The comparison theory of hod pairs below AD^++"The largest Suslin cardinal is a member of the Solovay sequence" *†

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Abstract

We develop the basic theory of hod mice below the *Largest Suslin Axiom* (LSA), which says that the largest Suslin cardinal is a member of the Solovay sequence. We also prove comparison theorem for such hod mice in the context of AD^+. This is the first paper in planned sequence of three papers that eventually will establish that *Mouse Set Conjecture* holds provided there is no inner model satisfying LSA.

Over the last two decades the *Mouse Set Conjecture* (MSC) has become one of the central open problems in inner model theory. One of the main reasons for this is that MSC is intimately connected with the inner model program, which is the program for constructing canonical inner models for large cardinals. Moreover,

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MSC lies in the heart of the core model induction, the most successful technique for calibrating lower bounds of set theoretic statements. An interested reader can consult [5] for more on the role of MSC in set theory.

In [3], the author developed the theory of *hod mice* below the theory $AD_{\mathbb{R}}$ + "Θ is regular" and used it to show that MSC holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular". In this paper, we extend the theory of hod mice much beyond up to the level of the *Largest Suslin Axiom* (LSA), which says that the largest Suslin cardinal is a member of the Solovay sequence. Our main theorem here is a universality result for fully backgrounded constructions (see Theorem 18.13). As a corollary we obtain that two hod mice with fullness preserving strategies can be compared via a normal iteration (see Corollary 18.14). However, here we do not prove that MSC holds in the minimal model of LSA. The proof of this result along with a proof that LSA is consistent relative to large cardinals will appear in subsequent publications.

It is expected that the work carried out here will be useful in core model induction applications just like [3] has been useful in core model induction applications (see for instance, [4] and [6]). In particular, we expect a positive resolution of the following weakening of the *PFA Conjecture*, which conjectures that PFA is equiconsistent with a supercompact cardinal.

Conjecture 0.1 *PFA implies that there is an inner model containing the reals and ordinals and satisfying LSA.*

In [4], the author showed that in the presence of some mild large cardinals PFA implies that there is an inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}}$ + "Θ is regular". We expect that the use of the large cardinal axiom is not essential, and that the result can just be obtained from PFA[1].

The general philosophy of organizing hod mice that we employ here is that of *layering*. Layers are where we activate new strategies. This means that we *skip over* non-Solovay pointclasses and *do not skip* over Solovay pointclasses. Recall that a pointclass Γ is called a Solovay pointclass if there is $\alpha < \Theta$ such that $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Layered hod mice have the following useful properly. If $(\mathcal{P}, \Sigma)$ is a hod pair then α is a layer of $\mathcal{P}$ if and only if $(\mathcal{P}|\alpha, \Sigma_\alpha)$ generates Solovay pointclass. The precise meaning of "skipping over" will become clear from reading the definition of hod mouse (see Definition 15.2), but it essentially says that if we have reached a level where our hod pair generates a Solovay pointclass then we activate the strategy. It is because of this principle that we have the equivalence mentioned above. We start our exposition by reviewing hybrid $\mathcal{J}$-structures.

[1] In fact, just from the failure of square at a singular strong limit cardinal.

1 Layered hybrid $\mathcal{J}$-structures

In what follows, given a transitive set M (or a structure) we will use $o(M)$ to denote the ordinal height of M. Also, given a set X, we let $trc(X)$ be the transitive closure of X. We also let $trc^X = (trc(X \cup \{X\}), X, \in)$.

Definition 1.1 (Definition of [3]) *Given a function f, we say f is amenable if the domain of f consists of transitive structures and for some formula ϕ and for all $a = (M, A, \in) \in dom(f)$*

1. *$f(a) \subseteq o(a)$ and $0 \in f(a)$,*
2. *letting $\beta = \sup f(a)$, $\beta < o(a)$ is the unique ordinal γ such that $a \vDash \phi[\gamma]$,*[2]
3. *whenever $\eta < \sup f(a)$, $f(a) \cap \eta \in M$.*

We let ϕ_f be the formula ϕ above.

We say f is a *shift of an amenable function* or a *shifted amenable function* if for all $a = (M, A, \in) \in dom(f)$, $f(a) \subseteq [min(f(a)), min(f(a)) + o(a))$ and there is an amenable function g such that (i) $dom(f) = dom(g)$ and (ii) for all $a \in dom(f)$ and $\gamma < o(a)$, $f(a) = \{min(f(a)) + \gamma : \gamma \in g(a)\}$. Notice that if f is a shift of an amenable function then it uniquely determines g. We say that g is the amenable component of f.

Jumping ahead, we remark that iteration strategies and mouse operators provide an ample source of amenable functions. For instance, let $\mathcal{M} = \mathcal{M}_1^\#$ and let Σ be its canonical iteration strategy. We define f as follows. Let first $dom(f)$ be the set of structures of the form $\mathcal{J}_\omega(\mathcal{T})$ where $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ of limit length and is according to Σ. Next, define $f(\mathcal{J}_\omega(\mathcal{T})) = b$ where $b = \Sigma(\mathcal{T})$. Then f is amenable. We will refer to such an f as *an amenable function given by an iteration strategy*. The reason we define the domain of f to be the set of $\mathcal{J}_\omega(\mathcal{T})$ instead of just the set of $trc^{\mathcal{T}}$ is that the later may not satisfy clause 2 of Definition 1.1.

Recall that a transitive structure $\mathcal{M} = (M, A)$ is called *amenable* if for every $X \in M$, $A \cap X \in M$. Following [10], we say $\mathcal{M}$ is a $\mathcal{J}$-structure over X if $\mathcal{M} = (\mathcal{J}_\alpha^A(X), B) = (\left|\mathcal{J}_\alpha^A(X)\right|, A, B)$ is an amenable structure. Keeping the notation, we also say $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure if for all $\beta < \alpha$ and for all $\tau < \omega\beta$, if $\wp(\tau) \cap \mathcal{J}_{\beta+1}^A \not\subseteq \mathcal{J}_\beta^A$ then there is a surjection $f : \tau \to \omega\beta$ in $\mathcal{J}_{\beta+1}^A$. Finally, we say X is self-well-ordered if there is a wellordering of X in $\mathcal{J}_1(X)$. We are now in a position to introduce the *hybrid $\mathcal{J}$-structures.*

[2] We seem to need this condition in order to develop fine structure of models of the form $\mathcal{J}^{\vec{E},f}$ where f is a shifted amenable function.

Definition 1.2 (Hybrid $\mathcal{J}$-structures) *We say $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a hybrid $\mathcal{J}$-structure over self-well-ordered set X with indexing scheme ϕ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure such that in $\mathcal{M}$, f is a shift of an amenable function with amenable component g such that*

1. *for all $a \in \mathcal{M}$, $a \in dom(f)$ if and only if in $\mathcal{M}$, there is β such that a is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}_\beta^{A,f}(X)$ such that*

$$\mathcal{J}_\beta^{A,f} \vDash \text{"ZFC-Replacement} + \phi[b]\text{"}$$

   and if γ is such that $b \vDash \phi_g[\gamma]$ then $\beta + \gamma \leq \alpha$, and

2. *for all $a \in \mathcal{M}$, if $a \in dom(f)$ then $min(f(a))$ is the least ordinal β satisfying clause 1 above.*

Remark 1.3 *Notice that it follows from clause 1 of Definition 1.2 that the function $a \to min(f(a))$ is injective on $dom(f)$.*

Hod mice are a special blend of *layered hybrid $\mathcal{J}$-structures* introduced below. Before introducing them we establish some notation. Suppose that $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a hybrid $\mathcal{J}$-structure over X and $\xi \leq \alpha$. Then we let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. Also, recall that if $\beta < \alpha$ then we write $\mathcal{J}_\beta^{\mathcal{M}}$ instead of $\mathcal{J}_\beta^{A,f}$ and, we say $\mathcal{N}$ is an (a proper) initial segment of $\mathcal{M}$ and write $\mathcal{N} \trianglelefteq \mathcal{M}$ ($\mathcal{N} \triangleleft \mathcal{M}$) if there is $\beta \leq \alpha$ ($\beta < \alpha$) such that $\mathcal{N} = \mathcal{J}_\beta^{\mathcal{M}}$.

Definition 1.4 (Layered hybrid $\mathcal{J}$-structure) *We say $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a layered hybrid $\mathcal{J}$-structure over X with indexing scheme $\phi(x, y)$ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure over X such that in $\mathcal{M}$, f is a function with domain $Y^{\mathcal{M}} \subseteq \alpha$ such that for all $\gamma \in Y^{\mathcal{M}}$, $f(\gamma)$ is a shift of an amenable function with amenable component g_γ such that*

1. *for all $a \in \mathcal{M}$, $a \in dom(f(\gamma))$ if and only if in $\mathcal{M}$, there is β such that a is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}_\beta^{A,f}(X)$ such that*

$$\mathcal{J}_\beta^{A,f} \vDash \text{"ZFC-Replacement} + \phi[\mathcal{M}|\gamma, b]\text{"}$$

   and if ξ is such that $b \vDash \phi_{g_\gamma}[\xi]$ then $\beta + \xi \leq \alpha$, and

2. *for all $a \in \mathcal{M}$, if $a \in dom(f(\gamma))$ then $min(f(\gamma)(a))$ is the least ordinal β satisfying clause 1 above.*

We will often omit ϕ when discussing a particular layered hybrid $\mathcal{J}$-structure. If $\mathcal{M}$ is a layered hybrid $\mathcal{J}$-structure then we let $f^{\mathcal{M}}$ and $Y^{\mathcal{M}}$ be as in Definition 1.4. We again have that for each $\gamma \in Y^{\mathcal{M}}$, the function $a \to min(f^{\mathcal{M}}(\gamma)(a))$ is injective on $dom(f(\gamma))$.

Notice that hybrid $\mathcal{J}$-structures can be viewed as a special case of layered hybrid $\mathcal{J}$-structures. Because of this, in the sequel we will only establish terminology for layered hybrid $\mathcal{J}$-structures though we might use the same terminology for hybrid $\mathcal{J}$-structures.

Typically, when discussing hybrid $\mathcal{J}$-structures, X will be an *iterable* structure and f will be the predicate coding its strategy. As mentioned above, hod mice are a special type of layered hybrid $\mathcal{J}$-structures: the f predicate of a hod mouse codes a strategy for its layers. When the A predicate of a layered hybrid $\mathcal{J}$-structure is a coherent sequence of extenders then the resulting model is called a *hybrid layered premouse.*

Definition 1.5 (Layered hybrid premouse) *Suppose $\mathcal{M} = \mathcal{J}^{\vec{E},f}(X)$ is a layered hybrid $\mathcal{J}$-structure over X. $\mathcal{M}$ is a called a layered hybrid premouse (lhp) if $\vec{E}$ is a fine extender sequence as in Definition 2.4 of [8].*

2 Layered strategy premice

In this paper, we are concerned with lhp whose f predicate codes a strategy. The goal of this section is to introduce the language used to describe such structures.

Suppose that $\mathcal{M}$ is an lhp. We then say that a shifted amenable function f *codes a partial strategy function* for $\mathcal{M}$ if (i) $dom(f) \subseteq \{\mathcal{J}_\omega(\vec{\mathcal{T}}) : \vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ without a last model$\}$, (ii) whenever $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ such that $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in dom(f)$ then whenever $\vec{\mathcal{U}}$ is an initial segment of $\vec{\mathcal{T}}$ without a last model, then $\mathcal{J}_\omega(\vec{\mathcal{U}}) \in dom(f)$ and (iii) if g is the amenable component of f then for all $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in dom(f)$, $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is a cofinal branch of $\vec{\mathcal{T}}$. Notice that we do not require that $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ be a well-founded branch of $\vec{\mathcal{T}}$ which is why we call the resulting function just a strategy function.

Suppose then a shifted amenable function f codes a partial strategy function for $\mathcal{M}$. We then let Σ^f be the partial strategy function coded by f. More precisely, letting g be the amenable component of f,

1. $dom(\Sigma^f) = dom(f)$ and

2. for all $\vec{\mathcal{T}} \in dom(\Sigma^f)$, $\Sigma^f(\vec{\mathcal{T}}) = g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$.

We say f *codes a partial strategy* if Σ^f chooses cofinal and well-founded branches. We say f *codes a total strategy* if Σ^f is a total strategy.

Recall that if $\mathcal{M}$ is an lhp, $\mathcal{N} \trianglelefteq \mathcal{M}$ and Σ is an iteration strategy for $\mathcal{M}$ then $\Sigma_{\mathcal{N}}$ is the strategy of $\mathcal{N}$ we get by the copy construction. More precisely, $\Sigma_{\mathcal{N}}$ is the id-pullback of Σ.

Definition 2.1 (Layered strategy premouse, lsp) *Suppose $\mathcal{M}$ is an lhp with an indexing scheme $\phi(x, y)$. We say $\mathcal{M}$ is a layered strategy premouse (lsp) if for all $\gamma \in Y^{\mathcal{M}}$, in $\mathcal{M}$,*

1. *$f(\gamma)$ codes a partial strategy function for $\mathcal{M}|\gamma$, and*

2. *if $\gamma_0 < \gamma_1 \in Y^{\mathcal{M}} - \{0\}$ then letting, for $i \in 2$, Σ_i be the partial strategy function coded by $f^{\mathcal{M}}(\gamma_i)$, then $(\Sigma_1)_{\mathcal{M}|\gamma_0} \subseteq \Sigma_0$.*

Notice that the fact that $\mathcal{M}$ is a layered strategy premouse depends on what ϕ says. Thus, the clauses above should be viewed as part of ϕ. The strategy premice are a special case of layered strategy premice, and we leave the details to the reader. We let $\Sigma^{\mathcal{M}}$ be the partial strategy function coded by $f^{\mathcal{M}}$. If $\gamma \in Y^{\mathcal{M}}$ then we let $\Sigma^{\mathcal{M}}_\gamma$ be the partial strategy function coded by $f(\gamma)$.

In most applications, lsp have a very canonical indexing scheme which is originally due to Woodin. At each stage the stack whose branch is being indexed by f is the least stack whose branch hasn't yet been indexed. We call this the *standard indexing scheme.*

Definition 2.2 (Standard indexing scheme) *We say $\phi(x, y)$ is the standard indexing scheme if whenever $\mathcal{M}$ is an lsp and $\gamma \in Y^{\mathcal{M}}$ then $\mathcal{M} \vDash \phi[\mathcal{M}|\gamma, a]$ if and only if*

1. *a is the $\mathcal{M}$-least set of the form $\mathcal{J}_\omega(\vec{\mathcal{T}})$ where $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}|\gamma$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{M}}_\gamma$, $\vec{\mathcal{T}}$ doesn't have a last model and $f(\nu)(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined and*

2. *for every $\zeta \geq \gamma$ such that $a \in \mathcal{M}|\zeta$ and $\mathcal{M}|\zeta \vDash ZFC - Replacement$, in $\mathcal{M}|\zeta$, (γ, a) isn't the lexicographically $\mathcal{M}|\zeta$-least set of the form $(\nu, \mathcal{J}_\omega(\vec{\mathcal{T}}))$ where $\nu \in Y^{\mathcal{M}|\zeta}$ and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}|\nu$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{M}|\zeta}_\nu$, $\vec{\mathcal{T}}$ doesn't have a last model and $f^{\mathcal{M}|\zeta}(\nu)(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined.*

Suppose $\mathcal{M}$ is an lsp and Σ is a (κ, θ)-iteration strategy for $\mathcal{M}||\gamma$ for some $\gamma \in Y^{\mathcal{M}}$. Then it can be the case that $\Sigma^{\mathcal{M}}_{\gamma} \subseteq \Sigma$. When this happens we get structures relative to Σ. Suppose $\mathcal{M}$ is an sp over X and Σ is a (κ, θ)-iteration strategy for X.

Definition 2.3 (Σ-premouse) *Then $\mathcal{M}$ is called a Σ-premouse if $\Sigma^{\mathcal{M}} \subseteq \Sigma \restriction \mathcal{M}$.*

Definition 2.4 (Σ-mouse) *We say $\mathcal{M}$ is a Σ-mouse or just Σ-mouse if $\mathcal{M}$ has an $\omega_1 + 1$-iteration strategy Λ such that whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$ then $\mathcal{N}$ is a Σ-premouse.*

Remark 2.5 *Suppose $\mathcal{M}$ is an lsp and $\gamma < \eta$ are two consecutive members of $Y^{\mathcal{M}}$. Then we can view $\mathcal{M}|\eta$ as a $\Sigma^{\mathcal{M}}_{\gamma}$-premouse over $\mathcal{M}|\gamma$.*

3 Iterations of Σ-mice

Suppose X is a hybrid premouse or a layered hybrid premouse and Σ is an (ω_1, ω_1)-iteration strategy for X. Given two Σ-mice, we can compare them using the usual comparison argument.

Theorem 3.1 (Theorem 3.11 of [8]) *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two countable k-sound Σ-mice with $(\omega_1 + 1)$-iteration strategies Λ and Γ respectively. Then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{M}$ and $\mathcal{N}$ respectively according to Λ and Γ respectively, having last models $\mathcal{M}^{\mathcal{T}}_{\alpha}$ and $\mathcal{N}^{\mathcal{N}}_{\eta}$ such that either*

1. *the iteration embedding $\pi^{\mathcal{T}}_{0,\alpha}$-exists[3], and $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_{\eta}$, or*
2. *the iteration embedding $\pi^{\mathcal{U}}_{0,\eta}$-exists, and $\mathcal{M}^{\mathcal{U}}_{\eta}$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_{\alpha}$.*

Comparison for lsp is more involved and we do not know how to do it in general. Below we recall our primary method of identifying the good branches of iteration trees. Recall that the strategy for a sound mouse projecting to ω is determined by *$\mathcal{Q}$-structures*. For $\mathcal{T}$ normal, let $\Phi(\mathcal{T})$ be the phalanx of $\mathcal{T}$ (see Definition 6.6 of [7]).

Definition 3.2 *Let $\mathcal{T}$ be a k-normal tree of limit length on a k-sound lsp, and let b be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $\mathcal{M}^{\mathcal{T}}_{b}$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ (i.e. $\rho(\mathcal{Q}) < \delta(\mathcal{T})$) or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

[3]In [8], this is stated in a somewhat stronger form, namely that $[0, \alpha]_T$ doesn't drop in model or degree.

Next we would like to state a general result stating that branches identified by $\mathcal{Q}$-structures are unique. Suppose that $\mathcal{M}$ is a lsp and Σ is a strategy for $\mathcal{M}$. If $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then we let $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ be the strategy of $\mathcal{N}$ given by $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

Definition 3.3 *Suppose $\mathcal{M}$ is an lsp and Σ is an iteration strategy for $\mathcal{M}$. We say $(\mathcal{M}, \Sigma)$ is an lsm pair if Σ has hull condensation (see Definition 1.30 of [3]) and whenever $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then $\Sigma^{\mathcal{N}} \subseteq \Sigma_{\mathcal{N},\vec{\mathcal{T}}}$.*

We say an iteration tree $\mathcal{T}$ is above η if all the extenders used in $\mathcal{T}$ have critical points $> \eta$.

Theorem 3.4 *Suppose $(\mathcal{M}, \Sigma)$ is an lsm pair. Suppose $\gamma < o(\mathcal{M})$ is such that $\sup(Y^{\mathcal{M}}) < \gamma$ and $\rho(\mathcal{M}) \leq \gamma$. Then $\mathcal{M}$ has at most one $(k, \omega_1 + 1)$ iteration strategy Λ which acts on iteration trees that are above γ and whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$ then $\Sigma^{\mathcal{N}} \subseteq \Sigma \restriction \mathcal{N}$. Moreover, any such strategy Λ is determined by: $\Lambda(\mathcal{T})$ is the unique cofinal b such that the phalanx $\Phi(\mathcal{T})^\frown(\delta(\mathcal{T}), \deg^{\mathcal{T}}(b), \mathcal{Q}(b, \mathcal{T}))$ is $\omega_1 + 1$-iterable (as a Σ-phalanx).*

In some cases, however, it is enough to assume that $\mathcal{Q}(b, \mathcal{T})$ is countably iterable. This happens, for instance, when $\mathcal{M}$ has no local Woodin cardinals with extenders overlapping it. While the lsp we will consider do have local overlapped Woodin cardinals, the lsp themselves will not have such Woodin cardinals. This simplifies our situation somewhat, and below we describe exactly how this will be used.

Definition 3.5 (Definition 2.1 of [9]) *Let $(\mathcal{M}, \Sigma)$ be an lsm pair and let $\gamma < o(\mathcal{M})$ be such that $\sup(Y^{\mathcal{M}}) < \gamma$. Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ above γ; then $\mathcal{Q}(\mathcal{T})$ is the unique $\oplus_{\nu \in Y^{\mathcal{M}}} \Sigma_{\mathcal{M}|\nu}$-mouse, if there is any, extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1 + 1$-iterable above $\delta(\mathcal{T})$ and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Countable iterability is usually enough to guarantee there is at most one hp with the properties of $\mathcal{Q}(\mathcal{T})$. If it exists, $\mathcal{Q}(\mathcal{T})$ might identify the good branch of $\mathcal{T}$, the one any sufficiently powerful iteration strategy must choose. This is the content of the next lemma which can be proved by analyzing the proof of Theorem 6.12 of [8]. To state it we need to introduce fatal drops and also the following useful notation.

Definition 3.6 ($\mathcal{O}^{\mathcal{P}}$-stack) *Suppose $\mathcal{P}$ is an lsp and $\eta, \gamma, \alpha < o(\mathcal{P})$. We then let*

$$\mathcal{O}^{\mathcal{P}}_{\eta,\gamma,\alpha} = \cup\{\mathcal{M} \trianglelefteq \mathcal{P} : \mathcal{P}|\eta \trianglelefteq \mathcal{M}, \rho(\mathcal{M}) \leq \eta,\ Y^{\mathcal{M}} \subseteq \gamma \text{ and for all } E \in \vec{E}^{M}, \text{ if } \eta \in [crit(E), lh(E)) \text{ then } crit(E) \leq \alpha\}.$$

Next we define the stack $(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha} : \xi \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha})$ *according to the following recursion:*

1. $\mathcal{O}^{\mathcal{P},0}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P}}_{\eta,\gamma,\alpha}$,
2. *for* $\xi + 1 \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$, $\mathcal{O}^{\mathcal{P},\xi+1}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P}}_{o(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha}),\gamma,\alpha}$,
3. *for limit* $\lambda \leq \Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$, $\mathcal{O}^{\mathcal{P},\lambda}_{\eta,\gamma,\alpha} = \cup_{\xi<\lambda}\mathcal{O}^{\mathcal{P},\xi}_{\eta,\gamma,\alpha}$, *and*
4. $\Omega^{\mathcal{P}}_{\eta,\gamma,\alpha}$ *is the least* ν *such that* $\mathcal{O}^{\mathcal{P},\nu+1}_{\eta,\gamma,\alpha} = \mathcal{O}^{\mathcal{P},\nu}_{\eta,\gamma,\alpha}$.

For $\xi \leq \Omega^{\mathcal{P}}_{\eta,\eta,\alpha}$, *we let* $\mathcal{O}^{\mathcal{P},\xi}_{\eta} = \mathcal{O}^{\mathcal{P},\xi}_{\eta,\eta,0}$.

We can now introduce fatal drops.

Definition 3.7 (Fatal drop) *Suppose* $\mathcal{P}$ *is a hod-like lhp and* $\mathcal{T}$ *is an iteration tree on* $\mathcal{P}$. *We say* $\mathcal{T}$ *has a fatal drop if for some* $\alpha < lh(\mathcal{T})$ *and some* $\eta < o(\mathcal{M}^{\mathcal{T}}_{\alpha})$, $\mathcal{T}_{\geq \mathcal{M}^{\mathcal{T}}_{\alpha}}$ *is a normal iteration tree on* $\mathcal{O}^{\mathcal{M}^{\mathcal{T}}_{\alpha}}_{\eta}$ *that is above* η. *We then say* $\mathcal{T}$ *has a fatal drop at* (α, η) *if the pair is the lexicographically least satisfying the above condition.*

The following is the lemma mentioned above.

Lemma 3.8 *Let* $(\mathcal{M}, \Sigma)$ *be an lsm pair and let* $\gamma < o(\mathcal{M})$ *be such that* $\sup(Y^{\mathcal{M}}) < \gamma$.

1. *Suppose* $\mathcal{T}$ *is a normal iteration tree on* $\mathcal{M}$ *above* γ *of limit length and suppose* $\mathcal{Q}(\mathcal{T})$ *exists. Then there is at most one cofinal branch* b *of* $\mathcal{T}$ *such that either* $\mathcal{Q}(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b$ *or* $\mathcal{Q}(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b|\xi$ *for some* ξ *in the wellfounded part of* $\mathcal{M}^{\mathcal{T}}_b$.
2. *Suppose further no measurable cardinal of* $\mathcal{M}$ *which is* $\geq \gamma$ *is a limit of Woodin cardinals. If then* $\mathcal{T}$ *is an iteration tree according to* Σ *above* γ *which doesn't have a fatal drop and* $b = \Sigma(\mathcal{T})$ *is such that* $\mathcal{Q}(b, \mathcal{T})$*-exists then* $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

$\mathcal{Q}(\mathcal{T})$ identifies b because it determines a canonical cofinal subset of $rng(\pi^{\mathcal{T}}_{\alpha,b} \cap \delta(\mathcal{T}))$, for some $\alpha \in b$, to which we can apply Lemma 1.13 of [3] (which is an immediate consequence of the zipper argument from [1]).

Remark 3.9 *Suppose* $\mathcal{M}$ *is an lsp and* $\gamma \in Y^{\mathcal{M}}$. *Let* $\xi = o(\mathcal{M})$ *if* γ *is the largest member of* $Y^{\mathcal{M}}$ *and otherwise, let* ξ *be the least member of* $Y^{\mathcal{M}}$ *which is bigger than* γ. *Suppose* $\mathcal{T}$ *is a tree on* $\mathcal{M}$ *which is above* γ *and is based on* $\mathcal{M}||\xi$. *Notice that in this case we can define* $\mathcal{Q}(\mathcal{T})$ *just as in Definition 3.5 by using* $\mathcal{M}||\xi$ *instead of* $\mathcal{M}$.

4 Hod-like layered hybrid premice

In this paper, we are concerned with lsp[4] whose f predicates code a fragment of their own strategy. The difference of the lsp considered here and those considered in [3] is that here we will have lsp whose predicate codes the *short tree strategy* of its initial segments. In order to introduce this concept it is useful to partition lsp into *windows*.

Suppose first that $\mathcal{P}$ is an lsp and κ is a cutpoint of $\mathcal{P}$. We let

$$X_\kappa^{\mathcal{P}} = \{\xi : \xi \text{ is an index of an extender with critical point } \kappa\}.$$

Let $I_\xi^{\mathcal{P}}$ be the closure of $X_{\delta_\xi^{\mathcal{P}}}^{\mathcal{P}}$ and let $(\nu_{\xi,\alpha}^{\mathcal{P}} : \alpha \in [1, \varsigma_\xi^{\mathcal{P}}])$ be the enumeration of $I_\xi^{\mathcal{P}}$ provided it is not empty. Notice that our enumeration starts with index 1.

Definition 4.1 *Suppose $\mathcal{P}$ is an lsp. We say $[\eta, \kappa]$ is a window of $\mathcal{P}$ if $\eta < \sup Y^{\mathcal{P}}$, $\eta < \kappa$ and one of the following holds:*

1. *η is a cutpoint Woodin cardinal of $\mathcal{P}$, $\kappa \leq \rho(\mathcal{P})$ and κ is the least Woodin cardinal of $\mathcal{P}$ that is $> \eta$.*
2. *η is a cutpoint limit of Woodin cardinals, $\rho(\mathcal{P}) > \eta$ and $\kappa = \sup X_\eta^{\mathcal{P}}$.*
3. *There is some $\nu < \eta$ such that ν is a cutpoint limit of Woodin cardinals, $\eta = \sup X_\nu^{\mathcal{P}}$ and κ is the least Woodin cardinal of $\mathcal{P}$ that is $> \eta$ and $\rho(\mathcal{P}) \geq \kappa$.*

We let $(\delta_\xi^{\mathcal{P}} : \xi \leq \lambda^{\mathcal{P}})$ be a closed increasing sequence of ordinals such that for all $\xi < \lambda^{\mathcal{P}}$, $[\delta_\xi^{\mathcal{P}}, \delta_{\xi+1}^{\mathcal{P}}]$ is a window of $\mathcal{P}$. We also let $\delta^{\mathcal{P}} = \delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$.

Suppose now that $\mathcal{P}$ is an lsp. We say $[\delta_\xi^{\mathcal{P}}, \delta_{\xi+1}^{\mathcal{P}}]$ is a *simple window* if $\delta_\xi^{\mathcal{P}}$ isn't a measurable cardinal. Next we let

$$w(\mathcal{P}) = \{\xi \leq \lambda^{\mathcal{P}} : \mathcal{P} \vDash \text{“}\delta_\xi^{\mathcal{P}} \text{ is a Woodin cardinal”}\},$$
$$lw(\mathcal{P}) = \{\xi \leq \lambda^{\mathcal{P}} : \mathcal{P} \vDash \text{“}\delta_\xi^{\mathcal{P}} \text{ is a limit of Woodin cardinals”}\},$$
$$mlw(\mathcal{P}) = \{\xi \leq \lambda^{\mathcal{P}} : \mathcal{P} \vDash \text{“}\delta_\xi^{\mathcal{P}} \text{ is a measurable limit of Woodin cardinals”}\}.$$

Suppose now $\xi < \lambda^{\mathcal{P}}$ is such that $\xi \in mlw(\mathcal{P})$. We then let

$$\alpha_\xi^{\mathcal{P}} = \begin{cases} \eta & : \text{ for some } \eta,\ \mathcal{P} = \mathcal{J}_\eta(\mathcal{P}||\xi) \\ \gamma & : \gamma \text{ is the least such that } crit(E_\gamma^{\mathcal{P}}) > \xi \end{cases}$$

[4]We write “lsp” for both layered hybrid premouse and layered hybrid premice.

and for $\alpha \in [1, \varsigma_\xi^{\mathcal{P}}]$, let $C_{\xi,\alpha}^{\mathcal{P}}$ be the closure of the set

$$\{\eta < o(\mathcal{P}) : \eta \geq \alpha_\xi^{\mathcal{P}} \wedge (\eta > \alpha_\xi^{\mathcal{P}} \rightarrow \rho(\mathcal{P}||\eta) \leq \nu_{\xi,\alpha}^{\mathcal{P}})\}.$$

Let then $(\mu_{\xi,\alpha,\gamma}^{\mathcal{P}} : \gamma \in [1, \zeta_{\xi,\alpha}^{\mathcal{P}}])$ be the increasing enumeration of $C_{\xi,\alpha}^{\mathcal{P}}$ where $\xi < \lambda^{\mathcal{P}}$ and $\alpha \leq \varsigma_\xi^{\mathcal{P}}$. Notice that our enumeration starts with index $(\xi, 1, 1)$. Below we will define $\mu_{\xi,0,0}^{\mathcal{P}}$ (see Definition 4.4).

The next definition isolates four types of lsp that we will encounter in this paper. The types are not necessarily disjoint.

Definition 4.2 *Suppose $\mathcal{P}$ is an lsp.*

1. ***(Successor type)*** *We say $\mathcal{P}$ has a successor type if $\lambda^{\mathcal{P}}$ is a successor ordinal and $[\delta_{\lambda^{\mathcal{P}}-1}^{\mathcal{P}}, \delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}]$ is a simple window.*

2. ***(Limit type)*** *We say $\mathcal{P}$ has a limit type if $\lambda^{\mathcal{P}}$ is a limit ordinal or $\lambda^{\mathcal{P}}$ is a successor ordinal and $\delta_{\lambda^{\mathcal{P}}-1}^{\mathcal{P}}$ is a measurable cardinal.*

3. ***(Lsa type)*** *We say $\mathcal{P}$ has an lsa type if $\mathcal{P} \vDash$ "ZFC-Powerset", $\lambda^{\mathcal{P}}$ is a successor ordinal, $[\delta_{\lambda^{\mathcal{P}}-1}^{\mathcal{P}}, \delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}]$ isn't simple and $\lambda^{\mathcal{P}} \in w(\mathcal{P})$.*

4. ***(Meek)*** *We say $\mathcal{P}$ is meek if either it has a successor type or $\lambda^{\mathcal{P}}$ is a limit ordinal.*

Next we introduce hod-like lsp. These will eventually turn into hod premice. To do this we need to impose conditions on layers of lsp, which recall are just the members of $Y^{\mathcal{P}}$ where $\mathcal{P}$ is an lsp.

Suppose M is a transitive structure and η is an ordinal. Then we let $(\eta^+)^M$ be the cardinal successor of η in M if it exists and otherwise, we let it be the ordinal height of M. The next definition isolates lsp that look like hod premice.

Definition 4.3 (Hod-like lsp) *Suppose $\mathcal{P}$ is an lsp. We say $\mathcal{P}$ is hod-like if the following conditions hold:*

1. *Suppose $\mathcal{P}$ is meek. Then $\mathcal{P} \vDash$ "$ZFC{-}Powerset$" and $o(\mathcal{P}) = \sup_{n<\omega}((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}}$.*

2. *Suppose $\lambda^{\mathcal{P}}$ is a successor ordinal and $\mathcal{P}$ is non-meek. It then follows that $\delta^{\mathcal{P}} = \sup(X_{\delta_{\lambda^{\mathcal{P}}-1}^{\mathcal{P}}}^{\mathcal{P}})$. Then either $\rho(\mathcal{P}) \leq \delta^{\mathcal{P}}$ or $o(\mathcal{P})$ is a limit of ordinals ξ such that $\rho(\mathcal{P}||\xi) \leq \delta^{\mathcal{P}}$.*

3. *For every limit $\gamma < \lambda^{\mathcal{P}}$, $o(\mathcal{O}_{\delta_\gamma^{\mathcal{P}}, \delta_\gamma^{\mathcal{P}}}^{\mathcal{P}}) = ((\delta_\gamma^{\mathcal{P}})^+)^{\mathcal{P}}$.*

4. $\xi \in Y^{\mathcal{P}}$ *if and only if one of the following conditions holds:*

   (a) For some $\alpha < \lambda^{\mathcal{P}}$, $\beta \in [0, \varsigma_\alpha^{\mathcal{P}}]$ *and* $\gamma \leq \zeta_{\alpha,\beta}^{\mathcal{P}}$, $\xi = \mu_{\alpha,\beta,\gamma}^{\mathcal{P}}$.

   (b) For some $\gamma < \lambda^{\mathcal{P}}$ *such that* $\delta_\gamma^{\mathcal{P}}$ *is a Woodin cardinal of* $\mathcal{P}$ *or is a limit of cutpoint Woodin cardinals of* $\mathcal{P}$, $\xi = o(\mathcal{O}^{\mathcal{P},\omega}_{\delta_\gamma^{\mathcal{P}},\delta_\gamma^{\mathcal{P}}})$.

The next definition isolates the layers of $\mathcal{P}$.

Definition 4.4 (Layers of hod-like lsp) *Suppose now* $\mathcal{P}$ *is a hod-like lsp. Given* $\xi < \lambda^{\mathcal{P}}$ *we let*

$$\varsigma_\xi^{\mathcal{P}} = \begin{cases} \varsigma_\xi^{\mathcal{P}} & : \xi \in mlw(\mathcal{P}) \\ 0 & : \text{otherwise.} \end{cases}$$

Now, given $\xi < \lambda^{\mathcal{P}}$ *and* $\alpha \leq \varsigma_\xi^{\mathcal{P}}$ *we let*

$$\zeta_{\xi,\alpha}^{\mathcal{P}} = \begin{cases} \zeta_{\xi,\alpha}^{\mathcal{P}} & : \xi \in mlw(\mathcal{P}) \text{ and } \alpha > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

Next, working in $\mathcal{P}$, *for* $\xi < \lambda^{\mathcal{P}}$, $\alpha \leq \varsigma_\gamma^{\mathcal{P}}$ *and* $\gamma \leq \zeta_{\xi,\alpha}^{\mathcal{P}}$, *we let*

$$\mu_{\xi,\alpha,\gamma}^{\mathcal{P}} = \begin{cases} \mu_{\xi,\alpha,\gamma}^{\mathcal{P}} & : \xi \in mlw(\mathcal{P}) \text{ and } \alpha > 0 \\ o(\mathcal{O}^{\mathcal{P},\omega}_{\delta_\xi^{\mathcal{P}},\delta_\xi^{\mathcal{P}}}) & : \text{otherwise} \end{cases}$$

Continuing with ξ, α, *and* γ, *we let* $\mathcal{P}(\xi,\alpha,\gamma) = \mathcal{P}|\mu_{\xi,\alpha,\gamma}^{\mathcal{P}}$ *and* $\mathcal{P}(\xi,\alpha) = \mathcal{P}(\xi,\alpha,\zeta_{\xi,\alpha}^{\mathcal{P}})$. *Also, for* $\xi < \lambda^{\mathcal{P}}$, *we let* $\mathcal{P}(\xi) = \mathcal{P}(\xi, \varsigma_\xi^{\mathcal{P}})$, *and finally we let* $\mathcal{P}(\lambda) = \mathcal{P}$. *The sequence*

$$(\mu_{\xi,\alpha,\gamma}^{\mathcal{P}} : \xi < \lambda^{\mathcal{P}} \wedge \alpha \leq \varsigma_\xi^{\mathcal{P}} \wedge \gamma \leq \varsigma_{\xi,\alpha}^{\mathcal{P}})$$

is called the sequence of layers of $\mathcal{P}$ *or just layers of* $\mathcal{P}$.

In this paper, all hod-like lsp that we will consider are *lsa-small*.

Definition 4.5 (Lsa small lsp) *Suppose* $\mathcal{P}$ *is a hod-like lsp. We say* $\mathcal{P}$ *is lsa-small if for all* $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha + 1)$ *isn't of lsa type.*

Remark 4.6 *From now on we tacitly assume that all lsp considered in this paper are lsa-small. We will, from time to time, remind the reader of this.*

Definition 4.7 (The internal strategy) *We also let $\Sigma^{\mathcal{P}}_{\xi,\alpha,\gamma}$ be the strategy of $\mathcal{P}(\xi,\alpha,\gamma)$ coded by $f^{\mathcal{P}}(\mu^{\mathcal{P}}_{\xi,\alpha,\gamma})$. We let $\Sigma^{\mathcal{P}}_{\xi} = \Sigma^{\mathcal{P}}_{\xi,\varsigma^{\mathcal{P}}_{\xi},\zeta^{\mathcal{P}}_{\xi,\varsigma^{\mathcal{P}}_{\xi}}}$.*

Next, we isolate the bottom part of non-meek limit type hod-like lsp. This is essentially the part of $\mathcal{P}$ that is below the largest measurable limit of cutpoint Woodins.

Definition 4.8 (The bottom part of lsp) *Given a non-meek limit type hod-like lsp $\mathcal{P}$, we let $\mathcal{P}^b = \mathcal{P}(\lambda^{\mathcal{P}} - 1, 0, 0)$ where "b" stands for "bottom". We say that $\mathcal{P}^b$ is the bottom part of $\mathcal{P}$.*

We end this section with the definition of hod initial segments of lsp.

Definition 4.9 (Hod initial segment) *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod-like lsp. We then write $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ and say $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$ if $\mathcal{P} = \mathcal{Q}(\xi,\alpha,\gamma)$ for some $\xi \leq \lambda^{\mathcal{Q}}$ and $\alpha \leq \varsigma^{\mathcal{Q}}_{\xi}$. We wright $\mathcal{P} \trianglelefteq_{chod} \mathcal{Q}$ and say $\mathcal{P}$ is a cutpoint hod initial segment of $\mathcal{Q}$ if either $\mathcal{Q} = \mathcal{P}$ or $\mathcal{P} = \mathcal{Q}(\xi, 0, 0)$ for some $\xi < \lambda^{\mathcal{Q}}$.*

5 Analysis of stacks

Here we review the analysis of stacks of iteration trees from Section 6.2 of [3]. Suppose M is a transitive structure and $\vec{\mathcal{T}}$ is a stack of iteration trees on M[5]. Let $\mathcal{S}$ and $\mathcal{R}$ be nodes in $\vec{\mathcal{T}}$. Then we write $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ for the component of $\vec{\mathcal{T}}$ that comes after stage $\mathcal{S}$ and $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ for the component of $\vec{\mathcal{T}}$ up to stage $\mathcal{S}$. In the case $\mathcal{R}$ appears in $\vec{\mathcal{T}}$ later than $\mathcal{S}$, we also write $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ for the part of $\vec{\mathcal{T}}$ that is between $\mathcal{S}$ and $\mathcal{R}$. Notice that neither $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ nor $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ might be a stack on $\mathcal{S}$.

Definition 5.1 (Cutpoint of a stack) *We say $\mathcal{S}$ is a cutpoint of $\vec{\mathcal{T}}$ if no normal component of $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ has a fatal drop and $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a stack on $\mathcal{S}$.*

Suppose now that $\mathcal{T}$ is a normal tree on M.

Definition 5.2 (Reducible and irreducible trees) *We say $\mathcal{T}$ is reducible if it has a cutpoint. Otherwise we say $\mathcal{T}$ is irreducible.*

[5]Recall that all trees are normal.

Suppose next that $\mathcal{P}$ is a hod-like lsp. In our current context we must consider stacks with more sever dropping patterns than those considered in [3]. However, we will rule out stacks with too many bad drops. The bad drops will consist of *fatal drops* and *non-continuable drops*. The stacks that we will consider can have at most one of each such drops. We have already introduced fatal drops (see Definition 3.7). Below we introduce non-continuable drops.

Continuing with our $\mathcal{P}$ suppose $\mathcal{T}$ is a normal irreducible tree on $\mathcal{P}$ which has a last model but $\pi^{\mathcal{T}}$ doesn't exist.

Definition 5.3 (Continuable drop) *We then say $\mathcal{T}$ has a continuable drop if $\mathcal{T}$ doesn't have a fatal drop and for some $\xi < \lambda^{\mathcal{P}}$ such that $\mathcal{P}(\xi+1)$ is of limit type, $\mathcal{T}$ is based on $\mathcal{P}(\xi+1)$ and is above $\delta_\xi^{\mathcal{P}}$.*

Definition 5.4 (Continuable stack) *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is continuable if for every two successive cutpoints $\mathcal{S}$ and $\mathcal{R}$, either $\pi^{\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}}$ exists or $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ (which is a normal irreducible tree) has a continuable drop.*

Say $\mathcal{T}$ has a non-continuable drop if $\mathcal{T}$ has a drop which is not a continuable drop. The next definition blocks iterations of hod-like lsp that have more than one non-continuable drops.

Definition 5.5 (Stack on hod-like lsp) *We say $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ if it is produced according to the rules of the usual iteration game except that for every $\alpha < \eta$, $\vec{\mathcal{T}} \restriction \alpha$ is continuable.*

Continuing with our $\mathcal{P}$, let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$. Given a node $\mathcal{R}$ in $\vec{\mathcal{T}}$ we say $\mathcal{R}$ is a *terminal node* in $\vec{\mathcal{T}}$ if I can legitimately continue $\vec{\mathcal{T}}_{\leq\mathcal{R}}$ by starting a new round of the iteration game. We say $\mathcal{R}$ is a *non-trivial* terminal node if it is a terminal node and the extender chosen from $\mathcal{R}$ is applied to $\mathcal{R}$. The following is an easy lemma.

Lemma 5.6 *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ and $\mathcal{S}$ is a cutpoint of $\vec{\mathcal{T}}$. Then $\mathcal{S}$ is a non-trivial terminal node of $\vec{\mathcal{T}}$.*

Suppose again that $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. We then let

$$\mathrm{tn}(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a terminal node in } \vec{\mathcal{T}}\}$$
$$\mathrm{ntn}(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a non-trivial terminal node in } \vec{\mathcal{T}}\}.$$

Given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$, we write $\mathcal{Q} \preceq^{\vec{\mathcal{T}},w} \mathcal{R}$ if[6], in $\vec{\mathcal{T}}$, $\mathcal{R}$ appears later than $\mathcal{Q}$.

Given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ if, in $\vec{\mathcal{T}}$, $\mathcal{Q}$-to-$\mathcal{R}$ iteration embedding exists. If $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ then we let $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the iteration embedding given by $\vec{\mathcal{T}}$. If $\mathcal{Q} = \mathcal{P}$ then we just write $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}}$.

Again given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}},s} \mathcal{R}$ if[7] $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ and $\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{R}}$ is a stack on $\mathcal{Q}$.

Continuing with $\mathcal{P}$ and $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$, suppose $C \subseteq tn(\vec{\mathcal{T}})$. We say C is *linear* if it is linearly ordered by $\preceq^{\vec{\mathcal{T}},s}$.

Suppose now that C is linear and $(\mathcal{R}_\alpha : \alpha < \eta)$ is a $\preceq^{\vec{\mathcal{T}},s}$-increasing enumeration of C. We let $lh(C) = \eta$. Suppose further that η is a limit ordinal. Then we let $\mathcal{R}^{\vec{\mathcal{T}}}_C$ be the direct limit of the $\mathcal{R}_\alpha$ under the iteration embeddings $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}_\alpha,\mathcal{R}_\beta}$. We then say $C \subseteq tn(\vec{\mathcal{T}})$ is *closed* if it is linear and for every limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha} \in C$. Notice that linearity implies that for each limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha}$ is a node in $\vec{\mathcal{T}}$.

Next, we say C is *cofinal* if for every node $\mathcal{S}$ of $\vec{\mathcal{T}}$ either $\mathcal{S} \in C$ or there are $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q} \in C$ such that $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$. The following is another easy lemma.

Lemma 5.7 *If C is cofinal then every node in C is a cutpoint.*

We say C is a *club* if it is closed and cofinal. Notice that if C is closed and cofinal and $\mathcal{S} \notin C$ then there is a $\preceq^{\vec{\mathcal{T}},s}$-largest $\mathcal{R} \in C$ such that for any $\mathcal{Q} \in C$ such that $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q}$, $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$.

Continuing with our fixed $\mathcal{P}$, suppose $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ is a stack on $\mathcal{P}$. If $\mathcal{R}$ is a non-trivial terminal node of $\vec{\mathcal{T}}$ then we let $\xi^{\vec{\mathcal{T}},\mathcal{R}}$ be the least $\xi < \lambda^{\mathcal{R}}$ such that $E^{\mathcal{T}_\alpha}_\beta \in \mathcal{R}(\xi + 1)$. We also let $\vec{\mathcal{T}}_{\mathcal{R}}$ be the largest initial segment of $\vec{\mathcal{T}}$ that can be regarded as a stack on $\mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}} + 1)$.

Notice that if $\vec{\mathcal{T}}$ doesn't have a last model but there is a club $C \subseteq tn(\vec{\mathcal{T}})$ then C uniquely identifies the branch of $\vec{\mathcal{T}}$. Indeed, let $D = \{\mathcal{S} \in tn(\vec{\mathcal{T}}) : \exists \mathcal{R}, \mathcal{Q} \in C(\mathcal{R} \preceq^{\vec{\mathcal{T}}} \mathcal{S} \preceq^{\vec{\mathcal{T}}} \mathcal{Q})\}$. Let $\mathcal{R} \in D$ be the $\preceq^{\vec{\mathcal{T}}}$-minimal member of D and let b be the set of indices of the nodes of $\vec{\mathcal{T}}$ between $\mathcal{P}$ and $\mathcal{R}$. Then the union of b with the indices of the nodes of D constitute a branch b_C of $\vec{\mathcal{T}}$. It is not hard to see that we have $\mathcal{M}^{\vec{\mathcal{T}}}_{b_C} = \mathcal{R}^{\vec{\mathcal{T}}}_C$.

Suppose now that $\vec{\mathcal{T}}$ doesn't have a last model and there is no club $C \subseteq tn(\vec{\mathcal{T}})$. Let then $D = \{\mathcal{S} \in tn(\mathcal{T}) : \mathcal{S}$ is a cutpoint$\}$. It follows from our discussion that D

[6] "w" stands for "weakly"

[7] "s" stands for "strongly".

has a $\preceq^{\vec{\mathcal{T}},s}$-largest element. Let $\mathcal{S}_{\vec{\mathcal{T}}}$ be this largest element. The following is our last easy lemma.

Lemma 5.8 *$\vec{\mathcal{T}}_{\geq \mathcal{S}_{\vec{\mathcal{T}}}}$ is a normal tree on $\mathcal{S}_{\vec{\mathcal{T}}}$ such that if it is reducible then it has either a fatal or a non-continuable drop.*

6 The iteration embedding $\pi^{\vec{\mathcal{T}},b}$

Continuing with $\mathcal{P}$ and $\vec{\mathcal{T}}$, assume that $\mathcal{P}$ is a limit type hod-like lsp which isn't meek. In some cases, regardless whether $\vec{\mathcal{T}}$ has a last model or not, it is possible to extract an embedding out of the iteration embeddings given by $\vec{\mathcal{T}}$ that acts on $\mathcal{P}^b$. We describe this embedding below. First we define it by assuming that $\vec{\mathcal{T}} = \mathcal{T}$ is a normal irreducible tree. Recall that our lsp are lsa-small (see Definition 4.5).

Definition 6.1 **($\pi^{\mathcal{T},b}$ fo irreducible trees)** *Let $\lambda = \lambda^{\mathcal{P}}$. Let $\mathcal{M} = \mathcal{M}(\mathcal{T})$ if $\mathcal{T}$ is of limit length and let $\mathcal{M}$ be the last model of $\mathcal{T}$ otherwise. Then letting $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$, we let $\pi^{\mathcal{T},b}$ be*

1. *undefined if $\mathcal{T}$ is below δ and $\pi^{\mathcal{T}}$ doesn't exist,*
2. *$\pi^{\mathcal{T}} \restriction \mathcal{P}^b$ if $\mathcal{T}$ is below δ and $\pi^{\mathcal{T}}$ exist,*
3. *$\pi^{\mathcal{T}} \restriction \mathcal{P}^b$ if all extenders in $\mathcal{T}$ have a critical point $\geq \delta$ and $\pi^{\mathcal{T}}$ exist,*
4. *id if $\mathcal{T}$ is above δ, $\pi^{\mathcal{T}}$ doesn't exist and $\mathcal{M}|(\delta^+)^{\mathcal{P}} = \mathcal{P}|(\delta^+)^{\mathcal{P}}$,*
5. *undefined if $\mathcal{T}$ is above δ, $\pi^{\mathcal{T}}$ doesn't exist and $\mathcal{M}|(\delta^+)^{\mathcal{P}} \neq \mathcal{P}|(\delta^+)^{\mathcal{P}}$.*

Remark 6.2 *Notice that in Definition 6.1, because $\mathcal{T}$ is irreducible and $\delta^{\mathcal{P}}_{\lambda-1}$ is a limit of cutpoints, it cannot be the case that for some $\alpha < lh(\mathcal{T})$, $crit(E^{\mathcal{T}}_\alpha) = \delta^{\mathcal{P}}_{\lambda-1}$ and $cp(E^{\mathcal{T}}_{\alpha+1}) \geq \pi_{E^{\mathcal{T}}_\alpha}(\delta^{\mathcal{P}}_\lambda)$ (this is because otherwise $\mathcal{T}_{\geq \mathcal{M}^{\mathcal{T}}_{\alpha+1}}$ would be a normal tree on $\mathcal{M}^{\mathcal{T}}_{\alpha+1}$). This observation makes clause 3 meaningful.*

Next we define $\pi^{\mathcal{T},b}$ for trees $\mathcal{T}$.

Definition 6.3 **($\pi^{\mathcal{T},b}$ for trees)** *Suppose $\mathcal{T}$ is a tree on $\mathcal{P}$. We define $\pi^{\mathcal{T},b}$ by induction on cutpoints of $\mathcal{T}$. If there is a cutpoint $\mathcal{R}$ of $\mathcal{T}$ such that $\pi^{\mathcal{T}_{\mathcal{P},\mathcal{R}},b}$ is undefined then let $\pi^{\mathcal{T},b}$ be undefined. Otherwise let $C = (\mathcal{R}_\alpha : \alpha < \eta)$ be the sequence of cutpoints of $\mathcal{T}$. If C is a club then letting c be the unique branch of $\mathcal{T}$, we let $\pi^{\mathcal{T},b} = \pi^{\mathcal{T}}_c \restriction \mathcal{P}^b$. Otherwise letting $\eta = \gamma + 1$*[8]*, let $\pi^{\mathcal{T},b} = \pi^{\mathcal{T}_{\geq \mathcal{R}_\gamma},b} \circ \pi^{\mathcal{T}_{\mathcal{P},\mathcal{R}_\gamma},b}$.*

[8] Notice that η is always a successor ordinal.

Finally we define $\pi^{\vec{\mathcal{T}},b}$ for stacks.

Definition 6.4 **($\pi^{\mathcal{T},b}$ for stacks)** *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$. If for some $\alpha < \eta$, $\pi^{\mathcal{T}_\alpha,b}$ is undefined then we let $\pi^{\vec{\mathcal{T}},b}$ be undefined. Suppose then for every $\alpha < \eta$, $\pi^{\mathcal{T}_\alpha,b}$ is defined. Then if η is a limit ordinal then, letting c be the unique branch of $\vec{\mathcal{T}}$, we let $\pi^{\vec{\mathcal{T}},b} = \pi^{\vec{\mathcal{T}}}_c \restriction \mathcal{P}^b$. If $\eta = \gamma + 1$ then let $\pi^{\vec{\mathcal{T}},b} = \pi^{\mathcal{T}_\gamma,b} \circ \pi^{\vec{\mathcal{T}}\restriction\gamma,b}$.*

Notice that in Definition 6.5 we are not assuming that the stack has a last model. The fragment of the eventual iteration embedding $\pi^{\vec{\mathcal{T}}}$ restricted to $\mathcal{P}^b$ can be seen without actually having the last branch. Notice also that the actual branch embedding may not agree with $\pi^{\vec{\mathcal{T}},b}$.

Definition 6.5 (Almost non-dropping stacks) *Suppose $\mathcal{P}$ is a non-meek hod-like lsp and $\vec{\mathcal{T}}$ is a stack of iteration trees on $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is almost non-dropping if $\pi^{\vec{\mathcal{T}},b}$ is defined on $\mathcal{P}^b$.*

Remark 6.6 *Notice that if $\vec{\mathcal{T}}$ is almost non-dropping then it may only have drops in some image of the top window of $\mathcal{P}$.*

The following notion will be used throughout this paper. We will use it to define short tree strategy mice (in particular, see Definition 12.5).

Definition 6.7 (Canonical singularizing sequences) *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is an almost non-dropping stack on $\mathcal{P}$. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}},b}(\mathcal{P}^b)$. Then $\mathcal{Q}$ is an lsp. For $\xi + 1 \leq \lambda^{\mathcal{Q}}$, we let*

$$s(\vec{\mathcal{T}}, \xi) = \{\alpha : \exists a \in (\delta^{\mathcal{Q}}_\xi)^{<\omega} \exists f \in \mathcal{P}^b(\alpha = \pi^{\vec{\mathcal{T}},b}(f)(a))\} \cap \delta^{\mathcal{Q}}_{\xi+1}$$

The following is an easy lemma.

Lemma 6.8 *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is an almost non-dropping stack on $\mathcal{P}$. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}},b}(\mathcal{P}^b)$. Then for any $\xi + 1 \leq \lambda^{\mathcal{Q}}$, $\sup(s(\vec{\mathcal{T}}, \xi)) = \delta^{\mathcal{Q}}_{\xi+1}$.*

7 The un-dropping game

Before we proceed, we explain the meaning of the un-dropping game. Imagine we are comparing the strategies of two lsa type hod-like lsp $\mathcal{P}$ and $\mathcal{Q}$. Let Σ be the strategy of $\mathcal{P}$ and Λ be the strategy of $\mathcal{Q}$. Lets assume that the pointclasses generated by $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are the same. We are then searching for $\mathcal{R}$ which is an iterate of $\mathcal{P}$ and $\mathcal{Q}$ and $\Sigma_{\mathcal{R}} = \Lambda_{\mathcal{R}}$. In this comparison we might be forced to consider iteration trees $\mathcal{T}$ and $\mathcal{U}$ with last models $\mathcal{M}$ and $\mathcal{N}$ such that $\pi^{\mathcal{T}}$ and $\pi^{\mathcal{U}}$ don't exist and for some $\xi < \min(\lambda^{\mathcal{M}}, \lambda^{\mathcal{N}})$, $\mathcal{M}(\xi + 1) = \mathcal{N}(\xi + 1)$ but $\Sigma_{\mathcal{M}(\xi+1)} \neq \Lambda_{\mathcal{N}(\xi+1)}$. We can continue the comparison by comparing $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$ and producing $(\mathcal{S}, \Phi)$ which is a common tail of $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$. However, $(\mathcal{S}, \Phi)$ cannot be thought of as a last model of a successful comparison of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ simply because $\pi^{\mathcal{T}}$ and $\pi^{\mathcal{U}}$ do not exist. What we need to do is to compare $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$ and then somehow get back to $\mathcal{P}$ and $\mathcal{Q}$. This is what the un-dropping game achieves.

To define the un-dropping game, we need to define the sequence of *main drops*. It is the sequence of stages in an iteration at which there is a drop below the top window.

Definition 7.1 (The main drops of a continuable stack) *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is a continuable stack. We say $md^{\vec{\mathcal{T}}} = (\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ is the sequence of main drops of $\vec{\mathcal{T}}$ if the following conditions hold:*

1. *$k < \omega$ and $\mathcal{R}_0 = \mathcal{P}$.*
2. *$(\mathcal{R}_i : i \leq k)$ is a $\preceq^{\vec{\mathcal{T}},s}$-increasing sequence of cutpoints of $\vec{\mathcal{T}}$.*
3. *For $i + 1 \leq k$, $\vec{\mathcal{T}}_i = \vec{\mathcal{T}}_{\mathcal{R}_i, \mathcal{R}_{i+1}}$ and $\vec{\mathcal{T}}_k = \vec{\mathcal{T}}_{\geq \mathcal{R}_k}$.*
4. *For each $i \leq k$, $\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1 < \lambda^{\mathcal{R}_i}$ and $\mathcal{R}_i(\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1)$ is a limit type hod-like lsp.*
5. *For each $i < k$, $\mathcal{R}_{i+1}$ is $\preceq^{\vec{\mathcal{T}},s}$-least cutpoint of $\vec{\mathcal{T}}$ such that $\mathcal{R}_i \preceq^{\vec{\mathcal{T}},s} \mathcal{R}_{i+1}$, $\mathcal{R}_i(\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1)$ is a limit type lsp, and if $\mathcal{W}$ is the largest initial segment of $\vec{\mathcal{T}}$ that is a stack on $\mathcal{R}_{i+1}(\xi^{\vec{\mathcal{T}}, \mathcal{R}_{i+1}} + 1)$ then $\pi^{\mathcal{W}, b}$ doesn't exist.*
6. *For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}_{\geq \mathcal{R}_k}$, $\pi^{(\vec{\mathcal{T}}_k)_{\mathcal{R}_k}, \mathcal{S}, b}$ exists.*

Notice that it is possible that in the above definition $\mathcal{R}_0 = \mathcal{R}_1$. This can happen, for instance, when I starts out with a drop. Next we define the un-dropping extender of $\vec{\mathcal{T}}$. This is essentially the extender given by dovetailing the embeddings $\pi^{\vec{\mathcal{T}}_i, b}$.

Definition 7.2 (The un-dropping extender of a stack) *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is a continuable stack. Let $(\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ be the sequence of the main drops of $\vec{\mathcal{T}}$. For $i \leq k$, let $\kappa_i = \delta^{\mathcal{R}_i}_{\xi^{\vec{\mathcal{T}}, \mathcal{R}_i}}$, and for $i + 1 \leq k$, let*

$$\sigma^{\vec{\mathcal{T}}}_{i,i+1} : (\wp(\kappa_i))^{\mathcal{R}_i} \to (\wp(\kappa_{i+1}))^{\mathcal{R}_{i+1}}$$

be given by

$$\sigma^{\vec{\mathcal{T}}}_{i,i+1}(A) = \pi^{\vec{\mathcal{T}}_i, b}(A) \cap \kappa_{i+1}.$$

Let $\sigma^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{T}}_k, b} \circ \sigma^{\vec{\mathcal{T}}}_{k-1,k} \circ \sigma^{\vec{\mathcal{T}}}_{k-2,k-1} \cdots \circ \sigma^{\vec{\mathcal{T}}}_{0,1}$. Let $E^{\vec{\mathcal{T}}}$ be the $(\kappa_0, \pi^{\vec{\mathcal{T}}_k, b}(\kappa_k))$-extender derived from $\sigma^{\vec{\mathcal{T}}}$. More precisely,

$E^{\vec{\mathcal{T}}} = \{(a, A) : a$ *is a finite subset of* $\pi^{\vec{\mathcal{T}}_k, b}(\kappa_k)$, $A \in (\wp(\kappa_i))^{\mathcal{P}}$, *and* $a \in \pi^{\vec{\mathcal{T}}_k, b}(\sigma^{\vec{\mathcal{T}}}(A))\}$.

When comparing hod premice we need to consider iterations in which at certain stages I is allowed to use the un-dropping extender of the resulting stack. The game producing such iterations is defined below.

Definition 7.3 (The un-dropping iteration game) *Suppose $\mathcal{P}$ is a hod-like lsp. The un-dropping iteration game on $\mathcal{P}$, $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$, is an iteration game satisfying the following conditions:*

1. *If any of the models produced during a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ is ill-founded then player II loses that run.*

2. *$\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ has at most κ main rounds. Player I starts the main rounds.*

3. *If p is a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ and $\mathcal{M}_\zeta$ is the model at the beginning of the ζth main round of p then the ζth main round of p is a run of $\mathcal{G}_k(\mathcal{M}_\zeta, \lambda, \alpha)$.*

4. *Suppose p is a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$. Then player I can start a main round in two different ways.*

    (a) *Suppose first p has $\zeta < \kappa$ main rounds where ζ is a limit ordinal and the main rounds are cofinal in p. Let $(\mathcal{M}_\alpha : \alpha < \zeta)$ be the sequence of the models at the beginning of the main rounds. Let then $\mathcal{M}_\zeta$ be the direct limit of $\mathcal{M}_\alpha$ under the iteration embeddings. Then the ζth main round is played on $\mathcal{M}_\zeta$.*

(b) *Next suppose $\zeta = \gamma + 1$. Then I can start a new main round only if the stack played in the γth main round is continuable. Let then $\vec{\mathcal{T}}_\gamma$ be the stack played in the γth main round and suppose $\vec{\mathcal{T}}_\gamma$ is continuable. Let $\mathcal{M}_\zeta = Ult(\mathcal{M}_\gamma, E^{\vec{\mathcal{T}}_\gamma})$. Then I can start a new main round, if she wishes so, on $\mathcal{M}_\zeta$.*

If $\vec{\mathcal{T}}$ is a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$, then we let $(\mathcal{M}_\varsigma, \vec{\mathcal{T}}_\varsigma, E_\varsigma : \varsigma < \eta)$ be such that $\mathcal{M}_\varsigma$ is the lsp at the beginning of the ςth main round, $\vec{\mathcal{T}}_\varsigma$ is the stack played in the ςth main round, and E_ς is defined iff $\varsigma + 1 < \eta$, and $E_\varsigma = E^{\vec{\mathcal{T}}_\varsigma}$. If Σ is a winning strategy for II in $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ then we say Σ is a $(\kappa, \lambda, \alpha)$-strategy.

We say $\vec{\mathcal{T}}$ is a generalized stack if it is produced via a run of the un-dropping game.

Definition 7.4 (Hod-like lsp pair) *We say $(\mathcal{P}, \Sigma)$ is a hod-like lsp pair if $\mathcal{P}$ is a hod-like lsp and Σ is a winning strategy in $\mathcal{G}^u_k(\mathcal{P}, \omega_1, \omega_1, \omega_1)$.*

8 Short tree component of a strategy

Suppose $(\mathcal{P}, \Sigma)$ is a hod-like lsp pair such that $\mathcal{P}$ is of lsa type. The next definition isolates the *short tree component* of Σ denoted by Σ^{stc}. Let $\kappa = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ and $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$.

Definition 8.1 (The normal short tree component of a strategy) *We first define the short tree component Σ^{nstc} on normal trees. Suppose $\mathcal{T}$ is a normal tree on $\mathcal{P}$ of limit length. Let $b = \Sigma(\vec{\mathcal{T}})$. We then let*

$$\Sigma^{nstc}(\mathcal{T}) = \begin{cases} b & : \ \pi^{\mathcal{T}}_b \text{ doesn't exists or } \pi^{\mathcal{T}}_b(\delta) > \delta(\mathcal{T}), \\ \mathcal{M}^{\mathcal{T}}_b & : \ \text{otherwise.} \end{cases}$$

Suppose $\mathcal{Q}$ is an iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}}$ exists. We define the short tree component of Σ by concatenating all $\Sigma^{nstc}_{\mathcal{Q},\vec{\mathcal{T}}}$.

To make the next definition more intuitive, we say $\mathcal{T}$ is Σ-*maximal* tree on $\mathcal{P}$ if $\mathcal{T}$ has a limit length, is according to Σ and case 2 above holds. Notice that maximality of $\mathcal{T}$ depends on Σ. Also, if $\mathcal{T}$ is a tree then we let $\mathcal{T}^-$ be $\mathcal{T}$ without its last model if it exists and $\mathcal{T}$ otherwise.

The next definitions describe when a stack is according to the short tree strategy component of Σ.

Definition 8.2 *We let*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \mathcal{U}_\alpha : \alpha < \eta) \in dom(\Sigma^{stc})$$

if there is $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta) \in dom(\Sigma)$ *such that* $\vec{\mathcal{U}}$ *is the same as* $\vec{\mathcal{T}}$ *except it doesn't have the maximal branches of* $\vec{\mathcal{T}}$*; more precisely,*

1. *for every* $\alpha < \eta$, $\mathcal{N}_\alpha = \mathcal{M}_\alpha$,

2. *for every* $\alpha < \eta$ *such that* $\pi^{\vec{\mathcal{T}}\restriction\alpha}$*-exists,*

$$\mathcal{U}_\alpha = \begin{cases} \mathcal{T}_\alpha^- & : \mathcal{T}_\alpha^- \text{ is } \Sigma_{\mathcal{N}_\alpha, \vec{\mathcal{T}}\restriction\alpha}\text{-maximal,} \\ \mathcal{T}_\alpha & : \text{ otherwise.} \end{cases}$$

3. *letting* α *be the least, if it exists, such that* $\pi^{\vec{\mathcal{T}}_\alpha}$*-doesn't exist, for all* $\beta \geq \alpha - 1$, $\mathcal{U}_\beta = \mathcal{T}_\beta$[9].

4. *there are finitely many* α *such that* $\mathcal{U}_\alpha \neq \mathcal{T}_\alpha$,

5. *either* η *is a limit ordinal or* $\mathcal{T}_{\eta-1}$ *has a limit length.*

If $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *are as above then we write* $\vec{\mathcal{U}} = \vec{\mathcal{T}}^{sc}$ *and say that* $\vec{\mathcal{U}}$ *is the short component of* $\vec{\mathcal{T}}$.

Finally, we define the domain of the short tree component of Σ on generalized stacks.

Definition 8.3 (The short tree component of a strategy: the domain) *We let*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \vec{\mathcal{U}}_\alpha, E_\alpha : \alpha < \eta) \in dom(\Sigma^{stc})$$

if there is $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, F_\alpha : \alpha < \eta) \in dom(\Sigma)$ *such that* $\vec{\mathcal{U}}$ *is the same as* $\vec{\mathcal{T}}$ *except it doesn't have the maximal branches of* $\vec{\mathcal{T}}$*; more precisely,*

1. *for every* $\alpha < \eta$, $\mathcal{N}_\alpha = \mathcal{M}_\alpha$ *and* $E_\alpha = F_\alpha$,

2. *for every* $\alpha < \eta$ *such that* $\vec{\mathcal{U}}_\alpha = \vec{\mathcal{T}}_\alpha^{sc}$,

3. *there are finitely many* α *such that* $\vec{\mathcal{U}}_\alpha \neq \vec{\mathcal{T}}_\alpha$,

4. *either* η *is a limit ordinal or the last normal component of* $\vec{\mathcal{T}}_{\eta-1}$ *has a limit length.*

[9] Notice that α is necessarily a successor ordinal.

If $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are as above then we write $\vec{\mathcal{U}} = \vec{\mathcal{T}}^{sc}$ and say that $\vec{\mathcal{U}}$ is the short component of $\vec{\mathcal{T}}$.

The next definition defines the short tree component of Σ. Recall that if $\vec{\mathcal{T}}$ is a stack of iteration trees then $\delta(\vec{\mathcal{T}})$ is the sup of the generators of $\vec{\mathcal{T}}$. It can be defined inductively on the number of normal components of $\vec{\mathcal{T}}$ (see Definition 1.15 of [3]).

Definition 8.4 (The short tree component of a strategy) *Given*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \mathcal{U}_\alpha, E_\alpha : \alpha < \eta) \in dom(\Sigma^{stc}),$$

letting $\vec{\mathcal{T}}$ be such that $\vec{\mathcal{T}}^{sc} = \vec{\mathcal{U}}$ and $b = \Sigma(\vec{\mathcal{T}})$, we let

$$\Sigma^{stc}(\vec{\mathcal{U}}) = \begin{cases} b & : \ \pi_b^{\vec{\mathcal{T}}} \textit{ doesn't exists or } \pi_b^{\mathcal{T}}(\delta) > \delta(\vec{\mathcal{T}}), \\ \mathcal{M}_b^{\vec{\mathcal{T}}} & : \ \textit{otherwise.} \end{cases}$$

Thus, $\Sigma^{stc}(\vec{\mathcal{T}})$ either returns the value of $\Sigma(\vec{\mathcal{T}})$ or $\mathcal{M}_b^{\vec{\mathcal{T}}}$ where $b = \Sigma(\vec{\mathcal{T}})$. In the sequel, we will use this notation even when Σ is a partial iteration strategy.

Notice the similarity with the short tree iterability for suitable mice in the context of core model induction or in the context of HOD analysis and Σ^{stc}. If $\mathcal{P}$ is a Σ_1^2-suitable premouse and Σ is fullness preserving iteration strategy for $\mathcal{P}$, Σ^{stc} is just the short tree iterability strategy of $\mathcal{P}$.

9 The short tree game

In order to define *short tree strategy mice*, we will need to define *short tree strategy* in a way that it is independent of a particular strategy. The short tree strategies are winning strategies for player II in the *short tree iteration game* introduced below. It will not be hard to see that if Σ is a strategy then Σ^{stc} is a short tree strategy.

Let $\mathcal{P}$ be a hod-like lsa type lsp. Let $\kappa = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ and $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$.

Definition 9.1 (The normal short tree game, $\mathcal{G}_k^{nst}(\mathcal{P}, \lambda)$) *Then the short tree game $\mathcal{G}_k^{nst}(\mathcal{P}, \lambda)$ is a two player game on $\mathcal{P}$ played as follows. Just like in $\mathcal{G}_k(\mathcal{P}, \lambda)$, I plays the successor steps in $\mathcal{G}_k^{nst}(\mathcal{P}, \lambda)$ according to the rules of $\mathcal{G}_k(\mathcal{P}, \lambda)$. Let then $\mathcal{T}$ be an iteration tree produced by a run of $\mathcal{G}_k^{nst}(\mathcal{P}, \lambda)$. Suppose $\mathcal{T}$ has a limit length. Then II has the following two options:*

Option 1. *$\pi^{\mathcal{T},b}$ exists, $\pi^{\mathcal{T},b}(\kappa) < \delta(\mathcal{T})$ and there is $\mathcal{M}$ such that*

1. $\mathcal{M}(\mathcal{T}) \trianglelefteq_{hod} \mathcal{M}$ *and* $\mathcal{M}$ *is* κ*-iterable above* $\delta(\mathcal{T})$*,*

2. $\mathcal{M}$ *is a hod-like lsa type lsp such that* $\delta^{\mathcal{M}} = \delta(\mathcal{T})$*,*

Option 2. *Otherwise.*

If Option 1 holds then II may choose to play $\mathcal{M}$ *satisfying the above clauses. In all other cases, II must play a cofinal well-founded branch b such that either* $\pi_b^{\mathcal{T}}$ *doesn't exists or* $\pi_b^{\mathcal{T}}(\delta^{\mathcal{P}}) > \delta(\mathcal{T})$*.*

If at any point II plays according to Option 1 then the game stops and II wins the game. In this case we also say that II plays a model (rather than a branch) or that II's move is a model and etc.

Next, we introduce the version of the normal short tree game that has at most ω main rounds.

Definition 9.2 (The short tree game) *The short tree game* $\mathcal{G}_k^{st}(\mathcal{P}, \lambda, \eta)$ *is an iteration game that has at most* ω *main rounds each of which consists of a run of the usual* (k, λ, η)*-iteration game (see Definition 5.5) with the following exceptions.*

1. *Suppose* $\mathcal{M}$ *is a model at the begging of the main round of some run of the game and* $\vec{\mathcal{T}}$ *is a run of* (k, λ, η) *on* $\mathcal{M}$*. Suppose* $\mathcal{R}$ *is a non-trivial terminal node in* $\vec{\mathcal{T}}$*. If* $\pi_{\mathcal{R}}^{\vec{\mathcal{T}}}$ *exists then the largest irreducible initial segment of* $\vec{\mathcal{T}}_{\geq\mathcal{R}}$ *is played according to the rules of* $\mathcal{G}^{nst}(\mathcal{R}, \eta)$*. If* $\pi_{\mathcal{R}}^{\vec{\mathcal{T}}}$ *doesn't exist then the largest irreducible initial segment of* $\vec{\mathcal{T}}_{\geq\mathcal{R}}$ *is played according to the rules of the usual iteration game.*

2. *If at any point during the run of a sub-round, II plays a model then I has to start a new main round on that model, and all main rounds are started in this fashion.*

If the play has ω *main rounds then II wins the game.*

Finally, we introduce the *un-dropping short tree game.*

Definition 9.3 (The un-dropping short tree game) *The un-dropping short tree game on* $\mathcal{P}$*,* $\mathcal{G}_k^{ust}(\mathcal{P}, \lambda, \eta, \alpha)$*, is an iteration game that has at most* ω *main rounds each of which consists of a run of* $\mathcal{G}_k^{u}(\mathcal{P}, \lambda, \eta, \alpha)$ *with the following exceptions.*

1. *Suppose* $\mathcal{M}$ *is a model at the begging of a main round of some play and* $\vec{\mathcal{T}}$ *is a run of* $\mathcal{G}_k^{u}(\mathcal{M}, \lambda, \eta, \alpha)$*. Suppose* $\vec{\mathcal{T}} = (\mathcal{M}_\varsigma, \vec{\mathcal{T}}_\varsigma, E_\varsigma : \varsigma < \eta)$*. Then for each* $\varsigma < \eta$*,* $\vec{\mathcal{T}}_\varsigma$ *is played according to the rules of* $\mathcal{G}^{st}(\mathcal{R}, \eta, \alpha)$*.*

2. *If at any point during the run of a sub-round, II plays a model then I has to start a new main round on that model, and all main rounds are started in this fashion.*

If the play has ω main rounds then II wins the game.

Definition 9.4 (Short tree strategy) *Suppose $\mathcal{P}$ is an lsa type lsp. We say Λ is a short tree (λ, η, α)-strategy for $\mathcal{P}$ if Λ is a wining strategy for II in $\mathcal{G}_k^{ust}(\mathcal{P}, \lambda, \eta, \alpha)$.*

Suppose now $\mathcal{P}$ and Λ are as in Definition 9.4. We let $b(\Lambda)$ be the set of all $\vec{\mathcal{T}} \in dom(\Lambda)$ such that $\vec{\mathcal{T}}$ has a last normal component of limit length and $\Lambda(\vec{\mathcal{T}})$ is a cofinal wellfounded branch of $\vec{\mathcal{T}}$. Let $m(\Lambda) = dom(\Lambda) - b(\Lambda)$. Given $\vec{\mathcal{U}} \in dom(\Lambda)$ such that the last component of $\vec{\mathcal{U}}$ has a limit length, we let

$$\mathcal{M}(\Lambda, \vec{\mathcal{U}}) = \begin{cases} \mathcal{M}_b^{\vec{\mathcal{U}}} & : \Lambda(\vec{\mathcal{U}}) = b \\ \Lambda(\vec{\mathcal{U}}) & : \text{otherwise.} \end{cases}$$

Remark 9.5 *In many situations, it is expected that winning $\mathcal{G}_k^{st}(\mathcal{M}, \kappa, \lambda)$ must be easy for II: II wins it as soon as she plays infinitely many models. However, we will be interested in strategies for II that have certain fullness preservation properties. For instance, suppose $\mathcal{M}$ is just a suitable mouse in the sense of $L(\mathbb{R})$. Suppose Λ is strategy for II in $\mathcal{G}_k^{nst}(\mathcal{M}, \omega_1)$ such that whenever $\mathcal{T}$ is a tree according to Λ then*

1. *if $\mathcal{T} \in b(\Lambda)$, $b = \Lambda(\mathcal{T})$ and $\pi_b^{\mathcal{T}}$ exists then $\mathcal{M}_b^{\mathcal{T}}$ is $(\Sigma_1^2)^{L(\mathbb{R})}$-full and*
2. *if $\mathcal{T} \in m(\Lambda)$ and $\mathcal{N} = \Lambda(\mathcal{T})$ then $\mathcal{N}$ is suitable in the sense of $L(\mathbb{R})$*

then Λ is in fact a "short tree iterability strategy" in the sense of $L(\mathbb{R})$, it is $L(\mathbb{R})$-fullness preserving. Such strategies are difficult to construct, and in our current situation, we will be interested in a notion of fullness preservation with respect to a much more complicated pointclasses then $L(\mathbb{R})$.

10 Lsa type pair

Suppose $\mathcal{P}$ is a hod-like lsa type lsp and suppose Λ is a short tree strategy for $\mathcal{P}$. We would like to introduce the notion of a short tree premouse and in particular, Λ-premouse. The main technical problem is that we do not have a reasonable notion of condensation for short tree strategies. In particular, if $\Lambda = \Sigma^{stc}$ for some strategy Σ, then it may well be that there is a tree $\mathcal{T}$ on $\mathcal{P}$ such that if $b = \Sigma(\mathcal{T})$ then b is non-dropping and $\pi_b^{\mathcal{T}}(\delta) = \delta(\mathcal{T})$ yet there is a hull $\mathcal{U}$ of $\mathcal{T}$ such that if $c = \Sigma(\mathcal{U})$ then in fact $\pi_c^{\mathcal{U}}(\delta) > \delta(\mathcal{T})$. Thus, $\Lambda(\mathcal{T}) = \mathcal{M}_b^{\mathcal{T}}$ while $\Lambda(\mathcal{U}) = c$.

The above scenario is the main difficulty with defining short tree strategy mice. We have to find a particular indexing of short tree strategies, or rather carefully skip over "bad trees", in a way that when $\mathcal{T}$ above is "cored down" to $\mathcal{U}$ above then our indexing is still preserved. In particular, the branch of $\mathcal{U}$ cannot be added too early. The idea is to wait until branches or rather the $\mathcal{Q}$-structures are *certified*. The particular way of certifying introduced in the next section is due to the first author. Before we define short tree hybrids, however, we have to make a few definitions that will be useful to us in the future.

First we define the *unambiguous stacks* which are essentially the stacks whose branches are easy to guess.

Definition 10.1 (Unambiguous stacks) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp and $\vec{\mathcal{T}}$ is a run of $\mathcal{G}_k^{st}(\mathcal{P}, \kappa, \lambda)$ without main rounds. We say $\vec{\mathcal{T}}$ is unambiguous if either, letting $md^{\vec{\mathcal{T}}} = (\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ be the sequence of main drops of $\vec{\mathcal{T}}$, $k \geq 1$ or one of the following holds:*

1. *There is a linear closed unbounded $C \subseteq ntn(\vec{\mathcal{T}})$.*
2. *$\vec{\mathcal{T}}$ has a last normal component of successor length.*
3. *Clauses 1 and 2 above fail, and $\vec{\mathcal{T}}$ has a last normal component of limit length such that letting $\mathcal{T}$ be this normal component, one of the following conditions hold:*
   - *(a) $\pi^{\mathcal{T},b}$ doesn't exist.*
   - *(b) $\pi^{\mathcal{T},b}$-exists and for some cutpoint $\mathcal{S}$ of $\mathcal{T}$ and some $\eta < o(\mathcal{S})$ such that $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1} < \eta$, $\mathcal{T}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{O}^{\mathcal{S}}_\eta$ (see Definition 3.6) and is above η.*
   - *(c) Clauses 3.a and 3.b fail, there is a cutpoint $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{T}_{\geq \mathcal{S}}$ is above $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}$, and there is $\mathcal{Q} \trianglelefteq \mathcal{J}(\mathcal{M}(\mathcal{T}))$ such that $\mathcal{Q} \vDash$ "$\delta(\mathcal{T})$ is Woodin" and $rud(\mathcal{Q}) \vDash$ "$\delta(\mathcal{T})$ isn't Woodin".*

We will only consider short tree strategies Λ with the property that whenever $\vec{\mathcal{T}} \in dom(\Lambda)$ is an unambiguous stack then $\Lambda(\vec{\mathcal{T}})$ is a branch. If Λ is a short tree strategy for $\mathcal{P}$ and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ according to Λ with last model $\mathcal{N}$ then we let $\Lambda_{\mathcal{N},\vec{\mathcal{T}}}$ be the short tree strategy of $\mathcal{N}$ induced by Λ, i.e., for every $\vec{\mathcal{U}}$ on $\mathcal{N}$, $\Lambda_{\mathcal{N},\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Lambda(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

Definition 10.2 (Faithful short tree strategy) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp and Λ is a short tree (κ, λ, η)-strategy for $\mathcal{P}$. We say Λ is a faithful short tree (κ, λ, η)-strategy if whenever $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq k < \omega) \in dom(\Lambda)$, and $\mathcal{R} \in tn(\vec{\mathcal{T}})$ then, letting $\vec{\mathcal{U}}$ be the largest initial segment of $\vec{\mathcal{T}}$ which is based on $\mathcal{R}$ and has no main rounds, then*

1. *if $\vec{\mathcal{U}}$ is unambiguous then $\vec{\mathcal{U}} \in b(\Lambda_{\mathcal{R}, \vec{\mathcal{T}}_{\leq \mathcal{R}}})$,*

2. *if clause 3.c of Definition 10.1 holds for $\vec{\mathcal{U}}$ then letting $\mathcal{S}$ be the cutpoint node of $\vec{\mathcal{U}}$ witnessing clause 3.c of Definition 10.1 then $\Lambda_{\mathcal{S}, \vec{\mathcal{T}}_{\leq \mathcal{S}}}(\vec{\mathcal{U}}_{\geq \mathcal{S}})$ is a branch of $\vec{\mathcal{U}}$ such that $\mathcal{Q}(b, \vec{\mathcal{U}}_{\geq \mathcal{S}})$ exists and $\mathcal{Q}(b, \vec{\mathcal{U}}_{\geq \mathcal{S}}) \trianglelefteq \mathcal{J}(\mathcal{M}(\vec{\mathcal{U}}_{\geq \mathcal{S}}))$.*

In the next section we will need to consider short tree iteration strategies that are partial and their range consists of branches. The next definition introduces this notion.

Definition 10.3 (Short tree strategy without a model component) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp. We say Λ is a partial short tree strategy for $\mathcal{P}$ if it is a partial winning strategy in $\mathcal{G}_k^{st}(\mathcal{P}, \omega_1, \omega_1, \omega_1)$. If Λ is a partial short tree strategy for $\mathcal{P}$ then we say it is without model component if $m(\Lambda) = \emptyset$.*

We can then also define faithful short tree strategies without model component.

Definition 10.4 (Lsa type pair) *We say $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair if $\mathcal{P}$ is a hod-like lsa type lsp and Λ is an $(\omega_1, \omega_1, \omega_1)$ faithful short tree strategy. We say $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair without model component if $\mathcal{P}$ is a hod-like lsa type lsp and Λ is an-$(\omega_1, \omega_1, \omega_1)$ faithful short tree strategy without model component.*

Suppose $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair. We then let

$$I(\mathcal{P}, \Lambda) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \vec{\mathcal{T}} \text{ is a run of the un-dropping short tree game on } \mathcal{P} \text{ according to } \Lambda \text{ such that } \pi^{\vec{\mathcal{T}}, b} \text{ exists and } \mathcal{Q} \text{ is the last model of } \vec{\mathcal{T}}\}.$$

Next we will need certain condensation properties for short tree strategies.

Definition 10.5 (Branch condensation for short tree strategies) *Suppose $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair. We say Λ has branch condensation if for any $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Lambda)$ and $(\vec{\mathcal{U}}, c)$ such that*

1. *$\vec{\mathcal{U}}$ is according to Λ and its last normal component has a limit length, and*

2. c is branch of $\vec{\mathcal{U}}$ such that there is $\sigma : \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{r\Sigma_1} \mathcal{Q}$ with the property that $\pi^{\vec{\mathcal{U}}^\frown \mathcal{M}_c^{\vec{\mathcal{U}}}, b}$ exists and $\pi^{\vec{\mathcal{T}}, b} = \sigma \circ \pi^{\vec{\mathcal{U}}, b}$,

$c = \Lambda(\vec{\mathcal{U}})$.

Suppose now that $(\mathcal{P}, \Lambda)$ is a hod-like lsa pair. Below we describe a full background construction that, if successful, constructs a Λ-iterate of $\mathcal{P}$. We say a (κ, λ)-extender E coheres Λ if $\mathcal{P} \in V_\kappa$, $V_\lambda \subseteq Ult(V, E)$ and $\pi_E(\Lambda) \cap V_\lambda = \Lambda \cap V_\lambda$.

Definition 10.6 ($(\mathcal{P}, \Lambda)$-coherent fully backgrounded constructions) *Suppose κ is an inaccessible cardinal and $(\mathcal{P}, \Lambda)$ is a hod-like lsa pair such that Λ is a (κ, κ, κ)-short tree strategy. Then for $\eta < \kappa$, we say $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma \leq \eta))$ is the output of the $(\mathcal{P}, \Lambda)$-coherent fully backgrounded construction if the following holds.*

1. *$\mathcal{M}_0 = \emptyset$.*
2. *$\mathcal{M}_\gamma$ is a hod-like lsp such that in the comparison of $\mathcal{P}$ with $\mathcal{M}_\gamma$, $\mathcal{M}_\gamma$ doesn't move and the comparison results in a tree $\mathcal{T}_\gamma$ on $\mathcal{P}$ according to Λ such that either $\mathcal{T}_\gamma$ has a last model $\mathcal{M}$ such that $\mathcal{M}_\gamma \trianglelefteq_{hod} \mathcal{M}$ or $\mathcal{M}_\gamma = \mathcal{M}(\mathcal{T}_\gamma)$.*
3. *Suppose $\gamma \leq \eta$ is such that either $\mathcal{T}_\gamma$ has a last model or $\mathcal{T}_\gamma \in b(\Lambda)$. Let $\mathcal{M}$ be the last model of $\mathcal{T}_\gamma$ if it exists and otherwise, letting $b = \Lambda(\mathcal{T}_\gamma)$, let $\mathcal{M} = \mathcal{M}_b^{\mathcal{T}_\gamma}$. Let ς be such that $\mathcal{M}_\gamma = \mathcal{M}|\varsigma$ and suppose $\mathcal{M}_\gamma = \mathcal{J}_\xi^{\vec{E}, f}$. Then the following statements hold.*
   - *(a) If $\mathcal{M}_\gamma = \mathcal{M}$ then $\gamma = \eta$.*
   - *(b) Suppose $\mathcal{M}_\gamma \triangleleft \mathcal{M}$. Suppose there is no pair (F^*, F) and an ordinal $\zeta < \xi$ such that $F^* \in V_\kappa$ is an extender over V cohering Λ, F is an extender over $\mathcal{M}_\gamma$, $V_{\zeta+\omega} \subseteq Ult(V, F^*)$ and*
   $$F \restriction \zeta = F^* \cap ([\zeta]^\omega \times \mathcal{J}_\xi^{\vec{E}, f})$$
   such that $(\mathcal{J}_\xi^{\vec{E}, f}, \in, \vec{E}, f, \tilde{F})$ is a hod-like lsp (here $\tilde{F}$ is the amenable code of F). Then $\mathcal{N}_\gamma = \mathcal{J}_1(\mathcal{M}_\gamma)$ and $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$.
   - *(c) Again suppose $\mathcal{M}_\gamma \triangleleft \mathcal{M}$ but there is a pair (F^*, F) and an ordinal ζ satisfying the above conditions. Then if $F^* \in \vec{E}^{\mathcal{M}}$ then we let*
   $$\mathcal{N}_\gamma = (\mathcal{J}_\xi^{\vec{E}, f}, \in, \vec{E}, f, \tilde{F})$$

where $\tilde{F}$ is the amenable code of F. Also, $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$. If $F^* \notin \vec{E}^{\mathcal{M}}$ then $\gamma = \eta$ and we stop the construction.

(d) *Again suppose $\mathcal{M}_\gamma \triangleleft \mathcal{M}$ and that $\mathcal{M}|\varsigma$ is an active $\mathcal{J}$-structure such that its last predicate is not an extender but is some A. Let then $\mathcal{N}_\gamma = (\mathcal{M}_\gamma, A, \in)$ and $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$.*

4. *Suppose $\gamma \leq \eta$ is such that $\mathcal{T}_\gamma$ is of limit length and $\mathcal{T}_\gamma \notin b(\Lambda)$. Then $\gamma = \eta$.*

Remark 10.7 *Notice that the constructions introduced in Definition 10.6 can be carried out even when $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair without model component. Also, if the background universe has a distinguished extender sequence then we tacitly assume that the extenders appearing in the $(\mathcal{P}, \Lambda)$-coherent fully background construction come from this distinguished extender sequence.*

11 A short tree strategy indexing scheme

Our goal here is to introduce the notion of a *short tree strategy premouse* (*sts premouse*). As we mentioned in the previous section, the difficulty with doing this lies in the fact that maximal trees might "core down" to short trees and thus, creating indexing issues. The idea behind the solution presented here is to add a branch for a tree as soon as we see a certificate, which in our case will be a $\mathcal{Q}$-structure, that it is short. As the $\mathcal{Q}$-structures that we will be looking for are themselves sts premice, this inevitably leads to an induction. In the course of this induction we will introduce sts ψ_α-premice which will correspond to the αth level of the induction.

Definition 11.1 (Closed hsp) *Suppose $\mathcal{M}$ is an hsp and $\alpha \leq o(\mathcal{M})$. Then we say $\mathcal{M}$ is closed under sharps below α if for all $\beta < \alpha$ there is $\gamma \in \mathrm{dom}(\vec{E}^{\mathcal{M}})$ such that $crit(E_\gamma^{\mathcal{M}}) > \beta$. We say $\mathcal{M}$ is closed under sharps if $\mathcal{M}$ is closed under sharps below $o(\mathcal{M})$.*

Definition 11.2 (Unambiguous hsp) *Suppose $\mathcal{M}$ is an hsp over some hod-like lsp $\mathcal{P}$ such that $\mathcal{P}$ is of lsa type. We say $\mathcal{M}$ is unambiguous if $\mathcal{M}$ is closed and whenever $\mathcal{U} \in \mathcal{M}$ is such that $\mathcal{M} \vDash$ "$\mathcal{U}$ is an unambiguous tree of limit length on $\mathcal{P}$ according to $\Sigma^{\mathcal{M}}$", $\mathcal{U} \in dom(\Sigma^{\mathcal{M}})$.*

The next definition introduces an indexing scheme which we will use to define short tree premice. The indexing scheme only defines the strategy on certain carefully chosen stacks. It turns out that this much information is enough to extend the strategy on all stacks. In the next two definitions, instead of explicitly writing what ψ says, we indicate the impact that it has on the structures satisfying it. We leave it to the reader to extract the actual formula from our description.

Definition 11.3 (Potential sts indexing scheme) *Suppose $\psi(x, y)$ and $\phi(x, y)$ are two formulas in the language of hsp. We say ψ is a potential sts indexing scheme for ϕ if whenever X is an swo and $\mathcal{N}$ is an hsp over X then $\mathcal{N} \vDash \psi[X, c]$ if and only if*

1. *X is a hod-like lsa type lsp and $\mathcal{N}$ is closed under sharps,*
2. *$\mathcal{N} \vDash$ "$\Sigma^{\mathcal{N}}$ is a partial faithful short tree strategy without model component"*
3. *$c = \mathcal{J}_\omega(\mathcal{T})$ for some tree $\mathcal{T} \in \mathcal{N}$ on X such that $\mathcal{T}$ is of limit length and $\mathcal{T}$ is according to $\Sigma^{\mathcal{N}}$,*
4. *letting $\mathcal{T}$ be as in clause 3 above, either*
   - *(a) $\mathcal{T}$ is unambiguous and is the $\mathcal{N}$-least unambiguous tree such that $\mathcal{T} \notin dom(\Sigma^{\mathcal{N}})$ or*
   - *(b) $\mathcal{N}$ is unambiguous, $\mathcal{T}$ is ambiguous and there is (ν, ξ) such that*
     - *i. letting $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma < \eta))$ be the output of the $(X, \Sigma^{\mathcal{N}})$-coherent fully background construction of $\mathcal{N}$ in which extenders used have critical points $> \nu$ (see Definition 10.6), $\mathcal{T}_\xi = \mathcal{T}$ and there is a unique branch $b \in \mathcal{N}$ of $\mathcal{T}$ such that $\phi[\mathcal{T}, b]$ holds, and*
     - *ii. for every $\alpha < o(\mathcal{N})$ and $(\nu^*, \xi^*) <_{lex} (\nu, \xi)$ such that $\mathcal{N}|\alpha$ is unambiguous, letting $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta^*), (F_\gamma : \gamma < \eta^*), (\mathcal{T}_\gamma : \gamma < \eta^*))$ be the output of the $(X, \Sigma^{\mathcal{N}|\alpha})$-coherent fully background construction of $\mathcal{N}|\alpha$ where extenders used have critical points $> \nu^*$, either $\mathcal{T}_{\xi^*} \neq \mathcal{T}$ or there is no branch $b \in \mathcal{N}|\alpha$ of $\mathcal{T}_{\xi^*}$ such that $\mathcal{N}|\alpha \vDash \phi[\mathcal{T}_{\xi^*}, b]$.*

Notice that for each ϕ there is at most one ψ such that ψ is a potential sts indexing scheme for ϕ. Suppose M is a transitive set. We then let $\mathcal{M}^+(M) = \mathcal{J}(M)|(o(M)^{+\omega})^{\mathcal{J}(\mathcal{M}(\mathcal{T}))}$.

Definition 11.4 (Sts indexing scheme) *Suppose $\psi(x, y)$ and $\phi(x, y)$ are two formulas in the language of hsp. We say ψ is an sts indexing scheme for ϕ if letting ψ^* be the potential sts indexing scheme for ϕ, whenever X is an swo and $\mathcal{N}$ is an hsp then $\mathcal{N} \vDash \psi[X, c]$ if and only if $\mathcal{N} \vDash \psi^*[X, c]$ or c is the $\mathcal{N}$-least set b such that there is a triple $(\mathcal{U}, \vec{\mathcal{U}}, \kappa)$ with the property that $\mathcal{U} \in dom(\Sigma^{\mathcal{N}})$, $\mathcal{N} \vDash$ "$\mathcal{U}$ is ambiguous", $\mathcal{N} \vDash \psi^*[X, \mathcal{J}_\omega(\mathcal{U})]$, $\kappa < \lambda^{\mathcal{M}(\mathcal{U})}$,*

$$\mathcal{N} \vDash \text{“}(\mathcal{U}, \mathcal{M}^+(\mathcal{U}), \vec{\mathcal{U}}) \textit{ is according to } \Sigma^{\mathcal{N}}\text{”},$$

$\Sigma^{\mathcal{N}}_{\mathcal{M}^+(\mathcal{U})}(\vec{\mathcal{U}})$ *is undefined,* $\vec{\mathcal{U}}$ *is based on* $\mathcal{M}(\mathcal{U})(\kappa)$ *and* $b = \mathcal{J}_\omega(\mathcal{U}, \mathcal{M}^+(\mathcal{U}), \vec{\mathcal{U}})$.

Notice that for each ϕ there is at most one ψ such that ψ is an sts indexing scheme for ϕ.

Definition 11.5 (Sts ϕ-premouse) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp and $\phi(x, y)$ is a formula in the language of hsp. Then $\mathcal{M}$ is an sts ϕ-premouse over $\mathcal{P}$ if $\mathcal{M}$ is an hsp over $\mathcal{P}$ with an indexing scheme ψ where ψ is the sts indexing scheme for ϕ.*

If $\phi(x, y) =$ “$0 = 1$” then we say $\mathcal{M}$ has a trivial indexing scheme and also say that $\mathcal{M}$ is a trivial sts premouse.

Suppose $\mathcal{M}$ is an sts ϕ-premouse over $\mathcal{P}$. We then let $H^{\mathcal{M}}$ be a function such that

$$dom(H^{\mathcal{M}}) = dom(\Sigma^{\mathcal{M}}) \cap \{\mathcal{T} \in \mathcal{M} : \mathcal{M} \vDash \text{“}\mathcal{T} \text{ is ambiguous”}\} \text{ and}$$
$$H^{\mathcal{M}}(\mathcal{T}) = (min(f^{\mathcal{M}}(\mathcal{J}_\omega(\mathcal{T}))), \nu, \xi)$$

where (ν, ξ) witness clause 4.b of Definition 11.3.

Definition 11.6 (Super closed sp) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, $\phi(x, y)$ is a formula in the language of hsp and ψ is the sts indexing scheme for ϕ. Suppose $\mathcal{M}$ is an sts ϕ-premouse. We say $\mathcal{M}$ is super closed if $\mathcal{M}$ is unambiguous and whenever $p = (\mathcal{U}, \mathcal{M}^+(\mathcal{U}), \vec{\mathcal{U}})$ is according to $\Sigma^{\mathcal{N}}$, $\Sigma^{\mathcal{N}}(p)$ is defined.*

12 Certified stacks

In this section, we describe the procedure that we will use to identify the correct $\mathcal{Q}$-structures of short trees. We start by introducing *terminal trees*, which are the trees for which we need to look for certified $\mathcal{Q}$-structures.

Definition 12.1 (Terminal tree) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, ϕ is a formula in the language of hsp and $\mathcal{N}$ is an sts ϕ-premouse over $\mathcal{P}$. Given $\mathcal{T} \in \mathcal{N}$ on $\mathcal{P}$, we say $\mathcal{T}$ is $\mathcal{N}$-terminal if $\mathcal{T}$ is according to $\Sigma^{\mathcal{N}}$ and $\mathcal{N} \vDash$ “$\mathcal{T}$ is ambiguous”.*

When $\mathcal{N}$ is clear from context we drop it from our notation and just say that $\mathcal{T}$ is terminal. The next notion is what allows us to interpret the internal strategy of an sts premouse on all trees. We will use it to make precise the meaning of certified $\mathcal{Q}$-structures. Essentially, a $\mathcal{Q}$-structure will be certified if the trees according to its own internal strategy have a successful coiteration with fully backgrounded constructions.

Definition 12.2 (Successful coiteration in sts premice) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, ϕ is a formula in the language of hsp and $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E},S}(\mathcal{P})$ is an sts ϕ-premouse over $\mathcal{P}$. Suppose further that $\mathcal{M}$ is closed and $\mathcal{T} \in \mathcal{M}$ is terminal. Suppose g is an $\mathcal{M}$-generic, $\mathcal{R} \in \mathcal{M}[g]$ is a hod-like lsa type lsp and (β, ν, ξ) are such that $(\nu, \xi) \in \beta^2$ and $\mathcal{M}|\beta$ is super closed. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma \leq \eta))$be the output of the $(\mathcal{P}, \Sigma^{\mathcal{M}|\beta})$-coherent fully backgrounded construction of $\mathcal{M}|\beta$ where the extenders used have critical points $> \nu$.*

Then we let $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\xi}$ be the tree on $\mathcal{R}$ which comes from comparing $\mathcal{R}$ with $\mathcal{N}_\xi$. More precisely, $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}$ is constructed as follows. Suppose we have constructed $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma$. As part of the inductive construction we assume that $\mathcal{N}_\xi$ hasn't moved yet. We proceed as follows.

1. *Suppose $\gamma = \alpha + 1$. If there is a disagreement on the $\mathcal{N}_\xi$ side then we stop the construction. More precisely, letting $\mathcal{Q}$ be the last model of $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma$, the following holds:*
   (a) *If there is ν such that $\mathcal{Q}|\nu = \mathcal{N}_\xi|\nu$, $\mathcal{N}_\xi|\nu \neq \mathcal{N}_\xi||\nu$ and $\mathcal{Q}||\nu \neq \mathcal{N}_\xi||\nu$, then we stop the construction.*
   (b) *If there is ν such that $\mathcal{Q}|\nu = \mathcal{N}_\xi|\nu$, $\mathcal{N}_\xi|\nu = \mathcal{N}_\xi||\nu$, $\mathcal{Q}||\nu \neq \mathcal{Q}|\nu$ and $\nu \notin dom(E^{\mathcal{Q}})$ then we stop the construction.*
   (c) *If there is ν such that $\mathcal{Q}|\nu = \mathcal{N}_\xi|\nu$, $\mathcal{N}_\xi|\nu = \mathcal{N}_\xi||\nu$, $\mathcal{Q}||\nu \neq \mathcal{Q}|\nu$ and $\nu \in dom(E^{\mathcal{Q}})$ then let $E^{\mathcal{Q}}_\nu$ be the next extender used in $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma$.*
2. *Suppose γ is limit. Suppose further that $\mathcal{N}_\xi \vDash$ "$\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma)$ is not Woodin" and there is a cofinal well-founded branch b of $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma$ such that $\mathcal{Q}(b, \mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\xi} \restriction \gamma)$-exists and $\mathcal{Q}(b, \mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma) \trianglelefteq \mathcal{N}_\xi$. Then II continues by playing b. If there is no such b then stop the construction.*
3. *Suppose then γ is limit but $\mathcal{N}_\xi \vDash$ "$\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma)$ is Woodin". Let μ be such that $\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\xi} \restriction \gamma) = \delta_\mu^{\mathcal{N}_\xi}$. Suppose there is a cofinal well-founded branch b of $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma$ such that for some $\alpha \in b$, $s(\mathcal{T}_\xi, \mu) \subseteq rng(\pi_{\alpha,b}^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi} \restriction \gamma})$. Then II plays b. If there is no such b then we stop the construction.*

We say $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}$ is successful if it has a last model $\mathcal{Q}$ such that $\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}}$ exists and there is $\beta < \lambda^{\mathcal{N}_\xi}$ such that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{N}_\xi(\beta)$. We also say that $(\mathcal{R}, \mathcal{N}_\xi)$ coiteration is successful.

Definition 12.3 (Certified iterations) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, ϕ is an sts-indexing scheme, and suppose $\mathcal{M}$ is an sts ϕ-premouse over $\mathcal{P}$. Suppose*

$(\mathcal{R},\vec{\mathcal{W}})\in\mathcal{M}$ is a pair such that $\mathcal{R}$ is a hod-like lsp and $\vec{\mathcal{W}}$ is a stack on $\mathcal{R}$. Then we say $(\mathcal{R},\vec{\mathcal{W}})$ is $\mathcal{M}$-certified if for some (β^*,ν^*,ξ^*), $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta^*,\nu^*,\xi^*}$ is successful and, letting

1. *(β,ν,ξ) be the least such that $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}$ is successful,*
2. *$((\mathcal{M}_\gamma,\mathcal{N}_\gamma:\gamma\leq\eta),(F_\gamma:\gamma<\eta),(\mathcal{T}_\gamma:\gamma\leq\eta))$ be the output of $(\mathcal{P},\Sigma^{\mathcal{M}|\beta})$-coherent fully backgrounded construction of $\mathcal{M}$ that uses extenders with critical points $>\nu$,*
3. *$\mathcal{N}\trianglelefteq\mathcal{N}_\xi$ be the last model of $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}$, and*
4. *$\pi=\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\beta,\nu,\xi}}$,*

$\vec{\mathcal{W}}$ is according to π-pullback of $(\Sigma^{\mathcal{M}})_{\mathcal{N}}$.

We will use our certification scheme to certify finite stacks.

Definition 12.4 (Finite stack) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp. We say $(\mathcal{T}_i,\mathcal{P}_i,\vec{\mathcal{U}}: i\leq n)$ is a finite stack on $\mathcal{P}$ if*

1. *$n<\omega$ and $\mathcal{P}_0=\mathcal{P}$,*
2. *For $i<n$, $\mathcal{T}_i$ is a normal ambiguous tree on $\mathcal{P}_i$ and $\mathcal{P}_{i+1}=\mathcal{M}^+(\mathcal{T}_i)$, and $\mathcal{T}_n$ is undefined,*
3. *$\vec{\mathcal{U}}$, if it is defined, is either*
   - *(a) a stack such that for some $\alpha+1<\lambda^{\mathcal{P}_n}$, $\vec{\mathcal{U}}$ is based on $\mathcal{P}_n(\alpha)$ and $\vec{\mathcal{U}}$ has a last normal component of limit length or*
   - *(b) a normal tree based on $\mathcal{P}_n$.*

Definition 12.5 (Certified finite stack) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, ϕ is an indexing scheme and $\mathcal{M}=\mathcal{J}^{\vec{E},f}_\alpha(\mathcal{P})$ is an sts ϕ-premouse over $\mathcal{P}$. Suppose further that $\mathcal{M}$ is super closed, g is $\mathcal{M}$-generic and $t=(\mathcal{T},\mathcal{U},\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2,\vec{\mathcal{W}})\in\mathcal{M}[g]$ is a finite stack on $\mathcal{P}$. We say t is an $\mathcal{M}$-certified finite stack if the following conditions are satisfied.*

1. *$\mathcal{T}$ is according to $\Sigma^{\mathcal{M}}$ and is $\mathcal{M}$-terminal.*
2. *Suppose $\mathcal{S}$ is a cutpoint of $\mathcal{U}$ such that for some $\alpha<\lambda^{\mathcal{U}}$ some initial segment of $\mathcal{U}_{\geq\mathcal{S}}$ is based on $\mathcal{S}(\alpha)$. Then $(\mathcal{S}(\alpha),\mathcal{W})$ is $\mathcal{M}$-certified where $\mathcal{W}$ be the longest component of $\mathcal{U}_{\geq\mathcal{S}}$ that is based on $\mathcal{S}(\alpha)$.*

3. *Suppose* $\mathcal{S}$ *is a cutpoint of* $\mathcal{U}$ *such that* $\pi^{\mathcal{U}_{\leq \mathcal{S}}, t}$ *exists and some initial segment of* $\mathcal{U}_{\geq \mathcal{S}}$ *is above* $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}$. *Let* $\mathcal{W}$ *be the largest such initial segment. Then the following conditions hold.*

   (a) *Suppose* $\mathcal{W}$ *doesn't have any fatal drops. Then for any limit* $\alpha < lh(\mathcal{W})$, *if* b *is the branch of* $\mathcal{W} \restriction \alpha$ *then* $\mathcal{Q}(b, \mathcal{W} \restriction \alpha)$ *exists and is the unique sp such that for some* (θ, γ, ζ), $\mathcal{U}^{\mathcal{M}}_{\mathcal{Q}(b,\mathcal{W}\restriction\alpha),\theta,\gamma,\zeta}$ *is successful.*

   (b) *Suppose* $\mathcal{W}$ *has a fatal drop at* (α, η). *Let* $\mathcal{Q} = \mathcal{O}^{\mathcal{M}^{\mathcal{W}}_{\alpha}}_{\eta}$. *Then* $(\mathcal{Q}, \mathcal{W}_{\geq \mathcal{Q}})$ *is* $\mathcal{M}$*-certified.*

4. *Letting* $\alpha < \lambda^{\mathcal{P}_2}$ *be such that* $\vec{\mathcal{U}}$ *is based on* $\mathcal{P}_2(\alpha)$, $(\mathcal{P}_2(\alpha), \vec{\mathcal{W}})$ *is* $\mathcal{M}$*-certified.*

Definition 12.6 (Certified branch) *Suppose* $\mathcal{P}$ *is a hod-like lsa type lsp,* ϕ *is an indexing scheme and* $\mathcal{M} = \mathcal{J}^{\vec{E},f}_{\alpha}(\mathcal{P})$ *is an sts* ϕ*-premouse over* $\mathcal{P}$. *Suppose further that* $\mathcal{M}$ *is closed,* g *is* $\mathcal{M}$*-generic and* $t = (\mathcal{T}, \mathcal{P}_0, \mathcal{P}_1, \vec{\mathcal{U}}) \in \mathcal{M}[g]$ *is an* $\mathcal{M}$*-certified finite stack on* $\mathcal{P}$. *If* b *is a branch of* $\vec{\mathcal{U}}$ *then we say* b *is* $\mathcal{M}$*-certified if* $(\mathcal{T}, \mathcal{P}_0, \mathcal{P}_1, \vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_b\})$ *is an* $\mathcal{M}$*-certified finite stack.*

13 Short tree strategy mice

We now have developed enough terminology and tools to define sts premouse. The definition is via induction. In the course of this induction, we define $(\psi^w_\alpha : \alpha < Ord)$ (the weak sts indexing schemeta) and $(\psi_\alpha : \alpha < Ord)$ (the sts indexing schemeta). We let $\psi^w_0 = \psi_0 =_{def}$ "$0 = 1$".

Definition 13.1 (Weak ψ^w_α-indexing scheme) *Suppose* $(\psi^w_\beta : \beta < \alpha)$ *and* $(\psi_\beta : \beta < \alpha)$ *have been defined. We let* ψ^w_α *be the following formula in the language of sp. Given an sp* $\mathcal{M}$ *over some hod-like lsa type lsp* $\mathcal{P}$, $\mathcal{M} \vDash \psi^w_\alpha[\mathcal{T}, b]$ *if and only if* $\mathcal{T}$ *is a terminal tree on* $\mathcal{P}$ *and* b *is a cofinal branch through* $\mathcal{T}$ *such that for some pair* (β, γ) *such that* $\gamma < \alpha$ *and* $\beta < o(\mathcal{M})$,

1. $\mathcal{M}|\beta$ *is super closed (see Definition 11.6) and* $\mathcal{M}|\beta \vDash ZFC +$ *"there are infinitely many Woodin cardinals* $> \delta(\mathcal{T})$*",*

2. $b \in \mathcal{M}|\beta$ *and* $\mathcal{M}|\beta \vDash$ *"b is well-founded branch",*

3. $\mathcal{M}|\beta \vDash$ *"*$\mathcal{Q}(b, \mathcal{T})$ *exists and is an sts* ψ_γ*-premouse over* $\mathcal{M}(\mathcal{T})$*" and*

4. *letting $(\delta_i : i < \omega)$ be the first ω Woodin cardinals $> \delta(\mathcal{T})$ of $\mathcal{M}|\beta$, $\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b, \mathcal{T})$ is $<$ Ord-iterable above $\delta(\mathcal{T})$ via a strategy Σ such that letting $\lambda = \sup_{i<\omega} \delta_i$, for every generic $g \subseteq Coll(\omega, < \lambda)$, Σ has an extension $\Sigma^+ \in D(\mathcal{M}|\beta, g)$ such that whenever $\mathcal{R} \in D(\mathcal{M}|\beta, g)$ is a Σ^+-iterate of $\mathcal{Q}(b, \mathcal{T})$ and $\vec{\mathcal{U}} \in dom(\Sigma^{\mathcal{R}})$ then $\vec{\mathcal{U}}$ is $\mathcal{M}|\beta$-certified".*

The lexicographically least pair (β, γ) satisfying the above conditions is called the least $(\mathcal{M}, \psi^w_\alpha)$-witness for $(\mathcal{T}, b)$. We also say that (β, γ, b) is an $\mathcal{M}$-minimal shortness witness for $\mathcal{T}$.

Notice that if (β_0, γ_0, b_0) and (β_1, γ_1, b_1) are two $\mathcal{M}$-shortness witnesses for $\mathcal{T}$ then we must have that $(\beta_0, \gamma_0) = (\beta_1, \gamma_1)$.

Definition 13.2 (ψ_α-indexing scheme) *Suppose $(\psi^w_\beta : \beta \leq \alpha)$ and $(\psi_\beta : \beta < \alpha)$ have been defined. We let ψ_α be the following formula in the language of sp. Given an sts ψ^w_α-premouse $\mathcal{M}$ over some hod-like lsa type lsp $\mathcal{P}$, $\mathcal{M} \vDash \psi_\alpha[\mathcal{T}, b]$ if and only if $\mathcal{M} \vDash$ "$\psi^w_\alpha[\mathcal{T}, b]$ and if (β, γ) is the least $(\mathcal{M}, \psi^w_\alpha)$-witness for $(\mathcal{T}, b)$ then for any branch c of $\mathcal{T}$ such that $c \neq b$, if (β^*, γ^*) is a $(\mathcal{M}, \psi^w_\alpha)$-witness for $(\mathcal{T}, c)$ then $(\beta, \gamma) <_{lex} (\beta^*, \gamma^*)$".*

Notice that at this point it is unclear that ψ_α-indexing scheme is meaningful. This is because it is possible that, keeping the notation of Definition 13.2, there could be two branches $b \neq c$ such that both (β, γ, b) and (β, γ, c) are $\mathcal{M}$-minimal shortness witnesses for $\mathcal{T}$. In the next section we will show that this cannot happen.

Next, we set $\psi_{Ord} =_{def} \psi_{sts}$ and refer to ψ_{sts} as the sts indexing scheme.

Definition 13.3 (Sts premouse) *Suppose $\mathcal{P}$ is an lsa type lsp. We say $\mathcal{M}$ is an sts premouse over $\mathcal{P}$ if $\mathcal{M}$ is ψ_{sts}-premouse over $\mathcal{P}$.*

Definition 13.4 (Sts mouse) *We say $\mathcal{M}$ is an sts mouse over $\mathcal{P}$ if $\mathcal{M}$ is ψ_{sts}-premouse over $\mathcal{P}$ which is $\omega_1 + 1$-iterable.*

Definition 13.5 (Λ-sts premouse) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, Λ is an sts strategy for $\mathcal{P}$ and $\mathcal{M}$ is an sts premouse over $\mathcal{P}$. Then we say $\mathcal{M}$ is a Λ-sts premouse over $\mathcal{P}$ if $\Sigma^M \subseteq \Lambda \restriction \mathcal{M}$.*

Definition 13.6 (Λ-sts mouse) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp, Λ is an sts strategy for $\mathcal{P}$ and $\mathcal{M}$ is a Λ-sts premouse over $\mathcal{P}$. Then we say $\mathcal{M}$ is a Λ-sts mouse over $\mathcal{P}$ if $\mathcal{M}$ has an $\omega_1 + 1$-iteration strategy Σ such that whenever $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via Σ, $\mathcal{N}$ is a Λ-sts premouse over $\mathcal{P}$.*

14 Uniqueness of shortness witnesses

The main contribution of this section is that the shortness witnesses are unique and hence, ψ_α-indexing scheme is meaningful. This is the content of the next lemma. For the purpose of this section we let P_α stand for the following statement.

$P_\alpha =$ "Whenever $\mathcal{P}$ is a hod-like lsa type lsp, $\mathcal{S}$ is an sts ψ^w_α-premouse over $\mathcal{P}$ and $\mathcal{T} \in \mathcal{S}$ is $\mathcal{S}$-terminal then $\mathcal{T}$ has at most one $\mathcal{S}$-minimal shortness witness".

Lemma 14.1 (The uniqueness of minimal shortness witnesses) *For all α, P_α is true.*

Proof. Let then $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{T} \in \mathcal{S}$ witness $\neg P_\alpha$. Recall that in particular $\mathcal{S}$ is an sts ψ^w_α-premouse and $\mathcal{T}$ is $\mathcal{S}$-terminal. Let (β, γ, b) and (β, γ, c) be two $\mathcal{S}$-minimal shortness witnesses for $\mathcal{T}$. We then must have that $\mathcal{Q}(b, \mathcal{T}) \neq \mathcal{Q}(c, \mathcal{T})$. Let $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ and $\mathcal{R} = \mathcal{Q}(c, \mathcal{T})$. Notice that we have that $\gamma < \alpha$, and $\mathcal{Q}$ and $\mathcal{R}$ are sts ψ_γ-premice.

We start working in $\mathcal{S}$. Let $\Sigma, \Lambda \in \mathcal{S}|\beta$ be the strategies of respectively $\mathcal{Q}$ and $\mathcal{R}$ with the property described in clause 4 of Definition 13.1. Notice that the comparison of $\mathcal{Q}$ with $\mathcal{R}$ (done inside $\mathcal{S}|\beta$) cannot be successful as otherwise standard fine structural facts would imply that $\mathcal{Q} = \mathcal{R}$.

It must then be the case that the comparison of $\mathcal{Q}$ and $\mathcal{R}$ doesn't only encounter extender disagreements (as otherwise the usual comparison argument would show that the comparison must halt). It then follows that, while working in $\mathcal{S}$, we can find $\mathcal{U}$ on $\mathcal{Q}$, $\mathcal{W}$ on $\mathcal{R}$ and a sequence $(\mathcal{S}_1, \alpha_1, \vec{\mathcal{T}}_1)$ such that the following holds:

1. $\mathcal{U}$ and $\mathcal{W}$ are according to respectively Σ and Λ, and are the result of comparing $\mathcal{Q}$ with $\mathcal{R}$ by hitting the least extender disagreement.
2. Let $d = \Sigma(\mathcal{U})$ and $e = \Lambda(\mathcal{W})$. Then $\mathcal{S}_1 = \mathcal{M}^{\mathcal{U}}_d|\alpha_1 = \mathcal{M}^{\mathcal{W}}_e|\alpha_1$.
3. $\vec{\mathcal{T}}_1 \in dom(\Sigma^{\mathcal{S}_1})$ is a stack on $\mathcal{M}^+(\mathcal{T})$ such that it has a last normal component of limit length and, letting $\Sigma^{\mathcal{M}^{\mathcal{U}}_d}(\vec{\mathcal{T}}_1) = b_1$ and $\Sigma^{\mathcal{M}^{\mathcal{W}}_e}(\vec{\mathcal{T}}_1) = c_1$, $b_1 \neq c_1$.
4. Both b_1 and c_1 are indexed at α_1 respectively in $\mathcal{M}^{\mathcal{U}}_d$ and $\mathcal{M}^{\mathcal{W}}_e$.

Claim 1. $\vec{\mathcal{T}}_1$ is a normal tree.

Proof. To see this assume otherwise. We then have that $\vec{\mathcal{T}}_1 = (\mathcal{T}_1, \mathcal{M}^+(\mathcal{T}_1), \vec{\mathcal{U}})$ as the only non-normal stacks in the domain of $\Sigma^{\mathcal{S}_1}$ have that form. We have that $\mathcal{T}_1$ is according to both $\Sigma^{\mathcal{M}^{\mathcal{U}}_d}$ and $\Sigma^{\mathcal{M}^{\mathcal{W}}_e}$. Let then $\mathcal{M} = \mathcal{M}^+(\mathcal{T}_1)$ and $\alpha + 1 < \lambda^{\mathcal{M}}$ be the least such that $\vec{\mathcal{U}}$ is a stack on $\mathcal{M}(\alpha + 1)$.

It follows from clause 4 of Definition 13.1 that $(\mathcal{M}(\alpha+1), \vec{\mathcal{U}})$ is $\mathcal{S}$-certified. It follows from Definition 12.5 that if (β, ν, ξ) is the least such that $\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha+1),\beta,\nu,\xi}$ is successful then letting

1. $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma \leq \eta))$ be the output of $(\mathcal{P}, \Sigma^{\mathcal{S}|\beta})$-coherent fully backgrounded construction of $\mathcal{S}|\beta$ that uses extenders with critical points $> \nu$,

2. $\mathcal{N} \trianglelefteq_{hod} \mathcal{N}_\xi$ be the last model of $\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha+1),\beta,\nu,\xi}$ and

3. $\pi = \pi^{\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha+1),\beta,\nu,\xi}}$

then both $\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_{b_1}\}$ and $\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_{c_1}\}$ are according to π-pullback of $(\Sigma^{\mathcal{S}|\beta})_{\mathcal{N}}$. It then follows that $b_1 = c_1$, contradiction!

□

Claim. $\mathcal{T}_1$ is both $\mathcal{M}^{\mathcal{U}}_d$-terminal and $\mathcal{M}^{\mathcal{W}}_e$-terminal.
Proof. Assume otherwise. Suppose $\mathcal{T}_1$ isn't $\mathcal{M}^{\mathcal{U}}_d$-terminal. Notice first that there cannot be a linear closed unbounded $C \subseteq ntn(\mathcal{T}_1)$ as otherwise we will have that $b_1 = b_C = c_1$.

Suppose then that for some cutpoint $\mathcal{M}$ of $\mathcal{T}_1$, there is $\alpha < \lambda^{\mathcal{S}}$ such that $(\mathcal{T}_1)_{\geq \mathcal{M}}$ is based on $\mathcal{M}(\alpha)$. It then follows from clause 4 of Definition 13.1 and clause 2 of Definition 12.5 that $b_1 = c_1$. Indeed, we have that if (β, ν, ξ) is the least such that $\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha),\beta,\nu,\xi}$ is successful then letting

1. $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma \leq \eta))$ be the output of $(\mathcal{P}, \Sigma^{\mathcal{S}|\beta})$-coherent fully backgrounded construction of $\mathcal{S}|\beta$ that uses extenders with critical points $> \nu$,

2. $\mathcal{N} \trianglelefteq_{hod} \mathcal{N}_\xi$ be the last model of $\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha),\beta,\nu,\xi}$, and

3. $\pi = \pi^{\mathcal{U}^{\mathcal{S}}_{\mathcal{M}(\alpha+1),\beta,\nu,\xi}}$,

$(\mathcal{T}_1)_{\geq \mathcal{M}}$ is according to π-pullback of $(\Sigma^{\mathcal{S}|\beta})_{\mathcal{N}}$. A similar argument handles the case when $\mathcal{T}_1$ has a fatal drop (see clause 3b of Definition 12.5).

It then follows that in $\mathcal{M}^{\mathcal{U}}_\delta|\alpha_1$, $\mathcal{J}(\mathcal{M}(\mathcal{T}_1)) \vDash$ "$\delta(\mathcal{T}_1)$ isn't Woodin". Because $\mathcal{M}^{\mathcal{U}}_d|\alpha_1 = \mathcal{M}^{\mathcal{W}}_e|\alpha_1$, it follows that in $\mathcal{M}^{\mathcal{W}}_e|\alpha_1$, $\mathcal{Y}(\mathcal{M}(\mathcal{T}_1)) \vDash$ "$\delta(\mathcal{T}_1)$ isn't Woodin". It then follows that $\mathcal{Q}(\mathcal{T}_1, b_1) = \mathcal{Q}(\mathcal{T}_1, c_1) \trianglelefteq (\mathcal{J}(\mathcal{M}(\mathcal{T}_1)))^{\mathcal{M}^{\mathcal{U}}_d|\alpha_1}$. Hence, $b_1 = c_1$, contradiction! □

We now have that $\mathcal{T}_1$ is both $\mathcal{M}^{\mathcal{U}}_d$-terminal and $\mathcal{M}^{\mathcal{W}}_e$-terminal. It follows that there is (β_1, γ_1) such that (β_1, γ_1, b_1) is a $\mathcal{M}^{\mathcal{U}}_d$-minimal shortness witness for $\mathcal{T}_1$.

Because $\mathcal{M}_d^{\mathcal{U}}|\alpha_1 = \mathcal{M}_e^{\mathcal{W}}|\alpha_1$, it follows that (β_1, γ_1, c_1) is a $\mathcal{M}_e^{\mathcal{W}}$-minimal shortness witness for $\mathcal{T}_1$. It then also follows that (β_1, γ_1, c_1) is a $\mathcal{M}_d^{\mathcal{U}}$-minimal shortness witness for $\mathcal{T}_1$.

Recall now that this entire construction was taking place in $\mathcal{S}$. Repeating the construction we get an infinite sequence $(\mathcal{U}_i, \mathcal{W}_i, \beta_i, \gamma_i, b_i, c_i, \mathcal{S}_i, \mathcal{T}_i : i < \omega)$ such that

1. $\mathcal{S}_0 = \mathcal{S}$, $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{U}_0 = \mathcal{U}$ and $\mathcal{W}_0 = \mathcal{W}$.
2. $(\mathcal{S}_1, \mathcal{T}_1, \beta_1, \gamma_1, b_1, c_1)$ are as above.
3. In $\mathcal{S}_{i+1}$, $\mathcal{T}_i$ is a terminal tree on $\mathcal{M}(\mathcal{T}_{i-1})$ and the two triples (β_i, γ_i, b_i) and (β_i, γ_i, c_i) are two distinct shortness witness for $\mathcal{T}_i$.
4. Let $\mathcal{Q}_{i+1} = \mathcal{Q}(b_i, \mathcal{T}_i)$ and $\mathcal{R}_{i+1} = \mathcal{Q}(c_i, \mathcal{T}_i)$. Working in $\mathcal{S}_i$, let Σ_i and Λ_i be the strategies of $\mathcal{Q}_{i+1}$ and $\mathcal{R}_{i+1}$ respectively that satisfy clause 4 of Definition 13.1. Let $\mathcal{U}_i$ and $\mathcal{W}_i$ be the trees on $\mathcal{Q}_{i+1}$ and $\mathcal{R}_{i+1}$ respectively that have been constructed via iterating away the least extender disagreement and are such that if $d_i = \Sigma_i(\mathcal{U}_i)$ and $e_i = \Sigma_i(\mathcal{W}_i)$ then there is α such that $\mathcal{M}_{d_i}^{\mathcal{U}_i}|\alpha = \mathcal{M}_{e_i}^{\mathcal{W}_i}|\alpha$, $\alpha \notin dom(\vec{E}^{\mathcal{M}_{d_i}^{\mathcal{U}_i}}) \cap dom(\vec{E}^{\mathcal{M}_{e_i}^{\mathcal{W}_i}})$ but $\mathcal{M}_{d_i}^{\mathcal{U}_i}||\alpha \neq \mathcal{M}_{e_i}^{\mathcal{W}_i}||\alpha$. Then $\alpha_{i+1} = \alpha$ and $\mathcal{S}_{i+1} = \mathcal{M}_{d_i}^{\mathcal{U}_i}|\alpha = \mathcal{M}_{e_i}^{\mathcal{W}_i}|\alpha$. $\mathcal{T}_{i+1}$ is then the normal tree whose branch is indexed at α_{i+1} (by the argument given above $\mathcal{T}_{i+1}$ is a terminal tree in $\mathcal{S}_{i+1}$). Then $(\beta_{i+1}, \gamma_{i+1}, b_{i+1})$ and $(\beta_{i+1}, \gamma_{i+1}, c_{i+1})$ are two $\mathcal{S}_{i+1}$-shortness witness for $\mathcal{T}_{i+1}$.
5. $\mathcal{S}_{i+1} \in \mathcal{S}_i$.

Clearly clause 5 leads to a contradiction!

□

15 Hod mice

We are now ready to introduce lsa small hod premice. Recall the definition of $\mu_{\gamma,\alpha}^{\mathcal{P}}$ (see Definition 4.4).

Definition 15.1 (Meek and lsa points) *Suppose $\mathcal{P} = \mathcal{J}_\alpha^{\vec{E},f}$ is an lsp.*

1. *Given $\alpha < \lambda^{\mathcal{P}}$, we say α is of meek type if either α is a successor or it is a limit but $\delta_\alpha^{\mathcal{P}}$ isn't a measurable cardinal in $\mathcal{P}$. Equivalently, α is of meek type if $\mathcal{P}(\alpha)$ is of meek type. We let meek$(\mathcal{P}) = \{\alpha < \lambda^{\mathcal{P}} : \alpha$ is of meek type$\}$.*

2. *Given $\alpha < \lambda^{\mathcal{P}}$ and $\xi \leq \varsigma_\alpha^{\mathcal{P}}$, we say (α, ξ) is of lsa type if $\mathcal{P} \vDash$ "$(\mathcal{P}|\nu_{\alpha,\xi})^\# \trianglelefteq \mathcal{P}$"[10] and $((\mathcal{P}|\nu_{\alpha,\xi})^\#)^{\mathcal{P}} \vDash$ "$\nu_{\alpha,\xi}$ is a Woodin cardinal". Equivalently, (α, ξ) is of lsa type if $\mathcal{P}(\alpha, \xi, 0)$ is of lsa type. We let $lsa(\mathcal{P}) = \{(\alpha, \xi) \in \lambda^{\mathcal{P}} \times \varsigma_\alpha^{\mathcal{P}} : (\alpha, \xi)$ is of lsa type$\}$. Also, we let $lsa(\mathcal{P}, \alpha, \xi) = \{\gamma < \zeta_{\alpha,\xi}^{\mathcal{P}} : \mathcal{P}|(\mu_{\alpha,\xi,\gamma}^{\mathcal{P}} + 1) \vDash$ "$\nu_{\alpha,\xi}^{\mathcal{P}}$ is a Woodin cardinal"$\}$.*

Notation: Given a hod-like lsp $\mathcal{P}$, we let $\mu^{\mathcal{P}} = \sup Y^{\mathcal{P}}$.

Definition 15.2 (Hod premouse) *Suppose $\mathcal{P} = \mathcal{J}_\alpha^{\vec{E},f}$ is a hod-like lsp. We say $\mathcal{P}$ is an lsa small hod premouse if the following holds:*

1. *Suppose $\mathcal{P}$ is meek. Then $\mathcal{P} = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}},\mu^{\mathcal{P}}+1}$.*
2. *(Lsa smallness) For every α such that $\alpha + 1 < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha + 1)$ isn't of lsa type.*
3. *Suppose $\alpha < \lambda$ is such that $\alpha \in mlw(\mathcal{P})$ and $\xi \leq \varsigma_\alpha^{\mathcal{P}}$. Then the following holds.*
   - *(a) For all $\gamma \in lsa(\mathcal{P}, \alpha, \xi)$, $\mathcal{P} \vDash$ "$\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is a partial faithful $(< \mu^{\mathcal{P}}_{\alpha,\xi,\gamma+1}, < \mu^{\mathcal{P}}_{\alpha,\xi,\gamma+1})$-short tree strategy for $\mathcal{P}(\alpha, \xi, \gamma)$ with hull and branch condensation, and $\mathcal{P}(\alpha, \xi, \gamma + 1)$ is a $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$-sts premouse over $\mathcal{P}(\alpha, \xi, \gamma)$"*
   - *(b) For all $\gamma < \zeta^{\mathcal{P}}_{\alpha,\xi}$ such that $\gamma \notin lsa(\mathcal{P}, \alpha, \xi)$, $\mathcal{P} \vDash$ "$\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is a partial $(< \mu^{\mathcal{P}}_{\alpha,\xi,\gamma+1}, < \mu^{\mathcal{P}}_{\alpha,\xi,\gamma+1})$-strategy for $\mathcal{P}(\alpha, \xi, \gamma)$ with hull and branch condensation, and $\mathcal{P}(\alpha, \xi, \gamma + 1)$ is a $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$-sts premouse over $\mathcal{P}(\alpha, \xi, \gamma)$"".*
   - *(c) Suppose $\nu^{\mathcal{P}}_{\alpha,\xi} < \delta^{\mathcal{P}}$. Then for $\gamma = \zeta^{\mathcal{P}}_{\alpha,\xi}$, $\mathcal{P} \vDash$ "$\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is a $(< \delta^{\mathcal{P}}, < \delta^{\mathcal{P}})$-strategy for $\mathcal{P}(\alpha, \xi, \gamma)$ with branch condensation and hull condensation", and if $\mathcal{P}$ is meek then $\mathcal{P} \vDash$ "$\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is a $(< Ord, < Ord)$-strategy for $\mathcal{P}(\alpha, \xi, \gamma)$ with branch condensation and hull condensation".*
4. *Suppose $\alpha < \lambda$ and $\xi \leq \varsigma_\alpha^{\mathcal{P}}$ are such that $\nu^{\mathcal{P}}_{\alpha,\xi} < \delta^{\mathcal{P}}$. Then the following holds.*
   - *(a) For all $\gamma < \zeta^{\mathcal{P}}_{\alpha,\xi}$, $\mathcal{P} \vDash$ "$\mathcal{P}(\alpha, \xi, \gamma + 1)$ is a $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$-sp over $\mathcal{P}(\alpha, \xi, \gamma)$".*
   - *(b) If $\gamma = \zeta^{\mathcal{P}}_{\alpha,\xi}$ and $\xi < \varsigma^{\mathcal{P}}_\alpha$ then $\mathcal{P} \vDash$ "$\mathcal{P}|\nu_{\alpha,\xi+1}$ is a $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$-sp over $\mathcal{P}|\nu^{\mathcal{P}}_{\alpha,\xi}$".*
   - *(c) If $\gamma = \zeta^{\mathcal{P}}_{\alpha,\xi}$ and $\xi = \varsigma^{\mathcal{P}}_\alpha$ then $\mathcal{P} \vDash$ "$\mathcal{P}(\alpha + 1)$ is a $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$-sp over $\mathcal{P}(\alpha)$".*
5. *Suppose η is a cutpoint of $\mathcal{P}$. Then the following holds.*
   - *(a) If $\mathcal{P}$ is meek then $\mathcal{P} \vDash$ "$\mathcal{O}^{\mathcal{P}}_{\eta,\eta+1}$is $<$Ord-iterable".*

[10]This condition should be read as saying that $\mathcal{P}$ has an extender whose critical point $> \nu_{\alpha,\xi}$.

(b) *If $\mathcal{P}$ is non-meek then if $\eta < \delta^{\mathcal{P}}$ then $\mathcal{P} \vDash$ "$\mathcal{O}^{\mathcal{P}}_{\eta,\eta+1}$ is $(<\delta^{\mathcal{P}}, <\delta^{\mathcal{P}})$-iterable".*

Definition 15.3 (Hod pairs) *We say $(\mathcal{P}, \Sigma)$ is a hod pair if $\mathcal{P}$ is a hod premouse and Σ is an $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ with hull condensation and such that whenever $\mathcal{Q}$ is a Σ-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{Q}} \times (\varsigma^{\mathcal{Q}}_{\alpha} + 1) \times (\zeta^{\mathcal{Q}}_{\alpha,\xi} + 1)$, either*

1. *$\gamma \notin lsa(\mathcal{Q}, \alpha, \xi)$ and $\Sigma^{\mathcal{Q}}_{\alpha,\xi,\gamma} \subseteq \Sigma_{\mathcal{Q}(\alpha,\xi,\gamma),\vec{\mathcal{T}}} \restriction \mathcal{Q}$ or*
2. *$\gamma \in lsa(\mathcal{Q}, \alpha, \xi)$ and $\Sigma^{\mathcal{Q}}_{\alpha,\xi,\gamma} \subseteq \Sigma^{stc}_{\mathcal{Q}(\alpha,\xi,\gamma),\vec{\mathcal{T}}} \restriction \mathcal{Q}$.*

Definition 15.4 (Sts hod pairs) *We say $(\mathcal{P}, \Sigma)$ is an sts hod pair if $\mathcal{P}$ is an lsa type hod premouse and Σ is a short tree $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ with hull condensation such that whenever $\mathcal{Q}$ is a Σ-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{Q}} \times (\varsigma^{\mathcal{Q}}_{\alpha} + 1) \times (\zeta^{\mathcal{Q}}_{\alpha,\xi} + 1)$, $\Sigma^{\mathcal{Q}}_{\alpha,\xi,\gamma} \subseteq \Sigma_{\mathcal{Q}(\alpha,\xi,\gamma),\vec{\mathcal{T}}} \restriction \mathcal{Q}$.*

Definition 15.5 ($\Gamma(\mathcal{P}, \Sigma)$ and $B(\mathcal{P}, \Sigma)$) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. We then let*

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \exists \mathcal{R}((\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \trianglelefteq_{hod} \mathcal{R}^b)\}$$

and

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists (\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}))\}.$$

16 Hod pair constructions

In this section we introduce the hod pair constructions which we will later use to produce hod pairs. We will perform such constructions inside sufficiently iterable structures we call *background models*. We start by introducing them.

Definition 16.1 (Background triple) *We say (M, δ, Σ) is a weak background triple if $M \vDash ZFC +$ "δ is a Woodin cardinal" and $\Sigma \in M$ is a $(\delta, \delta+1)$-iteration strategy for V^M_δ with hull condensation that acts on stacks which are in $\mathcal{J}_\omega(V^M_\delta)$. We say (M, δ, Σ) is a background triple if Σ is an (ω_1, ω_1)-strategy for M and $(M, \delta, \Sigma_{V_\delta} \restriction \mathcal{J}_\omega(V^M_\delta))$ is a weak background triple.*

Recall that if $(\mathcal{M}_\alpha : \alpha < \xi)$ is a sequence of $\mathcal{J}$-structures and ξ is a limit ordinal then $\mathcal{M} = lim_{\alpha \to \xi} \mathcal{M}_\alpha$ is a $\mathcal{J}$-structure such that for each β such that $\mathcal{J}^{\mathcal{M}}_\beta$ is defined, there is $\gamma < \xi$ such that for all $\alpha \in (\gamma, \xi)$, $\mathcal{J}^{\mathcal{M}_\alpha}_\beta = \mathcal{J}^{\mathcal{M}}_\beta$.

Definition 16.2 (Hod pair constructions) *Suppose (M, δ, Σ) is a weak background triple. Then the hod pair construction of M below δ is a sequence*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

that satisfies the following properties (in M).

1. *For all $(\alpha, \xi, \gamma) \in (\lambda+1)\times(\varsigma_\alpha+1)\times(\zeta_{\alpha,\xi}+1)$, $(\mathcal{P}_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma})$ and $(\mathcal{P}^*_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma})$ are hod pairs with the property that*

$$\lambda^{\mathcal{P}_{\alpha,\xi,\gamma}} = \lambda^{\mathcal{P}^*_{\alpha,\xi,\gamma}} = \begin{cases} \alpha+1 & : \varsigma^{\mathcal{P}}_\alpha > 0 \\ \alpha & : \varsigma^{\mathcal{P}}_\alpha = 0, \end{cases}$$

   and

   (a) $\mathcal{P}_{\alpha,\xi,\gamma}(\alpha, 0, 0) = \mathcal{P}^*_{\alpha,\xi,\gamma}(\alpha, 0, 0)$,

   (b) *if $(\alpha, \xi, \gamma) <_{lex} (\lambda, \varsigma_\alpha, \zeta_{\alpha,\varsigma_\alpha})$ then $\rho(\mathcal{P}^*_{\alpha,\xi,\gamma}) > \delta_\alpha^{\mathcal{P}^*_{\alpha,\xi,\gamma}}$, and*

   (c) *$\mathcal{P}_{\alpha,\xi,\gamma} = \mathcal{C}(\mathcal{P}^*_{\alpha,\xi,\gamma})$ and if $\pi : \mathcal{P}_{\alpha,\xi,\gamma} \to \mathcal{P}^*_{\alpha,\xi,\gamma}$ is the uncollapse map then $\Sigma_{\alpha,\xi,\gamma} = \pi$-pullback of $\Sigma^*_{\alpha,\xi,\gamma}$.*

2. ***The construction of** $(\mathcal{N}_{\alpha,0,0}, \mathcal{P}_{\alpha,0,0}, \mathcal{P}^*_{\alpha,0,0})$. We let $\mathcal{N}_{0,0,0} = (\mathcal{J}^{\vec{E}})^{V^M_\delta}$.*

   (a) *Assume $\alpha = 0$ or $\alpha = \theta + 1$ and that $\mathcal{N}_{\alpha,0,0}$ has been defined. As part of the inductive condition we assume that $\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}$ is not of lsa type. Let*

$$\nu = \begin{cases} \delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}} & : \alpha \neq 0 \\ 0 & : \text{otherwise} \end{cases}.$$

   Then if

      i. *θ is limit and $\mathcal{N}_{\alpha,0,0}$ doesn't project across $\delta_\theta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$,*

      ii. *$\mathcal{N}_{\alpha,0,0}$ has at least two Woodin cardinals $> \nu$,*

      iii. *$\alpha > 0$, θ is a successor and $\mathcal{N}_{\alpha,0,0} \vDash$ "ν is a Woodin cardinal"*

   then $\mathcal{P}^*_{\alpha,0,0} = \mathcal{P}_{\alpha,0,0} = \mathcal{N}_{\alpha,0,0}|(\mu^{+\omega})^{\mathcal{N}_{\alpha,0,0}}$ where μ is the least Woodin cardinal of $\mathcal{N}_{\alpha,0,0}$ bigger than ν.

   (b) *Assume α is a limit ordinal. Let $\mathcal{Q}_\alpha = \cup_{\theta<\alpha}\mathcal{P}_{\theta+1,0,0}$ and let Λ be the strategy of $\mathcal{Q}_\alpha$ induced by Σ. Let $\mathcal{M} = (\mathcal{J}^{\vec{E},\Lambda})^{V^M_\delta}$. Then the following holds.*

i. *If no initial segment of* $\mathcal{M}_\alpha$ *projects across* $o(\mathcal{Q}_\alpha)$ *then letting* $\kappa = o(\mathcal{Q}_\alpha)$, $\mathcal{P}^*_{\alpha,0,0} = \mathcal{P}_{\alpha,0,0} = (\kappa^{+\omega})^{\mathcal{M}_\alpha}$ *and* $\Sigma_{\alpha,0,0}$ *be the strategy of* $\mathcal{P}_{\alpha,0,0}$ *induced by* Σ.

ii. *Provided the above clause holds, let* $\mathcal{N}_\alpha = (\mathcal{J}^{\vec{E},\Sigma_{\alpha,0,0}})^{V^M_\delta}$. *If*

$$(\kappa^+)^{\mathcal{N}_\alpha} = (\kappa^+)^{\mathcal{P}_{\alpha,0,0}} \text{ and } \mathcal{N}_\alpha|(\kappa^+)^{\mathcal{N}_\alpha} = \mathcal{P}_{\alpha,0,0}|(\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$$

then $\mathcal{N}_\alpha = \mathcal{N}_{\alpha,0,0}$.

(c) ***The terminating condition.*** *Below we make a list of the conditions described above that will force the construction to stop. Stop the construction if any of the following happens.*

i. α *is a successor ordinal and there is a level of* $\mathcal{N}_{\alpha,0,0}$ *projecting across* $\delta_\theta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *where* $\theta = \alpha - 1$.

ii. $\alpha = 0$ *or* α *is a successor ordinal and* $\mathcal{N}_{\alpha,0,0}$ *doesn't have at least two Woodin cardinals* $> \nu$ *where* ν *is as in clause 2.a.*

iii. α *is a successor ordinal and* $\mathcal{N}_{\alpha,0,0} \vDash$ *"ν is not a Woodin cardinal" where* ν *is as in clause 2.a.*

iv. α *is a limit ordinal and some initial segment of* $\mathcal{M}_\alpha$ *projects across* $o(\mathcal{Q}_\alpha)$ *where* $\mathcal{M}_\alpha$ *and* $\mathcal{Q}_\alpha$ *are as in clause 2.b.i.*

v. α *is a limit ordinal and the above clause fails, but either* $(\kappa^+)^{\mathcal{N}_\alpha} \neq (\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$ *or* $\mathcal{N}_\alpha|(\kappa^+)^{\mathcal{N}_\alpha} \neq \mathcal{P}_{\alpha,0,0}|(\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$ *where* κ *and* $\mathcal{N}_\alpha$ *are as in clause 2.b.i and 2.b.ii.*

3. ***The construction of*** $(\mathcal{N}_{\alpha,\xi,0}, \mathcal{P}_{\alpha,\xi,0}, \mathcal{P}^*_{\alpha,\xi,0})$ ***for fixed*** α ***and*** $\xi > 0$***.*** *Suppose* $\alpha \in Ord$ *is fixed. We define* $(\mathcal{N}_{\alpha,\xi,0}, \mathcal{P}_{\alpha,\xi,0}, \mathcal{P}^*_{\alpha,\xi,0} : \xi > 0)$ *by induction on* ξ. *Suppose* $(\alpha,\xi) \in Ord^2$ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*) <_{lex} (\alpha,\xi) \wedge \gamma^* \leq \zeta_{\alpha^*,\xi^*})$$

has been defined.

(a) *Suppose* ξ *is limit. Let* $(\mathcal{Q}_{\alpha,\xi,i}, \mathcal{Q}^*_{\alpha,\xi,i} : i \leq \nu_{\alpha,\xi})$ *be a sequence defined as follows.*

i. $\mathcal{Q}^*_{\alpha,\xi,0} = lim_{\xi^* \to \xi} \mathcal{P}_{\alpha,\xi^*,\zeta_{\alpha,\xi^*}}$,

ii. *for all* $i \leq \nu_{\alpha,\xi}$, $\mathcal{Q}_{\alpha,\xi,i} = \mathcal{C}(\mathcal{Q}^*_{\alpha,\xi,i})$,

iii. *for all* $i < \nu_{\alpha,\xi}$, $\mathcal{Q}^*_{\alpha,\xi,i+1}$ *is the least initial segment* $\mathcal{M}$ *of* $\mathcal{J}(\mathcal{Q}_{\alpha,\xi,i})$, *if it exists, such that* $\rho(\mathcal{M}) < o(\mathcal{Q}_{\alpha,\xi,i})$,

iv. *for all limit* $i \leq \nu_{\alpha,\xi}$, $\mathcal{Q}_{\alpha,\xi,i} = lim_{j\to i}\mathcal{Q}_{\alpha,\xi,j}$,

v. $\nu_{\alpha,\xi}$ *is the least ordinal* β *such that either*

A. $\mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *has no level projecting across* $\mathcal{Q}_{\alpha,\xi,\beta}$ *or*

B. *there is* $\mathcal{M} \lhd \mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

Suppose that clause B doesn't hold. Let then $\mathcal{P}^*_{\alpha,\xi,0} = (\mathcal{Q}_{\alpha,\xi,\nu_{\alpha,\xi}})^\#$, *i.e.* $\mathcal{P}^*_{\alpha,\xi,0}$ *is the least active level of* $\mathcal{J}^{\vec{E}}[\mathcal{Q}_{\alpha,\xi,\nu_{\alpha,\xi}}]$. *We also let* $\Sigma^*_{\alpha,\xi,0}$ *be the strategy of* $\mathcal{P}^*_{\alpha,\xi,0}$ *induced by* Σ. *If* $\rho(\mathcal{P}^*_{\alpha,\xi,0}) > \delta^{\mathcal{P}_{\alpha,0,0}}$ *then let*

$$\mathcal{N}_{\alpha,\xi,0} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,0}})^{V^M_\delta} & : \mathcal{P}_{\alpha,\xi,0} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,0}})^{V^M_\delta} & : \mathcal{P}_{\alpha,\xi,0} \text{ is of lsa type.} \end{cases}$$

(b) *Suppose* $\xi = \xi^* + 1$.

i. *Suppose there is an extender* F *that coheres* Σ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*) <_{lex} (\alpha,\xi) \wedge \gamma^* \leq \zeta_{\alpha^*,\xi^*})$$

such that letting $E = F \cap \mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}$, $(\mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}, \tilde{E}, \in)$ *is a hod premouse where* $\tilde{E}$ *is the amenable code of* E. *Then* $\mathcal{P}^*_{\alpha,\xi,0} = (\mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}, \tilde{E}, \in)$.

ii. *Suppose there is no such* F. *Then stop the construction of* $\mathcal{P}_{\alpha,\xi,0}$ *and let* $\varsigma_\alpha = \xi^*$.

(c) ***The terminating condition.*** *Below we make a list of the conditions described above that will force the construction to stop. Stop the construction if any of the following happens.*

i. *Clause 3.a.v.B holds, i.e., there is* $\mathcal{M} \lhd \mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

ii. $\mathcal{P}^*_{\alpha,\xi,0}$ *is defined but* $\rho(\mathcal{P}^*_{\alpha,\xi,0}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

4. ***The description of*** $(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma})$ ***for fixed*** (α,ξ)***.*** *Fix* $(\alpha,\xi) \in Ord^2$. *We define* $(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma})$ *by induction. Suppose* $(\alpha,\xi,\gamma) \in Ord^3$ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*,\gamma^*) <_{lex} (\alpha,\xi,\gamma))$$

has been defined.

(a) *Suppose* γ *is a limit ordinal or is* 0. *Then we let* $\mathcal{P}^*_{\alpha,\xi,\gamma} = lim_{\gamma^*\to\gamma}\mathcal{P}_{\alpha,\xi,\gamma^*}$. *Also, let* $\Sigma^*_{\alpha,\xi,\gamma}$ *be the strategy of* $\mathcal{P}^*_{\alpha,\xi,\gamma}$ *induced by* Σ. *Then*

$$\mathcal{N}_{\alpha,\xi,\gamma} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,\gamma}})^{V_\delta^M} & : \mathcal{P}_{\alpha,\xi,\gamma} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,\gamma}})^{V_\delta^M} & : \mathcal{P}_{\alpha,\xi,\gamma} \text{ is of lsa type.} \end{cases}$$

(b) Suppose $\gamma = \gamma^* + 1$. Then

$$\mathcal{N}_{\alpha,\xi,\gamma} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,\gamma^*}})^{V_\delta^M} & : \mathcal{P}_{\alpha,\xi,\gamma^*} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,\gamma^*}})^{V_\delta^M} & : \mathcal{P}_{\alpha,\xi,\gamma^*} \text{ is of lsa type.} \end{cases}$$

(c) ***The terminating condition.*** *Terminate the construction if*

i. there is a $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha,\xi,\gamma}$ such that $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}^b_{\alpha,\xi,0}}$ or

ii. $o(\mathcal{P}^*_{\alpha,\xi,\gamma}) = \delta$.

(d) If γ is a limit ordinal and $\mathcal{N}_{\alpha,\xi,\gamma}$ exists but there is no $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha,\xi,\gamma}$ such that $\rho(\mathcal{M}) = \delta^{\mathcal{P}_{\alpha,\xi,\gamma}}$. Set $\zeta_{\alpha,\xi} = \gamma$.

5. ***The main termination condition*** *If the construction reaches (α, ξ) such that $\mathcal{P}_{\alpha,\xi,\zeta_{\alpha,\xi}}$ is of lsa type then stop the construction.*

The following can be proved by routine modifications of the proof of Theorem 12.1 of [2].

Theorem 16.3 (The iterability of hod pair constructions) *Suppose (M, δ, Σ) is a background triple and*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

are the models of the hod pair construction of M below δ. Then for each (α, ξ, γ), $\Sigma_{\alpha,\xi,\gamma}$ is a wining strategy for II in $\mathcal{G}_k(\mathcal{P}_{\alpha,\xi,\gamma}, \omega_1, \omega_1, \omega_1)$.

17 Iterability of backgrounded constructions

Our first definition is a game that we will use to show that hod pair constructions inherit an $(\omega_1, \omega_1, \omega_1)$-iteration strategies.

Definition 17.1 ($\mathcal{G}(M, \kappa, \lambda, \nu)$) *Suppose M is a transitive model of some fragment of ZFC. Then $\mathcal{G}(M, \kappa, \lambda, \nu)$ is the following iteration game on M. It has κ main rounds. If M_α is the model at the beginning of the αth main round then the αth main round is a run of $\mathcal{G}(M_\alpha, \lambda, \nu)$. I is the player starting the main rounds. Letting M_α be the model at the beginning of the αth round and $\vec{\mathcal{T}}_\alpha$ be the stack of iteration trees produced during αth round, if the last normal component of $\vec{\mathcal{T}}_\alpha$ has a successor length* *with last model Q then I may start a new round. She does so by choosing $\xi < o(Q)$ and letting $M_{\alpha+1} = Ult(M_\alpha, E_\alpha)$ where $E_\alpha = E_{\pi^{\vec{\mathcal{T}}}_\alpha} \restriction \xi$. As is usual, II wins the game if all the models produced in the iteration game are well-founded.*

We say M is (κ, λ, ν)-iterable if II has a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$. We also say Σ is a (κ, λ, ν)-strategy for M if Σ is a winning strategy for II in $\mathcal{G}(M, \kappa, \lambda, \nu)$. As is usual, when M has a distinguished extender sequence then player I can only play extenders from the images of the distinguished extender sequence of M.

As we show below a winning strategy in $\mathcal{G}(M, \kappa, \lambda)$ induces a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$. First we introduce the following terminology. Given an iteration strategy Σ let $dom^+(\Sigma) = \{\vec{\mathcal{T}} : \vec{\mathcal{T}}$ is according to $\Sigma\}$.

Definition 17.2 (Certified strategy) *Suppose M and N are two transitive models of $ZFC - Powerset$. Suppose Σ and Λ are iteration strategies for M and N respectively (in one of the iteration games that we have defined, not necessarily the same). We say Σ is certified by Λ if there is a set X and a function $F : dom^+(\Sigma) \to dom^+(\Lambda) \times X$ such that the following holds:*

1. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$, $\vec{\mathcal{U}}$ has a last model iff $(F(\vec{\mathcal{U}}))_0$ has a last model.*
2. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$, if $\vec{\mathcal{U}}$ has a last model then letting Q and R be the last models of $\vec{\mathcal{U}}$ and $(F(\vec{\mathcal{U}}))_0$, $(F(\vec{\mathcal{U}}))_1 = \sigma$ such that $\sigma : Q \to_{\Sigma_1} R$.*
3. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$ if $\alpha < lh(\vec{\mathcal{U}})$ then letting $\vec{\mathcal{T}} = (F(\vec{\mathcal{U}}))_0$ and $\vec{\mathcal{T}}^* = (F(\vec{\mathcal{U}} \restriction \alpha))_0$ then $\vec{\mathcal{T}}^*$ is an initial segment of $\vec{\mathcal{T}}$.*
4. *If $\vec{\mathcal{T}}$ is a stack on M according to Σ with last model Q and $\mathcal{U}$ is a normal tree on Q then letting R be the last model $(F(\vec{\mathcal{T}}))_0$ and $\mathcal{W}$ be such that $(F(\vec{\mathcal{T}}))_0{}^\frown\mathcal{W} = F(\vec{\mathcal{T}}{}^\frown\mathcal{U})$ then $\mathcal{W}$ is a normal tree such that $lh(\mathcal{U}) = lh(\mathcal{W})$, and for every $\alpha_0, \alpha_1 < lh(\mathcal{U})$, letting $\beta_0, \beta_1 < lh(\mathcal{W})$ be such that for $i = 0, 1$, $F(\vec{\mathcal{T}}{}^\frown\mathcal{U} \restriction \alpha_i + 1) = (F(\vec{\mathcal{T}}))_0{}^\frown\mathcal{W} \restriction \beta_i + 1$,*
   - *(a) $\alpha_0 <_U \alpha_1 \leftrightarrow \beta_0 <_W \beta_1$*
   - *(b) letting for $i = 0, 1$, $\sigma_i = (F(\vec{\mathcal{T}}{}^\frown\mathcal{U} \restriction \alpha_i))_i$, if $\alpha_0 <_U \alpha_1$ then $\pi^{\mathcal{W}}_{\beta_0,\beta_1} \circ \sigma_0 = \sigma_1 \circ \pi^{\mathcal{U}}_{\alpha_0,\alpha_1}$.*

Clearly pullback constructions produce certified strategies.

Proposition 17.3 *Suppose M is a transitive model of some fragment of ZFC and $\kappa \leq \lambda$. Then if II has a winning strategy Λ in $\mathcal{G}(M, \lambda, \nu)$ then II has a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$ certified by Λ.*

Proof. Suppose we have defined F as in Definition 17.2 on $\vec{\mathcal{T}} \in dom(\Sigma)$ which have $< \alpha$-many main rounds. We want to define F on $\vec{\mathcal{T}}$ with exactly α-many main rounds. We assume that α is a successor and leave the rest to the reader. Let $\alpha = \beta + 1$. Thus, we need to extend Σ to act on $\beta + 1$st round of $\mathcal{G}(M, \kappa, \lambda, \nu)$. Let then $\vec{\mathcal{T}} \in dom(\Sigma)$ be such that $\mathrm{lh}(\vec{\mathcal{T}}) = \beta + 1$ and $\vec{\mathcal{T}}$ has a last model Q. Let R be the last model $\vec{\mathcal{U}} = (F(\vec{\mathcal{T}}))_0$ and let $\sigma = (F(\vec{\mathcal{T}}))_1$.

Suppose that I wants to start a new main round by choosing $\xi < o(Q)$ and setting $M_{\beta+1} = Ult(M_\beta, E_{\pi^{\vec{\mathcal{T}}}} \restriction \xi)$. Let $k : M_{\beta+1} \to Q$ be the factor embedding, i.e., $\pi^{\vec{\mathcal{T}}} = k \circ \pi_{E_{\pi^{\vec{\mathcal{T}}}} \restriction \xi}$. Let $\pi = \sigma \circ k \circ \pi_{E_{\pi^{\vec{\mathcal{T}}}} \restriction \xi}$. We then let $F(\vec{\mathcal{T}}^\frown\{M_{\beta+1}\}) = (\vec{\mathcal{U}}, \pi)$ which clearly has the desired properties. Next we require that II plays the $\beta + 1$st round on $M_{\beta+1}$ according to $\Lambda^\pi_{R,\vec{\mathcal{U}}}$. □

18 Normal comparison theory of lsa hod mice

As in Theorem 2.2.2 of [3], under AD^+ and in several other contexts, we can prove a comparison theorem where comparison is achieved via normal trees. In this section we state a comparison theorem for hod pairs that can be applied inside models of AD^+ and also, inside models satisfying sufficiently rich extensions of ZFC, like hod mice themselves. Such comparison arguments, among other things, are useful in core model induction arguments and in the analysis of HOD of models of AD^+.

We start with some general definitions and facts. One warning is that our exposition differs from the one in [3] mainly because we would like to set up our arguments here in a more general setting than the ones stated in [3].

Definition 18.1 (Comparison) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then we say comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there is $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ such that one of the following holds:*

1. *$\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Lambda_{\mathcal{R},\vec{\mathcal{U}}} = \Sigma_{\mathcal{R},\vec{\mathcal{T}}}$.*
2. *$\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Sigma_{\mathcal{S},\vec{\mathcal{T}}} = \Lambda_{\mathcal{S},\vec{\mathcal{U}}}$.*

We say normal comparison for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ holds if we can take $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ to be normal.

As in [3], we can comparison for pairs whose corresponding strategies are fullness preserving.

18.1 Fullness preservation

We assume AD^+ throughout this section. Suppose that $\Gamma \subseteq \wp(\mathbb{R})$ is a pointclass, $X \in HC$ is a self-well-ordered set (swo)[11], $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P} \in HC$ and $Code(\Sigma) \in \Gamma$[12]. Recall also that $Lp^{\Gamma,\Sigma}(X)$ is the stack of all sound Σ-mice $\mathcal{M}$ over X[13] such that $\rho(\mathcal{M}) = X$ and $\mathcal{M}$ has a strategy in Γ. Let then

$$HP^\Gamma = \{(\mathcal{P}, \Sigma) : (\mathcal{P}, \Sigma) \text{ is a hod pair such that } Code(\Sigma) \in \Gamma\}$$
$$Mice^\Gamma = \{(a, (\mathcal{P}, \Sigma), \mathcal{M}) : a \in HC \wedge a \text{ is an swo} \wedge (\mathcal{P}, \Sigma) \in HP^\Gamma \wedge \mathcal{M} \trianglelefteq \mathcal{P} \in a \wedge \mathcal{M} \trianglelefteq Lp^{\Gamma,\Sigma}(a) \wedge \rho(\mathcal{M}) = a\}.$$

Definition 18.2 (Γ-Fullness preservation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $\mathcal{P} \in HC$ and Γ is a pointclass. We say Σ is Γ-fullness preserving if the following holds for all $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma)$:*

1. *For all $\alpha \leq \lambda^{\mathcal{Q}}$*

   (a) *letting $\mathcal{R} = \mathcal{Q}(\alpha, 0, 0)$,*
   $$\mathcal{R} = Lp_\omega^{\Gamma, \oplus_{\beta<\alpha-1}\Sigma_{\mathcal{R}(\beta)}, \vec{\mathcal{T}}}(\mathcal{R}|\delta^{\mathcal{R}}),$$

   (b) *if $\mathcal{Q}$ is of lsa type and $\alpha = \lambda^{\mathcal{P}}$ then*
   $$\mathcal{Q} = Lp_\omega^{\Gamma, \Sigma^{stc}_{(\mathcal{Q}|\delta^{\mathcal{Q}})^\#}, \vec{\mathcal{T}}}(\mathcal{Q}|\delta^{\mathcal{Q}}),$$

2. *If η is a cardinal cutpoint of $\mathcal{Q}$ such that for some $\alpha < \lambda^{\mathcal{Q}}$, $\eta \in (\delta^{\mathcal{Q}}_\alpha, \delta^{\mathcal{Q}}_{\alpha+1})$ then*
   $$\mathcal{Q}|(\eta^+)^{\mathcal{Q}} = Lp^{\Gamma, \Sigma_{\mathcal{Q}(\alpha)}}(\mathcal{Q}|\eta).$$

Theorem 18.3 *Assume AD^+. Suppose for some α such that $\theta_\alpha < \Theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$ and (M, δ, Σ) is a background triple that Suslin, co-Suslin captures Γ. Let*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \varsigma_{\alpha,\xi})$$

[11] Recall that X is a self-well-ordered set if $\mathcal{J}_\omega(X) \vDash$ "X is well-orderable".

[12] Recall that $Code(\Sigma)$ is the set of reals coding Σ.

[13] In case X isn't transitive or $\mathcal{P} \notin X$, "over X" means "over $Tc(\{X, \mathcal{P}\})$".

be the output of the hod pair constructions of M below δ. Then for all (α,ξ,γ), $\Sigma_{\alpha,\xi,\gamma}$ is Γ-fullness preserving.

Proof. If for some (α,ξ,γ), $\Sigma_{\alpha,\xi,\gamma}$ isn't fullness preserving then it follows by absoluteness that we can find a counterexample in $M[g]$ where g is $<\delta$-generic. Fix $(\alpha_0,\xi_0,\gamma_0)$ such that $\Sigma_{\alpha_0,\xi_0,\gamma_0}$ isn't fullness preserving. Let $\mathcal{P}=\mathcal{P}_{\alpha_0,\xi_0,\gamma_0}$ and $\Lambda=\Sigma_{\alpha_0,\xi_0,\gamma_0}$. Fix $<\delta$-generic g such that there $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Lambda)\cap M[g]$ witnessing that Λ isn't Γ-fulness preserving. All three clause of fullness preservation are very similar and follow from the universality of background constructions. Below we derive a contradiction from the failure of clause 1.a of Definition 18.2 and leave the rest to the reader.

Fix $\alpha\leq\lambda^{\mathcal{Q}}$ witnessing the failure of clause 1.a of Definition 18.2. Let $\mathcal{R}=\mathcal{Q}(\alpha,0,0)$ and $\kappa=\delta^{\mathcal{R}}$. We need to see that

$$\mathcal{R}=Lp_{\omega}^{\Gamma,\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

We only show that

$$\mathcal{R}|(\kappa^+)^{\mathcal{R}}=Lp^{\Gamma,\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

and leave the rest to the reader.

Suppose first that $\mathcal{M}\trianglelefteq\mathcal{R}|(\kappa^+)^{\mathcal{R}}$ is a $\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}$-mouse over $\mathcal{R}|\kappa$ such that $\rho(\mathcal{M})\leq\kappa$. Because $\mathcal{P}$ is constructed via backgrounded construction, it follows that $\mathcal{M}$ is ω_1-iterable as a a $\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}$-mouse and therefore,

$$\mathcal{M}\trianglelefteq Lp^{\Gamma,\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

Fix now $\mathcal{M}\trianglelefteq Lp^{\Gamma,\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}}(\mathcal{R}|\kappa)$ and let Φ be its ω_1-strategy. We let $\pi=\pi^{\vec{\mathcal{T}}}$. Let $\mathcal{N}=\mathcal{N}_{\alpha_0,0,0}$ and notice that if $E=E_\pi\restriction\pi(\delta^{\mathcal{P}})$ then $Ult(\mathcal{N},E)$ is δ-iterable. Let then $\pi^+=\pi^{\mathcal{N}}_{E_\pi\restriction\pi(\delta^{\mathcal{P}})}$ and let

$$\mathcal{S}=(\mathcal{J}^{\vec{E},\oplus_{\beta<\alpha-1}\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}})^{\pi^+(\mathcal{N})}.$$

It then follows that $\mathcal{S}$ too is δ-iterable and so we can compare $\mathcal{S}$ with $\mathcal{M}$. By universality of backgrounded constructions, $\mathcal{M}$ has to lose the comparison implying that $\mathcal{M}\trianglelefteq\mathcal{S}$. Therefore, $\mathcal{M}\in\pi^+(\mathcal{N})$. Since $\mathcal{M}$ is ω_1-iterable, it follows that $\mathcal{M}\trianglelefteq\mathcal{R}$. $\square$

18.2 Tracking disagreements

Here we introduce terminology that we will use to track down the disagreements between strategies. Given a stack $\vec{\mathcal{T}}$ on a hod premouse $\mathcal{P}$, we let $\delta(\vec{\mathcal{T}})$ be the sup of the generators of $\vec{\mathcal{T}}$ (see Definition 1.15 of [3]).

Definition 18.4 (Low level disagreement between strategies) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs. Suppose there is $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma) \cap B(\mathcal{P}, \Lambda)$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \neq \Lambda_{\mathcal{Q},\vec{\mathcal{T}}}$. Then we say that there is a low level disagreement between Σ and Λ. We say $(\vec{\mathcal{T}}, \mathcal{Q})$ is a minimal low level disagreement if $\mathcal{Q}$ is of successor type, for every $\alpha < \lambda^{\mathcal{Q}}$, $\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}} \neq \Lambda_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}$ and $\delta(\vec{\mathcal{T}}) \subseteq \mathcal{Q}(\lambda^{\mathcal{Q}} - 1)$.*

Lemma 18.5 (Disagreement implies low level disagreement) *Suppose Γ is a strategy invariant pointclass, and $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs such that both Σ and Λ are Γ-fullness preserving. Suppose that one of the following conditions holds:*

1. *$\mathcal{P}$ is of limit type and not of lsa type, and $\Sigma \neq \Lambda$.*
2. *$\mathcal{P}$ is of lsa type and $\Sigma^{stc} \neq \Lambda^{stc}$.*

Then there is a low level disagreement between Σ and Λ.

Proof. We give the proof from clause 2 and leave the proof from clause 1, which is easier, to the reader. Assume there is no low level disagreement between Σ and Λ. Let $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, E_\alpha : \alpha \leq \eta)$ be any disagreement between Σ^{stc} and Λ^{stc}. We must have that η is a successor ordinal, E_η is undefined, and the last normal component of $\vec{\mathcal{T}}_\eta$ is of limit length. Let E_η be the undropping extender of $\vec{\mathcal{T}}_\eta$ and $\mathcal{M} = Ult(\mathcal{M}_\eta, E)$. Notice that if $\vec{\mathcal{T}}$ has main drops then the claim follows. We then assume that $\vec{\mathcal{T}}$ has no main drops.

Notice that there cannot be a club $C \subseteq ntn(\vec{\mathcal{T}})$ as otherwise $\Sigma(\vec{\mathcal{T}}) = \Lambda(\vec{\mathcal{T}}) = b_C$. Let then $\mathcal{S} = \mathcal{S}_{\vec{\mathcal{T}}}$. Thus, $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{S}$. Notice that, because $\vec{\mathcal{T}}$ has no main drops, we must have that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists.

Let now $\mathcal{T} = \vec{\mathcal{T}}_{\geq \mathcal{S}}$. It then follows that $\mathcal{T}$ must be above $o(\mathcal{S}^b)$ as otherwise it will generate a low level disagreement. Without loss of generality, we can further assume that $\Sigma_{\mathcal{S}^b, \vec{\mathcal{T}}_{\leq \mathcal{S}}} = \Lambda_{\mathcal{S}^b, \vec{\mathcal{T}}_{\leq \mathcal{S}}}$ as otherwise, we get a low level disagreement. Notice that it follows from Γ-fullness preservation that $\vec{\mathcal{T}} \in b(\Sigma^{stc}) \cap b(\Lambda^{stc})$, i.e., $\Sigma(\vec{\mathcal{T}})$ and $\Lambda(\vec{\mathcal{T}})$ are branches rather than models. Let then $b = \Sigma(\vec{\mathcal{T}})$ and $c = \Lambda(\vec{\mathcal{T}})$. It follows that both $\mathcal{Q}(\mathcal{T}, b)$ and $\mathcal{Q}(\mathcal{T}, c)$ exist.

Because $b \neq c$, we have that $\mathcal{Q}(\mathcal{T}, b) \neq \mathcal{Q}(\mathcal{T}, c)$. It follows that

$$(\mathcal{M}(\mathcal{T}))^\# \lhd \mathcal{Q}(\mathcal{T}, b) \text{ and } (\mathcal{M}(\mathcal{T}))^\# \lhd \mathcal{Q}(\mathcal{T}, b).$$

Let $\mathcal{P}_1 = (\mathcal{M}(\mathcal{T}))^\#$. We then get that $\mathcal{Q}(\mathcal{T}, b)$ is a $\Sigma^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}}$-mouse over $\mathcal{P}_1$ and $\mathcal{Q}(\mathcal{T}, c)$ is a $\Lambda^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}}$-mouse over $\mathcal{P}_1$. It follows that $\Sigma^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}} \neq \Lambda^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}}$.

Repeating the above argument we get a sequence $(\mathcal{P}_i, \vec{\mathcal{T}}_i, b_i, c_i : i < \omega)$ with the following properties:

1. $\mathcal{P}_0 = \mathcal{P}$, $\vec{\mathcal{T}}_0 = \vec{\mathcal{T}}$, $b_0 = b$, $c_0 = c$ and $\mathcal{P}_1$ is as above.
2. $\mathcal{P}_i$ is a hod premouse of lsa type.
3. $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{P}_i$ witnessing that $\Sigma^{stc}_{\mathcal{P}_i,\oplus_{k<i}\vec{\mathcal{T}}_k} \neq \Lambda^{stc}_{\mathcal{P}_i,\oplus_{k<i}\vec{\mathcal{T}}_k}$.
4. $b_i = \Sigma(\oplus_{k\leq i}\vec{\mathcal{T}}_k)$ and $c_i = \Lambda(\oplus_{k\leq i}\vec{\mathcal{T}}_k)$.
5. Letting $\mathcal{S} = \mathcal{S}_{\vec{\mathcal{T}}_i}$ and $\mathcal{U} = (\vec{\mathcal{T}}_i)_{\geq\mathcal{S}}$, $\mathcal{P}_{i+1} = (\mathcal{M}(\mathcal{U}))^\#$.

It now follows that $\oplus_{i<\omega}\vec{\mathcal{T}}_i$ cannot be according to Σ as it either has infinitely many drops or an ill-founded direct limit. □

Just like in [3], we will prove that comparison holds by examining the minimal disagreements.

Definition 18.6 (Minimal disagreement) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs such that $\mathcal{P}$ is of limit type (see Definition 4.2). Suppose there is a low level disagreement between Σ and Λ. Then we say $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, E_\alpha : \alpha \leq \eta)$ constitutes a minimal disagreement between Σ and Λ if E_η is undefined and for some $\xi < \lambda^{\mathcal{M}_{\eta-1}}$,*

1. *$\Sigma_{\mathcal{M}_{\eta-1}(\xi),\vec{\mathcal{T}}\restriction\eta} = \Lambda_{\mathcal{M}_{\eta-1}(\xi),\vec{\mathcal{T}}\restriction\eta}$,*
2. *$\mathcal{M}_\eta$ is undefined, $\vec{\mathcal{T}}_\eta$ is a stack on $\mathcal{M}_{\eta-1}(\xi+1)$ such that $\vec{\mathcal{T}}_\eta \in dom(\Sigma_{\mathcal{M}_{\eta-1}(\xi+1),\vec{\mathcal{T}}\restriction\eta}) \cap dom(\Lambda_{\mathcal{M}_{\eta-1}(\xi+1),\vec{\mathcal{T}}\restriction\eta})$ and $\Sigma_{\mathcal{M}_{\eta-1}(\xi+1),\vec{\mathcal{T}}\restriction\eta}(\vec{\mathcal{T}}_\eta) \neq \Lambda_{\mathcal{M}_{\eta-1}(\xi+1),\vec{\mathcal{T}}\restriction\eta}(\vec{\mathcal{T}}_\eta)$, and*
3. *if ν is the sup of the generators of $E_{\eta-1}$ then $\nu \subseteq \mathcal{M}_{\eta-1}(\xi)$.*

Lemma 18.7 (The existence of minimal disagreements) *Suppose that Γ is a strategy invariant pointclass and $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs such that both Σ and Λ are Γ-fullness preserving and one of the following conditions hold:*

1. *$\mathcal{P}$ is of limit type but not of lsa type, and $\Sigma \neq \Lambda$.*

2. $\mathcal{P}$ is of lsa type and $\Sigma^{stc} \neq \Lambda^{stc}$.

Then there is a minimal disagreement between Σ and Λ.

Proof. Suppose not. We again do the proof by assuming clause 2. It follows from Lemma 18.5 that there is a low level disagreement between Σ and Λ. Let $(\vec{\mathcal{T}}_0, \mathcal{Q}_0^*) \in B(\mathcal{P}, \Sigma) \cap B(\mathcal{P}, \Lambda)$ be such that $\Sigma_{\mathcal{Q}_0^*, \vec{\mathcal{T}}_0} \neq \Lambda_{\mathcal{Q}_0^*, \vec{\mathcal{T}}_0}$. Let $\alpha_0 \leq \lambda^{\mathcal{Q}_0^*}$ be least such that letting $\mathcal{Q}_0 = \mathcal{Q}_0^*(\alpha_0)$, $\Sigma_{\mathcal{Q}_0, \vec{\mathcal{T}}_0} \neq \Lambda_{\mathcal{Q}_0, \vec{\mathcal{T}}_0}$. Notice that $\mathcal{Q}_0$ is not of lsa type (because we are working with lsa small hod mice). Moreover, notice that by modifying $\vec{\mathcal{T}}$ we can assume, without loss of generality, that if ν is the sup of generators of $E^{\vec{\mathcal{T}}}$ then for some $\gamma < \alpha_0$, $\nu \subseteq \mathcal{Q}_0^*(\gamma)$

Suppose first that α_0 is a successor ordinal. Let $\vec{\mathcal{U}}$ be such that $\Sigma_{\mathcal{Q}_0, \vec{\mathcal{T}}_0}(\vec{\mathcal{U}}) \neq \Lambda_{\mathcal{Q}_0, \vec{\mathcal{T}}_0}$. Then $\vec{\mathcal{T}}_0{}^\frown\vec{\mathcal{U}}$ is a minimal disagreement between Σ and Λ. Thus, we must have that α_0 is a limit ordinal. Repeating the above process, we can produce $(\mathcal{Q}_i, \mathcal{Q}_i^*, \vec{\mathcal{T}}_i, \alpha_i : i < \omega)$ such that

1. for every $i < \omega$, $(\vec{\mathcal{T}}_{i+1}, \mathcal{Q}_{i+1}^*) \in B(\mathcal{Q}_i, \Sigma_{\mathcal{Q}_i, \vec{\mathcal{T}}_i}) \cap B(\mathcal{Q}_i, \Lambda_{\mathcal{Q}_i, \vec{\mathcal{T}}_i})$
2. for every $i < \omega$, $\Sigma_{\mathcal{Q}_{i+1}^*, \oplus_{k \leq i} \vec{\mathcal{T}}_k} \neq \Lambda_{\mathcal{Q}_{i+1}^*, \oplus_{k \leq i} \vec{\mathcal{T}}_k}$,
3. for every $i < \omega$, $\alpha_i \leq \lambda^{\mathcal{Q}_i^*}$ is the least such that letting $\mathcal{Q}_i = \mathcal{Q}_i^*(\alpha_i)$, $\Sigma_{\mathcal{Q}_i, \oplus_{k<i} \vec{\mathcal{T}}_i} \neq \Lambda_{\mathcal{Q}_i, \oplus_{k<i} \vec{\mathcal{T}}_i}$.

It then follows that the direct limit along the main branch of $\oplus_{i<\omega} \vec{\mathcal{T}}_i$ is ill-founded, contradiction! □

Next we introduce several definitions that will be useful in the sequel.

Definition 18.8 (Comparison stack) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then we say $(\vec{\mathcal{T}}, \mathcal{R}, \vec{\mathcal{U}}, \mathcal{S})$ are comparison stacks for $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ with last models $(\mathcal{R}, \mathcal{S})$ if $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$, $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$, and for some (α, ξ, γ) one of the following holds:*

1. *$\mathcal{R}(\alpha, \xi, \gamma) = \mathcal{S}$ and $\Sigma_{\mathcal{S}, \vec{\mathcal{T}}} = \Lambda_{\mathcal{S}, \vec{\mathcal{U}}}$.*
2. *$\mathcal{S}(\alpha, \xi, \gamma) = \mathcal{R}$ and $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}} = \Lambda_{\mathcal{R}, \vec{\mathcal{U}}}$.*

Definition 18.9 (Agreement up to the top) *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then we say $\mathcal{P}$ and $\mathcal{Q}$ agree up to the top if $\lambda^{\mathcal{P}} = \lambda^{\mathcal{Q}}$ and $\mathcal{P}(\lambda^{\mathcal{P}} - 1, 0, 0) = \mathcal{Q}(\lambda^{\mathcal{Q}} - 1, 0, 0)$. Suppose further that Σ and Λ are such that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then we say $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ agree up to the top if $\mathcal{P}$ and $\mathcal{Q}$ agree up to the top and $\Sigma_{\mathcal{P}(\lambda^{\mathcal{P}}, 0, 0)} = \Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}}, 0, 0)}$.*

Definition 18.10 (Extender and strategy disagreement) *Given two hod premice $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \neq \mathcal{Q}$, we let $\beta(\mathcal{P}, \mathcal{Q})$ be the least ordinal γ such that $\mathcal{P}|\gamma = \mathcal{Q}|\gamma$ but $\mathcal{P}||\gamma \neq \mathcal{Q}||\gamma$. We say $\mathcal{P}$ and $\mathcal{Q}$ have an extender disagreement if $\beta(\mathcal{P}, \mathcal{Q}) \in dom(\vec{E}^{\mathcal{R}}) \cup dom(\vec{E}^{\mathcal{Q}})$. We say $\mathcal{P}$ and $\mathcal{Q}$ have a strategy disagreement if $\beta(\mathcal{P}, \mathcal{Q}) \notin dom(\vec{E}^{\mathcal{R}}) \cup dom(\vec{E}^{\mathcal{Q}})$. We let*

$$(\beta(\mathcal{P}, \mathcal{Q}), \xi(\mathcal{P}, \mathcal{Q}), \gamma(\mathcal{P}, \mathcal{Q}))$$

be lexicographically least triple such that if $\vec{\mathcal{T}} \in \mathcal{P} \cap \mathcal{Q}$ is the stack for which $\mathcal{P}$ and $\mathcal{Q}$ have a branch indexed at $\beta(\mathcal{P}, \mathcal{Q})$ then $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}(\alpha(\mathcal{P}, \mathcal{Q}), \xi(\mathcal{P}, \mathcal{Q}), \gamma(\mathcal{P}, \mathcal{Q})) = \mathcal{Q}(\alpha(\mathcal{P}, \mathcal{Q}), \xi(\mathcal{P}, \mathcal{Q}), \gamma(\mathcal{P}, \mathcal{Q}))$. We say

$$(\alpha(\mathcal{P}, \mathcal{Q}), \beta(\mathcal{P}, \mathcal{Q}), \xi(\mathcal{P}, \mathcal{Q}), \gamma(\mathcal{P}, \mathcal{Q}))$$

are the disagreement ordinals of $\mathcal{P}$ and $\mathcal{Q}$.

Definition 18.11 (Extender comparison) *Suppose that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs which agree up to the top. Then we say $(\mathcal{T}, \mathcal{R}, \mathcal{U}, \mathcal{S})$ are the trees of the extender comparison of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if $(\mathcal{T}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\mathcal{U}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ are the trees formed using the usual extender comparison process (i.e., by removing the least extender disagreements) for comparing the top windows of $\mathcal{P}$ and $\mathcal{Q}$ until a strategy disagreement appears.*

It follows that if in Definition 18.11, $\mathcal{R} \neq \mathcal{S}$ then $\mathcal{R}$ and $\mathcal{S}$ have a strategy disagreement.

18.3 Universality of background constructions

Here we show that the fully backgrounded constructions are universal in a sense that they win the comparison with hod pairs.

Definition 18.12 (Λ-coherent background construction) *Suppose (M, δ, Σ) is a weak background triple and $\Lambda \in M$ is a δ-strategy (or any other kind of strategy). We say*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

is the output of Λ-coherent hod pair construction if the extenders used during the construction cohere Λ[14]*.*

[14]Where we say E coheres Λ if $\pi_E(\Lambda) = \Lambda \restriction Ult(V, E)$.

Theorem 18.13 (Universality of background construction) *Assume AD^+. Suppose (M, δ, Σ) is a background triple and $(\mathcal{P}, \Lambda)$ is a hod pair such that for some α such that $\theta_\alpha < \Theta$, Λ is Γ_α-fullness preserving and (M, δ, Σ) Suslin, co-Suslin captures Γ_α. Let*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

be the output of Λ-coherent hod pair construction of M below δ. Then there is some $\alpha \leq \lambda$, $\xi \leq \varsigma_\alpha$ and $\gamma \leq \varsigma_{\alpha,\xi}$ such that $(\mathcal{P}_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma})$ is a normal tail of $(\mathcal{P}, \Lambda)$.

Proof. As in the proof of Lemma 1.1.27 of [3], in the comparison of $\mathcal{P}$ with the construction producing $\mathcal{P}_{\alpha,\xi,\gamma}$ no extender disagreement can appear on $\mathcal{P}_{\alpha,\xi,\gamma}$ side. It is then enough to show that

(1) for every $(\alpha, \xi, \gamma) \in (\lambda + 1) \times (\varsigma_\alpha + 1) \times (\zeta_{\alpha,\xi} + 1)$, letting $\mathcal{R} = \mathcal{P}_{\alpha,\xi,\gamma}$, if $\mathcal{T}$ is a tree on $\mathcal{P}$ according to Λ with last model $\mathcal{Q}$ such that there is $(\alpha^*, \xi^*, \gamma^*) \in (\lambda^{\mathcal{R}} + 1) \times (\varsigma^{\mathcal{R}}_{\alpha^*} + 1) \times (\zeta^{\mathcal{R}}_{\alpha^*,\xi^*} + 1)$ such that $\mathcal{R}(\alpha^*, \xi^*, \gamma^*) \trianglelefteq_{hod} \mathcal{Q}$ then

$$(\Sigma_{\alpha,\xi,\gamma})_{\mathcal{R}(\alpha^*,\xi^*,\gamma^*)} = \Lambda_{\mathcal{R}(\alpha^*,\xi^*,\gamma^*),\mathcal{T}}.$$

Towards a contradiction, we assume that (1) fails. Let (α, ξ, γ) and $(\alpha^*, \xi^*, \gamma^*)$ witness the failure of (1). We assume that $(\alpha^*, \xi^*, \gamma^*)$ is the lexicographically least witnessing the failure of (1). Let $\Phi = \Sigma_{\alpha,\xi,\gamma}$ and $\mathcal{R} = \mathcal{P}_{\alpha,\xi,\gamma}$.

Suppose first that $\alpha^* = \beta + 1$, $\xi^* = \gamma^* = 0$ and $\mathcal{R}(\alpha^*, \xi^*, \gamma^*)$ is of successor type. Then we get a contradiction using branch condensation of Λ. Let $\vec{\mathcal{U}}$ be a stack on $\mathcal{R}(\alpha^*, 0, 0)$ such that it is according to both Φ and $\Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$ but $\Phi(\vec{\mathcal{U}}) \neq \Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}(\vec{\mathcal{U}})$. Let $b = \Phi(\vec{\mathcal{U}})$ and $c = \Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}(\vec{\mathcal{U}})$. Let $\vec{\mathcal{U}}^*$ be the result of lifting $\vec{\mathcal{U}}$ to the background universe M. Then because extenders used to construct $\mathcal{R}$ cohere Λ, we have that $\pi^{\vec{\mathcal{U}}^*}(\mathcal{T})$ is according to Λ. Let N be the last model of $\vec{\mathcal{U}}^*$.

Notice now that it follows from Γ-fullness preservation and the fact that $\Phi_{\mathcal{R}(\beta)} = \Lambda_{\mathcal{R}(\beta),\mathcal{T}}$ that $\pi^{\vec{\mathcal{U}}}_b$ exists. Let then $\mathcal{Q}^* = Ult(\mathcal{Q}, E)$ where E is the $(\delta^{\mathcal{R}}_{\alpha^*}, \pi^{\vec{\mathcal{U}}}_b(\delta^{\mathcal{R}}_{\alpha^*}))$-extender derived from $\pi^{\vec{\mathcal{U}}}_b$. We then have $\sigma : \mathcal{Q}^* \to \pi^{\vec{\mathcal{U}}^*}_b(\mathcal{Q})$ such that $\pi^{\pi^{\vec{\mathcal{U}}^*}_b(\mathcal{T})} = \sigma \circ \pi_E \circ \pi^{\mathcal{T}}$. Notice, however, that π_E is just an iteration embedding obtained by applying $\vec{\mathcal{U}}$ to $\mathcal{Q}$. It then follows from branch condensation of Λ that $\vec{\mathcal{U}}^\frown\{\mathcal{M}^{\vec{\mathcal{U}}}_b\}$ is according to Λ implying that $b = c$, contradiction! Thus, $\mathcal{R}(\alpha^*, \xi^*, \gamma^*)$ cannot be of successor type.

Suppose next that α^* is a limit ordinal. Then by Lemma 18.7, we can fix some $\vec{\mathcal{U}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, E_\alpha : \alpha \leq \eta)$ which constitutes a minimal disagreement between Φ and $\Lambda_{\mathcal{R},\mathcal{T}}$. Let $b = \Phi(\vec{\mathcal{U}})$. Notice that again it follows from Γ-fullness preservation and the minimality of $\vec{\mathcal{T}}$ that $\pi_b^{\vec{\mathcal{U}}}$ exists. Let then $\mathcal{Q}^*$ be the result of applying $\vec{\mathcal{U}}$ and b to $\mathcal{Q}$. Let $\vec{\mathcal{U}}^*$ be the result of lifting $\vec{\mathcal{U}}$ to M and let $N = \mathcal{M}_b^{\vec{\mathcal{U}}^*}$. There is then $\sigma : \mathcal{Q}^* \to \pi_b^{\vec{\mathcal{U}}^*}(\mathcal{Q})$ such that $\pi^{\pi_b^{\vec{\mathcal{U}}^*}(\mathcal{T})} = \sigma \circ \pi_b^{\vec{\mathcal{U}}} \circ \pi^{\mathcal{T}}$. It then again follows from the branch condensation of Λ that $\Lambda(\vec{\mathcal{U}}) = b$, contradiction!

Suppose now that α^* is a successor ordinal, $\xi^* = \gamma^* = 0$ and $\mathcal{R}(\alpha^*, 0, 0)$ is of limit type. We must have that $\mathcal{R}(\alpha^*, 0, 0)$ is of lsa type. Assume first that $\Phi^{stc} \neq \Lambda^{stc}_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$. It follows from Lemma 18.5 that there is a minimal low level disagreement between Φ and $\Lambda_{\mathcal{R}(\alpha^*,0,0)}$. Let $(\vec{\mathcal{U}}, \mathcal{S})$ be one such minimal low level disagreement. Let $\vec{\mathcal{W}}$ on $\mathcal{S}$ be a minimal disagreement between $\Phi_{\mathcal{S},\vec{\mathcal{U}}}$ and $\Lambda_{\mathcal{S},\mathcal{T}^\frown\vec{\mathcal{U}}}$. Let $b = \Phi_{\mathcal{S},\vec{\mathcal{U}}}(\vec{\mathcal{W}})$. It again follows from Γ-fullness preservation and minimality of $(\vec{\mathcal{U}}, \mathcal{S})$ that $\pi_b^{\vec{\mathcal{W}}}$ exists. Repeating the two arguments presented above we again conclude that $\beta = \Lambda(\mathcal{T}^\frown\vec{\mathcal{U}}^\frown\vec{\mathcal{W}})$.

Finally, we assume that $\Phi^{stc} = \Lambda^{stc}_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$. It follows that there is no low level disagreement between Φ and $\Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$. Let $\vec{\mathcal{U}}$ be a disagreement between Φ and $\Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$ and $b = \Phi(\vec{\mathcal{U}})$. It follows from Γ-fullness preservation and the fact that there is no low level disagreement between Φ and $\Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}$ that $\pi_b^{\vec{\mathcal{U}}}$ exists. Repeating the above argument we again conclude that $\Lambda_{\mathcal{R}(\alpha^*,0,0),\mathcal{T}}(\vec{\mathcal{U}}) = b$. □

As a corollary to Theorem 18.13 we get that the comparison holds.

Corollary 18.14 (Comparison) *Assume AD^+. Suppose that α is such that $\theta_\alpha < \Theta$. Let $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that both Σ and Λ are Γ-fullness preserving and have branch condensation. Then the normal comparison for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ holds.*

Proof. Fix a good pointclass Γ_1 such that $\Gamma \cup \{Code(\Sigma), Code(\Lambda)\} \subsetneq \Delta_{\Gamma_1}$. Let F be as in Theorem 2.25 of [3] for Γ_1 and let $x \in dom(F)$ be such that $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures Γ, $Code(\Sigma)$ and $Code(\Lambda)$. Let

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

be the output of fully backgrounded hod pair construction of $\mathcal{N}_x^*|\delta_x$. It follows from Theorem 18.13 that there is (α, ξ, γ) and (β, ν, η) such that there are normal tree $\mathcal{T}$ and $\mathcal{U}$ such

1. $(\mathcal{T}, \mathcal{P}_{\alpha,\xi,\gamma}) \in I(\mathcal{P}, \Sigma)$ and $\Sigma_{\alpha,\xi,\gamma} = \Sigma_{\mathcal{P}_{\alpha,\xi,\gamma},\mathcal{T}}$ and
2. $(\mathcal{U}, \mathcal{Q}_{\beta,\nu,\eta}) \in I(\mathcal{Q}, \Lambda)$ and $\Lambda_{\beta,\nu,\eta} = \Sigma_{\mathcal{P}_{\beta,\nu,\eta},\mathcal{T}}$.

It then immediately follows that the normal comparison for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ holds. □

19 Branch condensation

In this subsection we prove that the hod pair constructions produce strategies with branch condensation and in fact more. In order, however, to prove that hod pair constructions converge, we will need to establish the solidity and universality of the standard parameter of the models appearing in such constructions. Establishing such fine structural facts wasn't an issue in [3] as the fine structure for hod mice considered in that paper was a routine generalization of the fine structure theory developed in [2]. Here the matters are somewhat more complicated as the fine structure of non-meek hod mice cannot be viewed as a routine generalization of the fine structure of [2]. Nevertheless, the matter isn't too complicated as a simple generalization of branch condensation, *strong branch condensation*, allows us to reduce our case to the one in [2].

In this subsection, we will establish that hod pair constructions produce strategies with *strong branch condensation.*

Definition 19.1 (Strong branch condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. We say Σ has strong branch condensation if Σ has branch condensation and whenever $(\vec{\mathcal{T}}, \mathcal{Q}, \pi, \mathcal{R}, \alpha, \sigma)$ is such that*

1. *$(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\mathcal{R}$ is a hod premouse,*
2. *$\pi : \mathcal{P} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{Q}$ and $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,*
3. *$\alpha + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\alpha+1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$*

then letting $\Lambda = \sigma$-pullback of $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$, whenever $\vec{\mathcal{W}}$ is such that $(\vec{\mathcal{W}}, \mathcal{R}(\alpha+1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$, if there is no low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1), \vec{\mathcal{W}}}$ then $\Lambda_{\mathcal{R}(\alpha+1)} = \Sigma_{\mathcal{R}(\alpha+1), \vec{\mathcal{W}}}$.

Theorem 19.2 *Suppose $\mathbb{M} = (M, \delta, \Sigma)$ is a background triple and*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \varsigma_{\alpha,\xi})$$

is the output of the hod pair construction of M below δ. Fix some $\alpha \leq \lambda$, $\xi \leq \varsigma_\alpha$ and $\gamma \leq \varsigma_{\alpha,\xi}$. Then both $\Sigma_{\alpha,\xi,\gamma}$ and $\Sigma^*_{\alpha,\xi,\gamma}$ have strong branch condensation.

Proof. Since the proofs are very similar we do the proof for $\Sigma_{\alpha,\xi,\gamma}$. Towards a contradiction, suppose that for some (α, ξ, γ), $\Sigma_{\alpha,\xi,\gamma}$ doesn't have strong branch condensation. We then let (α, ξ, γ) be the lexicographically least such that $\Sigma_{\alpha,\xi,\gamma}$ doesn't have strong branch condensation.

Let $\mathcal{Q} = \mathcal{P}_{\alpha,\xi,\gamma}$ and $\Lambda = \Sigma_{\alpha,\xi,\gamma}$. We start working in M. What we need to show is that whenever $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$ is such that

1. $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ and $\mathcal{R}$ is a hod premouse,

2. $\pi : \mathcal{Q} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{S}$ and $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,

3. $\beta + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$,

then letting $\Phi = \Lambda^{\sigma}_{\mathcal{S},\vec{\mathcal{T}}}$, whenever $\vec{\mathcal{U}}^*$ is such that $(\vec{\mathcal{U}}^*, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$, if there is no low level disagreement between $\Phi_{\mathcal{R}(\beta)}$ and $\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{W}}}$ then $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{W}}}$.

Fix then such a sequence $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$. Let $(\vec{\mathcal{U}}^*, \mathcal{W}) \in I(\mathcal{Q}, \Lambda)$ be such that $\mathcal{R}(\beta+1) = \mathcal{W}(\beta+1)$. Let $\Phi = \Lambda^{\sigma}_{\mathcal{S},\vec{\mathcal{T}}}$. We assume that there is no low level disagreement between $\Phi_{\mathcal{R}(\beta)}$ and $\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{U}}^*}$ and want to show that $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{W}(\beta+1),\vec{\mathcal{U}}^*}$. Towards a contradiction assume that $\Phi_{\mathcal{R}(\beta+1)} \neq \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{U}}^*}$.

It follows from Lemma 18.5 that either $\mathcal{R}(\beta+1)$ is of successor type or of lsa type. We then have two cases. Suppose first that $(\vec{\mathcal{U}}, \mathcal{R}(\beta+1)) \in I(\mathcal{Q}, \Lambda)$. Letting

$$\Lambda^* = \begin{cases} \Lambda & : \mathcal{Q} \text{ is of successor type} \\ \Lambda^{stc} & : \text{otherwise,} \end{cases}$$

set

$$\mathcal{Q}^* = (\mathcal{J}^{\vec{E},\Lambda^*})^{V^M_\delta} \text{ and } \mathcal{Q}^+ = \mathcal{S}^{\Lambda^*}(\mathcal{Q}^*)^{15}$$

Let E be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{R}})$-extender derived from π, F be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{S}})$-extender derived from $\pi^{\vec{\mathcal{T}}}$ and H be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{W}})$-extender derived from $\pi^{\vec{\mathcal{U}}^*}$. We let

$$\mathcal{R}^+ = Ult(\mathcal{Q}^+, E),\ \mathcal{S}^+ = Ult(\mathcal{Q}^+, F) \text{ and } \mathcal{W}^+ = Ult(\mathcal{Q}^+, H).$$

We then have that

$$\mathcal{R}^+ = \mathcal{S}^{\Phi}(\mathcal{R}^+|\delta),\ \mathcal{S}^+ = \mathcal{S}^{\Lambda_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{S}^+|\delta) \text{ and } \mathcal{W}^+ = \mathcal{S}^{\Lambda_{\mathcal{W},\vec{\mathcal{U}}^*}}(\mathcal{W}^+|\delta).$$

We also have $\sigma^+ : \mathcal{R}^+ \to \mathcal{S}^+$ such that $\pi_F = \sigma^+ \circ \pi_E$ and $\sigma^+ \restriction \mathcal{R} = \sigma$.

Let now $\vec{\mathcal{K}}$ be a stack on $\mathcal{R}$ such that $\Phi(\vec{\mathcal{K}}) \neq \Lambda_{\mathcal{R},\vec{\mathcal{U}}^*}(\vec{\mathcal{K}})$. Suppose that $b = \Phi(\vec{\mathcal{K}})$ and $c = \Lambda_{\mathcal{R},\vec{\mathcal{U}}^*}(\vec{\mathcal{K}})$. Notice that because of fullness preservation we must have that both $\pi^{\vec{\mathcal{K}}}_b$ and $\pi^{\vec{\mathcal{K}}}_c$ exist. Let now $\mathcal{R}^+_b$ and $\mathcal{W}^+_c$ be the last models of $\vec{\mathcal{K}}$ when its applied to $\mathcal{R}^+$ and $\mathcal{W}^+$ respectively. Comparing $\mathcal{R}^+_b$ and $\mathcal{W}^+_c$ we get a common model $\mathcal{M}$. Let $i : \mathcal{R}^+ \to M$ and $h : \mathcal{W}^+ \to \mathcal{M}$ be the two iteration embeddings.

[15] $\mathcal{S}$ here denotes the stack. See Section 1.1.6 of [3].

Let $C \subseteq (\delta^+)^M$ be an ω-club consisting of points β such that $\beta \in rng(i) \cap rng(j)$. Then we have that

(1) $\delta^{\mathcal{R}} = \sup(Hull^{\mathcal{R}^+}(\mathcal{R}(\beta), i^{-1}[C]) \cap \delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{W}^+}(\mathcal{R}(\beta), j^{-1}[C]) \cap \delta^{\mathcal{R}})$.

(2) $\pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{M}}(\pi_b^{\vec{\mathcal{K}}}(\mathcal{R}(\beta)), [C]) \cap \pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}))$.

(3) $\pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{M}}(\pi_c^{\vec{\mathcal{K}}}(\mathcal{R}(\beta)), [C]) \cap \pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}))$.

It follows from (1), (2) and (3) that $rng(\pi_b^{\vec{\mathcal{K}}}) \cap rng(\pi_c^{\vec{K}})$ is cofinal in $\pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}})$ implying that $b = c$.

The case $(\vec{\mathcal{U}}, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda)$ is very similar and we leave it to the reader.
□

20 Consequence of strong branch condensation

The following lemma will be used to prove solidity and universality of standard parameters of models appearing in hod pair constructions.

Definition 20.1 (Certified phalanxes) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is non-meek and $\mathcal{R}$ is a hod premouse. We say $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\pi, \mathcal{P}, \Sigma)$-certified phalanx if $\zeta > o(\mathcal{P}^b)$ and there is an embedding $\pi : \mathcal{R} \to \mathcal{P}$ such that $\zeta \leq crit(\pi)$. Given such a π, we let $\pi^+ : (\mathcal{P}, \mathcal{R}, \zeta) \to (\mathcal{P}, \mathcal{P}, \zeta)$ be given by (id, π) and also we let Σ^{π^+} be the strategy of $(\mathcal{P}, \mathcal{R}, \zeta)$ given by Σ.*

Lemma 20.2 (No strategy disagreement) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and $\mathcal{P}$ is non-meek. Suppose $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\mathcal{P}, \Sigma)$ certified phalanx as witnessed by $\pi : \mathcal{R} \to \mathcal{P}$. Let $\Lambda = \Sigma^{\pi^+}$. Then no strategy disagreement appears in the comparison of $\mathcal{P}$ and $(\mathcal{P}, \mathcal{R}, \zeta)$ where Σ is used on $\mathcal{P}$ side and Λ is used in $(\mathcal{P}, \mathcal{R}, \zeta)$.*

Proof. Towards a contradiction suppose not. We can the find iteration trees $\mathcal{T}$ and $\mathcal{U}$ on respectively $(\mathcal{P}, \mathcal{R}, \zeta)$ and $\mathcal{P}$ such that $\mathcal{T}$ is according to Λ, $\mathcal{U}$ is according to Σ, $\mathcal{T}$ and $\mathcal{U}$ have last models $\mathcal{S}$ and $\mathcal{Q}$ respectively and $\mathcal{S}$ and $\mathcal{Q}$ have a strategy disagreement. Let (α, ξ, γ) be lexicographically least such that $\mathcal{S}(\alpha, \xi, \gamma) = \mathcal{Q}(\alpha, \xi, \gamma)$ but $\Lambda_{\mathcal{S}(\alpha,\xi,\gamma),\mathcal{T}} \neq \Sigma_{\mathcal{Q}(\alpha,\xi,\gamma),\mathcal{U}}$ or $\Lambda^{sts}_{\mathcal{S}(\alpha,\xi,\gamma),\mathcal{T}} \neq \Sigma^{sts}_{\mathcal{Q}(\alpha,\xi,\gamma),\mathcal{U}}$.

Suppose first that (α, ξ) is not of lsa type. Then we must have that $\Lambda_{\mathcal{S}(\alpha,\xi,\gamma),\mathcal{T}} \neq \Sigma_{\mathcal{Q}(\alpha,\xi,\gamma),\mathcal{U}}$ and therefore, by Lemma 18.7, there is a minimal disagreement $\vec{\mathcal{K}}$ between $\Lambda_{\mathcal{S},\mathcal{T}}$ and $\Sigma_{\mathcal{Q},\mathcal{U}}$. Let $\mathcal{R}$ be the last model of $\vec{\mathcal{K}}$ and let $\vec{\mathcal{W}}$ be the stack on $\mathcal{P}$ obtained from $\mathcal{T}^\frown\vec{\mathcal{K}}$ by copying construction via π^+. Let $\mathcal{N}$ be the least model of $\vec{\mathcal{W}}$ and let $\sigma : \mathcal{R} \to \mathcal{N}$ be given by the copy construction. Let $\beta + 1 \leq \lambda^{\mathcal{R}}$ be the least such that $\Lambda_{\mathcal{R}(\beta+1),\mathcal{T}^\frown\vec{\mathcal{K}}} \neq \Sigma_{\mathcal{R}(\beta+1),\mathcal{T}^\frown\vec{\mathcal{K}}}$. Let now $E = E_{\mathcal{T}^\frown\vec{\mathcal{K}}}$ and let $\mathcal{R}^* = Ult(\mathcal{P}, E)$. σ then induces a map $\sigma^* : \mathcal{R}^* \to Ult(\mathcal{P}, E_{\vec{\mathcal{W}}})$. Applying strong branch condensation to $(\vec{\mathcal{W}}^\frown\{E_{\vec{\mathcal{W}}}, Ult(\mathcal{P}, E_{\vec{\mathcal{W}}})\}, Ult(\mathcal{P}, E_{\vec{\mathcal{W}}}), \pi_E, \mathcal{R}^*, \beta, \sigma^*)$ and $(\mathcal{U}^\frown\vec{\mathcal{T}}, \mathcal{R})$ we get a contradiction.

Suppose next that (α, ξ) is of lsa type. If $\Lambda^{sts}_{\mathcal{S}(\alpha,\xi,\gamma),\mathcal{T}} \neq \Sigma^{sts}_{\mathcal{Q}(\alpha,\xi,\gamma),\mathcal{U}}$ then we derive contradiction as before. So we assume that $\Lambda^{sts}_{\mathcal{S}(\alpha,\xi,\gamma),\mathcal{T}} = \Sigma^{sts}_{\mathcal{Q}(\alpha,\xi,\gamma),\mathcal{U}}$. It is easy to see that we must have that $\mathcal{S}(\alpha, \xi, \gamma) \vDash$ "ν_ξ is Woodin" and $\mathcal{Q}(\alpha, \xi, \gamma) \vDash$ "ν_ξ is Woodin" (as otherwise both strategies are strategies guided by the same kind of Q-structures.) It then follows that $\xi = \delta^{\mathcal{S}} = \delta^{\mathcal{Q}}$ and $\gamma = 0$, i.e., $\mathcal{S}(\alpha, \xi, \gamma) = \mathcal{S}$, $\mathcal{Q}(\alpha, \xi, \gamma) = \mathcal{Q}$ and both are lsa type. Let then $\mathcal{W} = \pi^+\mathcal{T}$ and let $\mathcal{N}$ be the last model of $\mathcal{W}$. Let $\sigma : \mathcal{S} \to \mathcal{N}$ come from the copying construction. Applying strong branch condensation to $(\mathcal{W}, \mathcal{N}, \pi^{\mathcal{T}}, \mathcal{S}, \alpha + 1, \sigma)$ and $(\mathcal{U}, \mathcal{Q})$, we get a contradiction.
□

Definition 20.3 (Certified pairs) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{R}$ is a hod premouse. Suppose that there is π such that if $\lambda^{\mathcal{P}}$ is a successor but $\mathcal{P}$ is not of limit type then $\pi : \mathcal{P} \to \mathcal{R}$ and otherwise, $\pi : \mathcal{P}^b \to \mathcal{R}^b$. We say the pair $(\pi, \mathcal{R})$ is $(\mathcal{P}, \Sigma)$-certified by $(\sigma, \vec{\mathcal{T}}, \mathcal{Q})$ if $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\sigma : \mathcal{R} \to \mathcal{Q}$ is such that*

1. *if $\lambda^{\mathcal{P}}$ is a successor but $\mathcal{P}$ is not of limit type then $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$, and*

2. *otherwise, $\pi^{\vec{\mathcal{T}},b} = (\sigma \restriction \mathcal{R}^b) \circ \pi$.*

We say $(\mathcal{R}, \Lambda)$ is a $(\mathcal{P}, \Sigma)$-certified hod pair if there is some π such that $(\pi, \mathcal{R})$ is certified by $(\sigma, \vec{\mathcal{T}}, \mathcal{Q})$ and $\Lambda = \Sigma^\sigma_{\mathcal{Q},\vec{\mathcal{T}}}$. In this case, we say $(\pi, \sigma, \vec{\mathcal{T}}, \mathcal{Q})$ is a $(\mathcal{P}, \Sigma)$-certificate for $(\mathcal{R}, \Lambda)$.

Lemma 20.4 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and $\mathcal{P}$ is non-meek. Suppose $(\pi, \mathcal{R})$ is certified by $(\sigma, \vec{\mathcal{T}}, \mathcal{Q})$ and that $\pi : \mathcal{P}^t \to \mathcal{R}^t$. Let $\Lambda = \Sigma^\sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ and let $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{R}, \Lambda)$ be such that for some $\alpha + 1 < \lambda^{\mathcal{S}}$ there is $\vec{\mathcal{W}}$ such that $(\vec{\mathcal{W}}, \mathcal{S}(\alpha+1, 0, 0)) \in I(\mathcal{P}, \Sigma)$ and $\Sigma_{\mathcal{S}(\alpha),\vec{\mathcal{W}}} = \Lambda_{\mathcal{S}(\alpha),\vec{\mathcal{U}}}$. Then $\Sigma_{\mathcal{S}(\alpha+1,0,0),\vec{\mathcal{W}}} = \Lambda_{\mathcal{S}(\alpha+1,0,0),\vec{\mathcal{U}}}$.*

Proof. Let E be the $(\delta^{\mathcal{P}}, \delta^{\mathcal{S}})$-extender derived from $\pi^{\vec{\mathcal{U}}} \circ \pi^t$ and let $F = E_{\vec{\mathcal{T}}^\frown \sigma\vec{\mathcal{U}}}$. We then have $\pi^{\vec{\mathcal{T}}^\frown \sigma\vec{\mathcal{U}}, t} = (k \restriction \mathcal{S}^t) \circ \pi_E$ where letting $\mathcal{S}^*$ be the last model of $\sigma\vec{\mathcal{U}}$, $k : \mathcal{S} \to \mathcal{S}^*$ comes from the copying construction. Let $\vec{\mathcal{T}}^* = \vec{\mathcal{T}}^\frown\vec{\mathcal{U}} \frown \{F, Ult(\mathcal{P}, F)\}$. Applying strong branch condensation to $(\vec{\mathcal{T}}^*, \mathcal{S}^*, \pi_E, Ult(\mathcal{P}, E), \alpha+1, k)$ and $(\vec{\mathcal{W}}, \mathcal{S}(\alpha+1, 0, 0))$ we get that $\Sigma_{\mathcal{S}(\alpha+1,0,0),\vec{\mathcal{W}}} = \Lambda_{\mathcal{S}(\alpha+1,0,0),\vec{\mathcal{U}}}$. □

Lemma 20.5 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and that $\mathcal{P}$ is non-lsa type non-meek hod premouse. Suppose $(\vec{\mathcal{T}}, \mathcal{P}_1) \in I(\mathcal{P}, \Sigma)$ is such that for some Λ, $(\mathcal{P}_1, \Lambda)$ is $(\mathcal{P}, \Sigma)$-certified by some $(\pi, \sigma, \vec{\mathcal{U}}, \mathcal{Q})$ such that $\pi : \mathcal{P} \to \mathcal{R}$. Then $\pi^{\vec{\mathcal{T}}}$ exists and $\pi^{\vec{\mathcal{T}}} \leq \pi$.*

Proof. Fix a $(\mathcal{P}, \Sigma)$-certificate $(\pi, \sigma, \vec{\mathcal{U}}, \mathcal{Q})$ for $(\mathcal{R}, \Lambda)$ such that $\pi : \mathcal{P} \to \mathcal{R}$. Since Σ has strong branch condensation, it follows that $\Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}} = \Lambda (= \Sigma^{\sigma}_{\mathcal{Q}, \vec{\mathcal{U}}})$. Let $\Phi = \Sigma^{\sigma\circ\pi}_{\mathcal{Q}, \vec{\mathcal{U}}}$. It follows from branch condensation that $\Phi^{sts} = \Sigma^{sts}$. Because $\mathcal{P}$ is not of lsa type, we get that $\Phi = \Sigma$. We can now apply the usual Dodd-Jensen argument to conclude that $\pi^{\vec{\mathcal{T}}}$ exists and that $\pi^{\vec{\mathcal{T}}} \leq \pi$. □

It follows from Lemma 20.5 that

Corollary 20.6 (Positional and commuting) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation. Then Σ is positional and commuting.*

We also get the following version of Lemma 20.5 for phalanxes which will be used in the solidity and universality proof.

Lemma 20.7 (Dodd-Jensen for certified phalanxes) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation. Suppose that $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\mathcal{P}, \Sigma)$-certified phalanx as witnessed by $\pi : \mathcal{R} \to \mathcal{P}$. Suppose that $(\mathcal{T}, \mathcal{Q}) \in I((\mathcal{P}, \mathcal{R}, \zeta), \Sigma^{\pi^+})$ and $(\mathcal{U}, \mathcal{S}) \in I(\mathcal{P}, \Sigma)$ are such that such hat the last branch of $\mathcal{T}$ is on $\mathcal{Q}$ and that either $\mathcal{Q} \trianglelefteq_{hod} \mathcal{S}$ and $\pi^{\mathcal{T}}$ exists or $\mathcal{S} \trianglelefteq_{hod} \mathcal{Q}$ and $\pi^{\mathcal{U}}$ exists. Then $\mathcal{Q} = \mathcal{S}$ and $\pi^{\mathcal{T}} = \pi^{\mathcal{U}}$.*

Proof. Let $\mathcal{T}^* = \pi^+\mathcal{T}$. Let $\mathcal{Q}^*$ be the last model of $\mathcal{T}^*$ and let $\sigma : \mathcal{Q} \to \mathcal{Q}^*$ come from the copying construction. Suppose first that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{S}$ and $\pi^{\mathcal{T}}$ exists. Applying Lemma 20.5 to σ, we get that $\pi^{\mathcal{U}}$ exists, $\mathcal{S} = \mathcal{Q}$ and $\pi^{\mathcal{U}} \leq \pi^{\mathcal{T}}$.

Suppose now $\mathcal{S} \trianglelefteq_{hod} \mathcal{Q}$ and $\pi^{\mathcal{U}}$ exists. Let Φ be $\pi^{\mathcal{U}} \circ \pi^+$-pullback of $\Sigma_{\mathcal{S},\mathcal{U}}$. We then have that $\Phi = \Sigma^{\pi^+}$. Applying Lemma 20.5 to $(\sigma \restriction \mathcal{S}) \circ \pi^{\mathcal{U}}$ and $(\mathcal{T}^*, \sigma(\mathcal{S}))$, we get $\sigma(\mathcal{S}) = \mathcal{Q}^*$ and $\pi^{\mathcal{T}^*}$ exists. It follows that $\Sigma^{\sigma}_{\mathcal{Q}^*, \mathcal{T}^*} = \Sigma_{\mathcal{S},\mathcal{U}}$. Therefore, the usual

Dodd-Jensen argument can be used to get that $\mathcal{S} = \mathcal{Q}$ and $\pi^{\mathcal{T}} \leq \pi^{\mathcal{U}}$. Putting the two arguments together we see that $\pi^{\mathcal{U}} = \pi^{\mathcal{T}}$. □

It is clear that it follows from Lemma 20.7 and from Lemma 20.2 that the usual proofs of condensation, universality and solidity go through for hod mice. We state the result without a proof (see [2] and [8] for the usual proofs of these results.)

Theorem 20.8 (Solidity and universality) *Suppose $k < \omega$ and $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is k-sound non-meek hod premouse which is not of lsa type, $\rho(\mathcal{P}) > ((\kappa^{\mathcal{P}})^+)^{\mathcal{P}}$ and Σ is a $(k, \lambda, \lambda, \lambda, \lambda)$ iteration strategy with strong branch condensation where $\lambda = |\mathcal{P}|^+$. Let r be the $k+1$st standard parameter of $(\mathcal{P}, u_k(\mathcal{P}))$; then r is $k+1$-solid and $k+1$-universal over $(\mathcal{P}, u_k(\mathcal{P}))$.*

Theorem 20.9 (Condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is non-meek hod premouse which is not of lsa type, $\rho(\mathcal{P}) > ((\kappa^{\mathcal{P}})^+)^{\mathcal{P}}$ and Σ is a $(k, \lambda, \lambda, \lambda, \lambda)$ iteration strategy with strong branch condensation where $\lambda = |\mathcal{P}|^+$. Suppose $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\mathcal{P}, \Sigma)$ certified phalanx as witnessed by $\pi : \mathcal{R} \to \mathcal{P}$ such that $\zeta = crit(\pi) = \rho_\omega^{\mathcal{R}}$. Then either*

1. *$\mathcal{R} \trianglelefteq_{hod} \mathcal{P}$ or*
2. *there is an extender E on the sequence of $\mathcal{P}$ such that $lh(E) = \rho_\omega^{\mathcal{R}}$ and $\mathcal{R} \trianglelefteq_{hod} Ult(\mathcal{P}, E)$.*

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