

# The mouse set conjecture for sets of reals<sup>\*†</sup>

Grigor Sargsyan<sup>‡</sup>

Department of Mathematics

Rutgers University

110 Frelinghuysen Rd.

Piscataway, NJ 08854

<http://math.rutgers.edu/~gs481>

[grigor@math.rutgers.edu](mailto:grigor@math.rutgers.edu),

John Steel

Department of Mathematics

University of California

Berkeley, California, 94720 USA

<http://math.berkeley.edu/~steel>

[steel@math.berkeley.edu](mailto:steel@math.berkeley.edu)

## Abstract

We show that the *Mouse Set Conjecture* for sets of reals is true in the minimal model of  $AD_{\mathbb{R}} + “\Theta$  is regular”’. As a consequence, we get that below  $AD_{\mathbb{R}} + “\Theta$  is regular”’, models of  $AD^+ + \neg AD_{\mathbb{R}}$  are hybrid mice over  $\mathbb{R}$ . Such a representation of models of  $AD^+$  is important in core model induction applications.

One of the central open problems in descriptive inner model theory is the conjecture known as the *Mouse Set Conjecture (MSC)*. It conjectures that under  $AD^+$

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ordinal definable reals are exactly those that appear in  $\omega_1$ -iterable mice. The counterpart of this conjecture for sets of reals conjectures that under  $AD^+$ , the sets of reals which are ordinal definable from a real are exactly those that appear in countably iterable mice over  $\mathbb{R}$ . In [3], The first author proved that  $MSC$  holds in the minimal model of  $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ , but  $MSC$  for sets of reals was left open. The goal of this paper is to establish that  $MSC$  for sets of reals holds in the minimal model of  $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ .

We will establish a stronger form of  $MSC$  known as the *Strong Mouse Set Conjecture* ( $SMSC$ ). We say  $\mathcal{M}$  is countably  $\kappa$ -iterable if all of its countable substructures are  $\kappa$ -iterable. We say  $\mathcal{M}$  is countably iterable if  $\mathcal{M}$  is countably  $\omega_1$ -iterable. Thus, under  $AD$ , if  $\mathcal{M}$  is countably iterable then  $\mathcal{M}$  is countably  $\omega_1 + 1$ -iterable.

In what follows, we will let “hod pair” stand for a hod pair below  $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ , i.e., the corresponding hod mouse cannot have inaccessible limit of Woodin cardinals. Given an iteration strategy  $\Sigma$ , we let  $Code(\Sigma)$  be the set of reals coding  $\Sigma$  for trees of length  $\omega_1$ . Given a hod pair  $(\mathcal{P}, \Sigma)$  we let

$$Lp^{\Sigma}(\mathbb{R}) = \cup\{\mathcal{M} : \mathcal{M} \text{ is a sound countably iterable } \Sigma\text{-mouse over } \mathbb{R} \text{ projecting to } \mathbb{R}\}.$$

The following is the statement of  $SMSC$  for sets of reals.

**The Strong Mouse Set Conjecture for sets of reals,  $SMSC(\mathbb{R})$ :** Assume  $AD^+$ . Suppose  $(\mathcal{P}, \Sigma)$  is a hod pair such that  $\Sigma$  has branch condensation and is fullness preserving. Then

$$\{A \subseteq \mathbb{R} : \exists x \in \mathbb{R} (A \text{ is } OD(\Sigma, x))\} = Lp^{\Sigma}(\mathbb{R}).$$

To following is the main theorem of this paper.

**Theorem 0.1** *Assume  $AD^+ + V = L(\wp(\mathbb{R}))$ . Suppose  $(\mathcal{P}, \Sigma)$  is a hod pair such that the following holds.*

1.  $\mathcal{P}$  does not have inaccessible limit of Woodin cardinals.
2.  $\Sigma$  has branch condensation and is fullness preserving.
3.  $MSC$  for  $\Sigma$  holds, i.e., for every  $x, y \in \mathbb{R}$ ,  $x \in OD(\Sigma, y)$  iff  $x$  is in a  $\Sigma$ -mouse over  $y$ .
4. Every set of reals  $A$  is  $OD(\Sigma, x)$  for some real  $x$ .

Then

$$\wp(\mathbb{R}) = \wp(\mathbb{R}) \cap Lp^\Sigma(\mathbb{R}).$$

In particular,  $V = L(Lp^\Sigma(\mathbb{R}))$ .

**Corollary 0.2** *SMSC*( $\mathbb{R}$ ) is true in the minimal model of  $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ .

*Proof.* Assume that  $V$  is the minimal model of  $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ . It is shown in [3] that if  $(\mathcal{P}, \Sigma)$  is as in the hypothesis of Theorem 0.1 then clause 3 holds in  $L(\Gamma_{\alpha+1})$  where  $\alpha$  is such that  $\theta_\alpha = w(\text{Code}(\Sigma))$  (here  $\text{Code}(\Sigma)$  is the set of reals coding  $\Sigma$ ) and  $\Gamma_{\alpha+1} = \{A \subseteq \mathbb{R} : w(A) < \theta_{\alpha+1}\}$ . It then follows from Theorem 0.1 that  $\Gamma_\alpha = \wp(\mathbb{R}) \cap Lp^\Sigma(\mathbb{R})$  implying that *SMSC*( $\mathbb{R}$ ) holds.  $\square$

All the background material that we will need in this paper is spelled out in [3]. We assume that our reader is familiar with some aspects of it. One important comment is that in general hybrid mice over  $\mathbb{R}$  or any non-self-wellordered set are not defined (recall that a set  $X$  is self-wellordered if there is a wellordering of it in  $\mathcal{J}_\omega(X)$ ). Given an iteration strategy  $\Sigma$  with hull condensation, the  $\Sigma$ -mice over self-wellordered sets are defined according to the following principle. At a typical stage where we would like to add more of  $\Sigma$  to the model, we choose the least tree  $\mathcal{T}$  for which  $\Sigma(\mathcal{T})$  hasn't been defined. However,  $\mathbb{R}$  isn't self-wellordered and hence, we cannot choose the least such  $\mathcal{T}$ .

In [3], the first author gave a definition of premice over any non-self-wellordered sets under the hypothesis that  $\mathcal{M}_1^{\#, \Sigma}$  exists, i.e., there is a minimal active  $\Sigma$ -mouse with one Woodin cardinal (see Definition 3.37 of [3]). This extra assumption is benign as under  $AD^+$  whenever  $(\mathcal{P}, \Sigma)$  is a hod pair such that  $\Sigma$  has branch condensation and is fullness preserving,  $\mathcal{M}_1^{\#, \Sigma}$  exists and is  $\Theta$ -iterable. The proof is the same as the proof that shows that  $AD^{L(\mathbb{R})}$  implies that  $\mathcal{M}_1^\#$  exists and is  $\Theta$ -iterable in  $L(\mathbb{R})$  (see [11]). One consequence of the indexing of the strategy introduced in Definition 3.37 of [3] is that it allows us to perform  $S$ -constructions, which we will use in this paper (see Chapter 3 of [3]).

Corollary 0.2 has been used in core model induction applications. See, for instance, [2], [4], [5] or Chapter 7 of [6]. Before we begin the proof of Theorem 0.1, we introduce Prikry tree forcing associated with Martin's measure on degrees.

## 1 Prikry tree forcing on degrees

We develop the notion of Prikry forcing that we need in a general context. Assume  $ZF - \text{Replacement} + AD$ . Let  $\mathcal{D}$  be the set of Turing degrees. Let  $f : \mathcal{D}^{<\omega} \rightarrow HC$  be some function. We would like to define *Prikry tree forcing* on degrees associated to  $f$ . Let  $\mu$  be Martin's measure. We let  $(p, A) \in \mathbb{P}^f$  if

1.  $p \in \mathcal{D}^{<\omega}$ ,
2. for any  $n < lh(p)$ ,  $(f(p \upharpoonright n), p \upharpoonright n) \in L[p(n)]$ ,
3.  $A \subseteq \bigcup_{n < \omega} \mathcal{D}^{<n}$  is a tree with stem  $p$  such that for every  $q \in A$ ,

$$\{d : q \frown d \in A\} \in \mu.$$

Given  $(p, A), (q, B) \in \mathbb{P}^f$  we let

$$(p, A) \preceq (q, B) \text{ iff } p \text{ end-extends } q, A \subseteq B, \text{ and } \forall n \in \text{dom}(p) \setminus \text{dom}(q) \\ (p|(n+1) \in B).$$

We say  $p \in \mathcal{D}^{<\omega}$  is a *precondition* if it satisfies 1 and 2 above. Given a precondition  $p$  and  $d \in \mathcal{D}$ , we say  $d$  is *valid* at  $p$  if  $p \frown d$  is a precondition.

Given a  $\mathbb{P}^f$ -generic  $G$  we let  $g = \bigcup \{p : \exists X(p, X) \in G\}$ . We then let

$$G^i =_{def} f(g \upharpoonright i + 1) \text{ and } f(G) =_{def} \bigcup_{i < \omega} G^i$$

The following is proved by a standard fusion argument.

**Lemma 1.1**  $\mathbb{P}^f$  has the Prikry property. More precisely, suppose  $Z$  is a countable set of  $\mathbb{P}^f$ -terms,  $\phi$  is a formula, and  $(p, A) \in \mathbb{P}^f$ . Then there is a condition  $(p, W) \in \mathbb{P}^f$  deciding  $\phi[\tau]$  for all  $\tau \in Z$  such that  $W \in OD_{Z, \{f, p, A\}}$ .

*Proof.* We will show that there is a condition  $(p, T_\tau)$  deciding  $\phi[\tau]$  such that  $\langle T_\tau : \tau \in Z \rangle \in OD_{Z, \{f, p, A\}}$ . It then follows that  $(p, \bigcap_{\tau \in Z} T_\tau)$  is as desired. We say  $q$  is *positive* if  $(\exists Y) ((q, Y) \Vdash \phi[\tau])$ , *negative* if  $(\exists Y) ((q, Y) \Vdash \neg \phi[\tau])$ , and *ambiguous* if it is neither positive nor negative. Notice that  $q$  cannot be both positive and negative.

We define a sequence of functions  $\langle H^i : i < \omega \rangle$  such that

$$\text{dom}(H^i) = \{q : p \trianglelefteq q \text{ and } q \text{ is a precondition}\}$$

and  $\text{rng}(H^i) \subseteq \{0, 1, 2\}$ . First define  $H$  on  $\{(q, d) : q \frown d \text{ is a precondition}\}$  by

$$H(q, d) = \begin{cases} 0 : & q \frown d \text{ is positive} \\ 1 : & q \frown d \text{ is negative} \\ 2 : & q \frown d \text{ is ambiguous.} \end{cases}$$

Now, let  $H^0(q) = i$  if for  $\mu$ -a.e.  $d$  is such that  $H(q, d) = i$ . Given  $\langle H^i : i \leq k \rangle$  define  $H^{k+1}$  by setting  $H^{k+1}(q) = i$  if for  $\mu$ -a.e.  $d$  is such that  $H^k(q \frown d) = i$ .

We then define a decreasing sequence of conditions  $(p, T^i)$  by induction as follows. We will have that  $(p, T^0) \preceq (p, A)$ . We define  $T^0$  by induction on the length of conditions. We let  $T^0 \upharpoonright m$  be  $T^0$  restricted to sequences of length  $m$ . Suppose we have defined  $T^0 \upharpoonright m + 1$  for  $m + 1 \geq lh(p)$ . Given  $q \in T^0 \upharpoonright m$  such that  $lh(q) = m$  we let

$$\{q \frown d \in A : H(q, d) = H^0(q)\}$$

be the one step extensions of  $q$  in  $T^0$ . This finishes our description of  $T^0$ .

Suppose now we have defined  $\langle (p, T^i) : i \leq k \rangle$  and  $T^{k+1} \upharpoonright m + 1$ . Given  $q \in T^{k+1} \upharpoonright m$  such that  $lh(q) = m$ , we let

$$\{q \frown d \in T^k : H^k(q \frown d) = H^{k+1}(q)\}$$

be the one step extensions of  $q$  in  $T^{k+1}$ . This finishes our description of  $\langle (p, T^i) : i \leq \omega \rangle$ . Let  $T_\tau = \bigcap_{i < \omega} T^i$ .

We claim that  $(p, T_\tau)$  decides  $\tau$ . Suppose not. We then have two conditions  $(q, X)$  and  $(r, Y)$  such that both are below  $(p, T_\tau)$  and

1.  $lh(q) = lh(r)$ ,
2.  $(q, X) \Vdash \phi[\tau]$ ,
3.  $(r, Y) \Vdash \neg\phi[\tau]$ .

Let now  $s$  be the common initial segment of  $q$  and  $r$ . Let  $s = (d_i : i \leq m)$ ,  $q = s \frown (q_i : i < n)$  and  $r = s \frown (r_i : i < n)$ . It follows from our construction that

$$\begin{aligned} H(s \frown (q_i : i < n - 1), q_{n-1}) &= H^0(s \frown (q_i : i \leq n - 1)) = H^1(s \frown (q_i : i < n - 2)) = \\ &\quad \dots = H^{n-1}(s) \\ H(s \frown (r_i : i < n - 1), r_{n-1}) &= H^0(s \frown (r_i : i \leq n - 1)) = H^1(s \frown (r_i : i < n - 2)) = \\ &\quad \dots = H^{n-1}(s). \end{aligned}$$

It then follows that  $H(s \frown (q_i : i < n - 1), q_{n-1}) = H(s \frown (r_i : i < n - 1), r_{n-1})$ , which is a contradiction.  $\square$

We now turn to proving Theorem 0.1.

## 2 The proof

We assume  $AD^+ + V = L(\wp(\mathbb{R}))$  and let  $(\mathcal{P}, \Sigma)$  be as in the hypothesis of Theorem 0.1. Given a good pointclass  $\Gamma$  and  $a \in HC$ , we let  $Lp^{\Gamma, \Sigma}(a)$  be the union of sound  $\Sigma$ -mice over  $a$  projecting to  $a$  whose iteration strategy is coded by a set in  $\Gamma$ . Our first lemma is an easy lemma. Below  $MC(\Sigma)$  (mouse capturing relative to  $\Sigma$ ) is the statement that for every  $x, y \in \mathbb{R}$ ,  $x \in OD_{\Sigma, y}$  if and only if there is an  $\omega_1$ -iterable sound  $\Sigma$ -mouse  $\mathcal{M}$  over  $y$  such that  $x \in \mathcal{M}$ .

**Lemma 2.1** *For any good pointclass  $\Gamma \neq \Sigma_1^2(Code(\Sigma))$  there is a good pointclass  $\Gamma_1 \neq \Sigma_1^2(Code(\Sigma))$  such that  $\Gamma \cup \{Code(\Sigma)\} \subseteq \Gamma_1$  and for any  $a \in HC$ ,*

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1, \Sigma}(a).$$

*Proof.* Fix a good pointclass  $\Gamma \neq \Sigma_1^2(Code(\Sigma))$ . Because  $MC(\Sigma)$  holds, using  $\Sigma_1(Code(\Sigma))$ -reflection, we can find  $\Gamma_1$  and  $\alpha$  such that  $\Gamma_1 \neq \Sigma_1^2(Code(\Sigma))$ ,  $\Gamma \cup \{Code(\Sigma)\} \subseteq \Delta_{\Gamma_1}$ ,  $L_\alpha(\Gamma_1, \mathbb{R}) \models ZF - Replacement$ ,  $\Gamma_1 = (\Sigma_1^2(Code(\Sigma)))^{\mathcal{J}_\alpha(\Gamma_1, \mathbb{R})}$  and

$$\mathcal{J}_\alpha(\Gamma_1, \mathbb{R}) \models MC(\Sigma).$$

□

Suppose now that  $\wp(\mathbb{R}) \neq Lp^\Sigma(\mathbb{R})$ . Using  $\Sigma_1$ -reflection we get  $\Gamma \subseteq \Delta_1^2(Code(\Sigma))$  and  $\alpha < \delta_1^2(Code(\Sigma))$  such that

1.  $\Gamma = \wp(\mathbb{R}) \cap \mathcal{J}_\alpha(\Gamma, \mathbb{R})$  and  $\alpha$  ends a  $\Sigma_1$ -gap,
2.  $\mathcal{J}_\alpha(\Gamma, \mathbb{R}) \models \phi$  where  $\phi$  is the conjunction of the following statements:
  - (a)  $ZF - Replacement + DC_{\mathbb{R}} + MC(\Sigma)$ ,
  - (b) there is an  $OD$  set of reals  $A$  such that  $A \notin Lp^\Sigma(\mathbb{R})$ .

We let  $N = \mathcal{J}_\alpha(\Gamma, \mathbb{R})$ . Let  $U$  be the set of pairs  $(x, y) \in \mathbb{R}^2$  such that  $y$  codes a sound  $\Sigma$ -mouse  $\mathcal{M}$  over  $x$  that projects to  $x$  and has an  $\omega_1$ -iteration strategy in  $N$ .

Since  $MC(\Sigma)$  holds in  $N$ ,  $U$  is a universal  $(\Sigma_1^2(Code(\Sigma)))^N$ -set. Let  $A \in N$  be an  $OD$  set of reals witnessing clause b of  $\phi$ . We assume that  $A$  has the minimal Wadge rank among the sets witnessing clause b of  $\phi$ . Using the results of Chapter 3 of [1], we can get  $\vec{B} = \langle B_i : i < \omega \rangle$  which is a semiscale on  $U^c$  such that each  $B_i \in (OD_\Sigma)^N$ . The following fact is a well known consequence of  $MC(\Sigma)$ .

**Proposition 2.2** *The following statements are true.*

1. There is a cone of  $x$  such that there is  $\mathcal{M} \trianglelefteq Lp^\Sigma(x)$  such that  $\rho(\mathcal{M}) = x$  and  $\mathcal{M}$  doesn't have an iteration strategy in  $N$ .
2. Let  $x$  be a base of the above cone. Then for every  $a \in HC$  such that  $x \in \mathcal{J}_\omega(a)$ , there is  $\mathcal{M} \trianglelefteq Lp^\Sigma(a)$  such that  $\rho(\mathcal{M}) = a$  and  $\mathcal{M}$  doesn't have an iteration strategy in  $N$ .

*Proof.* Clause 2 follows from clause 1. To see this, fix a real  $x$  such that it is base for the cone of clause 1. Then whenever  $a \in HC$  is such that  $x \in a$  and  $y$  is a real coding  $a$  generically over  $Lp^\Sigma(a)$ , then  $Lp^\Sigma(a)[y] = Lp^\Sigma(y)$ . Indeed, this follows from  $S$ -constructions (see Section 2.11 of [3]).

Clause 1 is an easy consequence of  $MC(\Sigma)$ . Indeed, suppose clause 1 fails. Then there is an  $x \in \mathbb{R}$  such that for all  $y \in \mathbb{R}$  such that  $x \leq_T y$ ,  $Lp^\Sigma(y) = (Lp^\Sigma(y))^N$ . However, because  $MC(\Sigma)$  holds, letting  $C$  be the set of pairs  $(y, z) \in \mathbb{R}^2$  such that  $x \leq_T y$  and  $z$  codes an  $\omega_1$ -iterable sound  $\Sigma$ -mouse  $\mathcal{M}$  over  $y$  projecting to  $y$ ,  $C$  is a universal  $\Sigma_1^2(\text{Code}(\Sigma))$  sets. Therefore,  $C \not\leq_n N$ , contradiction.  $\square$

Let now  $x$  be a base of the cone from clause 1 of Proposition 2.2. We say  $a$  is *good* if  $a \in HC$  and  $x \in \mathcal{J}_\omega(a)$ . For each good  $a$  let  $\mathcal{M}(a)$  be the least  $\Sigma$ -mouse with no iteration strategy in  $N$ . Let  $F$  be the set of pairs  $(a, \mathcal{M}(a))$ . It follows that if  $F^*$  is the set of reals coding  $F$  then  $F^* \in \underline{\Delta}_1^2(\text{Code}(\Sigma))$ . Furthermore, there is a set  $C \in \underline{\Delta}_1^2(\text{Code}(\Sigma))$  such that for every good  $a$ , the set of reals coding the unique iteration strategy of  $\mathcal{M}(a)$  is Wadge reducible to  $C$ . Let then

$$D = \{(y, \sigma) \in \mathbb{R}^2 : y \text{ codes a good } a \text{ and } \sigma \text{ codes a continuous function } f \text{ such that } f^{-1}[C] \text{ is the iteration strategy of } \mathcal{M}(a)\}.$$

We have that  $D \in \underline{\Delta}_1^2(\text{Code}(\Sigma))$ .

Let  $\Gamma_1$  be a good pointclass such that  $F^*, \text{Code}(\Sigma), \vec{B}, U, C, D \in \underline{\Delta}_{\Gamma_1}$ . Moreover, it follows from Lemma 2.1 that we can require that for any  $a \in HC$

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1, \Sigma}(a)$$

Let now  $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$  be as in Theorem 1.2.9 of [3] with the property that  $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$  Suslin, co-Suslin captures  $\text{Code}(\Sigma), \vec{B}, U, C, D$  (where Suslin capturing is defined on page 36 of [3]). We have that for any  $\eta < \delta_z$ ,  $C_{\Gamma_1}(\mathcal{N}_z^*|\eta) \in \mathcal{N}_z^*$ .

Let  $\Phi = (\Sigma_1^2)^N$ . We have that  $\Phi$  is a good pointclass. Because  $\vec{B}$  is Suslin captured by  $\mathcal{N}_z^*$ , we have  $(\delta_z^+)^{\mathcal{N}_z^*}$ -complementing trees  $T, S \in \mathcal{N}_z^*$  which capture  $\vec{B}$  in the sense that whenever  $i : \mathcal{N}_z^* \rightarrow \mathcal{N}$  is an iteration embedding according to  $\Sigma_z$  and  $g$  is a generic over  $\mathcal{N}$  for a poset of size  $\leq i(\delta_z)$  then

$$p[i(T)] \cap \mathcal{N}[g] = B \cap \mathcal{N}[g].$$

Let  $\kappa$  be the least cardinal of  $\mathcal{N}_z^*$  which is  $< \delta_z$ -strong in  $\mathcal{N}_z^*$ .

Next, we fix a notation. For each  $a \in HC$ , we let  $\mathcal{W}(a) = Lp^{\Gamma, \Sigma}(a)$ . Using the results of Section 2.11 of [3], we have that if  $g \subseteq Coll(\omega, a)$  is  $\mathcal{W}(a)$ -generic then

$$\mathcal{W}(a)[g] = \mathcal{W}(a, g) = \mathcal{W}(x_g) \quad (1).$$

where  $x_g$  is the generic real coding  $a$ . The following claim is standard.

**Lemma 2.3**  $\mathcal{N}_z^* \models$  “ $\kappa$  is a limit of cardinals  $\eta$  such that  $\eta$  is a Woodin cardinal in  $\mathcal{W}(\mathcal{N}_z^*|\eta)$ ”.

*Proof.* Working in  $\mathcal{N}_z^*$ , let  $\lambda = \delta_z^{++}$  and let  $\pi : M \rightarrow \mathcal{N}_z^*|\lambda$  be an elementary substructure such that

1.  $T, S \in \text{ran}(\pi)$  and
2. letting  $\text{crit}(\pi) = \eta$ ,  $V_\eta^{\mathcal{N}_z^*} \subseteq M$ ,  $\pi(\eta) = \delta_z$  and  $\eta > \kappa$ .

By elementarity, we have that  $M \models$  “ $\eta$  is Woodin”. Letting  $\pi^{-1}(\langle T, S \rangle) = \langle \bar{T}, \bar{S} \rangle$ , we have that  $(\bar{T}, \bar{S})$  Suslin captures  $\vec{B}$  over  $M$  at  $(\eta^+)^M$ . This implies that whenever  $a \in M|(\eta^+)^M$ ,  $\mathcal{W}(a) \in M$ . To see this, first note that we have that whenever  $g \subseteq Coll(\omega, a)$  is  $M$ -generic and  $x_g$  is the generic real then  $\mathcal{W}(x_g) \in M$ . But using (1) above, we have that  $\mathcal{W}(x_g) = \mathcal{W}(a)[g]$ . Therefore,  $\mathcal{W}(a) \in M[g]$ . Since  $g$  was arbitrary, we have that  $\mathcal{W}(a) \in M$ .

We now have that  $\mathcal{W}(\mathcal{N}_z^*|\eta) \in M$  and since  $M \models$  “ $\eta$  is Woodin”, we have that  $\mathcal{W}(\mathcal{N}_z^*|\eta) \models$  “ $\eta$  is Woodin”. Because  $\kappa$  is  $< \delta_z$ -strong in  $\mathcal{N}_z^*$  and because  $a \rightarrow \mathcal{W}(a)$  is definable over  $\mathcal{N}_z^*$ , we have that for unboundedly many  $\nu < \eta$ ,  $\mathcal{W}(\mathcal{N}_z^*|\nu) \models$  “ $\nu$  is Woodin”.  $\square$

## 2.1 A $\Sigma$ -mouse beyond $N$

In this section, we prove the following important lemma.

**Lemma 2.4** *There is a  $\Sigma$ -mouse  $\mathcal{N}$  such that there is a sequence  $(\gamma_i : i < \omega)$  with the property that*

1.  $(\gamma_i : i < \omega)$  is the sequence of Woodin cardinals of  $\mathcal{N}$ ,

2. letting  $\gamma = \sup_{i < \omega} \gamma_i$ ,  $\rho_\omega(\mathcal{N}) < \gamma$ ,
3. for some  $k < \omega$ ,  $\mathcal{N}$  is a sound  $\Sigma$ -mouse over  $\mathcal{N}|\gamma_k$ ,
4. for every cutpoint cardinal  $\eta < \gamma$ ,  $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathcal{W}(\mathcal{N}|\eta)$ , and
5. letting  $\Lambda$  be the  $(\omega, \omega_1, \omega_1)$ -strategy of  $\mathcal{N}$ ,  $\text{Code}(\Lambda) \notin \Gamma$  and  $\Lambda$  is  $\Gamma$ -fullness preserving (i.e., clause 4 holds for any  $\Lambda$ -iterate of  $\mathcal{N}$ ).

We now begin the proof of Lemma 2.4. We continue with previous subsection's notation and start working in  $\mathcal{N}_z^*$ . Our aim is to use fully backgrounded constructions of  $\mathcal{N}_z^*$  to produce a mouse  $\mathcal{N}^*$  such that for some  $l$ ,  $\mathcal{N} = \mathcal{C}_l(\mathcal{N}^*)$  has the desired properties. Let  $\langle \eta_i : i < \omega \rangle$  be the first  $\omega$  cardinals below  $\kappa$  such that for every  $i < \omega$ ,  $\mathcal{W}(\mathcal{N}_z^*|\eta_i) \models$  “ $\eta_i$  is a Woodin cardinal” (it follows from Lemma 2.3 that there are such cardinals). Let now  $\langle \mathcal{N}_i : i < \omega \rangle$  be a sequence constructed according to the following rules:

1.  $\mathcal{N}_0 = (\mathcal{J}^{\vec{E}, \Sigma})^{\mathcal{N}_z^*|\eta_0}$ ,
2.  $\mathcal{N}_{i+1} = (\mathcal{J}^{\vec{E}, \Sigma}[\mathcal{N}_i])^{\mathcal{N}_z^*|\eta_{i+1}}$ .

Let  $\mathcal{N}_\omega = \cup_{i < \omega} \mathcal{N}_i$ .

*Claim 1.* For every  $i < \omega$ ,  $\mathcal{N}_\omega \models$  “ $\eta_i$  is a Woodin cardinal” and  $\mathcal{N}_\omega|(\eta_i^+)^{\mathcal{N}_\omega} = \mathcal{W}(\mathcal{N}_i)$ .

*Proof.* It is enough to show that

1.  $\mathcal{N}_{i+1} \models$  “ $\eta_i$  is a Woodin cardinal”,
2. no level of  $\mathcal{N}_{i+1}$  projects across  $\eta_i$ , and
3.  $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = \mathcal{W}(\mathcal{N}_i)$ .

To show 1-3, it is enough to show that if  $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$  is such that  $\rho_\omega(\mathcal{Q}) \leq \eta_i$  then the fragment of the iteration strategy of  $\mathcal{Q}$  that acts on trees above  $\eta_i$  is coded by a set in  $\Gamma$ . Fix then  $i$  and let  $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$  be such that  $\rho_\omega(\mathcal{Q}) \leq \eta_i$ . Let  $\xi$  be such that if  $\mathcal{S}$  is the  $\xi$ th model of the fully backgrounded construction producing  $\mathcal{N}_{i+1}$  then  $\mathcal{Q}$  is the core of  $\mathcal{S}$ . Let  $\pi : \mathcal{Q} \rightarrow \mathcal{S}$  be the uncollapse map. It is a fine structural map but that is irrelevant and we suppress this point.

Let  $\nu < \eta_{i+1}$  be a cardinal such that  $\mathcal{S}$  is the  $\xi$ th model of the full background construction of  $\mathcal{N}_z^*|\nu$ . Let  $\Psi$  be the fragment of  $\Sigma_z$  that acts on non-dropping trees that are based on  $\mathcal{N}_z^*|(\nu^+)^{\mathcal{N}_z^*}$  and are above  $\eta_i$ . We have that  $\Psi$  induces an iteration

strategy  $\Psi^*$  for  $\mathcal{S}$  and that  $\pi$ -pullback of  $\Psi^*$  is an iteration strategy for  $\mathcal{Q}$ . It is then enough to show that  $Code(\Psi) \in \Gamma$ .

Notice that whenever  $\mathcal{T}$  is a tree on  $\mathcal{N}_z^* | (\nu^+)^{\mathcal{N}_z^*}$  according to  $\Psi$  and  $b = \Psi(\mathcal{T})$  then  $\mathcal{Q}(b, \mathcal{T})$  is defined. Also, notice that because of our choice of  $\eta_{i+1}$ , for any such  $\mathcal{T}$  and  $b$ ,  $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$ . Because the function  $a \rightarrow \mathcal{W}(a)$  is coded by a set in  $\Gamma$ , we have that  $Code(\Psi) \in \Gamma$ . □

*Claim 2.* There is  $\mathcal{Q} \trianglelefteq (\mathcal{J}^{\vec{E}, \Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$  such that  $\rho_\omega(\mathcal{Q}) < \eta_\omega$ .

*Proof.* To see this suppose not. Let  $\mathcal{R} = (\mathcal{J}^{\vec{E}, \Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$ . By universality of  $\mathcal{R}$ , we have that

$$Lp^{\Gamma_1, \Sigma}(\mathcal{N}_\omega) \trianglelefteq \mathcal{R}.$$

It follows from our choice of  $\Gamma_1$  and from our hypothesis that  $Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$ . Notice that if  $\mathcal{M}(a)$  is defined for some  $a$  then because of our choice of  $\Gamma_1$ ,  $\mathcal{M}(a) \trianglelefteq Lp^{\Gamma_1, \Sigma}(a)$ .

We claim that  $\mathcal{M}(\mathcal{N}_0)$  is defined. To see this, notice that  $x$  is generic over  $\mathcal{J}[\mathcal{N}_0]$  for the extender algebra at  $\eta_0$ . Hence, if  $g \subseteq Coll(\omega, \eta_0)$  is  $Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0)$ -generic such that  $x \in Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0)[g]$ , then by the results of Section 2.11 of [3], we have that

$$Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0)[g] = Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0[g]).$$

But now, because  $x \in \mathcal{J}[\mathcal{N}_0][g]$ , we have that  $\mathcal{M}(\mathcal{N}_0[g])$  is defined, and by our choice of  $\Gamma_1$ , we have that  $\mathcal{M}(\mathcal{N}_0[g]) \trianglelefteq Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0[g])$ . Again using the results of Section 2.11 of [3], we have that some initial segment of  $Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0)$  has an iteration strategy which is not coded by a set of reals in  $\Gamma$ . Hence,  $\mathcal{M}(\mathcal{N}_0)$  is defined.

Because  $\mathcal{M}(\mathcal{N}_0)$  is defined, we have that  $\mathcal{M}(\mathcal{N}_0) \trianglelefteq Lp^{\Gamma_1, \Sigma}(\mathcal{N}_0)$  and therefore,  $\mathcal{M}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$ . However, it follows from the proof of Claim 1 that all initial segments of  $\mathcal{N}_\omega$  projecting to  $\eta_0$  have an iteration strategy coded by a set in  $\Gamma$ . This implies that  $\mathcal{M}(\mathcal{N}_0)$  has an iteration strategy coded by a set in  $\Gamma$ , contradiction! □

Let now  $\mathcal{N}^* \trianglelefteq Lp(\mathcal{N}_\omega)$  be least such that  $\rho_\omega(\mathcal{N}^*) < \eta_\omega$ . Let  $l$  be least such that  $\rho_l(\mathcal{N}^*) < \eta_\omega$  and let  $k$  be least such that  $\rho_l(\mathcal{N}^*) < \eta_k$ . In what follows, we will regard  $\mathcal{N}^*$  as a  $\Sigma$ -mouse over  $\mathcal{N}^* | \eta_k$ . We let  $\mathcal{N} = \mathbb{C}_l(\mathcal{N}^*)$ . Thus,  $\mathcal{N}$  is sound (as a  $\Sigma$ -mouse over  $\mathcal{N} | \eta_k$ ). We let  $\langle \gamma_i : i < \omega \rangle$  be the Woodin cardinals of  $\mathcal{N}$  and  $\gamma = \sup_{i < \omega} \gamma_i$ . Let  $\Lambda$  be the  $(\omega, \omega_1, \omega_1)$ -strategy of  $\mathcal{N}$  induced by  $\Sigma_z$ . Notice that  $Code(\Lambda) \notin \Gamma$  because otherwise, since  $\mathcal{N}_\omega$  is  $\Gamma$ -full,  $\mathcal{N} \trianglelefteq \mathcal{N}_\omega \trianglelefteq \mathcal{N}^*$ .

*Claim 3.*  $\Lambda$  is  $\Gamma$ -fullness preserving.

*Proof.* To see this fix  $\mathcal{N}_1$  which is a  $\Lambda$ -iterate of  $\mathcal{N}$  via  $\vec{\mathcal{T}}$  such that the iteration embedding  $i : \mathcal{N} \rightarrow \mathcal{N}_1$  exists. If  $\mathcal{N}_1$  isn't  $\Gamma$ -full then there is a cutpoint  $\nu$  of  $\mathcal{N}_1$  and a sound  $\Sigma$ -mouse  $\mathcal{Q}$  over  $\mathcal{N}_1|\nu$  with  $(\omega, \omega_1)$ -iteration strategy  $\Psi$  such that  $\text{Code}(\Psi) \in \Gamma$ ,  $\rho_\omega(\mathcal{Q}) = \nu$  and  $\mathcal{Q} \not\leq \mathcal{N}_1$ .

*Subclaim.*  $\Psi$  can be extended to an  $(\omega, \omega_1, \omega_1)$ -iteration strategy.

*Proof.* We can find a good pointclass  $\Gamma^*$  such that  $\text{Code}(\Psi) \in \underline{\Delta}_{\Gamma^*}$ . Using Theorem 1.2.9 of [3], we can find  $(\mathcal{N}_y^*, \Sigma_y, \delta_y)$  that Suslin captures  $\text{Code}(\Psi)$ . Notice that  $\Sigma_y$  is an  $(\omega, \omega_1, \omega_1)$ -iteration strategy. It follows from universality that  $\mathcal{Q} \leq (\mathcal{J}^{\vec{E}, \Sigma}[\mathcal{N}_1|\nu])^{\mathcal{N}_y^*|\delta_y}$ . Hence,  $\mathcal{Q}$  has an  $(\omega, \omega_1, \omega_1)$ -iteration strategy  $\Psi^+$ . Because  $\Psi$  is the unique  $(\omega, \omega_1)$ -iteration strategy of  $\mathcal{Q}$ , we have that  $\Psi^+$  extends  $\Psi$ .  $\square$

We now compare  $\mathcal{Q}$  with  $\mathcal{N}_1$ . Let  $\mathcal{S}$  be the comparison tree on the  $\mathcal{Q}$  side with last model  $\mathcal{Q}^*$  and  $\mathcal{T}$  be the comparison tree on the  $\mathcal{N}_1$  side with last model  $\mathcal{N}_1^*$ . Because  $\mathcal{Q} \not\leq \mathcal{N}_1$ , we must have that  $\mathcal{N}_1^* \leq \mathcal{Q}^*$  and  $\pi^{\mathcal{T}} : \mathcal{N}_1 \rightarrow \mathcal{N}_1^*$  exists. Because the  $(\omega, \omega_1)$ -fragment of  $\Lambda$  is the unique  $(\omega, \omega_1)$ -iteration strategy of  $\mathcal{N}$ , we must have that it is the  $\pi^{\mathcal{T}} \circ i$ -pullback of  $\Psi_{\mathcal{N}_1^*, \vec{\mathcal{T}} \restriction \mathcal{S}}$ . This implies that  $\Lambda \in \Gamma$ , contradiction.  $\square$

It is now clear that  $(\mathcal{N}, \Lambda)$  is as desired. This completes the proof of Lemma 2.4.

## 2.2 A Prikry generic

In this subsection, while working in  $N$ , we define a Prikry forcing with the property that the generic object produces a sound countably iterable  $\Sigma$ -mouse  $\mathcal{R}$  over  $\mathbb{R}$  such that  $\mathcal{R} \in N$  and extends  $(Lp^\Sigma(\mathbb{R}))^N$ . Clearly this is a contradiction.

We now start working in  $N$ . We now describe a function  $f : \mathcal{D}^{<\omega} \rightarrow HC$  such that if  $G \subseteq \mathbb{P}^f$  is  $N$ -generic then  $f(G)$  is a  $\Sigma$ -premouse such that certain  $\mathcal{J}^{\vec{E}, \Sigma}$ -construction of it is an initial segment of some  $\Lambda$ -iterate of  $\mathcal{N}$ .

Following [3], we say  $\mathcal{Q}$  is  $\Sigma$ -suitable (in  $N$ ) if for some ordinal  $\delta$

1.  $\delta$  is the unique Woodin cardinal of  $\mathcal{Q}$ ,
2.  $o(\mathcal{Q}) = \sup_{n < \omega} (\delta^{+n})^{\mathcal{Q}}$ ,
3.  $\mathcal{Q}$  is full with respect to  $\Sigma$ -mice, i.e., for any cutpoint  $\eta$ ,  $Lp^\Sigma(\mathcal{Q}|\eta) \leq \mathcal{Q}$ .

We let  $\delta^{\mathcal{Q}}$  be the Woodin cardinal of  $\mathcal{Q}$ . Similarly we can define the notion of a  $\Sigma$ -suitable  $\mathcal{Q}$  over any set  $a$ . In particular, if  $\mathcal{Q}$  is  $\Sigma$ -suitable and  $\mathcal{R}$  is  $\Sigma$ -suitable over

$\mathcal{Q}$  then  $\mathcal{R} \models \text{“}\delta^{\mathcal{Q}} \text{ is a Woodin cardinal”}$ . Because we will only deal with  $\Sigma$ -suitable structures, we omit  $\Sigma$  and just say suitable instead of  $\Sigma$ -suitable.

A normal iteration tree  $\mathcal{U}$  on a suitable  $\mathcal{P}$  is *short* if for all limit  $\xi \leq lh(\mathcal{U})$ ,

$$Lp^{\Sigma}(\mathcal{M}(\mathcal{U}|\xi)) \models \text{“}\delta(\mathcal{U}|\xi) \text{ is not Woodin”}.$$

Otherwise, we say that  $\mathcal{U}$  is maximal. We say that a suitable  $\mathcal{P}$  is *short tree iterable* if for any short tree  $\mathcal{T}$  on  $\mathcal{P}$ , there is a cofinal wellfounded branch  $b$  such that  $\mathcal{Q}(b, \mathcal{T})$  exists and if  $\pi_b^{\mathcal{T}} : \mathcal{P} \rightarrow \mathcal{M}_b^{\mathcal{T}}$  exists then  $\mathcal{M}_b^{\mathcal{T}}$  is suitable.

Write  $\mathcal{P}_y$  for the premouse coded by the real  $y$ . Let  $a$  be countable transitive and  $d \in \mathcal{D}$  be such that  $a$  is coded by a real recursive in  $d$ . Put

$$\mathcal{F}_a^d = \{\mathcal{P}_z : z \leq_T d, \mathcal{P}_z \text{ is a short-tree iterable suitable premouse over } a\}$$

**Lemma 2.5** *For any fixed  $a$ , there is a cone of  $d$  such that  $\mathcal{F}_a^d \neq \emptyset$ .*

*Proof.* If not, the failure of the statement in the claim is a  $\Sigma_1$  statement. Call this statement  $\phi[a]$ . Using  $\Sigma_1^2(\text{Code}(\Sigma))$ -reflection, we get a transitive model

$$H \models ZF^- + \Theta = \Theta_{\text{Code}(\Sigma)} + \phi[a],$$

$\mathbb{R} \subseteq H$  and  $\wp(\mathbb{R}) \cap H \subsetneq \Delta_1^2(\text{Code}(\Sigma))$ .

Let  $\Gamma^*$  be a good pointclass beyond  $H$ . Such a  $\Gamma^*$  exists by our assumption on  $H$ . We use Theorem 1.2.9 of [3] to get a triple  $\langle N_w^*, \delta_w, \Sigma_w \rangle$  (for some real  $w$ ) that Suslin captures the universal  $\Gamma^*$  set. Using universality of fully backgrounded constructions and the proofs of the claims from the proof of Lemma 2.4 (or the results of Section 3.2.2 of [3]), we conclude that the  $(\mathcal{J}^{\vec{E}, \Sigma}[a])^{N_w^*|\delta_w}$  reaches a premouse  $\mathcal{Q}_a$  such that in  $H$ ,  $\mathcal{Q}_a$  is short tree iterable and suitable (with respect to  $H$ ). This contradicts our assumptions on  $H$ .  $\square$

For each  $a$  and for each Turing degree  $d$  from the cone of Lemma 2.5, we can simultaneously compare all  $\mathcal{Q} \in \mathcal{F}_a^d$  while doing the generic genericity iteration to make  $d$  generic over the common part of the final model  $\mathcal{Q}_a^{d,-}$ . This process (hence  $\mathcal{Q}_a^{d,-}$ ) depends only on  $d$ . Set

$$\mathcal{Q}_a^d = Lp_{\omega}^{\Sigma}(\mathcal{Q}_a^{d,-}) \text{ and } \delta_a^d = o(\mathcal{Q}_a^{d,-}).$$

Recall that we are working in  $N$  (thus, we really have that  $\mathcal{Q}_a^d = Lp_{\omega}^{\Gamma, \Sigma}(\mathcal{Q}_a^{d,-})$ ).

**Lemma 2.6** *The following statements are true (in  $N$ ).*

1.  $\mathcal{Q}_a^d$  and  $\delta_a^d$  depend only on  $d$ .

2.  $\mathcal{Q}_a^{d,-}$  is  $\Sigma$ -full (no levels of  $\mathcal{Q}_a^d$  project strictly below  $\delta_a^d$ ).
3.  $\mathcal{Q}_a^d \models \delta_a^d$  is Woodin.
4.  $\wp(a) \cap \mathcal{Q}_a^d = \wp(a) \cap OD_\Sigma(a \cup \{a\})$  and  $\wp(\delta_a^d) \cap \mathcal{Q}_a^d = \wp(\delta_a^d) \cap OD_\Sigma(Q_a^{d,-} \cup \{Q_a^{d,-}\})$ .
5.  $\delta_a^d = \omega_1^{L[S,d]}$ .

*Proof.* 1-4 just follow from our definitions. We consider 5. Let  $S$  be the tree of a  $(\Sigma_1^2(\text{Code}(\Sigma)))^N$  scale on a universal  $(\Sigma_1^2(\text{Code}(\Sigma)))^N$  set  $U$ . Suppose that in  $L[S, d]$ , the process producing  $\mathcal{Q}_a^d$  stops at stage  $\alpha < \omega_1^{L[S,d]}$ . We then have that  $\mathcal{Q}_a^d$  is countable in  $L[S, d]$ . The suitability of  $\mathcal{Q}_a^d$  then implies that  $\mathbb{R} \cap L[S, d] \subseteq \mathcal{Q}_a^d[d]$ . It then follows that  $\delta_a^d$ , the Woodin of  $\mathcal{Q}_a^d$ , is countable in  $L[S, d]$  while it is a cardinal in  $\mathcal{Q}_a^d[d]$  (because the extender algebra of  $\mathcal{Q}_a^d$  at  $\delta_a^d$  is  $\delta_a^d$ -cc). Hence,  $\omega_1^{L[S,d]}$  is countable in  $L[S, d]$ , contradiction!  $\square$

We now define  $f : \mathcal{D}^{<\omega} \rightarrow HC$  by induction on  $\mathcal{D}^n$ . Fix  $(\mathcal{N}, \Lambda)$  as in Lemma 2.4 and let  $k$  be as in clause 3. Below we use the notation of Lemma 2.4. We let  $f(\emptyset) = \mathcal{N} \upharpoonright \gamma_k$ . Suppose we have defined  $f \upharpoonright \mathcal{D}^{n+1}$ . Given  $p \in \mathcal{D}^{n+2}$ , we let

$$f(d) = \begin{cases} \mathcal{Q}_{f(p \upharpoonright n+1)}^d & : f(p \upharpoonright n+1) \text{ is countable in } L[d] \\ \emptyset & : \text{otherwise} \end{cases}$$

Suppose now that  $G \subseteq \mathbb{P}^f$  is  $N$ -generic. Let  $\mathcal{Q}_i = G^i$  and let  $\mathcal{Q}_\omega = f(G)$ . We let  $\delta_i$  be the largest Woodin cardinal of  $\mathcal{Q}_i$ . Without loss of generality, we assume that if  $(\langle d \rangle, X) \in G$  then  $\mathcal{N}$  is countable in  $L[d]$ .

Given an increasing function  $h : \omega \rightarrow \omega$ , we define  $\langle \mathcal{Q}_i^h, \mathcal{Q}_i^{h,*} : i < \omega \rangle$  according to the following procedure:

1.  $\mathcal{Q}_0^{h,*}$  is the output of  $\mathcal{J}^{\vec{E}, \Sigma}[a]$  construction done in  $\mathcal{Q}_{h(0)+1}$  using extenders with critical point  $> \delta_{h(0)}$ .
2.  $\mathcal{Q}_0^h = (Lp^\Sigma(\mathcal{Q}_0^{h,*}))_{\mathcal{Q}_{h(0)+2}}$ .
3.  $\mathcal{Q}_{i+1}^{h,*}$  is the output of  $\mathcal{J}^{\vec{E}, \Sigma}[\mathcal{Q}_i^h]$  construction done in  $\mathcal{Q}_{h(i+1)+1}$  using extenders with critical point  $> \delta_{h(i+1)}$ .
4.  $\mathcal{Q}_{i+1}^h = (Lp^\Sigma(\mathcal{Q}_{i+1}^{h,*}))_{\mathcal{Q}_{h(i+1)+2}}$ .

We let  $\mathcal{Q}_\omega^h = \cup_{i < \omega} \mathcal{Q}_i^h$ .

**Lemma 2.7** *For some increasing function  $h : \omega \rightarrow \omega$  such that  $h \in V$ ,  $\mathcal{Q}_\omega^h$  is an initial segment of a  $\Lambda$ -iterate of  $\mathcal{N}$ .*

*Proof.* Let  $\vec{d} = \langle d_i : i < \omega \rangle$  be the generic sequence of degrees given by  $G$ . We define  $h$  recursively. It will have the property that  $\mathcal{Q}_i^h$  is a  $\Lambda$ -iterate of  $\mathcal{N} | (\gamma_{k+i+1}^+)^{\mathcal{N}}$ . While defining  $h$ , we also define a sequence  $\vec{H} = \langle \mathcal{N}_i, \mathcal{U}_i, b_i : i \in [-1, \omega) \rangle$  such that

1.  $\mathcal{N}_{-1} = \mathcal{N}$ ,
2. for each  $i$ ,  $\mathcal{U}_i$  is an iteration tree on  $\mathcal{N}_i$  and  $b_i = \Lambda(\oplus_{m < i+1} \mathcal{U}_m)$ ,
3. for each  $i$ ,  $\mathcal{N}_{i+1} = \mathcal{M}_{b_i}^{\mathcal{U}_i}$ ,
4. for each  $i$ ,  $\pi_{b_i}^{\mathcal{U}_i}$ -exists and letting  $\pi_{i,j} : \mathcal{N}_i \rightarrow \mathcal{N}_j$  be the composition of iteration embeddings,  $\mathcal{U}_i$  is a tree based on  $\mathcal{N}_i | [\pi_{-1,i}(\gamma_{k+i}), \pi_{-1,i}(\gamma_{k+i+1})]$ ,
5.  $\mathcal{Q}_i^h = \mathcal{N}_i | (\pi_{-1,i}(\gamma_{k+i+1}^+))^{\mathcal{N}}$
6. for each  $i$ ,  $h(i) = m + 1$  where  $m$  is the least integer such that  $\vec{H} \upharpoonright i + 1$  is countable in  $L[d_m]$ .

1-6 above tell us how to define the sequence. To see that we can always arrange 6, recall that  $\vec{d}$  is cofinal in the set of degrees. To see that  $h \in V$ , recall that Prikry property implies that  $\mathbb{P}^f$  doesn't add new reals. To see 5, notice that by our construction,  $\vec{H} \upharpoonright i$  is generic over  $\mathcal{Q}_i$  for the extender algebra at  $\delta_i$ .  $\square$

We let  $h$  be as in Lemma 2.7. We let  $\mathcal{S}_i = \mathcal{Q}_i^h$  and  $\mathcal{S}_\omega = \mathcal{Q}_\omega^h$ . Also, let  $\mathcal{S}$  be the  $\Lambda$ -iterate of  $\mathcal{N}$  such that  $\mathcal{S}_\omega \trianglelefteq \mathcal{S}$ . Because  $\rho(\mathcal{N}) \leq \gamma_k$ , we have that  $\rho(\mathcal{S}) \leq \gamma_k$ . Let  $\langle \eta_n : n < \omega \rangle$  be the Woodin cardinals of  $\mathcal{S}$ . Let  $\eta_\omega = \sup_{n < \omega} \eta_n$ . Notice that in  $V[G]$ ,  $\mathcal{S}_\omega$  is  $(\omega, \omega_1)$ -iterable for short trees.

We now have that there is  $g \subseteq \text{Col}(\omega, < \eta_\omega)$ -generic over  $\mathcal{S}$  such that

$$\bigcup_{n < \omega} \mathbb{R}^{\mathcal{S}[g \cap \eta_n]} = \mathbb{R}.$$

Next we perform an  $S$ -constructions (see Section 2.11 of [3], [7], [8] or [10]) to translate  $\mathcal{S}$  to a  $\Sigma$ -mouse over  $\mathbb{R}$ . To see that the translation procedure works, let  $\lambda = \Theta^{\mathcal{J}_\alpha(U, \mathbb{R})}$ . Notice that  $\mathbb{P}^f \in \mathcal{J}_\lambda(\mathbb{R})$  and that all extenders of  $\mathcal{S}$  above  $\eta_\omega$  have critical point  $> \lambda$ . Thus, we can translate  $\Sigma$ -premise over  $\mathcal{J}_\lambda[\mathcal{S}_\omega]$  to  $\Sigma$ -premise over  $\mathcal{J}_\lambda(\mathbb{R})$ . Let then  $\mathcal{W}$  be the  $\Sigma$ -premouse over  $\mathbb{R}$  that is the result of translating  $\mathcal{S}$  into a  $\Sigma$ -premouse over  $\mathbb{R}$ .

**Lemma 2.8**  $(Lp^\Sigma(\mathbb{R}))^N \triangleleft \mathcal{W}$ .

*Proof.* Suppose  $\mathcal{W} \trianglelefteq (Lp^\Sigma(\mathbb{R}))^N$ . Notice that  $\mathcal{W}$  is  $OD_\Sigma^N$ . Notice that  $\mathcal{N}$  is the  $\delta_k$ -core of  $\mathcal{S}$ . Let then  $\tau$  be a name for a sound  $\Sigma$ -premouse over  $\mathcal{N}|\delta_k$  projecting to  $\delta_k$  such that it is always realized as the  $\delta_k$ -core of the translation of  $\mathcal{W}$  into an extension of  $\mathcal{Q}_\omega^h$ . Then  $\tau$  is  $OD_\Sigma^N$  and hence, there is  $OD_\Sigma^N$  condition  $(\emptyset, X)$  that decides  $\tau$ . It then follows that if  $\mathcal{N}^*$  is the premouse given by  $\tau$  and  $(\emptyset, X)$  then  $\mathcal{N}^* = \mathcal{N}$ . But this implies that  $\mathcal{N} \in OD_{\mathcal{N}|\delta_k, \Sigma}^N$  and hence, by  $N$ -fullness of  $\mathcal{N}$ ,  $\mathcal{N} \in \mathcal{N}$ , contradiction.  $\square$

Let then  $\mathcal{R} \trianglelefteq \mathcal{W}$  be the first level of  $\mathcal{W}$  such that

$$(Lp^\Sigma(\mathbb{R}))^N \trianglelefteq \mathcal{R} \text{ and } \rho_\omega(\mathcal{R}) = \mathbb{R}.$$

The next two lemmas finishes the prove of Theorem 0.1.

**Lemma 2.9**  $\mathcal{R} \in V$ .

*Proof.* Suppose not. Using *DC* we can find  $\pi : H \rightarrow \mathcal{J}_\mu(\wp(\mathbb{R}))$  such that  $\mu > \Theta$  and  $H$  is countable. We can further assume that  $\Sigma, \Lambda, N \in \text{rng}(\pi)$ . Let then  $\bar{N} = \pi^{-1}(N)$ . Let  $g \subseteq \text{Coll}(\omega, \bar{N})$  be  $H$ -generic and let  $g_1, g_2 \subseteq \pi^{-1}(\mathbb{P}^f) \in H[g]$  be two different  $\bar{N}$ -generics. Let  $\mathcal{R}_1$  and  $\mathcal{R}_2$  be the versions of  $\mathcal{R}$  defined for  $\bar{N}$  using  $g_1$  and  $g_2$  respectively. Because both are  $(\omega, \omega_1)$ -iterable, we have that  $\mathcal{R}_1 = \mathcal{R}_2$ . Hence, the version of  $\mathcal{R}$  for  $\bar{N}$  is  $OD_{\Sigma^H}^{H[g]}$  and hence, it is in  $H$ .  $\square$

It remains to show that  $\mathcal{R}$  is in  $N$  and countably iterable in  $N$ . Granted this, we obtain the desired contradiction, hence complete the proof of Theorem 0.1.

**Lemma 2.10**  $\mathcal{R} \in N$  and  $\mathcal{R}$  is countably iterable in  $N$ .

*Proof.* First, we show  $\mathcal{R} \in N$ . We can assume  $o(\mathcal{R})$  is limit and  $\rho_1(\mathcal{R}) = \mathbb{R}$  (if not, look at the mastercode structure of  $\mathcal{R}$ ). In  $V$ , we can write  $\mathcal{R} = \cup_{\xi < o(\mathcal{R})} \text{Th}^{\mathcal{R}|\xi}(\mathbb{R})$ . Notice that for all  $\xi < o(\mathcal{R})$ ,  $|\text{Th}^{\mathcal{R}|\xi}(\mathbb{R})|_w < |A|_w$  (where  $A$  is the least  $OD_\Sigma^N$  set of reals such that  $A \notin Lp^\Sigma(\mathbb{R})^N$ ). Since  $\mathcal{R}$  is a well-ordered union of sets Wadge reducible to  $A$ , it follows from a theorem of Kechris that  $\mathcal{R}$  is projective in  $A$ . This implies that  $\mathcal{R} \in N$ .

It remains to show  $\mathcal{R}$  is countably iterable in  $N$ . Working in  $N$ , given  $\sigma \in \wp_{\omega_1}(\mathbb{R})$  we say  $\sigma$  is bad if there is a non-iterable sound  $\Sigma$ -premouse  $\mathcal{W}$  over  $\sigma$  projecting to  $\sigma$  and an embedding  $\pi : \mathcal{W} \rightarrow \mathcal{R}$ . Notice that (in  $V$ )  $\mathcal{R}$  is countably  $(\omega, \omega_1)$ -iterable and hence, for each  $\sigma \in \wp_{\omega_1}(\mathbb{R})$  there is at most one such  $\mathcal{W}$ . We denote it by  $\mathcal{W}(\sigma)$ .

To show that  $\mathcal{R}$  is countably iterable in  $N$ , it is enough to show that for stationary many  $\sigma$ ,  $\mathcal{W}(\sigma)$  is undefined. Towards a contradiction assume that for a club  $C$  of  $\sigma$ ,  $\mathcal{W}(\sigma)$  is defined. Then the set

$$B = \{(\sigma, \mathcal{W}(\sigma)) : \sigma \in C\}.$$

is  $OD_{\Sigma, u}^N$  for some real  $u$ . It follows that for every  $\sigma \in C$  such that  $u \in \sigma$ ,  $\mathcal{W}(\sigma) \in (Lp^\Sigma(\sigma))^N$ . Because for every  $\sigma$ ,  $\mathcal{W}(\sigma)$  has an  $(\omega, \omega_1)$ -iteration strategy in  $V$ , we get that  $\mathcal{W}(\sigma) \leq (Lp^\Sigma(\sigma))^N$ , which is a contradiction.  $\square$

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