not. Then for cone of $z$, there is a $\Sigma$-mouse $\mathcal{M}_z$ over $z$ such that $\rho(\mathcal{M}_z)=z$ and if $\Phi$ is the unique strategy of $\mathcal{M}_z$ then $Code(\Phi)\notin\Gamma$. We then claim that this implies that $y$ is in a $\Sigma$-mouse over $(\mathcal{P},x)$. Let $\Gamma^* = (\Sigma^2_1(Code(\Sigma))^{L_\alpha(\Gamma,\mathbb{R})}$.

*Claim.* $y$ is in a $\Sigma$-mouse over $(\mathcal{P},x)$.

PROOF. Let $\Phi_z$ be the iteration strategy of $\mathcal{M}_z$ and let $w$ be such that for any $z$ coding $w$, $Code(\Phi_z)\notin\Gamma$. Notice that what we have is that for any $z$ Turing above $w$, $C_{\Gamma^*}(z)\subseteq Lp^{\Gamma,\Sigma}(z)$ (one way to see this is to use a result due to Steel on jump operators, see [**38**]). Let $A$ be a set of reals coding the function $z\to\Phi_z$. Let $\Gamma_1<_w\Gamma_2$ be good pointclasses such that $Code(\Sigma),A\in\Delta_{\Gamma_1}$ and $\Gamma\subseteq\Delta_{\underset{\sim}{\Gamma}_1}$. Let $F$ be as in Theorem 2.25 for $\Gamma_2$ and let $z$ be such that $w,y\leq_T z$ and $(\mathcal{N}_z^*,\delta_z,\Sigma_z)$ Suslin, co-Suslin captures $Code(\Sigma)$, $A$ and the set coding the function $u\to C_{\Gamma_1}(u)$. Let $\mathcal{N}=(\mathcal{J}^{\vec{E},\Sigma}(\mathcal{P},x))^{\mathcal{N}_z^*|\delta_z}$. We must then have $\eta<\delta_z$ such that $C_{\Gamma_1}(\mathcal{N}_z^*|\eta)\vDash$ "$\eta$ is Woodin". Let $\eta$ be the least such and let $\mathcal{Q}=Lp^{\Gamma_1,\Sigma}_\omega(\mathcal{N}|\eta)$. We have that $\mathcal{Q}\trianglelefteq\mathcal{N}$

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and $\mathcal{Q} \vDash$ "$\eta$ is Woodin". Using the proof of Lemma 5.5, it is not hard to check that $\mathcal{Q}$ is a $\Sigma$-hod premouse over $(\mathcal{P}, x)$ with $\lambda^{\mathcal{Q}} = 0$.

We have that $z$ is generic for the extender algebra of $\mathcal{Q}$ at $\eta$ and moreover, $\mathcal{Q}[z]$ is $\Sigma$-closed and in fact, if $u$ is any $\mathcal{Q}$-generic code of $(\mathcal{Q}|\eta, z)$ then if $\mathcal{S}$ is the output of the $\mathcal{S}$-construction of $\mathcal{Q}[u]$ over $u$ with respect to $\Sigma$ then $\mathcal{M}_u \trianglelefteq \mathcal{Q}^*$. Because iterability of $\mathcal{Q}^*$ is certified by the extenders of $\mathcal{Q}[u]$ that have critical points $> \eta$, there must be some $\nu < (\eta^+)^{\mathcal{Q}}$ such that if $\Psi$ is the iteration strategy of $\mathcal{Q}|\nu$ that acts on trees that are above $\eta$ then $Code(\Psi) \notin \Gamma$. Thus, $\mathcal{Q}|\nu \notin C_{\Gamma^*}(\mathcal{Q}|\eta[z])$ implying that $C_{\Gamma^*}(\mathcal{Q}|\eta) \subseteq \mathcal{Q}$. But because $y \in C_{\Gamma^*}(\mathcal{Q}|\eta[z])$, we have that $y \in \mathcal{Q}$ as $Lp^{\Gamma,\Sigma}(\mathcal{Q}|\eta) \trianglelefteq \mathcal{Q}$. This then implies that $y$ is in a $\Sigma$-mouse over $(\mathcal{P}, x)$.

□

By the same proof, we in fact have that whenever $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, $y$ is in a $\Sigma_{\mathcal{Q}}$-mouse over $(\mathcal{Q}, x)$. We now use Theorem 6.5 to get a contradiction. For $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, let $\mathcal{M}_{\mathcal{Q}}$ be the least $\Sigma_{\mathcal{Q}}$-mouse over $(\mathcal{Q}, x)$ containing $y$ and let $\Phi_{\mathcal{Q}}$ be its strategy. Let $A \subseteq \mathbb{R}$ code the set $\{(\mathcal{M}_{\mathcal{Q}}, \Phi_{\mathcal{Q}}) : \mathcal{Q} \in pI(\mathcal{P}, \Sigma)\}$. Let $\Gamma_1$ be a good pointclass such that $A \in \Delta_{\underset{\sim}{\Gamma}_1}$. Let $F$ be as in Theorem 2.25 for $\Gamma_1$ and let $u$ be such that $(\mathcal{N}_u^*, \delta_u, \Sigma_u)$ Suslin, co-Suslin captures $A$, $Code(\Sigma)$ and the function $u \to Lp(u)$. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}}(x))^{\mathcal{N}_u^* | \delta_u}$. Then using Theorem 5.20 and Theorem 6.5, we get that there is $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in L[\mathcal{N}]$. Then by the universality of $\mathcal{N}$, we get that $\mathcal{M}_{\mathcal{Q}} \in \mathcal{N}$, and hence, $y \in \mathcal{N}$. Therefore, $y$ is in a $x$-mouse, contradiction! □

Our strategy now is to show that there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma \subseteq L(\Sigma, \mathbb{R})$ and $\Sigma$ has branch condensation and is fullness preserving. Once we do this, it will follow that $y$ is in a $\Sigma$-mouse over $(\mathcal{P}, x)$. Then using our capturing result, Theorem 6.5, we will get that $y$ is in some $x$-mouse.

LEMMA 6.22. *Suppose $(\mathcal{M}, \Sigma)$ is either an anomalous hod pair of type II or III, or a hod pair such that $\lambda^{\mathcal{M}}$ is a successor ordinal. Suppose further that for all $\alpha < \lambda^{\mathcal{M}}$, $\Sigma_{\mathcal{M}(\alpha+1)}$ is fullness preserving in $L(\Sigma, \mathbb{R})$. Then for all $\alpha < \lambda^{\mathcal{M}}$, $L(\Sigma, \mathbb{R}) \vDash$ "$MC$ with respect to $\Sigma_{\mathcal{M}(\alpha)}$".*

PROOF. Towards a contradiction, suppose not. Let $\alpha$ be least such that $L(\Sigma, \mathbb{R}) \vDash$ "$MC$ fails for $\Sigma_{\mathcal{M}(\alpha)}$". Because $\Gamma(\mathcal{M}, \Sigma)$ is mouse full and $\Gamma(\mathcal{M}, \Sigma) \vDash SMC$, it follows from (the relativized version of) Theorem 6.1 that there is a $\Sigma_{\mathcal{M}(\alpha)}$-hod pair, perhaps an anomalous one, $(\mathcal{R}, \Lambda)$ such that $\Gamma(\mathcal{M}, \Sigma) = \Gamma(\mathcal{R}, \Lambda)$. But then $L(\Sigma, \mathbb{R}) = L(\Lambda, \mathbb{R})$ (this follows from Theorem 5.9).

We now start working entirely in $L(\Lambda, \mathbb{R})$. Suppose that $x, y \in \mathbb{R}$ are such that $y$ is $OD(\Sigma_{\mathcal{M}(\alpha)}, x)$. We have that by Theorem 6.20 whenever $\mathcal{Q} \in pI(\mathcal{R}, \Lambda)$, $MC$ holds with respect to $\Lambda_{\mathcal{Q}}$. Because $\Sigma_{\mathcal{M}(\alpha)}$ is $OD(\Lambda)$, whenever $\mathcal{Q} \in pI(\mathcal{R}, \Lambda)$, there is a $\Lambda_{\mathcal{Q}}$-mouse $\mathcal{S}$ over $x$ such that $y \in \mathcal{S}$. We let $\mathcal{S}_{\mathcal{Q}}$ be the largest sound $\Lambda_{\mathcal{Q}}$-mouse over $x$ such that $y \notin \mathcal{S}_{\mathcal{Q}}$ and $\rho(\mathcal{S}_{\mathcal{Q}}) = x$. Let $\Psi_{\mathcal{Q}}$ be the strategy of $\mathcal{S}_{\mathcal{Q}}$ as a $\Lambda_{\mathcal{Q}}$-mouse.

Let $\beta$ be least such that $L_\beta(\Lambda, \mathbb{R}) \vDash$ "$y$ is $OD(\Sigma_{\mathcal{Q}}, x)$" and let $\Gamma$ be a good pointclass beyond $L_\beta(\Lambda, \mathbb{R})$ such that $Code(\Lambda) \in \Delta_{\underset{\sim}{\Gamma}}$. Let $A \subseteq \mathbb{R}$ code $L_\beta(\Lambda, \mathbb{R})$. Let $B$ be the set of reals that codes a continuous function $\sigma$ such that for some $\mathcal{Q} \in pI(\mathcal{R}, \Lambda)$, $\sigma^{-1}[A]$ is a code for $(\mathcal{Q}, \Lambda_{\mathcal{Q}}, \Psi_{\mathcal{Q}})$. Then let $F$ be as in Theorem 2.25 and let $w$ be such that $(\mathcal{N}_w^*, \delta_w, \Sigma_w)$ Suslin, co-Suslin captures $A$, $B$, $Code(\Lambda)$, $Code(\Sigma)$ and $Code(\Sigma_{\mathcal{M}(\alpha)})$.

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Let $\mathcal{N} = (\mathcal{J}^{\vec{E},\Sigma_{\mathcal{M}(\alpha)}}(x))^{\mathcal{N}^*_w|\delta_w}$. If $\Lambda$ is fullness preserving then by Theorem 6.5, we have that there is some $\mathcal{Q} \in \mathcal{N}|\delta_w$ such that $\Lambda_{\mathcal{Q}} \restriction \mathcal{N} \in L[\mathcal{N}]$. By absoluteness between $\mathcal{N}^*_w$ and $V$ and by universality of $\mathcal{N}$, we then have that $\mathcal{S}_{\mathcal{Q}} \in \mathcal{N}$. This implies that $y \in \mathcal{N}$ and because $\mathcal{N}$ is $\Sigma_{\mathcal{M}(\alpha)}$-mouse over $x$, we get a contradiction. Hence, assume that $\Lambda$ isn't fullness preserving. Notice that by the same argument no tail of $\Lambda$ is fullness preserving.

*Claim.* There is $\mathcal{W} \in pI(\mathcal{R},\Lambda)$ such that for some $\xi < \lambda^{\mathcal{W}}$, $\Lambda_{\mathcal{W}(\xi)}$ is fullness preserving but for any $\mathcal{S} \in pI(\mathcal{W}(\xi+1),\Lambda_{\mathcal{W}(\xi+1)})$ there is $\eta > \delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}$ which is a strong cutpoint of $\mathcal{S}$ and

$$\mathcal{S}|(\eta^+)^{\mathcal{S}} \neq Lp^{\Lambda_{\mathcal{S}^-}}(\mathcal{S}|\eta).$$

or

$$\mathcal{S}|(\delta^+_{\lambda^{\mathcal{S}}-1})^{\mathcal{S}} \neq Lp^{\oplus_{\gamma<\lambda^{\mathcal{S}}-1}\Lambda_{\mathcal{S}(\gamma+1)}}(\mathcal{S}^-).$$

PROOF. Let $\Gamma^*$ be a good pointclass such that $Code(\Lambda) \in \Delta_{\widetilde{\Gamma^*}}$ and let $F$ be as in Theorem 2.25 for $\Gamma^*$. Let $w$ be such that $(\mathcal{N}^*_w, \delta_w, \Sigma_w)$ Suslin, co-Suslin captures $Code(\Lambda)$. Suppose $(\mathbb{C}_\gamma, \mathcal{P}_\gamma, \Sigma_\gamma, \delta_\gamma, \eta_\gamma : \gamma < \zeta)$ is the output of the $\Gamma(\mathcal{R},\Lambda)$-hod pair construction of $\mathcal{N}^*_w|\delta_w$. Then it follows from Theorem 2.33 that there is $\gamma$ such that $(\mathcal{P}_\gamma, \Sigma_\gamma)$ is a tail of $(\mathcal{R},\Lambda)$. Let then $\mathcal{N} = (\mathcal{J}^{\vec{E},\Sigma_\gamma})^{\mathcal{N}^*_w|\delta_w}$.

If $\mathcal{N}$ has no level projecting across $\delta_\gamma$ then by Lemma 5.7, we get that $\Sigma_\gamma$ is fullness preserving. Because no tail of $\Lambda$ is fullness preserving, this cannot happen. Therefore, there must be some largest initial segment $\mathcal{K} \trianglelefteq \mathcal{N}$ such that either $\rho(\mathcal{K}) < \delta_\gamma$.

It follows from the proof of Lemma 6.1 that we cannot have that for every $\xi < \gamma$, $\Sigma_\xi$ is fullness preserving (otherwise there would be $OD$ subset of HOD that is not in HOD, namely the image of $\mathcal{K}$ under $\pi^{\Sigma_\xi}_{\mathcal{P}_\xi,\infty}$). Let then $\nu$ be the least such that $\Sigma_\nu$ isn't fullness preserving. Suppose first $\nu$ is limit. Repeating the argument just given we have that $(\mathcal{J}^{\vec{E},\Sigma_\nu})^{\mathcal{N}^*_w|\delta_w}$ must have a level projecting across $\delta_\nu$ implying that there is $\zeta < \nu$ such that $\Sigma_\zeta$ isn't fullness preserving. It then follows that $\nu$ is a successor.

Let then $\mathcal{M} = (\mathcal{J}^{\vec{E},\Sigma_\nu})^{\mathcal{N}^*_w|\delta_w}$. Notice that $\mathcal{M} \vDash$ "$\delta_\nu$ is Woodin" then it follows from Lemma 5.7 that $\Sigma_\nu$ is fullness preserving. Let then $\mathcal{K} \trianglelefteq \mathcal{M}$ be the largest such that $\mathcal{K} \vDash$ "$\delta_\nu$ is Woodin" but $\mathcal{J}_1(\mathcal{K}) \vDash$ "$\delta_\nu$ isn't Woodin". Because $\Sigma_\nu$ carries $\mathcal{K}$ with it, it follows that letting $\xi = \nu - 1$ and $\mathcal{W} = \mathcal{P}_\nu$, $(\xi, \mathcal{W})$ satisfy the claim. □

By absoluteness then we can assume that $\mathcal{W}$ of the claim is in $\mathcal{N}^*_x$. Using Theorem 5.20 and Theorem 6.5, we can fix $\mathcal{Q} \in pI(\mathcal{W}(\xi+1),\Lambda_{\mathcal{W}(\xi+1)})$ such that $\mathcal{Q}(\lambda^{\mathcal{Q}}-1) \in \mathcal{N}$ and $\Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)} \restriction \mathcal{N} \in L[\mathcal{N}]$. Let $\Phi = \Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)} \restriction \mathcal{N}$. Let $\mathcal{N}^* = (\mathcal{J}^{\vec{E},\Phi})^{\mathcal{N}}$.

Repeating the proof of the claim in the proof of Lemma 6.21 or using the proof of Lemma 5.13, we get that the least strong cardinal of $\mathcal{N}^*$ is a limit of Woodin cardinals. Moreover, if $\eta$ is the least Woodin of $\mathcal{N}^*$, by the proof of Lemma 5.8, letting $((\mathcal{M}_\zeta, \mathcal{N}_\zeta : \zeta \leq \eta), (F_\zeta : \zeta < \eta))$ be the output of the $(\mathcal{J}^{\vec{E},\Phi})^{\mathcal{N}|\eta}$-construction, there is $\xi$ such that $\mathcal{Q}$ iterates to $\mathcal{N}_\xi$ via a normal tree which is according to $\Lambda_{\mathcal{Q}}$. Let $\mathcal{K} = \mathcal{N}_\xi$.

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It follows from the claim that either there is $\nu > o(\mathcal{Q}^-)$ which is a strong cutpoint of $\mathcal{K}$ and there is a sound $\Lambda_{\mathcal{Q}^-}$-mouse $\mathcal{S}$ over $\mathcal{K}|\nu$ such that $\rho(\mathcal{S}) = \nu$ and $\mathcal{S} \not\trianglelefteq \mathcal{K}$ or there is a sound $\oplus_{\gamma<\lambda^{\mathcal{K}}-1}\Lambda_{\mathcal{K}(\gamma+1)}$-mouse $\mathcal{S}$ over $\mathcal{K}^-$ such that $\rho(\mathcal{S}) = \delta^{\mathcal{K}^-}$ and $\mathcal{S} \not\trianglelefteq \mathcal{K}$. We assume that $\mathcal{S}$ is the least such. By universality of $\mathcal{N}^*$, $\mathcal{S} \in \mathcal{N}^*$. Let $\Psi$ be the strategy of $\mathcal{S}$ above $\nu$. By the proof of Lemma 5.8, we have that $\Psi \restriction \mathcal{N} \in L[\mathcal{N}]$. Because $\Lambda$ is $\Gamma(\mathcal{R}, \Lambda)$-fullness preserving, $\Psi \notin \Gamma(\mathcal{R}, \Lambda)$ and therefore, $L(\Lambda, \mathbb{R}) = L(\Psi, \mathbb{R})$.

Let now $\kappa$ be the least strong cardinal of $\mathcal{N}^*$. Let $g \subseteq Coll(\omega, < \kappa)$ be $\mathcal{N}^*_w$-generic. Let $M$ be the derived model of $\mathcal{N}^*$ at $\kappa$ as computed in $\mathcal{N}^*_w[g]$. Then by the proof of Lemma 5.8 and Lemma 5.13, $\Psi \restriction HC^M \in M$. This then implies that $\Lambda \restriction HC^M \in M$. Let then $\eta^* < \kappa$ be such that $\mathcal{N}^*[g \cap Coll(\omega, \eta^*)]$ has a $\kappa$-UB code for $\Lambda$. Because $\mathcal{N}^*[g \cap Coll(\omega, \eta^*)]$ has infinitely many Woodin cardinals, it follows that $\mathcal{M}^{\#,\Lambda}_\omega$-exists, which is a contradiction as we are in $L(\Lambda, \mathbb{R})$. This completes the proof of Lemma 6.22. □

Our goal now is to prove the following lemma.

LEMMA 6.23. *There is a hod pair $(\mathcal{P}, \Sigma)$ such that*

(1) *$\Sigma$ is fullness preserving and has branch condensation,*
(2) *$\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma$ and $\Gamma \subseteq L(\Sigma, \mathbb{R})$.*

PROOF. Suppose not. Using Theorem 6.1 we get that either $\Gamma$ isn't mouse full or $L_\alpha(\Gamma, \mathbb{R}) \vDash \neg SMC$. Without loss of generality we can assume that $L_\alpha(\Gamma, \mathbb{R}) \vDash SMC$ (otherwise we can drop to a smaller pointclass). Thus, suppose $\Gamma$ isn't mouse full. Let $A$ be the set of hod pairs $(\mathcal{P}, \Sigma)$ such that $Code(\Sigma) \in \Gamma$ and $\Sigma$ is fullness preserving and has branch condensation.

*Claim 1.* $A \neq \emptyset$.

PROOF. Let $\Gamma^*$ be a good pointclass such that $Mice \in \Delta_{\Gamma^*}$ and there is sjs $\vec{C} = (C_i : i < \omega) \in \Delta_{\underset{\sim}{\Gamma^*}}$ such that $C_0 = Mice$. Then let $F$ be as in Theorem 2.25 for $\Gamma^*$ and let $z$ be such that $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Mice$ and $\vec{C}$. Then the first model of hod pair construction of $\mathcal{N}^*_x$ exists (see the proof of Lemma 5.13). Let $(\mathcal{P}, \Sigma) = (\mathcal{P}_0, \Sigma_0)$. We have that $\Sigma$ is fullness preserving and has branch condensation. Moreover, $Code(\Sigma) \in \Gamma$ as otherwise $\Gamma \subseteq L(\Sigma, \mathbb{R})$. Hence, $A \neq \emptyset$. □

*Claim 2.* Suppose $(\mathcal{P}, \Sigma) \in A$. Then there is hod pair $(\mathcal{Q}, \Lambda) \in A$ such that $\lambda^{\mathcal{Q}}$ is a successor, $\Lambda$ has branch condensation and is fullness preserving and $\mathcal{Q}^- \in pI(\mathcal{P}, \Sigma)$.

PROOF. Let $\Gamma^*$ be a good pointclass such that $Mice_\Sigma \in \Delta_{\Gamma^*}$ and there is sjs $\vec{C} = (C_i : i < \omega) \in \Delta_{\underset{\sim}{\Gamma^*}}$ such that $C_0 = Mice_\Sigma$. Then let $F$ be as in Theorem 2.25 for $\Gamma^*$ and let $z$ be such that $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Mice_\Sigma$ and $\vec{C}$. Then there is $\beta$ such that the $\beta$th model of the hod pair construction of $\mathcal{N}^*_z$ exists and $(\mathcal{P}_\beta, \Sigma_\beta)$ is a tail of $(\mathcal{P}, \Sigma)$. We claim that $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$ exists. The proof of Lemma 5.13 implies that $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$-exists provided

(1) $\mathcal{N}_{\beta+1}$ doesn't project across $\mathcal{P}_\beta$,
(2) if $\beta = \gamma + 1$ then $\mathcal{N}_{\beta+1} \vDash$ "$\delta_\beta$ is Woodin",

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(3) if $\beta$ is limit then no level of $\mathcal{N}_{\beta+1}$ projects across $\delta_\beta$ and $(\delta^+_\beta)^{\mathcal{N}_\beta} = (\delta^+_\beta)^{\mathcal{P}_\beta}$.

Suppose first that 1, 2 or the first half of 3 fail. Let $\mathcal{M} \trianglelefteq \mathcal{N}_{\beta+1}$ be the least level of $\mathcal{N}_{\beta+1}$ witnessing the failure of 1, 2 of the first half of 3. Let $\Sigma$ be the strategy of $\mathcal{M}$. Then we have that $\Sigma$ is fullness preserving in $L(\Sigma, \mathbb{R})$. Because by Lemma 6.22, $L(\Sigma, \mathbb{R}) \vDash SMC$, we get a contradiction as in the proof of Theorem 6.1. If it is the second half of 3 that fails then we finish as in the proof of Theorem 6.1. Thus, 1-3 hold, and therefore, $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$-exists and $\Sigma_{\beta+1}$ has branch condensation and is fullness preserving (see Lemma 5.11 and Lemma 5.7). □

Let then $\Gamma_1 = \cup_{(\mathcal{P},\Sigma)\in A}\Gamma(\mathcal{P}, \Sigma)$. Notice that if $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda) \in A$ then either $\Gamma(\mathcal{P}, \Sigma) \trianglelefteq_\theta \Gamma(\mathcal{Q}, \Lambda)$ or vice a versa. Therefore, $\Gamma_1$ is a mouse full pointclass such that $\Gamma_1 \vDash SMC$. Because of Claim 2, $\Omega^\Gamma$ must be limit. It follows from Theorem 6.1 that there must be possibly an anomalous hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma_1$ and $\Sigma$ has branch condensation. It then follows that $\Sigma$ is fullness preserving. Suppose first that $\mathcal{P}$ is not an anomalous hod premouse. Then because $\Gamma \not\subseteq L(\Sigma, \mathbb{R})$, we have that in fact $(\mathcal{P}, \Sigma) \in A$. It then follows from Claim 2 that $\Gamma(\mathcal{P}, \Sigma) \subsetneq \Gamma_1$, which is a contradiction.

Suppose then $(\mathcal{P}, \Sigma)$ is an anomalous hod pair of type II or III. We can now repeat the proof of Theorem 6.1 in $L(\Sigma, \mathbb{R})$ and get a contradiction ($(\mathcal{P}, \Sigma)$ produces an $OD^{L(\Sigma,\mathbb{R})}$-subset of $\mathrm{HOD}^{L(\Sigma,\mathbb{R})}$ which is not in $\mathrm{HOD}^{L(\Sigma,\mathbb{R})}$). This completes the proof of the lemma. □

To finish the proof of Theorem 6.19, fix $(\mathcal{P}, \Sigma)$ such that

(1) $\Sigma$ is fullness preserving and has branch condensation,
(2) $\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma$ and $\Gamma \subseteq L(\Sigma, \mathbb{R})$.

Because $L(\Sigma, \mathbb{R}) \vDash SMC$, we get a contradiction. This completes the proof of Theorem 6.19.

## 6.5. A last word

In the hypothesis of Theorem 6.19 we assume that there is no proper class model containing the reals and satisfying $AD_\mathbb{R}$ + "$\Theta$ is regular". How does this hypothesis relate to "there is no mouse with a superstrong cardinal" hypothesis, which is what was used in the statement of the Mouse Set Conjecture? It turns out that $AD_\mathbb{R}$ + "$\Theta$ is regular" is much weaker than "there is a mouse with a superstrong cardinal". Our goal in this short section is to establish this. First, we state the following unpublished theorems of Woodin.

THEOREM 6.24 (Woodin). *It is consistent relative to a Woodin limit of Woodins that there are $A_0, A_1 \subseteq \mathbb{R}$ such that $L(A_i, \mathbb{R}) \vDash AD^+$ but $A_0$ and $A_1$ are not Wadge comparable.*

THEOREM 6.25 (Woodin). *If $A_0, A_1 \subseteq \mathbb{R}$ are as in Theorem 6.24 then if $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(A_0, \mathbb{R}) \cap L(A_1, \mathbb{R})$ then $L(\Gamma, \mathbb{R}) \vDash AD_\mathbb{R}$.*

We can now establish that a Woodin limit of Woodins is an upper bound for $AD_\mathbb{R}$ + "$\Theta$ is regular".

THEOREM 6.26. *It is consistent relative to a Woodin limit of Woodins that there is a model of $AD_\mathbb{R}$ + "$\Theta$ is regular".*

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PROOF. Without loss of generality, we assume that there are $A_0, A_1 \subseteq \mathbb{R}$ as in Theorem 6.24. Suppose there is no model of $AD_{\mathbb{R}}$ + "$\Theta$ is regular". Then it follows from Theorem 6.19 that $L(A_i, \mathbb{R}) \vDash SMC$. Let $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(A_0, \mathbb{R}) \cap L(A_1, \mathbb{R})$. Then using Theorem 6.1 we get hod pairs $(\mathcal{P}_i, \Sigma_i) \in L(A_i, \mathbb{R})$ such that $\Sigma_i$ has branch condensation and is $\Gamma$-fullness preserving, and $\Gamma(\mathcal{P}_i, \Sigma_i) = \Gamma$. It then follows that $\Sigma_0 \notin L(A_1, \mathbb{R})$ and $\Sigma_1 \notin L(A_0, \mathbb{R})$ (this is simply because we have that $\Sigma_i \notin \Gamma$). Then using Theorem 2.47, we can get $(\mathcal{Q}, \Lambda)$ which is a common tail of $(\mathcal{P}_0, \Sigma_0)$ and $(\mathcal{P}_1, \Sigma_1)$.

This then gives a contradiction as follows. For $i < 2$ let $\pi_i : \mathcal{P}_i \to \mathcal{Q}$ be the iteration embedding according to $\Sigma_i$. We then have that $\Lambda = (\pi_i\text{−pulback of } \Sigma_i)$. It then follows that for each $i < 2$, $Code(\Lambda) \in L(A_i, \mathbb{R})$. On the other hand, since $\Gamma(\mathcal{Q}, \Lambda) = \Gamma$, we have that $Code(\Lambda) \notin \Gamma$ . □

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APPENDIX A

# Descriptive set theory primer

Here we review some descriptive set theoretic notions. The standard reference is [**21**]. Unless otherwise specified, we assume $ZF + DC_{\mathbb{R}} + AD$ and let $\mathbb{R}$ stand for the Baire space $\omega^\omega$. We also fix one of the standard ways of (i) coding hereditarily countable sets by reals and (ii) coding subsets of $\mathbb{R}^n$ for $n \in \omega$ by subsets of $\mathbb{R}$. Throughout this book, we work relative to these coding methods.

## A.1. Pointclasses

Following [**21**] we say $\Gamma$ is a pointclass if it is a collection of sets of reals (that is, we are not requiring pointclasses to be closed under any operation). If $\Gamma$ is a pointclass then $\breve{\Gamma} = \{A \subseteq \mathbb{R} : A^c \in \Gamma\}$ is the dual pointclass and $\Delta_\Gamma = \Gamma \cap \breve{\Gamma}$.

Given two sets of reals $A$ and $B$, $A$ is Wadge reducible to $B$ if there is a real $\sigma$ that codes a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $A = f^{-1}[B]$. We write $A \leq_w B$ for "$A$ is Wadge reducible to $B$". We also say that $B$ is Wadge above $A$. Martin showed that $\leq_w$ is a well founded relation (see [**46**]). Given a set of reals $A$ we let $w(A) = |A|_{\leq_w}$. It can be shown that $w(A) = w(A^c)$ (see [**46**]). If $\Gamma$ is a pointclass then we let $w(\Gamma) = \sup\{w(A) : A \in \Gamma\}$.

Recall that a relation $\leq$ is a *prewellordering* if it is transitive, reflexive, connected and wellfounded. Given a set of reals $A$, $\phi$ is a norm on $A$ if $\phi : A \to \text{Ord}$. For each norm $\phi$ on $A$, we let $\leq^\phi$ be the binary relation on $A$ given by $x \leq^\phi y$ iff $\phi(x) \leq \phi(y)$. Then $\leq^\phi$ is a prewellordering of $A$. The opposite is true as well, given a prewellordering $\leq$ of $A$ there is an associated norm $\phi$ defined on $A$ such that $\leq = \leq^\phi$. If $\Gamma$ is a pointclass then $\phi$ is a $\Gamma$-norm if there are relations $\leq^\phi_\Gamma \in \Gamma$ and $\leq^\phi_{\breve{\Gamma}} \in \breve{\Gamma}$ such that for every $y \in dom(\phi)$ and for any $x \in \mathbb{R}$,

$$[x \in dom(\phi) \wedge \phi(x) \leq \phi(y)] \leftrightarrow x \leq^\phi_\Gamma y \leftrightarrow x \leq^\phi_{\breve{\Gamma}} y.$$

A sequence of norms $\vec{\phi} = \langle \phi_i : i < \omega \rangle$ on $A$ is a *scale* on $A$ if whenever $\langle x_i : i < \omega \rangle \subseteq A$ is a sequence of reals converging to $x$ such that for each $i$ the sequence $\langle \phi_i(x_k) : k < \omega \rangle$ is eventually constant then $x \in A$ and for each $i$, $\phi_i(x) \leq \lambda_i$ where $\lambda_i$ is the eventual value of $\langle \phi_i(x_k) : k < \omega \rangle$. The later property is called *lower semi-continuity*. We write $x_i \to x (mod \vec{\phi})$ if $\langle x_i : i < \omega \rangle$ converges to $x$ in the above sense. $\vec{\phi}$ is a $\Gamma$-scale on $A$ if there are relations $R \in \Gamma$ and $S \in \breve{\Gamma}$ such that for all $y \in A$, for any $x \in \mathbb{R}$ and for any $n < \omega$

$$[x \in A \wedge \phi_n(x) \leq \phi_n(y)] \leftrightarrow R(n, x, y) \leftrightarrow S(n, x, y).$$

We say $\Gamma$ has the *prewellordering property* if every set in $\Gamma$ has a $\Gamma$-norm. We say $\Gamma$ has the *scale property* if every set in $\Gamma$ has a $\Gamma$-scale. A sequence of norms $\langle \phi_i : k < \omega \rangle$ is called a *semi scale* if it has all the properties of a scale except lower semi-continuity. We also say that a sequence of sets of reals $(A_i : i \in \omega)$ is a (semi) scale if for some (semi) scale $(\phi_i : i \in \omega)$, $A_i$ codes $\leq_{\phi_i}$.



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DEFINITION A.1. *$(A_i : i < \omega)$ is a self-justifying system (sjs) if (i) $A_i \subseteq \mathbb{R}$, (ii) for every $n$, there exists $k$ such that $A_k = A_n^c$, and (iii) for each $n$ there is a subsequence $(A_{k_i} : i < \omega)$ such that $(A_{k_i} : i < \omega)$ is a scale on $A_n$ (i.e., $A_k$'s code prewellorderings $\phi_n$ such that $(\phi_n : n < \omega)$ is a scale). $(A_i : i < \omega)$ is a semi-self-justifying system (ssjs) if (i) $A_i \subseteq \mathbb{R}$, (ii) for every $n$, there exists $k$ such that $A_k = A_n^c$, and (iii) for each $n$ there is a subsequence $(A_{k_i} : i < \omega)$ such that $(A_{k_i} : i < \omega)$ is a semi scale on $A_n$.*

Suppose $\kappa$ is a cardinal. $T \subseteq \cup_{n<\omega}\omega^n \times \kappa^n$ is a tree if whenever $s \in T$ then $s \restriction i \in T$ for any $i < lh(s)$. For $(x, f) \in \omega^\omega \times \kappa^\omega$ is a branch of $T$ if $(x \restriction i, f \restriction i) \in T$ for any $i < \omega$. $[T]$ is the set of branches of $T$. $p[T]$ is the projection of $[T]$ on the first coordinate, i.e., $x \in p[T]$ if and only if there is $f \in \kappa^\omega$ such that $(x, f) \in T$.

A set of reals $A$ is $\kappa$-Suslin if there is a tree $T \subseteq \cup_{n<\omega}\omega^n \times \kappa^n$ such that $A = p[T]$. $A$ is Suslin if it is $\kappa$-Suslin for some $\kappa$. Given a scale $\vec{\phi}$ on $A$ one can construct a tree $T$ such that $p[T] = A$. More precisely, let $T$ be the set of pairs $(s, f)$ such that there is some real $x \in A$ such that $s \lhd x$ and $f(i) = \phi_i(x)$ for each $i < lh(f)$. Given a tree $T$ such that $p[T] = A$, one can get a scale $\vec{\phi}$ on $A$ by considering the leftmost branches of $T$. Thus, carrying a scale and being Suslin are equivalent. Finally, we say that $\kappa$ is a *Suslin cardinal* if there is a set of reals $A$ which is $\kappa$-Suslin but $A$ is not $\eta$-Suslin for any $\eta < \kappa$.

A pointclass $\Gamma$ is called *good* if it is closed under recursive preimages, it is $\omega$-parameterized, it is closed under $\exists^{\mathbb{R}}$ quantification and it has the scale property. The smallest good pointclass is $\Sigma^1_2$. Notice that good pointclasses are lightface pointclasses. However, if $\Gamma$ is good and $U^k \subseteq \omega \times \mathbb{R}^k$ is a universal $\Gamma$ set then the pointclass

$$\underset{\sim}{\Gamma} = \{A \subseteq \mathbb{R} : \exists n, k \in \omega \exists x \in \mathbb{R}(A = U^k_{n,x})\}$$

is a boldface pointclass and is closed under both continuous preimages and $\exists^{\mathbb{R}}$.

Given a pointclass $\Gamma$, we say $x$ is a $\Gamma$-singleton if $\{x\} \in \Gamma$. We say $x$ is $\Gamma$-definable from $y$ if there is a set $B \in \Gamma$ such that

$$x(n) = m \leftrightarrow B(n, m, y).$$

We say $x$ is $\Gamma$-definable from $y$ in a countable ordinal if there is $B \in \Gamma$ such that for some countable ordinal $\alpha$, whenever $z$ is a real coding $\alpha$, for any $n, m \in \omega$

$$x(n) = m \leftrightarrow B(n, m, y, z).$$

If $\Gamma$ is good then one can define $C_\Gamma(x)$, the largest countable $\Gamma$ set as follows

$$C_\Gamma(x) = \{y : y \text{ is } \Gamma\text{-definable from } x \text{ in some countable ordinal }\}.$$

By a result of Becker-Kechris (see [**1**]), if $T$ is a tree of a $\Gamma$-scale on a universal $\Gamma$ set then $L[T, x]$ for any real $x$ is independent of both $T$ and the choice of the universal set. By a result of Harrington and Kechris (see [**4**]),

$$C_\Gamma(x) = \mathbb{R}^{L[T,x]}.$$

Note that $C_\Gamma(x)$ is Turing invariant, i.e., if $x \equiv_T y$ then $C_\Gamma(x) = C_\Gamma(y)$. Using ideas from [**11**], one can extend $x \to C_\Gamma(x)$ to act on $HC$, the set of hereditary countable sets. Given a countable transitive set $a$, a set $b \subseteq a$ and a real $x$ coding $a$ we let $b_x$ be the real given by $x$ that codes $b$, i.e., if $E_x$ is the extensional well-founded relation coded by $x$ and $\pi_x : (a, \in) \to (x, E_x)$ is the collapsing map then $n \in b_x \leftrightarrow \pi_x^{-1}(n) \in b$. We can then define $C_\Gamma(a)$ by

$$C_\Gamma(a) = \{b \subseteq a : \text{for all } x \text{ coding } a,\ b_x \in C_\Gamma(x)\}.$$

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Note that it follows that $C_\Gamma(a) = (\mathcal{P}(a))^{L[T,a]}$ because we can define $C_\Gamma(a)$ in $L[T,a]$ (see [**31**] and [**42**]). Again using some elementary facts about forcing, it is not hard to show that in fact

$$C_\Gamma(a) = \{b \subseteq a : \text{for comeager many } x \text{ coding } a,\ b_x \in C_\Gamma(x)\}.$$

Kechris, in [**12**], showed that $C_\Gamma(x)$ has a $\Gamma$-good well ordering uniform in $x$. A well ordering $\leq \subseteq \mathbb{R}^2$ is $\Gamma$-good if for every $x \in dom(\leq)$, $\leq\restriction x = \{y \in dom(\leq) : y \leq x\}$ is countable and there are relations $R, S \in \Gamma$ such that for every $y \in dom(\leq)$ and for every $x \in \mathbb{R}$,

$$\{x_i : i < \omega\} = \{z : z < y\} \leftrightarrow R(x,y) \leftrightarrow \neg S(x,y).$$

We let $<_{\Gamma,x}$ be one such uniform $\Gamma$-good well ordering of $C_\Gamma(x)$.

## A.2. $Env(\Gamma)$

Suppose $\Gamma$ is a good scaled pointclass and there are scaled pointclasses beyond $\Gamma$. Martin and Woodin independently found an exact description of the next scaled pointclass after $\Gamma$. Martin's work is presented in [**6**] while Woodin's work is unpublished and uses inner model theoretic methods. Essentially the next scaled pointclass after $\Gamma$ is the pointclass of $\lambda$-Suslin sets where $\lambda = w(Env(\Gamma))$ where $Env(\Gamma)$, the *envelope* of $\Gamma$, is defined below. Our presentation below follows [**42**].

Definition A.2. *Suppose $\Gamma$ is a good pointclass, and $A \subseteq \mathbb{R}$. We say $A$ is countably captured over $\Gamma$ just in case there is a real $x$ such that for all countable $\sigma \subseteq \mathbb{R}$ with $x \in \sigma$, $A \cap \sigma \in C_\Gamma(\sigma \cup \{\sigma\})$. We call such a real $x$ a $\Gamma$-good parameter for $A$. We call $x$ a $\Gamma$-good parameter for a sequence $\vec{A}$ if $x$ is a $\Gamma$-good parameter for $A_i$ for each $i$.*

Let now

$$Env(\Gamma) = \{A \subseteq \mathbb{R} : A \text{ is countably captured over } \Gamma\}.$$

If $\Gamma$ is a good pointclass and $\vec{A}$ is a sjs or a ssjs such that $\vec{A} \subseteq Env(\Gamma)$ and $A_0$ is a universal $\Gamma$ set then we say $\vec{A}$ *seals* $\Gamma$.

Theorem A.3 (Martin, [**6**]). *If $\Gamma$ is closed under real quantifiers, then*

1. *$Env(\Gamma)$ is closed under real quantifiers, and hence, $Env(\Gamma)$ is a projectively closed boldface pointclass, and*
2. *if there are scaled pointclasses beyond $\Gamma$ then there is a semi-scale on a universal $\Gamma$-dual set whose sequence of associated prewellorders is a subset of $Env(\Gamma)$.*

Theorem A.4 (Martin and Jackson, [**5**], [**6**])). *Suppose $\Gamma$ is a good pointclass closed under real quantification and there are scaled pointclasses beyond $\Gamma$. Then there is a sjs $\vec{A}$ which seals $\Gamma$.*

Trevor Wilson proved a lightface version of this theorem (see Theorem 4.1.4 of [**47**]).

## A.3. $AD^+$

The axioms of $AD^+$ were isolated and extensively investigated by Woodin. Unfortunately, most of his work is unpublished. Our outline of $AD^+$ is based on [**26**], [**36**], [**49**] and [**48**].

A set of reals $A$ is called $^\infty$Borel if there is a set of ordinals $S$, an ordinal $\alpha$ and a formula $\phi(x_0, x_1)$ such that

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$$x \in A \leftrightarrow L_\alpha[S, x] \vDash \phi[S, x].$$

$(S, \phi)$ is called an $^\infty$Borel code for $A$. An equivalent definition can be given in terms of an infinitary logic that allows infinite conjunctions and disjunctions, in which case the resemblance to ordinary Borel sets becomes apparent. Note that if $A$ is Suslin and $T$ is such that $p[T] = A$ then $T$ witnesses that $A$ is $^\infty$Borel. Thus, all Suslin sets are $^\infty$Borel.

THEOREM A.5 (Woodin). *Assume ZF+ AD+DC$_\mathbb{R}$. Suppose $A \subseteq \mathbb{R}$ and $A$ is $^\infty$Borel. Then $A$ has an $^\infty$Borel code $S$ coded by a set of reals projective in $A$.*

COROLLARY A.6 (Woodin). *Suppose $M \subseteq N$ are two models of AD such that $\mathbb{R} \in M$, $N \vDash ZF + AD + DC_\mathbb{R}$, and $A \in \mathcal{P}(\mathbb{R}) \cap M$ is $^\infty$Borel in $N$. Then $A$ is $^\infty$Borel in $M$.*

Suppose $\lambda$ is an ordinal and $A \subseteq \lambda^\omega$. Then $A$ is determined if one of the players has a winning strategy in the two player game on $\lambda$ with payoff set $A$. Let $G_A^\lambda$ be this game. Ordinal determinacy is the statement that

**Ordinal Determinacy:** for any $\lambda < \Theta$, for any continuous function $\pi : \lambda^\omega \to \omega^\omega$, and for any set $A \subseteq \omega^\omega$ the set $\pi^{-1}[A]$ is determined.

The following is a positive result on ordinal determinacy.

THEOREM A.7 (Moschovakis and independently Woodin, [**13**] and [**22**]). *Suppose $\kappa < \Theta$ and $\lambda \in (\kappa, \Theta)$ is a Suslin cardinal. Then for any $A \subseteq \mathbb{R}$, and for any continuous $\pi : \kappa^\omega \to \omega^\omega$, $G^\kappa_{\pi^{-1}[A]}$ is determined.*

DEFINITION A.8 (Woodin). *$AD^+$ is the following theory*

1. *ZF+ AD+DC$_\mathbb{R}$.*
2. *All sets of reals are $^\infty$Borel.*
3. *Ordinal determinacy.*

The following is an important consequence of $AD^+$.

THEOREM A.9 (Steel and Woodin, [**36**]). *Assume $AD^+$. Then the class of Suslin cardinals is closed below $\Theta$.*

Under $V = L(\mathcal{P}(\mathbb{R}))$, $AD^+$ implies a very useful $\Sigma_1$-reflection property. Whether such a reflection holds under just $AD$ is an important open problem. Suppose first that $\Gamma$ is a pointclass such that for any $A \in \Gamma$,

$$L_{\eta_A}(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \subseteq \Gamma.$$

where $\eta_A$ is the least $A$-admissible ordinal, i.e., it is the least ordinal $\alpha$ such that $L_{\eta_A}(A, \mathbb{R}) \vDash KP$. We then let

$$M_\Gamma = N_\Gamma = \cup\{L_{\eta_A}(A, \mathbb{R}) : A \in \Gamma\}.$$

Recall that if $\Gamma$ is a pointclass then $\delta(\Gamma) = \sup\{|\leq^*| : \leq^* \in \Delta_\Gamma \text{ is a pre-wellordering}\}$. If now $A$ is a set of reals then we let $\underset{\sim}{\delta}^2_1(A) = \delta(\underset{\sim}{\Sigma}^2_1(A))$.

THEOREM A.10 (Woodin, unpublished but see Theorem 9.7 of [**48**]). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $A$ is a set of reals such that there is a Suslin cardinal in the interval $(w(A), \theta_A)$. Then*

1. *For any real $x \in \mathbb{R}$ the pointclass $\Sigma^2_1(A, x)$ has the scale property and hence, $\Sigma^2_1(A, x)$ is a good pointclass.*

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(2) $\underset{\sim}{\delta}^2_1 < \theta_A$, *and* $\underset{\sim}{\delta}^2_1(A)$ *is the largest Suslin cardinal below* $\theta_A$.
(3) *For any real* $x$, $\Delta^2_1(A,x)$ *is a basis for* $\Sigma^2_1(A,x)$, *i.e., every* $\Sigma^2_1(A,x)$ *set has a* $\Delta^2_1(A,x)$*-member. Furthermore, for any formula* $\phi$,

$$\exists B \subseteq \mathbb{R}\langle HC, A, B, \in\rangle \vDash \phi[x] \leftrightarrow \exists B \in \Delta^2_1(A,x)\langle HC, A, B, \in\rangle \vDash \phi[x].$$

(4) $M_{\underset{\sim}{\Delta}^2_1(A)} \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.
(5) $L_\Theta(\mathcal{P}(\mathbb{R})) \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.

It is not hard to see, using part 2 of Theorem A.10, that if $A, B \subseteq \mathbb{R}$ are such that there is a Suslin cardinal in the interval $(w(A), \theta_A)$ and in the interval $(w(B), \theta_B)$ and $B$ is $OD(A,x)$ for some $x$ then $\underset{\sim}{\delta}^2_1(B) \leq \underset{\sim}{\delta}^2_1(A)$ (the assumption that there is a Suslin cardinal in those intervals is used to make sense of $\underset{\sim}{\delta}^2_1(A)$ and $\underset{\sim}{\delta}^2_1(B)$). Also, if $A$ is a set of Wadge rank $\underset{\sim}{\delta}^2_1(B)$ for some $B$ for which there is a Suslin cardinal in the interval $(w(B), \theta_B)$, then $\theta_B = \theta_A$. Notice that it follows from Moschovakis' coding lemma that the statement "there is a Suslin cardinal in the interval $(w(B), \theta_B)$" is upward absolute in the sense that if $\mathbb{R} \subseteq M \subseteq N$ are models of $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ and $B \in \mathcal{P}(\mathbb{R}) \cap M$ then

$$M \vDash \text{"there is a Suslin cardinal in the interval } (w(B), \theta_B)\text{"}$$

if and only if

$$N \vDash \text{"there is a Suslin cardinal in the interval } (w(B), \theta_B)\text{"}$$

This is because $w(B)^M = w(B)^N$, $\mathcal{P}(\mathbb{R})^M = \{A \in N : w(A) < \Theta^M\}$ is ordinal definable in $N$, and if $\mu < \Theta^M$ is a Suslin cardinal in $N$ then it is a Suslin cardinal in $M$.

## A.4. The derived model theorem

THEOREM A.11 (The Derived Model Theorem, Woodin, [**49**]). *Suppose that* $\delta$ *is a limit of Woodin cardinals. Suppose* $g \subseteq Coll(\omega, < \delta)$ *is* $V$*-generic and let*

$$\mathbb{R}_g = \cup\{(\mathbb{R})^{V[g\restriction\alpha]} : \alpha < \delta\}.$$

*Let* $\Gamma$ *be the set of all*

$$A \in \mathcal{P}(\mathbb{R}_g) \cap V(\mathbb{R}_g)$$

*such that* $L(A, \mathbb{R}_g) \vDash AD^+$. *Then in* $V(\mathbb{R}_g)$ *the following hold.*

(1) $L(\Gamma, \mathbb{R}_g) \vDash AD^+$.
(2) *For each* $A \in \mathcal{P}(\mathbb{R}_g) \cap V(\mathbb{R}_g)$ *the following are equivalent.*
   (a) $A$ *is Suslin, co-Suslin in* $V(\mathbb{R}_g)$.
   (b) $A \in \Gamma$ *and* $A$ *is Suslin, co-Suslin in* $L(\Gamma, \mathbb{R}_g)$.

The model $L(\Gamma, \mathbb{R}_g)$ is called the derived model at $\delta$. While it depends on $g$, because of the homogeneity of the Levy collapse, its theory does not.

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