The analysis of HOD below the theory AD^++ "The largest Suslin cardinal is a member of the Solovay sequence" *†

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Abstract

The main contribution of this paper is a the analysis of HOD below the theory AD^+ + "The largest Suslin cardinal is a member of the Solovay sequence". It is shown that below the aforementioned theory, HOD of models of AD^+ are fine structural models and hence, satisfy GCH. We also show that the aforementioned theory is consistent relative to a Woodin cardinal which is a limit of Woodin cardinals. This was stated as an open problem in [6]. The paper is a continuation of [4].

In this paper, we continue the work that has started in [4]. Here our goal is to carry out the analysis of HOD of models of $AD^+ + SMC + V = L(\wp(\mathbb{R}))$ under the following minimality assumption. Given α such that θ_α is defined, we let $\Gamma_\alpha = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. We say Γ is a *Solovay pointclass* if $\Gamma = \Gamma_\alpha$ for some α.

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Recall from [4] that LSA stands for the theory AD^+ + "the largest Suslin cardinal is a member of the Solovay sequence". We let

$\#_{lsa}$**:** There is α such that $\theta_{\alpha+2} \leq \Theta$ and $L(\Gamma_{\alpha+1}) \vDash LSA$.

Minimality Assumption: $\neg\#_{lsa}$ holds.

Recall from [3] that SMC stands for *Strong Mouse Capturing*, i.e., Mouse Capturing relative to any strategy of a hod pair. It is known that SMC is not needed for the computation of HOD as it follows from $AD^+ + V = L(\wp(\mathbb{R}))$ and our Minimality Assumption. However, this is a subject of another paper. This paper is a continuation of [4], and we assume that the reader is familiar with [4].

Our computation of HOD follows the general outline of the computation of HOD that can be found in [5]. The proof is by induction on the Solovay pointclasses. Suppose we have shown that for all $\beta < \alpha$, there is a hod pair $(\mathcal{P}, \Sigma)$ such that

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\beta\} \qquad (*).$$

Then we show that if $\theta_\alpha < \Theta$ then there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. We then use this fact to show that $\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\alpha$ has $V^{\mathrm{HOD}}_{\theta_\alpha}$ as its universe, and then using this, we complete induction by showing that if $\theta_{\alpha+1} < \theta$ then there is $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_{\alpha+1}\}$.

Recall from [3] or from [5] that (*) is known as *Generation of Pointclasses*. To prove it, we need to introduce hod pair constructions that produce hod pairs whose strategies are Γ-fullness preserving. However, for technical reasons, it is convenient to introduce such constructions for arbitrary pointclasses.

1 Γ-hod pair constructions

Fix a pointclass Γ closed under continuous preimages and images. Below if Λ is an iteration strategy then we let M_Λ be the structure that Λ iterates. We use $Code(\Lambda)$ for the set of reals coding $\Lambda \restriction HC$. Also, recall that we say that a is self-wellordered if there is a well ordering of a in $\mathcal{J}_\omega(a)$.

Fix a pointclass Γ. Let

$$HP^\Gamma = \{(\mathcal{P}, \Lambda) : (\mathcal{P}, \Lambda) \text{ is a hod pair and } Code(\Lambda) \in \Gamma\}$$

and

$Mice^\Gamma = \{(a, \Lambda, \mathcal{M}) : a \in HC$, a is self-wellordered transitive set, Λ is an iteration strategy such that $(M_\Lambda, \Lambda) \in HP^\Gamma$, $M_\Lambda \in a$, and $\mathcal{M} \trianglelefteq Lp^{\Gamma,\Lambda}(a)\}$.

Suppose $(\mathcal{P}, \Sigma) \in HP^\Gamma$. We then let

$$Mice^\Gamma_\Sigma = \{(a, \mathcal{M}) : (a, \Sigma, \mathcal{M}) \in Mice^\Gamma\}.$$

Suppose now $A \subseteq \mathbb{R}$ is such that $w(A) \geq w(\Gamma)$. We let A_Γ be the set of reals σ that code a pair (σ_0, σ_1) of continuous functions such that $\sigma_0^{-1}[A]$ codes some $(\mathcal{P}, \Lambda) \in HP^\Gamma$ and $\sigma_1^{-1}[A]$ codes some $(a, \mathcal{M}, \Psi)$ such that $(a, \Lambda, \mathcal{M}) \in Mice^\Gamma$ and Ψ is the unique strategy of $\mathcal{M}$. If $\Gamma = \wp(\mathbb{R})$ then we let $HP = HP^\Gamma$ and $Mice = Mice^\Gamma$.

It is convenient to introduce the notion of Γ-hod pair construction while working inside *self-capturing background triples* (see Definition 2.24 of [3]).

Definition 1.1 (Self-capturing background triples) *Suppose (M, δ, Σ) is a background triple. We say (M, δ, Σ) is self-capturing if for every M-inaccessible cardinal $\lambda < \delta$ and for any $g \subseteq Coll(\omega, \lambda)$, there is a set $X \in M$ such that for every $M[g]$-cardinal η which is countable in V, $(M[g], \Sigma)$ Suslin, co-Suslin captures $Code(\Sigma_{V^M_\lambda})$ at η as witnessed by a pair $(T, S) \in OD^{M[g]}_X$.*

Suppose now that (M, δ, Σ) is a self-capturing background triple such that (M, Σ) Suslin, co-Suslin captures the pair (A_Γ, A). Suppose $B \subseteq \mathbb{R}$ and suppose $\lambda < \delta$ is an M-inaccessible cardinal such that whenever $g \subseteq Coll(\omega, \lambda)$, $(M[g], \Sigma)$ Suslin, co-Suslin captures B. We then write $(M, \lambda, \Sigma) \vDash B \in \Gamma$ if whenever $g \subseteq Coll(\omega, \lambda)$ is M-generic, there is $\sigma \in M[g] \cap A_\Gamma$ such that if h is $M[g]$-generic and (σ_0, σ_1) is the pair coded by σ then, letting $B^* = B \cap M[g * h]$ and $A^* = A \cap M[g * h]$, $M[g * h] \vDash B^* = \sigma_0^{-1}[A^*]$. Notice that the definition of $(M, \lambda, \Sigma) \vDash B \in \Gamma$ depends on the pair (A_Γ, A). In our exposition the pair (A_Γ, A) will always be clear. The following lemma isn't difficult to show.

Lemma 1.2 (Lemma 2.29 of [3]) *Suppose (M, δ, Σ) is a self-capturing background triple such that (M, Σ) Suslin, co-Suslin captures the pair (A_Γ, A). Suppose $B \subseteq \mathbb{R}$ and suppose $\lambda < \delta$ is an M-inaccessible cardinal such that whenever $g \subseteq Coll(\omega, \lambda)$, $(M[g], \Sigma)$ Suslin, co-Suslin captures B. Then $(M, \lambda, \Sigma) \vDash B \in \Gamma$ if and only if $B \in HP^\Gamma$.*

The next lemma shows that mouse operators are definable over self-capturing triples.

Lemma 1.3 (Lemma 2.30 of [3]) *Suppose that (M, δ, Σ) is a self-capturing background triple such that (M, Σ) Suslin, co-Suslin captures (A_Γ, A). Suppose further that $(\mathcal{P}, \Lambda) \in HP^\Gamma$ and $\lambda < \delta$ is an M-inaccessible cardinal such that $(M, \lambda, \Sigma) \vDash Code(\Lambda) \in \Gamma$. Let $F : HC \to HC$ be given by $F(a) = Lp^{\Gamma,\Sigma}(a)$. Suppose $g \subseteq Coll(\omega, \lambda)$ is M-generic and $h \in HC$ is $V_\delta^M[g]$-generic. Then $F \restriction V_\delta^M[g][h]$ is definable over $M[g][h]$ from the pair $(A_\Gamma \cap M[g], A \cap M[g])$ uniformly in h.*

Lemma 1.4 (Lemma 2.31 of [3]) *Assume the hypothesis of Lemma 1.3. Suppose further that the function $\eta \to \Lambda \restriction V_\eta^M$ is definable over M (from parameters) and that there is some set $X \in M$ such that there is an invariant $\tau \in M^{Coll(\omega,\lambda)}$ such that $\tau \in OD_X^M$ and whenever $g \subseteq Coll(\omega, \lambda)$ is M-generic then $\tau_g = \{(x, y) \in \mathbb{R}^2 : x$ codes $\mathcal{P}$ and $y \in Code(\Lambda)\}$. Then $F \restriction V_\delta^M$ is definable over V_δ^M.*

We can now introduce Γ-hod pair constructions. Such constructions are modifications of the hod pair constructions introduced in [4] (see Definition 16.2 of [4]). Below we write $M \vDash B \in \Gamma$ if for some $\lambda < \delta$, $(M, \lambda, \Sigma) \vDash B \in \Gamma$.

Definition 1.5 (Γ-hod pair constructions) *Suppose Γ is a pointclass closed under continuous preimages and images and suppose that $A \subseteq \mathbb{R}$ is such that $w(A) = w(\Gamma)$. Suppose further (M, δ, Σ) is a self-capturing background triple such that M locally Suslin, co-Suslin captures (A_Γ, A). Then*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

is the output of the Γ-hod pair construction of M if it satisfies the following properties (in M).

1. *For all $(\alpha, \xi, \gamma) \in (\lambda+1) \times (\varsigma_\alpha+1) \times (\zeta_{\alpha,\xi}+1)$, $(\mathcal{P}_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma})$ and $(\mathcal{P}^*_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma})$ are hod pairs with the property that*

$$\lambda^{\mathcal{P}_{\alpha,\xi,\gamma}} = \lambda^{\mathcal{P}^*_{\alpha,\xi,\gamma}} = \begin{cases} \alpha + 1 & : \varsigma_\alpha^{\mathcal{P}} > 0 \\ \alpha & : \varsigma_\alpha^{\mathcal{P}} = 0, \end{cases}$$

   and

   (a) *$M \vDash (\mathcal{P}_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}) \in \Gamma$ and $M \vDash (\mathcal{P}^*_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma}) \in \Gamma$,*

   (b) *$\mathcal{P}_{\alpha,\xi,\gamma}(\alpha, 0, 0) = \mathcal{P}^*_{\alpha,\xi,\gamma}(\alpha, 0, 0)$,*

   (c) *if $(\alpha, \xi, \gamma) <_{lex} (\lambda, \varsigma_\alpha, \zeta_{\alpha,\varsigma_\alpha})$ then $\rho(\mathcal{P}^*_{\alpha,\xi,\gamma}) > \delta_\alpha^{\mathcal{P}^*_{\alpha,\xi,\gamma}}$, and*

(d) $\mathcal{P}_{\alpha,\xi,\gamma} = \mathcal{C}(\mathcal{P}^*_{\alpha,\xi,\gamma})$ *and if* $\pi : \mathcal{P}_{\alpha,\xi,\gamma} \to \mathcal{P}^*_{\alpha,\xi,\gamma}$ *is the uncollapse map then* $\Sigma_{\alpha,\xi,\gamma} = \pi$*-pullback of* $\Sigma^*_{\alpha,\xi,\gamma}$.

2. ***The construction of*** $(\mathcal{N}_{\alpha,0,0}, \mathcal{P}_{\alpha,0,0}, \mathcal{P}^*_{\alpha,0,0})$***.*** *We let* $\eta_0 < \eta_1$ *be the first two* M*-cardinals such that for each* $i \in \{0,1\}$, $Lp^\Gamma(V^M_{\eta_i}) \vDash$ "η_i *is a Woodin cardinal", and* $\mathcal{N}_{0,0,0} = (\mathcal{J}^{\vec{E}})^{V^M_{\eta_1}}$.

   (a) *We let* $\eta_0 < \eta_1$ *be the first two* M*-cardinals such that for each* $i \in \{0,1\}$, $Lp^\Gamma(V^M_{\eta_i}) \vDash$ "η_i *is a Woodin cardinal". We then let* $\mathcal{N}_{0,0,0} = (\mathcal{J}^{\vec{E}})^{V^M_{\eta_1}}$. *Next suppose* $\alpha = \theta + 1$. *If* $\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}$ *is of lsa type then stop the construction. Otherwise let*

   $$\nu = \begin{cases} \delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}} & : \alpha \neq 0 \\ 0 & : \textit{otherwise} \end{cases}.$$

   Then if

      i. *no level of* $\mathcal{N}_{\alpha,0,0}$ *projects across* $\delta_\theta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$,

      ii. $\mathcal{N}_{\alpha,0,0}$ *has at least two Woodin cardinals* $> \delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$,

      iii. *if* $\alpha = \theta + 1$ *and* $\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}} \vDash$ "$\delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *is a Woodin cardinal" then*

      $$\mathcal{N}_{\alpha,0,0} \vDash \text{"}\delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}} \textit{ is a Woodin cardinal"}$$

   then $\mathcal{P}^*_{\alpha,0,0} = \mathcal{P}_{\alpha,0,0} = \mathcal{N}_{\alpha,0,0}|(\mu^{+\omega})^{\mathcal{N}_{\alpha,0,0}}$ *where* μ *is the least Woodin cardinal of* $\mathcal{N}_{\alpha,0,0}$ *bigger than* $\delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$.

   (b) *Assume* α *is a limit ordinal. Let* $\mathcal{Q}_\alpha = \cup_{\theta<\alpha}\mathcal{P}_{\theta+1,0,0}$ *and let* Λ *be the strategy of* $\mathcal{Q}_\alpha$ *induced by* Σ. *Let* $\eta_0 < \eta_1$ *be the first two Woodin cardinals of* M *such that for every* $i \in \{0,1\}$, $Lp^{\Gamma,\Lambda}(V^M_{\eta_i}) \vDash$ "η_i *is a Woodin cardinal". Let* $\mathcal{M}_\alpha = (\mathcal{J}^{\vec{E},\Lambda})^{V^M_{\eta_1}}$. *Then the following holds.*

      i. *If no initial segment of* $\mathcal{M}_\alpha$ *projects across* $o(\mathcal{Q}_\alpha)$ *then letting* $\kappa = o(\mathcal{Q}_\alpha)$, $\mathcal{P}^*_{\alpha,0,0} = \mathcal{P}_{\alpha,0,0} = (\kappa^{+\omega})^{\mathcal{M}_\alpha}$ *and* $\Sigma_{\alpha,0,0}$ *be the strategy of* $\mathcal{P}_{\alpha,0,0}$ *induced by* Σ.

      ii. *Provided the above clause holds, let* $\nu_0 < \nu_1$ *be the first two cardinals of* M *such that for every* $i \in \{0,1\}$, $Lp^{\Gamma,\Sigma_{\alpha,0,0}}(V^M_{\nu_i}) \vDash$ "ν_i *is a Woodin cardinal". Let* $\mathcal{N}_\alpha = (\mathcal{J}^{\vec{E},\Sigma_{\alpha,0,0}})^{V^M_{\nu_1}}$. *If*

      $$(\kappa^+)^{\mathcal{N}_\alpha} = (\kappa^+)^{\mathcal{P}_{\alpha,0,0}} \textit{ and } \mathcal{N}_\alpha|(\kappa^+)^{\mathcal{N}_\alpha} = \mathcal{P}_{\alpha,0,0}|(\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$$

      then $\mathcal{N}_\alpha = \mathcal{N}_{\alpha,0,0}$.

(c) ***The terminating condition.*** *Below we make a list of the conditions described above that will force the construction to stop. Stop the construction if any of the following happens.*

   i. α *is a successor ordinal and there is a level of* $\mathcal{N}_{\alpha,0,0}$ *projecting across* $\delta_\theta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *where* $\theta = \alpha - 1$.

   ii. $\alpha = 0$ *or* $\alpha = \theta + 1$ *is a successor ordinal and* $\mathcal{N}_{\alpha,0,0}$ *doesn't have at least two Woodin cardinals* $> \nu$ *where* $\nu = 0$ *if* $\alpha = 0$ *and* $\nu = \delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *otherwise.*

   iii. $\alpha = \theta + 1$, $\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}} \vDash$ "$\delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *is a Woodin cardinal" and* $\mathcal{N}_{\alpha,0,0} \vDash$ "$\delta^{\mathcal{P}_{\theta,\varsigma_\theta,\zeta_{\theta,\varsigma_\theta}}}$ *is not a Woodin cardinal".*

   iv. α *is a limit ordinal and some initial segment of* $\mathcal{M}_\alpha$ *projects across* $o(\mathcal{Q}_\alpha)$ *where* $\mathcal{M}_\alpha$ *and* $\mathcal{Q}_\alpha$ *are as in clause 2.b.i.*

   v. α *is a limit ordinal and the above clause fails, but either* $(\kappa^+)^{\mathcal{N}_\alpha} \neq (\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$ *or* $\mathcal{N}_\alpha|(\kappa^+)^{\mathcal{N}_\alpha} \neq \mathcal{P}_{\alpha,0,0}|(\kappa^+)^{\mathcal{P}_{\alpha,0,0}}$ *where* κ *and* $\mathcal{N}_\alpha$ *are as in clause 2.b.i and 2.b.ii.*

3. ***The construction of*** $(\mathcal{N}_{\alpha,\xi,0}, \mathcal{P}_{\alpha,\xi,0}, \mathcal{P}^*_{\alpha,\xi,0})$ ***for fixed*** α ***and*** $\xi > 0$***.*** *Suppose* $\alpha \in Ord$ *is fixed. We define* $(\mathcal{N}_{\alpha,\xi,0}, \mathcal{P}_{\alpha,\xi,0}, \mathcal{P}^*_{\alpha,\xi,0} : \xi > 0)$ *by induction on* ξ. *Suppose* $(\alpha, \xi) \in Ord^2$ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*) <_{lex} (\alpha,\xi) \wedge \gamma^* \leq \zeta_{\alpha^*,\xi^*})$$

has been defined.

(a) *Suppose* ξ *is limit. Let* $(\mathcal{Q}_{\alpha,\xi,i}, \mathcal{Q}^*_{\alpha,\xi,i} : i \leq \nu_{\alpha,\xi})$ *be a sequence defined as follows.*

   i. $\mathcal{Q}^*_{\alpha,\xi,0} = lim_{\xi^* \to \xi} \mathcal{P}_{\alpha,\xi^*,\zeta_{\alpha,\xi^*}}$,

   ii. *for all* $i \leq \nu_{\alpha,\xi}$, $\mathcal{Q}_{\alpha,\xi,i} = \mathcal{C}(\mathcal{Q}^*_{\alpha,\xi,i})$,

   iii. *for all* $i < \nu_{\alpha,\xi}$, $\mathcal{Q}^*_{\alpha,\xi,i+1}$ *is the least initial segment* $\mathcal{M}$ *of* $\mathcal{J}(\mathcal{Q}_{\alpha,\xi,i})$, *if it exists, such that* $\rho(\mathcal{M}) < o(\mathcal{Q}_{\alpha,\xi,i})$,

   iv. *for all limit* $i \leq \nu_{\alpha,\xi}$, $\mathcal{Q}_{\alpha,\xi,i} = lim_{j \to i} \mathcal{Q}_{\alpha,\xi,j}$,

   v. $\nu_{\alpha,\xi}$ *is the least ordinal* β *such that either*

      A. $\mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *has no level projecting across* $\mathcal{Q}_{\alpha,\xi,\beta}$ *or*

      B. *there is* $\mathcal{M} \lhd \mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

Suppose that clause B doesn't hold. Let then $\mathcal{P}^*_{\alpha,\xi,0} = (\mathcal{Q}_{\alpha,\xi,\nu_{\alpha,\xi}})^\#$, *i.e.* $\mathcal{P}^*_{\alpha,\xi,0}$ *is the least active level of* $\mathcal{J}^{\vec{E}}[\mathcal{Q}_{\alpha,\xi,\nu_{\alpha,\xi}}]$. *We also let* $\Sigma^*_{\alpha,\xi,0}$ *be the strategy of* $\mathcal{P}^*_{\alpha,\xi,0}$ *induced by* Σ. *Suppose* $\rho(\mathcal{P}^*_{\alpha,\xi,0}) > \delta^{\mathcal{P}_{\alpha,0,0}}$. *Then let* $\eta_0 < \eta_1$ *be the first two cardinals of* M *such that for* $i \in \{0,1\}$, *if* $\mathcal{P}_{\alpha,\xi,0}$ *is not of lsa type then* $Lp^{\Gamma,\Sigma_{\alpha,\xi,0}}(V^M_{\eta_i}) \vDash$ "η_i *is a Woodin cardinal", and if* $\mathcal{P}_{\alpha,\xi,0}$ *is of lsa type then* $Lp^{\Gamma,\Sigma^{stc}_{\alpha,\xi,0}}(V^M_{\eta_i}) \vDash$ "η_i *is a Woodin cardinal". Let then*

$$\mathcal{N}_{\alpha,\xi,0} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,0}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,0} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,0}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,0} \text{ is of lsa type.} \end{cases}$$

(b) *Suppose* $\xi = \xi^* + 1$.

i. *Suppose there is an extender* F *that coheres* Σ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*) <_{lex} (\alpha,\xi) \wedge \gamma^* \leq \zeta_{\alpha^*,\xi^*})$$

such that letting $E = F \cap \mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}$, $(\mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}, \tilde{E}, \in)$ *is a hod premouse where* $\tilde{E}$ *is the amenable code of* E. *Then* $\mathcal{P}^*_{\alpha,\xi,0} = (\mathcal{P}_{\alpha,\xi,\varsigma_{\alpha,\xi}}, \tilde{E}, \in)$.

ii. *Suppose there is no such* F. *Then stop the construction of* $\mathcal{P}_{\alpha,\xi,0}$ *and let* $\varsigma_\alpha = \xi^*$.

(c) ***The terminating condition.*** *Below we make a list of the conditions described above that will force the construction to stop. Stop the construction if any of the following happens.*

i. *Clause 3.a.v.B holds, i.e., there is* $\mathcal{M} \lhd \mathcal{J}(\mathcal{Q}_{\alpha,\xi,\beta})$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

ii. $\mathcal{P}^*_{\alpha,\xi,0}$ *is defined but* $\rho(\mathcal{P}^*_{\alpha,\xi,0}) \leq \delta^{\mathcal{P}_{\alpha,0,0}}$.

4. ***The description of*** $(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma})$ ***for fixed*** (α,ξ). *Fix* $(\alpha,\xi) \in Ord^2$. *We define* $(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma})$ *by induction. Suppose* $(\alpha,\xi,\gamma) \in Ord^3$ *and*

$$(\mathcal{N}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}_{\alpha^*,\xi^*,\gamma^*}, \mathcal{P}^*_{\alpha^*,\xi^*,\gamma^*}, F_{\alpha^*,\xi^*,\gamma^*}, \Sigma_{\alpha^*,\xi^*,\gamma^*}, \Sigma^*_{\alpha^*,\xi^*,\gamma^*} : (\alpha^*,\xi^*,\gamma^*) <_{lex} (\alpha,\xi,\gamma))$$

has been defined.

(a) *Suppose* γ *is a limit ordinal or is* 0. *Then we let* $\mathcal{P}^*_{\alpha,\xi,\gamma} = lim_{\gamma^* \to \gamma} \mathcal{P}_{\alpha,\xi,\gamma^*}$. *Also, let* $\Sigma^*_{\alpha,\xi,\gamma}$ *be the strategy of* $\mathcal{P}^*_{\alpha,\xi,\gamma}$ *induced by* Σ. *Let* $\eta_0 < \eta_1$ *be the first two cardinals of* M *such that for* $i \in \{0,1\}$, *if* $\mathcal{P}_{\alpha,\xi,0}$ *is not of lsa type* *then* $Lp^{\Gamma,\Sigma_{\alpha,\xi,0}}(V^M_{\eta_i}) \models$ *"*η_i *is a Woodin cardinal", and if* $\mathcal{P}_{\alpha,\xi,0}$ *is of lsa type then* $Lp^{\Gamma,\Sigma^{stc}_{\alpha,\xi,0}}(V^M_{\eta_i}) \models$ *"*η_i *is a Woodin cardinal". Then*

$$\mathcal{N}_{\alpha,\xi,\gamma} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,\gamma}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,\gamma} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,\gamma}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,\gamma} \text{ is of lsa type.} \end{cases}$$

We then let $\mathcal{P}^*_{\alpha,\xi,\gamma+1}$ *be the first level of* $\mathcal{N}_{\alpha,\xi,\gamma}$*, if exists, that projects across* $o(\mathcal{P}_{\alpha,\xi,\gamma})$*. If there isn't such a level then stop the construction.*

(b) *Suppose* $\gamma = \gamma^* + 1$*. Let* $\eta_0 < \eta_1$ *be the first two cardinals of* M *such that for* $i \in \{0, 1\}$*, if* $\mathcal{P}_{\alpha,\xi,0}$ *is not of lsa type then* $Lp^{\Gamma,\Sigma_{\alpha,\xi,0}}(V^M_{\eta_i}) \models$ *"*η_i *is a Woodin cardinal", and if* $\mathcal{P}_{\alpha,\xi,0}$ *is of lsa type then* $Lp^{\Gamma,\Sigma^{stc}_{\alpha,\xi,0}}(V^M_{\eta_i}) \models$ *"*η_i *is a Woodin cardinal". Then*

$$\mathcal{N}_{\alpha,\xi,\gamma} = \begin{cases} (\mathcal{J}^{\vec{E},\Sigma_{\alpha,\xi,\gamma^*}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,\gamma^*} \text{ is not of lsa type} \\ (\mathcal{J}^{\vec{E},\Sigma^{stc}_{\alpha,\xi,\gamma^*}})^{V^M_{\eta_1}} & : \mathcal{P}_{\alpha,\xi,\gamma^*} \text{ is of lsa type.} \end{cases}$$

(c) ***The terminating condition.*** *Terminate the construction if*

i. *there is a* $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha,\xi,\gamma}$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}^b_{\alpha,\xi,0}}$ *or*

ii. $o(\mathcal{P}^*_{\alpha,\xi,\gamma}) = \delta$.

(d) *If* γ *is a limit ordinal and* $\mathcal{N}_{\alpha,\xi,\gamma}$ *is defined but there is no* $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha,\xi,\gamma}$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,\xi,\gamma}}$*. Set* $\zeta_{\alpha,\xi} = \gamma$.

(e) *If* $\gamma = \gamma^* + 1$ *and* $\mathcal{N}_{\alpha,\xi,\gamma}$ *is defined but there is no* $\mathcal{M} \trianglelefteq \mathcal{N}_{\alpha,\xi,\gamma}$ *such that* $\rho(\mathcal{M}) \leq \delta^{\mathcal{P}_{\alpha,\xi,\gamma^*}}$*. Set* $\zeta_{\alpha,\xi} = \gamma^*$.

5. ***The main termination condition*** *If the construction reaches* (α, ξ) *such that* $\mathcal{P}_{\alpha,\xi,\zeta_{\alpha,\xi}}$ *is of lsa type then stop the construction.*

The next theorem, which is the generalization of the equivalent theorem in [4], shows that Γ-hod pair constructions produce winning strategies for II in the undropping game (see Definition 7.1 of [4]).

Theorem 1.6 (Theorem 17.3 of [4]) *Suppose* Γ *is a pointclass closed under continuous preimages and images and suppose that* $A \subseteq \mathbb{R}$ *is such that* $w(A) = w(\Gamma)$*. Suppose further* (M, δ, Σ) *is a self-capturing background triple such that* M *locally Suslin, co-Suslin captures* (A_Γ, A)*. Let*

$$(\mathcal{N}_{\alpha,\xi,\gamma}, \mathcal{P}_{\alpha,\xi,\gamma}, \mathcal{P}^*_{\alpha,\xi,\gamma}, F_{\alpha,\xi,\gamma}, \Sigma_{\alpha,\xi,\gamma}, \Sigma^*_{\alpha,\xi,\gamma} : \alpha \leq \lambda, \xi \leq \varsigma_\alpha, \gamma \leq \zeta_{\alpha,\xi})$$

be the output of the Γ*-hod pair construction of* M*. Then for every* (α, ξ, γ) *such that* $\Sigma_{\alpha,\xi,\gamma}$ *is defined,* $\Sigma_{\alpha,\xi,\gamma}$ *is a winning strategy for* II *in* $\mathcal{G}^u(\mathcal{P}_{\alpha,\xi,\gamma}, \omega_1, \omega_1, \omega_1)$.

Just like in [4], the notion of Γ-hod pair construction leads to comparison of two Γ-fullness preserving hod pairs. We state the result without the proof as it can be proved using the same proof as the corresponding result from [4] (see Theorem 18.11 and Corollary 18.12 of [4]).

Corollary 1.7 (Comparison) *Assume AD^+ and suppose Γ is a pointclass closed under continuous images and preimages. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that both Σ and Λ are Γ-fullness preserving and have branch condensation. Then the normal comparison[1] for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ holds.*

Recall that the proof of Corollary 1.7 is by first choosing a self-capturing background triple $(\mathcal{M}, \delta, \Sigma)$ that Suslin, co-Suslin captures some pair (A, A_Γ) and then showing that the Γ-hod pair construction of M produces a common iterate of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$.

2 The internal theory of hod mice

In this section we generalize the result of Section 3.1 of [3] to our current context. As in [3], these results lead towards showing that given a hod pair $(\mathcal{P}, \Sigma)$, $\Gamma(\mathcal{P}, \Sigma)$ is an OD-full pointclass (see Definition 3.16 of [3]).

2.1 The uniqueness of the internal strategy

The first theorem, Theorem 2.2, is just a direct generalization of Theorem 3.3 of [3]. It says that the internal strategies are unique. First we prove a useful lemma.

Lemma 2.1 *Suppose $\mathcal{P}$ is a hod premouse and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{P}} \times (\varsigma_\alpha^{\mathcal{P}} + 1) \times (\zeta_{\alpha,\xi}^{\mathcal{P}} + 1)$. Suppose further that if $\alpha + 1 = \lambda^{\mathcal{P}}$ then $(\xi, \gamma) \in \varsigma_\alpha^{\mathcal{P}} \times (\zeta_{\alpha,\xi}^{\mathcal{P}} + 1)$. Suppose $\vec{\mathcal{U}} \in \mathcal{P}$ is a stack on $\mathcal{P}(\alpha, \xi, \gamma)$ and suppose $\mathcal{R}$ is its last model. Then for all $\nu + 1 \leq \lambda^{\mathcal{R}}$ such that $\mathcal{R} \vDash$ "$\delta_{\nu+1}^{\mathcal{R}}$ is a Woodin cardinal", $\mathrm{cf}^{\mathcal{P}}(\delta_{\nu+1}^{\mathcal{R}}) > \omega$.*

Proof. Towards a contradiction, assume not. Let $(\mathcal{N}_\alpha, \vec{\mathcal{U}}_\alpha, E_\alpha : \alpha \leq \eta)$ be the components of $\vec{\mathcal{U}}$. Without loss of generality we can assume that for every cutpoint $\mathcal{S}$ of $\vec{\mathcal{U}}$, $\vec{\mathcal{U}}_{\leq \mathcal{S}}$ is not a counterexample to our claim.

Let $\mathcal{S}$ be the least model in $\vec{\mathcal{U}}$ such that $\pi_{\mathcal{S},\mathcal{R}}^{\vec{\mathcal{U}}}$ exists and $\delta_{\nu+1}^{\mathcal{R}} \in rng(\pi_{\mathcal{S},\mathcal{R}}^{\vec{\mathcal{U}}})$. It follows that there is $\mathcal{N}$ in $\vec{\mathcal{U}}$ such that for some extender F in $\vec{\mathcal{U}}$, F is applied to $\mathcal{M}$

[1] Recall that this means that the comparison can be achieved via a normal tree.

and $\mathcal{S} = Ult(\mathcal{N}, F)$. Let μ be such that $\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}}(\mu) = \nu$. Since $\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}}$ is cofinal at $\delta^{\mathcal{S}}_{\mu+1}$, we have that $\mathrm{cf}^{\mathcal{P}}(\delta^{\mathcal{S}}_{\mu+1}) = \omega$.

Let E be the undropping extender of $\vec{\mathcal{U}}_{\leq \mathcal{M}}$ and let $\mathcal{M}^+ = Ult(\mathcal{P}, E)$. Hence, there is a sequence $(h_i : i < \omega) \in \mathcal{P}$ such that for some $(a_i : i < \omega) \in (\nu_F^{<\omega})^\omega$,

$$\sup_{i<\omega} \pi^{\mathcal{M}}_F(\pi_E(h_i))(a_i) = \delta^{\mathcal{S}}_{\mu+1}.$$

But $(\pi^{\mathcal{M}^+}_F(\pi_E(h_i)) : i < \omega) \in Ult(\mathcal{M}, F)$. Hence,

$$Ult(\mathcal{M}^+, F) \vDash \delta^{\mathcal{S}}_{\mu+1} = \sup_{a \in \nu_F^{<\omega}, i<\omega} \pi^{\mathcal{M}^+}_F(\pi_E(h_i))(a)$$

implying that $Ult(\mathcal{M}^+, F) \vDash \mathrm{cf}(\delta^{\mathcal{S}}_{\mu+1}) \leq \nu_F$ contradicting the fact that $Ult(\mathcal{M}^+, F) \vDash$ "$\delta^{\mathcal{S}}_{\mu+1}$ is a Woodin cardinal". □

Theorem 2.2 (Uniqueness of internal strategies) *Suppose $\mathcal{P}$ is a hod premouse and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{P}} \times (\varsigma^{\mathcal{P}}_\alpha + 1) \times (\zeta^{\mathcal{P}}_{\alpha,\xi} + 1)$. Suppose further that if $\alpha + 1 = \lambda^{\mathcal{P}}$ then $(\xi, \gamma) \in \varsigma^{\mathcal{P}}_\alpha \times (\zeta^{\mathcal{P}}_{\alpha,\xi} + 1)$. Then $\mathcal{P} \vDash$ "$\mathcal{P}(\alpha, \xi, \gamma)$ has a unique iteration strategy".*

Proof. It follows from the proof of Lemma 2.1 that whenever $\vec{\mathcal{U}} \in \mathcal{P}$ is a stack on $\mathcal{P}(\alpha, \xi, \gamma)$ with last model $\mathcal{R}$ and $\beta + 1 \leq \lambda^{\mathcal{R}}$ then $\mathrm{cf}^{\mathcal{P}}(\delta^{\mathcal{R}}_{\beta+1}) > \omega$. Uniqueness of the strategy is an easy consequence of this fact. To show it, we start working in $\mathcal{P}$.

Suppose $\Lambda \neq \Sigma_{\alpha,\xi,\gamma}$ is another iteration strategy for $\mathcal{P}_{\alpha,\xi,\gamma}$. Since $\mathcal{P}(\alpha, \xi, \gamma)$ is not of lsa type, it follows from Lemma 18.5 of [4] that we can fix $\vec{\mathcal{T}}$ on $\mathcal{P}(\alpha, \xi, \gamma)$ which constitutes a minimal low-level disagreement between Λ and $\Sigma_{\alpha,\xi,\gamma}$. Let $b = \Sigma_{\alpha,\xi,\gamma}(\vec{\mathcal{T}})$ and $c = \Lambda(\vec{\mathcal{T}})$. Let $\mathcal{Q}$ be a cutpoint of $\vec{\mathcal{T}}$ such that $\vec{\mathcal{T}}_{\leq \mathcal{Q}}$ is a continuable stack on $\mathcal{P}(\alpha, \xi, \gamma)$ (see Definition 5.4 of [4]) and $\mathcal{U} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{Q}}$ is an irreducible normal tree on $\mathcal{Q}$. Let $\nu + 1 \leq \lambda^{\mathcal{Q}}$ be least such that $\mathcal{U}$ is based on $\mathcal{Q}(\nu+1, 0, 0)$. We then have that

(1) $(\Sigma_{\alpha,\xi,\gamma})_{\mathcal{Q}(\nu),\vec{\mathcal{T}}_{\leq \mathcal{Q}}} = \Lambda_{\mathcal{Q}(\nu),\vec{\mathcal{T}}_{\leq \mathcal{Q}}}$ and that $\mathcal{U}$ is above $\delta^{\mathcal{Q}}_\nu$.

It then follows that $\mathrm{cf}(\delta(\mathcal{T})) = \omega$. Notice that because of (1) it cannot be the case that

$$\delta(\mathcal{T}) < min(\pi^{\mathcal{U}}_b(\delta^{\mathcal{Q}}_{\nu+1}), \pi^{\mathcal{U}}_c(\delta^{\mathcal{Q}}_{\nu+1}))$$

as in that case, both $\mathcal{Q}(\beta, \mathcal{U})$ and $\mathcal{Q}(c, \mathcal{U})$ exist and are fully iterable, and hence the same, implying that $b = c$. It then follows from (1) that $\mathrm{cf}(\delta^{\mathcal{Q}}_{\nu+1}) = \omega$ contradicting Lemma 2.1. □

The proof of Theorem 2.2 can be used in the context of lsa hod premice as well. We will state this result after proving the fullness preservation of the internal strategies. Essentially the internal short tree strategy is the unique short tree strategy which is internally fullness preserving. For now, we state the following corollary of the proof of Theorem 2.2.

Corollary 2.3 *Suppose $\mathcal{P}$ is an lsa type hod premouse and Λ is its internal short tree strategy. Suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Lambda)$ and $\beta + 1 < \lambda^{\mathcal{Q}}$. Then $\mathcal{P} \vDash$ "$\Lambda_{\mathcal{Q}(\beta+1), \vec{\mathcal{T}}}$ is the unique strategy of $\mathcal{Q}(\beta + 1)$".*

2.2 Generic interpretability

We now move to generic interpretability. We start by recalling and generalizing the definition of a pre-hod pair.

Definition 2.4 (Prehod pair) *$(\mathcal{P}, \Sigma)$ is a prehod pair if*

1. *$\mathcal{P}$ is a countable hod premouse,*
2. *$\lambda^{\mathcal{P}}$ is a successor but $\mathcal{P}$ is not of lsa type,*
3. *Σ is an $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ acting on stacks based on $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$ such that $(\mathcal{P}(\lambda^{\mathcal{P}} - 1), \Sigma)$ is a hod pair and that whenever $i : \mathcal{P} \to \mathcal{Q}$ comes from an iteration according to Σ, $\Sigma^{\mathcal{Q}}_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)} = \Sigma \restriction \mathcal{Q}$,*
4. *$\mathcal{P}$ is a Σ-mouse over $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$,*
5. *for any $\mathcal{P}$-cardinal $\eta \in (\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \delta^{\mathcal{P}}_{\lambda})$, considering $\mathcal{P}|\eta$ as a Σ-mouse over $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$, there is an ω_1-strategy Λ for $\mathcal{P}|\eta$.*

Notice that there must be a unique strategy Λ as in 5 of Definition 2.4. Also, recall the definition of Generic Interpretability from [3] (Definition 3.8). In our current context it takes the following form.

Definition 2.5 (Generic Interpretability) *Suppose $(\mathcal{P}, \Sigma)$ is a pre-hod pair or a hod pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal. We say generic interpretability holds for $(\mathcal{P}, \Sigma)$ if there is a function F such that*

1. *F is definable over $\mathcal{P}$ with no parameters,*
2. *dom(F) consists of triples (α, ξ, κ) such that $\alpha < \lambda^{\mathcal{P}}$, $\xi < \varsigma^{\mathcal{P}}_{\alpha}$, $\mu^{\mathcal{P}}_{\alpha,\xi,\zeta^{\mathcal{P}}_{\alpha,\xi}} < \delta^{\mathcal{P}}$ and $\kappa \in (\mu^{\mathcal{P}}_{\alpha,\xi,\zeta^{\mathcal{P}}_{\alpha,\xi}}, \delta^{\mathcal{P}})$ is a $\mathcal{P}$-cardinal,*

3. *for* $(\alpha, \xi, \kappa) \in dom(F)$*,* $F(\alpha, \xi, \kappa) = (\dot{T}, \dot{S})$ *such that letting* $\mu = \mu^{\mathcal{P}}_{\alpha,\xi,\zeta^{\mathcal{P}}_{\alpha,\xi}}$*,*

   (a) $\dot{T}, \dot{S} \in \mathcal{P}^{Coll(\omega,\mu)}$*,*

   (b) $\mathcal{P} \vDash$ *"* $\Vdash_{Coll(\omega,\mu)} \dot{T}$ *and* $\dot{S}$ *are* κ*-complementing",*

   (c) *for any* $\nu \in (\mu, \kappa)$ *and any* $\mathcal{P}$*-generic* $g \subseteq Coll(\omega, \nu)$*,*

$$\mathcal{P}[g] \vDash \text{“}p[\dot{T}_g] \textit{ is an } (\omega_1, \omega_1, \omega_1)\textit{-iteration strategy for } \mathcal{P}(\alpha, \xi) \textit{ which extends } \Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha,\xi)}\text{”}$$

   and

$$(p[\dot{T}_g])^{\mathcal{P}[g]} = \Sigma_{\mathcal{P}(\alpha,\xi)} \restriction HC^{\mathcal{P}[g]}.$$

The proof that the generic interpretability holds is just like the proof of Theorem 3.10 of [3] using Theorem 18.3 of [4] and Theorem 2.2 instead of Theorem 3.3 and Lemma 2.15 of [3]. First the proof of Lemma 3.9 of [3] can be used with no changes to establish the following useful lemma.

Lemma 2.6 *Suppose* $(\mathcal{P}, \Sigma)$ *is a prehod pair and* $\alpha + 1 = \lambda^{\mathcal{P}}$*. Let* $\kappa < \delta^{\mathcal{P}}$ *be a* $\mathcal{P}$*-cardinal such that* $\mathcal{P}$ *has no extenders with critical point* $\delta^{\mathcal{P}}_\alpha$ *and index greater than* κ*. Let* Λ^* *be the iteration strategy of* $\mathcal{P}|\kappa$ *as in 5 of Definition 2.4. Let* Λ *be the fragment of* Λ^* *that acts on non-dropping stacks. Let* $g \subseteq Coll(\omega, \kappa)$ *be* $\mathcal{P}$*-generic. Then* $\mathcal{P}[g]$ *locally Suslin captures* $Code(\Lambda)$ *and its complement at any cardinal of* $\mathcal{P}$ *greater than* κ*.*

Fix now a prehod pair $(\mathcal{P}, \Sigma)$ and let $\alpha + 1 \leq \lambda^{\mathcal{P}}$. Let $\kappa < \delta^{\mathcal{P}}$ be a $\mathcal{P}$-cardinal such that $\mathcal{P}$ has no extenders with critical point $\delta^{\mathcal{P}}_\alpha$ and index greater than κ. Fix $\xi \leq \varsigma^{\mathcal{P}}_\alpha$ and let

$$(\mathcal{N}_{\beta,\nu,\zeta}, \mathcal{P}_{\beta,\nu,\zeta}, \mathcal{P}^*_{\beta,\nu,\zeta}, F_{\beta,\nu,\zeta}, \Phi_{\alpha,\nu,\zeta}, \Phi^*_{\alpha,\nu,\zeta} : \beta \leq \lambda, \nu \leq \varsigma_\alpha, \zeta \leq \varsigma_{\alpha,\nu})$$

be the output of hod pair construction of $\mathcal{P}|\delta^{\mathcal{P}}$ in which extenders used have critical point $> \kappa$. It follows from Theorem 18.11 of [4], Lemma 2.2 and Lemma 2.6 that for some (α, ν, ζ), $(\mathcal{N}_{\beta,\nu,\zeta}, \Phi_{\beta,\nu,\zeta})$ is a tail of $(\mathcal{P}(\alpha, \xi), \Sigma_{\alpha,\xi})$. We then set

$$\mathcal{N}^{\mathcal{P}}_{\kappa,\alpha,\xi} = \mathcal{N}_{\beta,\nu,\zeta} \text{ and } \Lambda_{\kappa,\alpha,\xi} = \Phi_{\beta,\nu,\zeta}.$$

Also let $\pi^{\mathcal{P}}_{\kappa,\alpha,\xi} : \mathcal{P}(\alpha, \xi) \to \mathcal{N}^{\mathcal{P}}_{\kappa,\alpha,\xi}$ be the iteration embedding according to $\Sigma_{\alpha,\xi}$ and let $\mathcal{T}_{\kappa,\alpha,\xi}$ be the tree on $\mathcal{P}(\alpha, \xi)$ with last model $\mathcal{N}^{\mathcal{P}}_{\kappa,\alpha,\xi}$. It then follows from Lemma 2.6, hull condensation of Σ and the proof of Theorem 18.33 of [4] that

Corollary 2.7 *whenever $\eta \in (\kappa, \delta^{\mathcal{P}})$ is such that $\eta > o(\mathcal{N}^{\mathcal{P}}_{\kappa,\beta,\xi})$ and $n < \omega$, there are names $(\dot{T}, \dot{S}) \in \mathcal{P}^{Coll(\omega,\eta)}$, such that*

1. *$\dot{T}, \dot{S} \in \mathcal{P}^{Coll(\omega,\eta)}$,*
2. *$\mathcal{P} \models$ "$\Vdash_{Coll(\omega,\mu^{\mathcal{P}}_{\beta,\xi,\gamma})} \dot{T}$ and $\dot{S}$ are $(\delta^{\mathcal{P}})^{+n}$-complementing",*
3. *for any $\lambda < (\eta, ((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}})$ and any $\mathcal{P}$-generic $g \subseteq Coll(\omega, \lambda)$,*

$$\mathcal{P}[g] \models \text{“}p[\dot{T}_g] \text{ is an } (\omega_1, \omega_1, \omega_1)\text{-iteration strategy for } \mathcal{N}_{\kappa,\alpha,\xi}\text{”}$$

and letting Φ be the $\pi^{\mathcal{P}}_{\kappa,\alpha,\xi}$-pullback of the strategy given by $(p[\dot{T}_g])^{\mathcal{P}[g]}$ then

$$\Phi = \Sigma_{\mathcal{P}(\alpha,\xi)} \restriction HC^{\mathcal{P}[g]}.$$

Our generic interpretability result now can be proved using the tree production lemma (Theorem 3.3.15 of [1]) and Corollary 2.7. We leave the details to the reader.

Theorem 2.8 (The generic interpretability) *Suppose $(\mathcal{P}, \Sigma)$ is a prehod pair or is a hod pair such that $\lambda^{\mathcal{P}}$ is limit. Assume that for every $\alpha < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)}$ has branch condensation. Then generic interpretability holds for $(\mathcal{P}, \Sigma)$.*

Next, we present our result on internal fullness preservation. The proof follows the same line of thought as the proof of Theorem 3.12 of [3] and because of that we omit it.

Definition 2.9 *Suppose $\mathcal{P}$ is a hod premouse and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{P}} \times (\varsigma^{\mathcal{P}}_{\alpha} + 1) \times (\zeta^{\mathcal{P}}_{\alpha,\xi} + 1)$. We say $\Lambda = \Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is internally fullness preserving if whenever $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}(\alpha,\xi,\gamma), \Lambda) \cap \mathcal{P}$ is a stack such that $(|\vec{\mathcal{T}}|^{+})^{\mathcal{P}}$ exists, $\beta < \lambda^{\mathcal{R}}$ and $\eta \in (\delta_{\beta}, \delta_{\beta+1}]^{\mathcal{R}}$ is a cardinal cutpoint of $\mathcal{R}$,*

1. *if $\mathcal{M} \in \mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}} + 1, (|\vec{\mathcal{T}}|^{+})^{\mathcal{P}})$-iterable $\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}$-mouse over $\mathcal{R}|\eta$ then $\mathcal{M} \trianglelefteq \mathcal{R}$,*
2. *if $\mathcal{M} \in \mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}} + 1, (|\vec{\mathcal{T}}|^{+})^{\mathcal{P}})$-iterable $\Lambda_{\mathcal{R}(\beta),\vec{\mathcal{T}}}$-mouse over $\mathcal{R}(\beta, 0, 0)$ then $\mathcal{M} \trianglelefteq \mathcal{R}$.*

Theorem 2.10 (Internal fullness preservation) *Suppose $\mathcal{P}$ is a hod premouse and $(\alpha, \xi, \gamma) \in \lambda^{\mathcal{P}} \times (\varsigma^{\mathcal{P}}_{\alpha} + 1) \times (\zeta^{\mathcal{P}}_{\alpha,\xi} + 1)$. Then $\Sigma^{\mathcal{P}}_{\alpha,\xi,\gamma}$ is internally fullness preserving.*

3 Diamond comparison

Our goal here is to provide another comparison argument, *diamond comparison*, that doesn't rely on branch condensation as heavily as our other argument (see Corollary 1.7). The new comparison argument follows the same line of thought as the proof of a similar comparison argument from [3] (see Theorem 2.47 of [3]).

We have two applications in mind for such a comparison argument. First we will use it to show that in some instances tails of strategies with hull condensation get branch condensation and strong branch condensation. This will appear as Theorem 6.7.

Next, as in [3], the diamond comparison argument can be used to show that $AD^+ + LSA$ is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals. This will appear as Theorem 13.1. In [3], a similar argument gave the consistency of $AD_{\mathbb{R}}$ + "Θ is regular" relative to a Woodin cardinal that is a limit of Woodin cardinals.

Following the proof of Theorem 2.47 of [3], we first define a *bad block* and a *bad sequence* and show that there cannot be such a bad sequence of length ω_1. We then show that the failure of comparison produces such bad sequences of length ω_1.

3.1 Bad sequences

For the purposes of this subsection, we make a definition of a bad block and a bad sequence. In later subsections, we will redefine these names for different objects. For the duration of this subsection, we fix a Γ-fullness preserving

Definition 3.1 (Bad block) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs of limit type. Then*

$$B = (((\mathcal{P}_i, \mathcal{Q}_i, \Sigma_i, \Lambda_i) : i < 5), (\vec{\mathcal{T}}_i, \vec{\mathcal{U}}_i : i < 4), (c, d))$$

is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ if the following holds:

1. *$(\mathcal{P}_0, \Sigma_0) = (\mathcal{P}, \Sigma)$ and $(\mathcal{Q}_0, \Lambda_0) = (\mathcal{Q}, \Lambda)$.*
2. *$\vec{\mathcal{T}}_0$ is a stack according to Σ_0 on $\mathcal{P}$.*
3. *$\vec{\mathcal{U}}_0$ is a stack according to Λ_0 on $\mathcal{Q}$.*
4. *Let $\vec{\mathcal{T}}_0 = (\mathcal{M}_\beta, \vec{\mathcal{T}}_\beta, E_\beta : \beta \leq \nu)$ and $\vec{\mathcal{U}}_0 = (\mathcal{N}_\beta, \vec{\mathcal{U}}_\beta, F_\beta : \beta \leq \nu)$. Then $\vec{\mathcal{T}}_\nu$ and $\vec{\mathcal{U}}_\nu$ are undefined, $\mathcal{P}_1 = \mathcal{M}_\nu$ and $\mathcal{Q}_1 = \mathcal{N}_\nu$.*

5. *There is some* $\beta+1 < min(\lambda^{\mathcal{P}_1}, \lambda^{\mathcal{Q}_1})$ *such that* $\mathcal{P}_1(\beta+1) = \mathcal{Q}_1(\beta+1)$, $\mathcal{P}_1(\beta+1)$ *is of successor type,* $\Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0} \neq \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0}$ *and*

$$\Sigma_{\mathcal{P}_1(\beta),\vec{\mathcal{T}}_0} = \Sigma_{\mathcal{Q}_1(\beta),\vec{\mathcal{U}}_0}.$$

6. $\vec{\mathcal{T}}_1 = \vec{\mathcal{U}}_1$ *is a comparison tree for* $(\mathcal{P}_1(\beta+1), \Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0})$ *and* $(\mathcal{Q}_1(\beta+1), \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0})$, $c = \Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0}(\vec{\mathcal{T}}_1)$*, and* $d = \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0}(\vec{\mathcal{U}}_1)$*. We let* $\mathcal{P}_2 = \mathcal{M}_c^{\vec{\mathcal{T}}_1}$ *and* $\mathcal{Q}_2 = \mathcal{M}_d^{\vec{\mathcal{U}}_1}$ *where we apply the stack* $\vec{\mathcal{T}}_1$ *and* $\vec{\mathcal{U}}_1$ *to* $\mathcal{P}_1$ *and* $\mathcal{Q}_1$ *respectively.*

7. $\Sigma_1 = \Sigma_{\mathcal{P}_1,\vec{\mathcal{T}}_0}$, $\Sigma_2 = \Sigma_{\mathcal{P}_2,\vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\{\mathcal{P}_2\}}$, $\Lambda_1 = \Sigma_{\mathcal{Q}_1,\vec{\mathcal{U}}_0}$*, and* $\Lambda_2 = \Sigma_{\mathcal{Q}_2,\vec{\mathcal{U}}_0^\frown\vec{\mathcal{U}}_1^\frown\{\mathcal{Q}_2\}}$,

8. $\vec{\mathcal{T}}_2$ *is a stack according to* Σ_2 *on* $\mathcal{P}_2$ *with last model* $\mathcal{P}_3$ *and* $\Sigma_3 = (\Sigma_2)_{\mathcal{P}_3,\vec{\mathcal{T}}_2}$.

9. $\vec{\mathcal{U}}_2$ *is a stack according to* Λ_2 *on* $\mathcal{Q}_2$ *with last model* $\mathcal{Q}_3$ *and* $\Lambda_3 = (\Lambda_2)_{\mathcal{Q}_3,\vec{\mathcal{U}}_2}$.

10. $\vec{\mathcal{T}}_3$ *is a normal tree according to* Σ_3 *on* $\mathcal{P}_3$ *with last model* $\mathcal{P}_4$ *and* $\Sigma_4 = (\Sigma_3)_{\mathcal{P}_4,\vec{\mathcal{T}}_3}$.

11. $\vec{\mathcal{U}}_3$ *is a normal tree according to* Λ_3 *on* $\mathcal{Q}_3$ *with last model* $\mathcal{Q}_4$ *and* $\Lambda_4 = (\Lambda_3)_{\mathcal{Q}_4,\vec{\mathcal{U}}_3}$.

12. $\mathcal{P}_3^b = \mathcal{Q}_3^b$ *and* $(\Sigma_3)_{\mathcal{P}_3^b} = (\Lambda_3)_{\mathcal{Q}_3^b}$.

13. $\vec{\mathcal{T}}_3$ *and* $\vec{\mathcal{U}}_3$ *are the trees produced via extender comparison between* $\mathcal{P}_3$ *and* $\mathcal{Q}_3$.

We set $\vec{\mathcal{T}}^B = \vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\{\mathcal{P}_2\}^\frown\vec{\mathcal{T}}_2^\frown\vec{\mathcal{T}}_3$ *and* $\vec{\mathcal{U}}^B = \vec{\mathcal{U}}_0^\frown\vec{\mathcal{U}}_1^\frown\{\mathcal{Q}_2\}^\frown\vec{\mathcal{U}}_2^\frown\vec{\mathcal{U}}_3$*. We say* $\vec{\mathcal{T}}^B$ *is the stack on the top of* B *and* $\vec{\mathcal{U}}^B$ *is the stack in the bottom of* B*.*

Next we show that there cannot be a bad sequence of length ω_1.

Lemma 3.2 (No bad sequences) *Suppose* $(\mathcal{P}, \Sigma)$ *and* $(\mathcal{Q}, \Lambda)$ *are two hod pairs of limit type such that* $\mathcal{P}$ *and* $\mathcal{Q}$ *are countable, and both* Σ *and* Λ *are* $(\omega_1, \omega_1, \omega_1)$*-strategies. There is then no bad sequence, i.e., a sequence* $(B_\beta : \beta < \omega_1)$ *satisfying the following holds:*

1. *For all* $\beta < \omega_1$, $B_\beta = (((\mathcal{P}_{\beta,i}, \mathcal{Q}_{a,i}, \Sigma_{\beta,i}, \Lambda_{\beta,i}) : i < 5), (\vec{\mathcal{T}}_{\beta,i}, \vec{\mathcal{U}}_{\beta,i} : i < 4), (c_\beta, d_\beta))$.

2. *For all* $\beta < \omega_1$, B_β *is a bad block on* $((\mathcal{P}_{\beta,0}, \Sigma_{\beta,0}), (\mathcal{Q}_{\beta,0}, \Lambda_{\beta,0}))$.

3. *For all* $\beta < \omega_1$, $\mathcal{P}_{\beta+1,0} = \mathcal{P}_{\beta,4}$ *and* $\mathcal{Q}_{\beta+1,0} = \mathcal{Q}_{\beta,4}$.

4. *For $\beta < \alpha < \omega_1$, let $\pi_{\beta,\alpha} : \mathcal{P}_{\beta,0} \to \mathcal{P}_{\alpha,0}$ be the composition of the embeddings on the "top" and $\sigma_{\beta,\alpha} : \mathcal{Q}_{\beta,0} \to \mathcal{Q}_{\alpha,0}$ be the composition of the embeddings in the "bottom". Then for all limit $\lambda < \omega_1$, $\mathcal{P}_{\lambda,0}$ is the direct limit of $(\mathcal{P}_\beta : \beta < \lambda)$ under the maps $\pi_{\beta,\alpha}$. Similarly, for all limit $\lambda < \omega_1$, $\mathcal{Q}_{\lambda,0}$ is the direct limit of $(\mathcal{Q}_\beta : \beta < \lambda)$ under the maps $\sigma_{\beta,\alpha}$.*

5. *For all $\beta < \omega_1$, $\Sigma_{\beta,0} = \Sigma_{\mathcal{P}_{\beta,0}, \oplus_{\gamma<\beta} \vec{\mathcal{T}}^{B_\gamma}}$ and $\Lambda_{\beta,0} = \Sigma_{\mathcal{Q}_{\beta,0}, \oplus_{\gamma<\beta} \vec{\mathcal{U}}^{B_\gamma}}$.*

Proof. Towards a contradiction, suppose $\vec{B} = (B_\beta : \beta < \omega_1)$ is a bad sequence. Let $\mathcal{P}_{\omega_1}$ be the direct limit of $(\mathcal{P}_{\beta,0} : \beta < \omega_1)$ under the embeddings $\pi_{\beta,\alpha}$ and $\mathcal{Q}_{\omega_1}$ be the direct limit of $(\mathcal{Q}_{\beta,0} : \beta < \omega_1)$ under the embeddings $\sigma_{\beta,\alpha}$. Let X be a countable submodel of H_{ω_3} such that letting $\tau : M \to H_{\omega_3}$ be the uncollapse map, $\vec{B} \in rng(\sigma)$. Let $\kappa = \omega_1^{\mathcal{M}}$ and notice that for every $\beta < \kappa$,

$$B_\beta^- =_{def} (((\mathcal{P}_{\beta,i}, \mathcal{Q}_{\beta,i}) : i < 5), (\vec{\mathcal{T}}_{\beta,i}, \vec{\mathcal{U}}_{\beta,i} : i < 4), (c_\beta, d_\beta)) \in M$$

and B_β^- is countable in M. It then follows that $\tau^{-1}(\mathcal{P}_{\omega_1}) = \mathcal{P}_{\kappa,0}$ and $\tau^{-1}(\mathcal{Q}) = \mathcal{Q}_{\kappa,0}$. Let

$$\pi_\beta : \mathcal{P}_{\beta,0} \to \mathcal{P}_{\omega_1} \text{ and } \sigma_\beta : \mathcal{Q}_{\beta,0} \to \mathcal{Q}_{\omega_1}$$

be the direct limit embeddings.

Standard arguments show that for all $x \in \mathcal{P}_{\kappa,0} \cap \mathcal{Q}_{\kappa,0}$,

$$\pi_\kappa(x) = \tau(x) = \sigma_\kappa(x).$$

Notice that we have that $\lambda^{\mathcal{P}_{\kappa,0}} = \lambda^{\mathcal{Q}_{\kappa,0}}$. Letting $\lambda = \lambda^{\mathcal{P}_{\kappa,0}}$, notice that $\delta_{\lambda-1}^{\mathcal{P}_{\kappa,0}} = \delta_{\lambda-1}^{\mathcal{Q}_{\kappa,0}}$. Let then $\delta = \delta_{\lambda-1}^{\mathcal{P}_{\kappa,0}}$. Let $\phi = \pi^{\vec{\mathcal{T}}_{\kappa,0}}$ and $\psi = \pi^{\vec{\mathcal{U}}_{\kappa,0}}$. It then follows that

(1) $\wp(\delta)^{\mathcal{P}_{\kappa,0}} = \wp(\delta)^{\mathcal{Q}_{\kappa,0}}$.

Let β be such that $\vec{\mathcal{T}}_{\kappa,1} = \vec{\mathcal{U}}_{\kappa,1}$ is based on $\mathcal{P}_{\kappa,1}(\beta+1) = \mathcal{Q}_{\kappa,1}(\beta+1)$. Notice that

(2) $\delta^{\mathcal{P}_{\kappa,1}(\beta+1)} = \sup\{\phi(f)(a) : f \in \mathcal{P}_{\kappa,0} \wedge f : \delta \to \delta \wedge a \in (\mathcal{P}_{\kappa,1}(\beta))^{<\omega}\}$
(3) $\delta^{\mathcal{Q}_{\kappa,1}(\beta+1)} = \sup\{\psi(f)(a) : f \in \mathcal{Q}_{\kappa,0} \wedge f : \delta \to \delta \wedge a \in (\mathcal{Q}_{\kappa,1}(\beta))^{<\omega}\}$

Let now $p = \pi_{c_\kappa}^{\vec{\mathcal{T}}_{\kappa,1}}$, $q = \pi_{d_\kappa}^{\vec{\mathcal{T}}_{\kappa,1}}$, $j : \mathcal{P}_{\kappa,2} \to \mathcal{P}_{\omega_1}$ and $i : \mathcal{Q}_{\kappa,2} \to \mathcal{Q}_{\omega_1}$ be the iteration embeddings along the top and bottom of $\vec{B}$. Notice that because

$$(\Sigma_{\kappa,2})_{\mathcal{P}_{\kappa,2}(p(\beta)+1)} = (\Lambda_{\kappa,2})_{\mathcal{Q}_{\kappa,2}(p(\beta)+1)},$$

we have that

(4) $j \restriction \mathcal{P}_{\kappa,2}(p(\beta)+1) = i \restriction \mathcal{Q}_{\kappa,2}(q(\beta)+1)$.

Let then

$$s = \{\gamma < \delta_{\beta+1}^{\mathcal{P}_{\kappa,1}} : \exists f \in (\delta^\delta)^{\mathcal{P}_{\kappa,0}} \exists a \in (\mathcal{P}_{\kappa,1}(\beta))^{<\omega}(\gamma = \phi(f)(a))\}$$
$$t = \{\gamma < \delta_{\beta+1}^{\mathcal{Q}_{\kappa,1}} : \exists f \in (\delta^\delta)^{\mathcal{Q}_{\kappa,0}} \exists a \in (\mathcal{Q}_{\kappa,1}(\beta))^{<\omega}(\gamma = \psi(f)(a))\}.$$

(1) then implies that

(5) $j \circ p[s] = i \circ q[t]$.

(4) then implies that

(6) $p[s] = q[t]$

and because (by (2) and (3)) s and t are cofinal in $\delta^{\mathcal{P}_{\kappa,0}(\beta+1)}$, we have that $c_\kappa = d_\kappa$, contradiction. □

3.2 The comparison argument

In this subsection we prove the following comparison theorem under the hypothesis that the *lower level comparison* holds. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving and $\mathcal{P}$ and $\mathcal{Q}$ are of limit type. We then let "lower level comparison" stand for the following statement.

Lower Level Comparison: for every $(\vec{\mathcal{T}}, \mathcal{P}_1) \in B(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{Q}_1) \in B(\mathcal{Q}, \Lambda)$, comparison holds for $(\mathcal{P}_1, \Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}})$ and $(\mathcal{Q}_1, \Lambda_{\mathcal{Q}_1, \vec{\mathcal{U}}})$.

The following is then the comparison theorem we will prove in this subsection.

Theorem 3.3 (Diamond comparison) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving $(\omega_1, \omega_1, \omega_1)$-strategies, $\mathcal{P}$ and $\mathcal{Q}$ are countable and are of limit type, and lower level comparison holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. Then there are $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$ such that either*

1. $\mathcal{P}$ and $\mathcal{Q}$ are of lsa type and $\Sigma^{sts}_{\mathcal{R},\vec{\mathcal{T}}} = \Lambda^{sts}_{\mathcal{R},\vec{\mathcal{U}}}$ or

2. $\mathcal{P}$ and $\mathcal{Q}$ are not of lsa type and $\Sigma_{\mathcal{R},\vec{\mathcal{T}}} = \Lambda_{\mathcal{R},\vec{\mathcal{U}}}$.

We prove the theorem by showing that the failure of its conclusion produces a bad sequence of length ω_1. Towards showing this, we prove two useful lemmas.

In the sequel, we say that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ satisfy the *lower level comparison* if $(*)$ above holds. We say that *weak comparison* holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there is $(\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{R}, \mathcal{S})$ such that

1. $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$,
2. $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$,
3. $\mathcal{R}^b = \mathcal{S}^b$ and $\Sigma_{\mathcal{R}^b,\vec{\mathcal{T}}} = \Lambda_{\mathcal{S}^b,\vec{\mathcal{U}}}$.

Our first lemma says that lower level comparison implies that weak comparison holds.

Lemma 3.4 *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving, $\mathcal{P}$ and $\mathcal{Q}$ are of limit type and that lower level comparison holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. Then weak comparison holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$.*

Proof. We inductively construct $(\mathcal{P}_i, \vec{\mathcal{T}}_i : i < \omega)$ and $(\mathcal{Q}_i, \vec{\mathcal{U}}_i : i < \omega)$ such that the following conditions hold.

1. $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$.
2. Suppose $i = 2n$. Then the following holds.
   (a) $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{P}^b_i$ according to $\Sigma_{\mathcal{P}^b_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$ with last model $\mathcal{P}_{i+1}$ (when we apply $\vec{\mathcal{T}}_i$ to $\mathcal{P}_i$).
   (b) $\vec{\mathcal{U}}_i$ is a stack on $\mathcal{Q}_i$ according to $\Lambda_{\mathcal{Q}_i, \oplus_{k<i}\vec{\mathcal{U}}_i}$ with last model $\mathcal{Q}_{i+1}$.
   (c) $\mathcal{P}^b_{i+1} \trianglelefteq_{hod} \mathcal{Q}^b_{i+1}$ and $\Lambda_{\mathcal{P}^b_{i+1}, \oplus_{k\leq i}\vec{\mathcal{U}}_k} = \Sigma_{\mathcal{P}^b_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$.
3. Suppose $i = 2n + 1$. Then the following holds.
   (a) $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{P}_i$ according to $\Sigma_{\mathcal{P}_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$ with last model $\mathcal{P}_{i+1}$.
   (b) $\vec{\mathcal{U}}_i$ is a stack on $\mathcal{Q}^b_i$ according to $\Lambda_{\mathcal{Q}^b_i, \oplus_{k<i}\vec{\mathcal{U}}_i}$ with last model $\mathcal{Q}_{i+1}$ (when we apply $\vec{\mathcal{U}}_i$ to $\mathcal{P}_i$).

(c) $\mathcal{Q}^b_{i+1} \trianglelefteq_{hod} \mathcal{P}^b_{i+1}$ and $\Lambda_{\mathcal{Q}^b_{i+1}, \oplus_{k\leq i}\vec{\mathcal{U}}_k} = \Sigma_{\mathcal{Q}^b_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$.

We show how to carry out the inductive step. Suppose $i = 2n$ and we have constructed $(\mathcal{P}_i, \vec{\mathcal{T}}_i : i \leq 2n)$ and $(\mathcal{Q}_i, \vec{\mathcal{U}}_i : i \leq 2n)$. We now consider three cases.

Suppose first that $\mathcal{P}_i = \mathcal{P}^b_i$. Next suppose that either $\mathrm{cf}^{\mathcal{P}_i}(\delta^{\mathcal{P}_i}) = \omega$ or $\mathrm{cf}^{\mathcal{P}_i}(\delta^{\mathcal{P}_i}) = \delta^{\mathcal{P}_i}$. Let $(\alpha_i : i < \omega)$ be such that $\sup(\alpha_k : k < \omega) = \delta^{\mathcal{P}^b_i}$. By induction we construct a sequence $(\vec{\mathcal{T}}^*_k, \mathcal{W}_k, \vec{\mathcal{S}}_k, \mathcal{R}_k, \vec{\mathcal{S}}^*_k, \mathcal{R}^*_k, \beta_k : k < \omega)$ such that the following holds.

1. $(\vec{\mathcal{S}}^*_0, \mathcal{R}^*_0) \in I(\mathcal{Q}_i, \Lambda_{\mathcal{Q}_i, \oplus_{m<i}\vec{\mathcal{U}}_m})$ and

$$\Gamma(\mathcal{P}_i(\alpha_0), \Sigma_{\mathcal{P}_i(\alpha_0), \oplus_{m<i}\vec{\mathcal{T}}_m}) = \Gamma(\mathcal{R}^*_0(\beta_0), \Lambda_{\mathcal{R}^*_0(\beta_0), (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown \vec{\mathcal{S}}^*_0}).$$

   Moreover, $(\vec{\mathcal{T}}^*_0, \mathcal{W}_0) \in I(\mathcal{P}_i, \Sigma_{\mathcal{P}_i, \oplus_{m<i}\vec{\mathcal{T}}_m})$, $(\vec{\mathcal{S}}_0, \mathcal{R}_0) \in I(\mathcal{R}^*_0, \Lambda_{\mathcal{R}^*_0, (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown \vec{\mathcal{S}}^*_0})$ and $\vec{\mathcal{T}}^*_0$ and $\vec{\mathcal{S}}_0$ are some stacks that come from comparing $(\mathcal{P}_i(\alpha_0), \Sigma_{\mathcal{P}_i(\alpha_0), \oplus_{m<i}\vec{\mathcal{T}}_m})$ with $(\mathcal{R}^*_0(\beta_0), \Lambda_{\mathcal{R}^*_0(\beta_0), (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown \vec{\mathcal{S}}^*_0})$.

2. For $k+1 < \omega$, $(\vec{\mathcal{S}}^*_{k+1}, \mathcal{R}^*_{k+1}) \in I(\mathcal{R}_k, \Lambda_{\mathcal{R}_k, (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown (\oplus_{m\leq k}(\vec{\mathcal{S}}^*_m{}^\frown \vec{\mathcal{S}}_m))})$ and

$$\Gamma(\mathcal{W}_k(\alpha^*_{k+1}), \Sigma_{\mathcal{W}_k(\alpha^*_{k+1}), (\oplus_{m<i}\vec{\mathcal{T}}_m)^\frown \oplus_{m\leq k}\vec{\mathcal{T}}^*_m}) =$$
$$\Gamma(\mathcal{R}^*_{k+1}(\beta_{k+1}), \Lambda_{\mathcal{R}^*_{k+1}(\beta_{k+1}), (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown (\oplus_{m\leq k}(\vec{\mathcal{S}}^*_m{}^\frown \vec{\mathcal{S}}_m))}).$$

   where α^*_k is the image of α_{k+1} in $\mathcal{W}_k$. Moreover,

$$(\vec{\mathcal{T}}^*_{k+1}, \mathcal{W}_{k+1}) \in I(\mathcal{W}_k, \Sigma_{\mathcal{W}_k, (\oplus_{m<i}\vec{\mathcal{T}}_m)^\frown \oplus_{m\leq k}\vec{\mathcal{T}}^*_m}),$$
$$(\vec{\mathcal{S}}_{k+1}, \mathcal{R}_{k+1}) \in I(\mathcal{R}^*_{k+1}, \Lambda_{\mathcal{R}^*_{k+1}, (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown (\oplus_{m\leq k}(\vec{\mathcal{S}}^*_m{}^\frown \vec{\mathcal{S}}_m))^\frown \vec{\mathcal{S}}^*_{k+1}})$$

   and $\vec{\mathcal{T}}^*_{k+1}$ and $\vec{\mathcal{S}}_{k+1}$ are some stacks that come from comparing

$$(\mathcal{W}_k(\alpha^*_k), \Sigma_{\mathcal{W}_k(\alpha^*_{k+1}), (\oplus_{m<i}\vec{\mathcal{T}}_m)^\frown \oplus_{m\leq k}\vec{\mathcal{T}}^*_m}) \text{ and}$$
$$(\mathcal{R}^*_{k+1}(\beta_{k+1}), \Lambda_{\mathcal{R}^*_{k+1}(\beta_{k+1}), (\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown (\oplus_{m\leq k}(\vec{\mathcal{S}}^*_m{}^\frown \vec{\mathcal{S}}_m))^\frown \vec{\mathcal{S}}^*_{k+1}})$$

We then let $\vec{\mathcal{T}}_{i+1} = \oplus_{k<\omega}\vec{\mathcal{T}}^*_k$ and $\vec{\mathcal{U}}_i = \oplus_{m<\omega}\vec{\mathcal{S}}^*_k{}^\frown \vec{\mathcal{S}}$. Also, we let $\mathcal{P}_{i+1}$ be the last model of $\vec{\mathcal{T}}_{i+1}$ and $\mathcal{Q}_{i+1}$ be the last model of $\vec{\mathcal{U}}_i$. We carry out the odd induction step similarly by reversing the roles of $\mathcal{P}_i$ and $\mathcal{Q}_i$ in the above construction. Notice that because $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda)$, we must have that $\mathrm{cf}^{\mathcal{Q}_i}(\delta^{\mathcal{Q}_i}) = \omega$ or $\mathrm{cf}^{\mathcal{Q}_i}(\delta^{\mathcal{Q}_i}) = \delta^{\mathcal{P}_i}$

The construction of $(\mathcal{P}_i, \vec{\mathcal{T}}_i : i < \omega)$ and $(\mathcal{Q}_i, \vec{\mathcal{U}}_i : i < \omega)$ in the case $\mathrm{cf}^{\mathcal{P}}(\delta^{\mathcal{P}})$ is measurable is very similar to the above construction. Just let $(\alpha_i : i < \omega)$ in the above construction be such that $\alpha_i = \delta^{\mathcal{W}_i^b}$.

We then clearly have that if $\vec{\mathcal{T}} = \oplus_{i<\omega}\vec{\mathcal{T}}_i$, $\vec{\mathcal{U}} = \oplus_{i<\omega}\vec{\mathcal{U}}_i$, $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}$ and $\mathcal{S}$ is the last model of $\vec{\mathcal{U}}$ then $(\vec{\mathcal{T}}, \mathcal{R})$ and $(\vec{\mathcal{U}}, \mathcal{S})$ witness weak comparison between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. □

Lemma 3.5 *Suppose* $(\mathcal{P}, \Sigma)$ *and* $(\mathcal{Q}, \Lambda)$ *are two hod pairs such that* $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, *both* Σ *and* Λ *are* Γ*-fullness preserving, both* $\mathcal{P}$ *and* $\mathcal{Q}$ *are of limit type and low level comparison holds. Suppose further that* $\mathcal{P}^b = \mathcal{Q}^b$ *and for all* $\beta < \lambda^{\mathcal{P}} - 1$, $\Sigma_{\mathcal{P}(\beta+1)} = \Lambda_{\mathcal{Q}(\beta+1)}$. *Let* $(\mathcal{T}, \mathcal{R}, \mathcal{U}, \mathcal{S})$ *be the trees of the extender comparison of* $\mathcal{P}$ *and* $\mathcal{Q}$[2]. *Suppose that either*

1. $\mathcal{R} \neq \mathcal{S}$ *or*

2. $\mathcal{R} = \mathcal{S}$ *and* $\Sigma_{\mathcal{R},\mathcal{T}} \neq \Lambda_{\mathcal{S},\mathcal{U}}$.

Then there is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$.

Proof. It follows from Lemma 18.5 of [4] that we can find minimal low level disagreement $(\vec{\mathcal{T}}^*, \vec{\mathcal{U}}^*, \mathcal{W})$ between $(\mathcal{R}, \Sigma_{\mathcal{R},\mathcal{T}})$ and $(\mathcal{S}, \Lambda_{\mathcal{S},\mathcal{U}})$. We then let $\mathcal{P}_1$ and $\mathcal{Q}_1$ be the last models of $\vec{\mathcal{T}}^*$ and $\vec{\mathcal{U}}^*$ when we regard them as stacks on $\mathcal{R}$ and $\mathcal{S}$ respectively. Let $\vec{\mathcal{T}}_1$ be a comparison stack for $(\mathcal{W}, \Sigma_{\mathcal{W},\mathcal{T}^\frown\vec{\mathcal{T}}^*})$ and $(\mathcal{W}, \Lambda_{\mathcal{W},\mathcal{U}^\frown\vec{\mathcal{U}}^*})$. Let $b = \Sigma(\mathcal{T}^\frown\vec{\mathcal{T}}^{*\frown}\vec{\mathcal{T}}_2)$, $c = \Lambda(\mathcal{U}^\frown\vec{\mathcal{U}}^{*\frown}\vec{\mathcal{U}}_2)$, $\mathcal{P}_2 = \mathcal{M}_b^{\vec{\mathcal{T}}_2}$ and $\mathcal{Q}_2 = \mathcal{M}_c^{\vec{\mathcal{T}}_2}$ (here we apply the stacks to $\mathcal{P}_1$ and $\mathcal{Q}_1$ respectively).

Next let $(\vec{\mathcal{T}}_2, \mathcal{P}_3)$ and $(\vec{\mathcal{U}}_2, \mathcal{Q}_3)$ witness that weak comparison holds between

$$(\mathcal{P}_2, \Sigma_{\mathcal{P}_2,\mathcal{T}^\frown\vec{\mathcal{T}}^{*\frown}\vec{\mathcal{T}}_1}), \text{ and}(\mathcal{Q}_2, \Lambda_{\mathcal{Q}_2,\mathcal{U}^\frown\vec{\mathcal{U}}_1^{*\frown}\vec{\mathcal{U}}_1}).$$

Finally, let $(\mathcal{T}_3, \mathcal{P}_4)$ and $(\mathcal{U}_3, \mathcal{Q}_4)$ be the result of extender comparison between $\mathcal{P}_3$ and $\mathcal{Q}_3$.

Next let $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{Q}_0 = \mathcal{Q}$, $\Sigma_0 = \Sigma$, $\Lambda_0 = \Lambda$, $\vec{\mathcal{T}}_0 = \mathcal{T}^\frown\vec{\mathcal{T}}^*$, and $\vec{\mathcal{U}}_0 = \mathcal{U}^\frown\vec{\mathcal{U}}^*$. Also, for $i \in \{1, 2, 3, 4\}$ let $\Sigma_i = \Sigma_{\mathcal{P}_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$ and $\Lambda_i = \Lambda_{\mathcal{Q}_i, \oplus_{k<i}\vec{\mathcal{U}}_k}$, . It is then easy to see that

$$(((\mathcal{P}_i, \mathcal{Q}_i, \Sigma_i, \Lambda_i) : i < 5), (\vec{\mathcal{T}}_i, \vec{\mathcal{U}}_i : i < 4), (\pi_i, \sigma_i : i < 5), (b, c))$$

[2]Thus, $\mathcal{T}$ is on $\mathcal{P}$ with last model $\mathcal{R}$ and $\mathcal{U}$ is on $\mathcal{Q}$ with last model $\mathcal{S}$.

is a bad block on $((\mathcal{P},\Sigma),(\mathcal{Q},\Lambda))$. $\square$

We now start proving Theorem 3.3. Suppose that the conclusion of Theorem 3.3 fails. This means that

(1) whenever $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{P},\Sigma)$ and $(\vec{\mathcal{U}},\mathcal{R})\in I(\mathcal{Q},\Lambda)$,

1. if $\mathcal{P}$ and $\mathcal{Q}$ are of lsa type then $\Sigma^{sts}_{\mathcal{R},\vec{\mathcal{T}}}\neq\Lambda^{sts}_{\mathcal{R},\vec{\mathcal{U}}}$ or
2. if $\mathcal{P}$ and $\mathcal{Q}$ are not of lsa type then $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}\neq\Lambda_{\mathcal{R},\vec{\mathcal{U}}}$.

It follows from Lemma 3.4 that, without loss of generality, we can assume that $\mathcal{P}^b=\mathcal{Q}^b$ and for all $\beta+1<\lambda^{\mathcal{P}^b}$, $\Sigma_{\mathcal{P}(\beta+1)}=\Lambda_{\mathcal{Q}(\beta+1)}$. We now by induction construct a bad sequence $(B_\alpha:\alpha<\omega_1)$ on $((\mathcal{P},\Sigma),(\mathcal{Q},\Lambda))$.

It follows from Lemma 3.5 that there is a bad block on $((\mathcal{P},\Sigma),(\mathcal{Q},\Lambda))$. Let B_0 be any bad block on $((\mathcal{P},\Sigma),(\mathcal{Q},\Lambda))$. Suppose next that we have constructed $(B_\beta:\beta<\lambda)$ for λ a limit. Let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\lambda$ be the direct limit of respectively $(\mathcal{P}_\beta:\beta<\lambda)$ and $(\mathcal{Q}_\beta:\beta<\lambda)$ under the corresponding iteration embeddings. Then letting $\Sigma_{\lambda,0}$ and $\Lambda_{\lambda,0}$ be the corresponding tails of Σ and Λ, we have that $(\mathcal{P}_\lambda,\Sigma_\lambda)$ and $(\mathcal{Q}_\lambda,\Lambda_\lambda)$ satisfy the hypothesis of Lemma 3.5. Let then B_λ be a bad block on $((\mathcal{P}_\lambda,\Sigma_\lambda),(\mathcal{Q}_\lambda,\Lambda_\lambda))$.

Next suppose that we have constructed $(B_\beta:\beta<\lambda+1)$. Let $\mathcal{P}_{\lambda+1}=\mathcal{P}_{\lambda,4}$, $\mathcal{Q}_{\lambda+1}=\mathcal{Q}_{\lambda,4}$ and let $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ be the stacks respectively on the top of $(B_\beta:\beta<\lambda+1)$ and in the bottom of $(B_\beta:\beta<\lambda+1)$. We then again can find, using Lemma 3.5, a bad block on $B_{\lambda+1}$ on $((\mathcal{P}_{\lambda+1},\Sigma_{\mathcal{P}_{\lambda+1},\vec{\mathcal{T}}_\lambda}),(\mathcal{Q}_{\lambda+1},\Lambda_{\mathcal{Q}_{\lambda+1},\vec{\mathcal{U}}_\lambda}))$. It then follows that the resulting sequence $(B_\beta:\beta<\omega_1)$ is a bad sequence on $((\mathcal{P},\Sigma),(\mathcal{Q},\Lambda))$, contradiction!

4 The derived models of hod mice

First we state the version of Theorem 3.19 of [3]. First recall (see Corollary 20.6 of [4]) that if $(\mathcal{P},\Sigma)$ is a hod pair and $\mathcal{Q}\in pI(\mathcal{P},\Sigma)$ such that Σ has strong branch condensation, the the strategy of $\mathcal{Q}$ induced by Σ is independent of the particular iteration producing $\mathcal{Q}$. We denote this strategy by $\Sigma_{\mathcal{Q}}$.

Given a hod pair $(\mathcal{P},\Sigma)$ and $(\beta,\xi,\gamma)\in\lambda^{\mathcal{P}}\times(\varsigma^{\mathcal{P}}_\beta+1)\times(\zeta^{\mathcal{P}}_{\beta,\varsigma^{\mathcal{P}}_\beta}+1)$ such that β is limit, we let $D(\mathcal{P},\Sigma,\beta)$ be the derived model of $\mathcal{P}(\beta,0,0)$ as computed by $\Sigma_{\beta,0,0}$ and we let

$$D(\mathcal{P},\Sigma,(\beta,\xi,\gamma))=\cup_{\mathcal{Q}\in pI(\mathcal{P}(\beta,\xi,\gamma),\Sigma_{\mathcal{P}(\beta,\xi,\gamma)})}D(\mathcal{Q},\Sigma_{\mathcal{Q}},\pi^{\Sigma}_{\mathcal{P},\mathcal{Q}}(\beta)).$$

Next recall (see Definition 15.5 of [3]) that if $(\mathcal{P}, \Sigma)$ is a hod pair then we let

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \exists \mathcal{R}((\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \trianglelefteq_{hod} \mathcal{R}^b)\}$$

and

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}})\}.$$

Theorem 4.1 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and is fullness preserving. Suppose further that there is a good pointclass Γ such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma}$. Then*

1. $\Gamma(\mathcal{P}, \Sigma) = \cup_{\mathcal{Q} \in pI(\mathcal{P}, \Sigma), \beta < \lambda^{\mathcal{Q}}} D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \beta)$.
2. *For any $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, if $\beta + \omega < \lambda^{\mathcal{P}}$ then $D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \beta)$ is completely mouse full and if $\beta + \omega = \lambda^{\mathcal{P}}$ then $D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \beta)$ is mouse full*[3].
3. *For any $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, if $\beta < \lambda^{\mathcal{P}}$ then letting $\Gamma^* = D(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \beta + \omega)$, if ξ is such that $\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta)})} = \theta^{\Gamma}_{\xi}$ then for every n,*

$$\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta+n)})} = \theta^{\Gamma}_{\xi+n} \text{ and } \Omega^{\Gamma} = \xi + \omega.$$

4. *$\Gamma(\mathcal{P}, \Sigma)$ is a mouse full pointclass.*

Theorem 4.2 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and Σ has strong branch condensation and is fullness preserving. Suppose further that there is a good pointclass Γ such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma}$. Then for every $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, $\Gamma(\mathcal{P}, \Sigma) \vDash$ "MC for $\Sigma_{\mathcal{Q}}$".*

5 Anomalous hod premice

The use of anomalous hod premice here is the same as it is in [3]. We will use them to produce pointclasses that are not completely OD-full (see Definition 3.14 of [3]).

[3]See Definition 3.18 of [3]. More precisely, a pointclass is completely mouse-full if the next model of determinacy has the same mice relative to common iteration strategies. Given two pointclasses Γ_1 and Γ_2, we write $\Gamma_1 \trianglelefteq_{mouse} \Gamma_2$ if $\Gamma_1 \subseteq \Gamma_2$ and Γ_2 has the same mice as Γ_1 relative to common iteration strategies. Finally, Γ is mouse full if either it is completely mouse full or is a union of completely mouse full pointclasses $(\Gamma_\alpha : \alpha < \Omega)$ such that for all α, $\Gamma_\alpha \trianglelefteq_{mouse} \Gamma_{\alpha+1}$ and for all limit α, $\Gamma_\alpha = \cup_{\beta<\alpha}\Gamma_\beta$.

Definition 5.1 (Anomalous hod premouse of type I) *$\mathcal{P}$ is an anomalous hod premouse of type I if for some successor ordinal λ and some δ there is a sequence $\langle \mathcal{P}_\beta : \beta < \lambda \rangle$ such that*

1. *$\mathcal{P}_\beta$ is a hod premouse such that $\lambda^{\mathcal{P}_\beta} = \beta$,*
2. *for all $\beta < \alpha < \lambda$, $\mathcal{P}_\beta \trianglelefteq_{hod} \mathcal{P}_\alpha$ and $\mathcal{P}_\beta = \mathcal{P}_\alpha(\beta)$,*
3. *$\mathcal{P} \models$ "$\forall \xi > o(\mathcal{P}(\lambda - 1))(\mathcal{P}||\xi$ is a $\Sigma_{\mathcal{P}(\lambda-1)}$-premouse over $\mathcal{P}(\lambda - 1))$",*
4. *$\mathcal{P} \models$ "δ is Woodin",*
5. *for every $\xi \in (\delta, o(\mathcal{P}))$, $\rho(\mathcal{P}|\xi) \geq \delta$,*
6. *$\rho(\mathcal{P}) < \delta^{\mathcal{P}}$.*

Definition 5.2 (Anomalous hod premouse of type II) *$\mathcal{P}$ is an anomalous hod premouse of type II if $\mathcal{P}$ is meek and for some limit ordinal λ and some δ there is a sequence $\langle \mathcal{P}_\beta : \beta < \lambda \rangle$ such that*

1. *$\mathcal{P}_\beta$ is a hod premouse such that $\lambda^{\mathcal{P}_\beta} = \beta$,*
2. *for all $\beta < \alpha < \lambda$, $\mathcal{P}_\beta \trianglelefteq_{hod} \mathcal{P}_\alpha$ and $\mathcal{P}_\beta = \mathcal{P}_\alpha(\beta)$,*
3. *$\mathcal{P}|\delta = \cup_{\beta<\lambda} \mathcal{P}_\beta$,*
4. *$\mathcal{P} \models$ "$\forall \xi > o(\mathcal{P}(\lambda - 1))(\mathcal{P}||\xi$ is a $\oplus_{\beta<\lambda}\Sigma_{\mathcal{P}(\beta)}$-premouse over $\mathcal{P}|\delta)$",*
5. *for every $\xi \in (\delta, o(\mathcal{P}))$, $\rho(\mathcal{P}|\xi) \geq \delta$,*
6. *$\rho(\mathcal{P}) < \delta^{\mathcal{P}}$.*

Definition 5.3 (Anomalous hod premouse of type III) *$\mathcal{P}$ is an anomalous hod premouse of type III if it is limit type, it is not anomalous hod premouse of type II and $\rho(\mathcal{P}) < \delta^{\mathcal{P}}$.*

We say $\mathcal{P}$ is an anomalous hod premouse if it is an anomalous hod premouse of some type. If $\mathcal{P}$ is an anomalous hod premouse then we let $\delta^{\mathcal{P}}$ and $\lambda^{\mathcal{P}}$ be as in the above definitions. We then let $\Sigma^{\mathcal{P}}$ be the strategy that is on the sequence of $\mathcal{P}$.

Definition 5.4 (Anomalous hod pair) *$(\mathcal{P}, \Sigma)$ is an anomalous hod pair if $\mathcal{P}$ is an anomalous hod premouse, Σ is an iteration strategy with hull condensation and whenever $\mathcal{Q}$ is a Σ iterate of $\mathcal{P}$, $\Sigma^{\mathcal{Q}} = \Sigma \cap \mathcal{Q}$.*

The following lemma is due to Mitchell and Steel. It appears as Claim 5 in the proof of Theorem 6.2 of [2]. In the current work, the lemma is used to show that certain hod pair constructions converge, which leads to showing that generation of pointclasses holds (see Theorem 12.2).

Lemma 5.5 *Suppose $(\mathcal{P}, \Sigma)$ is a an anomalous hod pair, $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and n is least such that $\rho_n(\mathcal{P}) < \delta^{\mathcal{P}}$. Then $\rho_n(\mathcal{Q}) < \delta^{\mathcal{Q}}$.*

The next theorem is the equivalent of Theorem 3.27 of [3].

Theorem 5.6 *Suppose $(\mathcal{P}, \Sigma)$ is an anomalous hod pair of type II or III. Suppose that there is a pointclass Γ such that for any $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$ there is a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has branch condensation and is Γ-fullness fullness preserving, and there is $\pi : \mathcal{Q} \to \mathcal{R}$ such that $\Lambda^{\pi} = \Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$. Then*

1. *For every $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has branch condensation, is positional and is commuting.*

2. *Σ is $\Gamma(\mathcal{P}, \Sigma)$-fullness preserving and $\Gamma(\mathcal{P}, \Sigma)$ is a mouse full pointclass.*

We omit the proof of Theorem 5.6 as it is only notationally more complicated then the proof of Theorem 3.10 of [3]. It can be proved using the same proof. We remind the reader that the proof of Theorem 3.27 of [3] depended on generic interpretability result, which appeared as Theorem 3.10 in [3]. In our current context we need to use Theorem 2.8. The general idea is that we can translate the properties of Σ into the derived model of $\mathcal{P}$ as computed via Σ. This fact then just gets preserved under pull-back embeddings.

The following is an easy corollary of Theorem 5.6.

Corollary 5.7 (Branch condensation pulls back) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and Σ has branch condensation. Suppose $\pi : \mathcal{Q} \to \mathcal{P}$ is elementary. Then for every $\beta < \lambda^{\mathcal{Q}}$, $(\Sigma^{\pi})_{\mathcal{Q}(\beta)}$ has branch condensation.*

6 Getting strong branch condensation

In this section we show that strategies with branch condensation acquire strong branch condensation on a tail.

Theorem 6.1 (From condensation to strong condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and $\mathcal{P}$ is of limit type. Then there is some $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $(\mathcal{Q}, \Sigma_{\mathcal{Q}, \vec{\mathcal{T}}})$ has strong branch condensation.*

We spend the rest of this section proving Theorem 6.1. We prove Theorem 6.1 by proving three useful general lemmas. The idea is just like the idea behind the comparison proof of the previous section. If there is no tail with strong branch condensation then we obtain a certain bad sequence of length ω_1. As is expected, such sequences cannot exist. We start by describing the blocks of our bad sequences.

Definition 6.2 (A bad diamond) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit. We say $((\mathcal{P}_i : i < 2), (\vec{\mathcal{T}}_i : i < 3), (\vec{\mathcal{U}}_i : i < 3), (\mathcal{R}_i : i < 2), (\mathcal{S}_i : i < 2), k, \xi)$ is a bad diamond on $(\mathcal{P}, \Sigma)$ if it satisfies the following conditions:*

1. *$\mathcal{P}_0 = \mathcal{P}$, for $i < 2$, $\mathcal{P}_i, \mathcal{R}_i$ and $\mathcal{S}_i$ are hod premice and $k : \mathcal{P}_0 \to \mathcal{R}_0$.*
2. *$(\vec{\mathcal{U}}_0, \mathcal{S}_0) \in I(\mathcal{P}, \Sigma)$, for $i \in [1, 3)$, $(\vec{\mathcal{U}}_i, \mathcal{S}_i) \in I(\mathcal{S}_{i-1}, \Sigma_{\mathcal{S}_{i-1}})$[4], $\vec{\mathcal{U}}_1$ is a normal tree on $\mathcal{S}_0$ and $\mathcal{P}_1$ is the last model of $\vec{\mathcal{U}}_2$.*
3. *$\vec{\mathcal{T}}_0 = \emptyset$, $\vec{\mathcal{T}}_1$ is a normal tree on $\mathcal{R}_0$ with last model $\mathcal{R}_1$ and $\vec{\mathcal{T}}_2$ is a stack on $\mathcal{R}_1$ with last model $\mathcal{P}_1$,.*
4. *$\xi + 1 < \lambda^{\mathcal{S}_0}$, $\mathcal{S}_0(\xi + 1) = \mathcal{R}_0(\xi + 1)$, $\vec{\mathcal{T}}_1^- = \vec{\mathcal{U}}_1^-$[5] is a normal tree based on $\mathcal{S}(\xi+1)$ such that it has $\preceq^{\vec{\mathcal{T}}_1^-, s}$-maximal cutpoint $\mathcal{N}$ such that $(\vec{\mathcal{T}}_1^-)_{\geq \mathcal{N}}$ is based on $\mathcal{N}(\nu + 1)$ where $\nu = \pi^{(\vec{\mathcal{T}}_1^-)_{\mathcal{N}}}(\xi)$.*
5. *If b is the last branch of $\vec{\mathcal{T}}_1^-$ in $\vec{\mathcal{T}}_1$ then $b \neq \Sigma_{\mathcal{S}_0}(\vec{\mathcal{U}}_1^-)$.*
6. *Letting $\gamma = \pi^{\vec{\mathcal{T}}_1}(\xi) = \pi^{\vec{\mathcal{U}}_1}(\xi)$, $\mathcal{R}_1(\gamma + 1) = \mathcal{S}_1(\gamma + 1)$. If $\vec{\mathcal{W}}$ is the part of $\vec{\mathcal{T}}_2$ based on $\mathcal{R}_1(\gamma + 1)$ then $\vec{\mathcal{W}}$ is according to $\Sigma_{\mathcal{S}_1(\gamma+1)}$.*

Lemma 6.3 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and $\mathcal{P}$ is of limit type. Suppose further that Σ doesn't have strong branch condensation. Then there is a bad diamond on $(\mathcal{P}, \Sigma)$.*

---

[4]Recall that because Σ has branch condensation, Σ is positional and the strategy of $\mathcal{S}_{i-1}$ is independent of the particular iteration producing it (see Corollary 20.6 of [4]).

[5]Recall that this is just $\vec{\mathcal{T}}_1$ without its last model.

Proof. Let $(\vec{\mathcal{T}}, \mathcal{Q}, \pi, \mathcal{R}, \beta, \sigma)$ be a witness for the failure of strong condensation of $(\mathcal{P}, \Sigma)$. Let $(\vec{\mathcal{U}}_0, \mathcal{S}_0) \in I(\mathcal{P}, \Sigma)$ be such that $\mathcal{R}(\beta+1) = \mathcal{S}_0(\beta+1)$. We let $\pi = k$, $\mathcal{R}_0 = \mathcal{R}$ and $\xi = \beta$. Let $\Lambda = \Sigma^\sigma_{\mathcal{Q}}$. Let $\mathcal{T}$ be a normal tree on $\mathcal{R}(\beta+1)$ according to both $\Sigma_{\mathcal{S}_0}$ and Λ and such that $\Sigma_{\mathcal{S}_0}(\mathcal{T}) \neq \Lambda(\mathcal{T})$ but letting $b = \Sigma_{\mathcal{S}_0}(\mathcal{T})$, $c = \Lambda(\mathcal{T})$, $\mathcal{S}_1 = \mathcal{M}^{\vec{\mathcal{T}}}_b$ and $\mathcal{R}_1 = \mathcal{M}^{\vec{\mathcal{T}}}_c$ then $\Sigma_{\mathcal{S}_1(\pi^{\mathcal{T}}_b(\beta+1))} = \Lambda_{\mathcal{R}_1(\pi^{\mathcal{T}}_c(\beta+1)),\mathcal{T}}$. Such a $\mathcal{T}$ can be found using the Theorem 1.7. Notice that Theorem 1.7 is applicable because both Σ and Λ are $\Gamma(\mathcal{S}_0(\beta+\omega), \Sigma_{\mathcal{S}_0})$-fullness preserving (here we need to use Corollary 5.7 to conclude that $\Lambda_{\mathcal{R}(\beta+\omega)}$ has branch condensation). Let $\vec{\mathcal{T}}_1 = \mathcal{T}^\frown\{\mathcal{M}^{\mathcal{T}}_c\}$, $\vec{\mathcal{U}}_1 = \mathcal{T}^\frown\{\mathcal{M}^{\mathcal{T}}_b\}$, $\mathcal{R}_1 = \mathcal{M}^{\mathcal{T}}_c$ and $\mathcal{S}_1 = \mathcal{M}^{\mathcal{T}}_b$.

Next we would like to compare $(\mathcal{R}_1, \Lambda_{\mathcal{R}_1, \vec{\mathcal{T}}_1})$ and $(\mathcal{S}_1, \Sigma_{\mathcal{S}_1})$. To do this, we can use Corollary 5.7 and Theorem 3.3. Let then $(\vec{\mathcal{T}}_2, \mathcal{P}_1) \in I(\mathcal{R}_1, \Lambda_{\mathcal{R}_1, \vec{\mathcal{T}}_1})$ and $(\vec{\mathcal{U}}_2, \mathcal{P}_1) \in I(\mathcal{S}_1, \Sigma_{\mathcal{S}_1})$ be such that $\Sigma_{\mathcal{P}_1} = \Lambda_{\mathcal{P}_1, \vec{\mathcal{T}}_1{}^\frown \vec{\mathcal{T}}_2}$. It is then not hard to see that

$$((\mathcal{P}_i : i < 2), (\vec{\mathcal{T}}_i : i < 3), (\vec{\mathcal{U}}_i : i < 3), (\mathcal{R}_i : i < 2), (\mathcal{S}_i : i < 2), k, \xi)$$

is a bad diamond on $(\mathcal{P}, \Sigma)$. □

Now we want to show that there cannot be an ω_1-sequence of bad diamonds on $\mathcal{P}$.

Definition 6.4 (A bad diamond sequence of length β) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit. We say $\vec{D} = \langle D_\alpha : \alpha < \beta\rangle$ is a bad diamond sequence of length β if $D_\alpha = ((\mathcal{P}^\alpha_i : i < 2), (\vec{\mathcal{T}}^\alpha_i : i < 3), (\vec{\mathcal{U}}^\alpha_i : i < 3), (\mathcal{R}^\alpha_i : i < 2), (\mathcal{S}^\alpha_i : i < 2), k^\alpha, \xi^\alpha)$ and the following holds:*

1. *D_0 is a bad diamond on $(\mathcal{P}, \Sigma)$ and $\mathcal{P}^1_0 = \mathcal{P}^0_1$.*
2. *For all $\alpha < \beta$, $\mathcal{P}^0_\alpha \in pI(\mathcal{P}, \Sigma)$, D_α is a bad diamond on $(\mathcal{P}^\alpha_0, \Sigma_{\mathcal{P}^\alpha_0})$ and $\mathcal{P}^{\alpha+1}_0 = \mathcal{P}^\alpha_1$.*
3. *For $\nu < \alpha < \beta$, let $\pi_{\nu,\alpha} : \mathcal{P}^\nu_0 \to \mathcal{P}^\alpha_0$ be the embedding obtained by composing κ^γ with the iteration embeddings given by $\vec{\mathcal{T}}^\gamma_1{}^\frown\vec{\mathcal{T}}^\gamma_2$'s and $\sigma_{\nu,\alpha} : \mathcal{P}^\nu_0 \to \mathcal{P}^\alpha_0$ be the iteration embedding given by $\vec{\mathcal{U}}^\gamma_0{}^\frown\vec{\mathcal{U}}^\gamma_1{}^\frown\vec{\mathcal{U}}^\gamma_2$. Then for limit $\lambda < \beta$, $\mathcal{P}^\lambda_0$ is the direct limit of $(\mathcal{P}^\gamma_0 : \gamma < \lambda)$ under $\sigma_{\nu,\alpha}$, and $(\mathcal{P}^\lambda_0)^b$ is the direct limit of $((\mathcal{P}^\gamma_0)^b : \gamma < \lambda)$ under $\pi_{\nu,\alpha}$.*

We say that π embeddings are the top embeddings of $\vec{D}$ and σ embeddings are the bottom embeddings of $\vec{D}$.

Lemma 6.5 (No bad diamond sequence of length ω_1) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has a branch condensation. Then there is no bad diamond sequence of length ω_1 based on $(\mathcal{P}, \Sigma)$.*

Proof. Suppose not and let $\vec{D} = (D_\beta : \beta < \omega_1)$ be a bad diamond sequence of length ω_1. Let $\tau : H \to H_{\omega_2}$ be a countable submodel such that $\{\vec{D}, (\mathcal{P}, \Sigma)\} \in rng(\tau)$. Let $\kappa = \omega_1^H$. Notice that $\kappa = crit(\tau)$. Let for $\xi < \beta \leq \omega_1$, $\pi_{\xi,\beta} : \mathcal{P}_0^\xi \to \mathcal{P}_0^\beta$ be the composition of the top embedding of $\vec{D}$ and let $\sigma_{\xi,\beta} : \mathcal{P}_0^\xi \to \mathcal{P}_0^\beta$ be the composition of the bottom embeddings of $\vec{D}$. Let $\mathcal{P}^{\omega_1} = \tau(\mathcal{P}_0^\kappa)$. Standard arguments show that

$$\tau \restriction \mathcal{P}_0^\kappa = \pi_{\kappa,\omega_1} = \sigma_{\kappa,\omega_1}.$$

Let $j : \mathcal{R}_1^\kappa \to \mathcal{P}^{\omega_1}$ and $k : \mathcal{S}_1^\kappa \to \mathcal{P}^{\omega_1}$ be the composition of respectively the top and the bottom embeddings of $\vec{D}$. Let $\gamma = \pi^{\vec{\mathcal{T}}_1^\kappa}(\xi^\kappa)$. We then have that

(1) $j \restriction \mathcal{R}_1^\kappa(\gamma + 1) = k \restriction \mathcal{S}_1^\kappa(\gamma + 1)$.

Notice then

(2) $\delta_{\gamma+1}^{\mathcal{R}_1^\kappa} = \sup\{\pi^{\vec{\mathcal{T}}_1^\kappa} \circ k^\kappa(f)(a) : a \in (\mathcal{R}_1^\kappa(\gamma))^{<\omega} \wedge f \in \mathcal{P}_0^\kappa\}$ and $\delta_{\gamma+1}^{\mathcal{S}_1^\kappa} = \sup\{\pi^{\vec{\mathcal{U}}_0^\kappa \frown \vec{\mathcal{U}}_1^\kappa}(f)(a) : a \in (\mathcal{S}_1^\kappa(\gamma))^{<\omega} \wedge f \in \mathcal{P}_0^\kappa\}$

and because of (1),

(3) for all $f \in \mathcal{P}_0^\kappa$ and $a \in (\mathcal{S}_1^\kappa(\gamma))^{<\omega}$, $\pi^{\vec{\mathcal{T}}_0^\kappa \frown \vec{\mathcal{T}}_1^\kappa}(f)(a) = \pi^{\vec{\mathcal{U}}_0^\kappa \frown \vec{\mathcal{U}}_1^\kappa}(f)(a)$.

It then follows from (2) and (3) that

(4) $\delta_{\gamma+1}^{\mathcal{R}_1} = \sup(rng(\pi^{\vec{\mathcal{T}}_1}) \cap rng(\pi^{\vec{\mathcal{U}}_1}))$

contradicting the fact that $\vec{\mathcal{T}}_1$ isn't according to $\Sigma_{\mathcal{S}_0^\kappa}$. □

The next lemma finishes the proof of Theorem 6.1. Its proof is straightforward and we leave it to the reader.

Lemma 6.6 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and for every $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ doesn't have strong branch condensation. Then there is a bad diamond sequence on $(\mathcal{P}, \Sigma)$ of length ω_1.*

6.1 Getting branch condensation

In this section, we state a theorem that shows how to get branch condensation and on a tail by starting with a pair that has only hull condensation. This result will be used when proving generation of pointclasses (Theorem 12.2). The proof is very much like the proof of Theorem 6.1 and because of that we omit it.

Theorem 6.7 (Getting branch condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an anomalous hod pair of type II or III with the property that $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$. Suppose further that whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. Then there is $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation.*

7 Σ-closed mice

In this section, our goal is to prove a finer versions of generic interpretability than the one presented in Theorem 2.8 (see for instance Theorem 7.9). We start with the definition of *super fullness preservation*, which first appeared in [3] as Definition 3.33. Recall from [4] that given a transitive set X, we let $\mathcal{M}^+(X)$ be the least sound active mouse over X.

Definition 7.1 (Super fullness preservation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Σ is super fullness preserving if it is fullness preserving and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\alpha < \lambda^{\mathcal{Q}}$ is such that if $\mathcal{Q}$ is of limit type then $\alpha + 1 < \lambda^{\mathcal{Q}}$, the two sets*

$$U^{\Sigma}_{\mathcal{Q}(\alpha)} = \{(x, y) \in \mathbb{R}^2 : x \text{ codes a transitive set } a \in HC \text{ and } y \text{ codes } \mathcal{M} \text{ such that } \mathcal{M} \trianglelefteq Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) \text{ and } \rho(\mathcal{M}) = a\}$$

$$W^{\Sigma}_{\mathcal{Q}(\alpha)} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U^{\Sigma}_{\mathcal{Q}(\alpha)} \text{ and if } \mathcal{M} \text{ is the } \Sigma_{\mathcal{Q}(\alpha)}\text{-mouse coded by } y \text{ then } z \text{ codes a tree according to the unique strategy of } \mathcal{M}\}.$$

are term captured by $(\mathcal{Q}[g], \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ whenever $g \subseteq Coll(\omega, \mathcal{Q}(\alpha))$ is $\mathcal{Q}$-generic. We let $u^{\Sigma}_{\mathcal{Q}(\alpha)}$ and $w^{\Sigma}_{\mathcal{Q}(\alpha)}$ be the term relations locally capturing $U^{\Sigma}_{\mathcal{Q}(\alpha)}$ and $W^{\Sigma}_{\mathcal{Q}(\alpha)}$.

Notice that if $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is super fullness preserving then whenever $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, $\alpha < \lambda^{\mathcal{Q}}$ is such that if $\mathcal{Q}$ is of limit type then $\alpha + 1 < \lambda^{\mathcal{Q}}$, $g \subseteq Coll(\omega, \mathcal{Q}(\alpha))$, $a \in HC$ and x is a real coding a such that x is generic over $\mathcal{Q}[g]$ then

$$Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) = \{\mathcal{M} : \text{there is } y \in \mathbb{R}^{\mathcal{Q}[x]} \text{ such that } y \text{ codes } \mathcal{M} \text{ and } (x, y) \in (u^{\Sigma}_{\mathcal{Q}(\alpha)})_{g*x}\}.$$

Thus, $Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(a) \in \mathcal{Q}[x]$. Moreover, if h is $\mathcal{Q}[g]$-generic then the restriction of the function $b \rightarrow Lp^{\Sigma_{\mathcal{Q}(\alpha)}}(b)$ to $\mathcal{Q}[g * h]$ is definable over $\mathcal{Q}[g * h]$. Also, continuing with the above setting, if $\mathcal{M}$ is a sound $\Sigma_{\mathcal{Q}(\alpha)}$-mouse over a projecting to a and Λ is its unique iteration strategy then whenever h is $\mathcal{Q}[g][x]$ generic and κ is a cardinal of $\mathcal{Q}[g][x]$,

$$\Lambda \restriction H_\kappa^{\mathcal{Q}[g*x*h]} \in \mathcal{Q}[g * x * h].$$

This is because $\Lambda \restriction H_\kappa^{\mathcal{Q}[g*x*h]}$ can be defined over $\mathcal{Q}[g * x * h]$ using $w^{\Sigma}_{\mathcal{Q}(\alpha)}$.

Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is non-meek and N is Σ-closed (i.e., $\Sigma \restriction N$ is definable over N). We would like to show that generic extension of N are also Σ-closed. In order to show this, we need a way of finding $\mathcal{Q}$-structures for trees that are based on the top window of $\mathcal{P}$. Some of these $\mathcal{Q}$-structures would involve short tree strategy mice. The super fullness preservation cannot be used to find such $\mathcal{Q}$-structures (the way it was used in the proof of Lemma 3.34 or Lemma 3.35 of [3]). Here to prove an analogous result, we only consider sufficiently closed Σ-mice (where Σ is allowed to be an sts strategy).

Definition 7.2 (Σ-closed mouse) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair (possibly an sts hod pair) and $\mathcal{N}$ is a Σ-premouse. We say $\mathcal{N}$ is Σ-closed if for every $\mathcal{N}$-cardinal κ there is $\mathcal{M} \trianglelefteq \mathcal{N}$ such that $\mathcal{M} \vDash ZFC - Replacement$, $\mathcal{N}||\kappa \trianglelefteq \mathcal{M}$ and some $(\mathcal{P}, \Sigma^{\mathcal{M}})$-hod pair construction of $\mathcal{M}$ in which extenders used have critical points $> \kappa$ reaches a $\Sigma^{\mathcal{M}}$-iterate of $\mathcal{P}$. We say $\mathcal{N}$ is a Σ-closed mouse if it has a $(k, |\mathcal{P}|^+ + 1)$-iteration strategy Λ witnessing that $\mathcal{N}$ is a Σ-mouse. We say $\mathcal{M}$ witnesses Σ-closure of $\mathcal{N}$ at κ.*

Suppose $\mathcal{N}$ is a Σ-closed mouse, κ is an $\mathcal{N}$-cardinal and $\mathcal{M}$ is as in Definition 7.2. We then let $\mathcal{S}^{\mathcal{M}}_\kappa$ be the $\Sigma^{\mathcal{M}}$-iterate of $\mathcal{P}$ constructed via a fully backgrounded construction where critical points of extenders used are $> \kappa$.

We cannot in general hope to prove that if $(\mathcal{P}, \Sigma)$ is an sts pair and $\mathcal{N}$ is Σ-closed mouse then its generic extensions are also Σ-closed (not as mice). The reason is that the statement that an iterate of $\mathcal{P}$ has been constructed by the construction of $\mathcal{M}$ is too weak. What it means in the above definition is that $\mathcal{S}^{\mathcal{M}}_\kappa$ is a $\Sigma^{\mathcal{M}}$-iterate of $\mathcal{P}$, i.e., there is a normal tree $\mathcal{T}$ on $\mathcal{P}$ such that $\mathcal{M}^+(\mathcal{T}) \trianglelefteq \mathcal{S}^{\mathcal{M}}_\kappa$ and in $\mathcal{M}$, $\mathcal{S}^{\mathcal{M}}_\kappa$ is the stack of all $\Sigma^{\mathcal{M}}_{\mathcal{M}^+(\mathcal{T})}$-mice (consult clause 4 of Definition 13.1 of [4] for an idea on how $\mathcal{M}$ certifies the mice in this stack). However, it is quite possible that $\mathcal{N}$ may just not be full enough to find all $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-mice. We will prove our generic interpretability result for Σ-closed mice that have a fullness preserving iteration strategy in the following sense.

Keeping the notation and terminology of Definition 7.2, suppose Λ is an iteration strategy for $\mathcal{N}$ (witnessing that $\mathcal{N}$ is a Σ-mouse). Suppose Λ is an iteration strategy for $\mathcal{N}$. We then let $\Gamma(\mathcal{N}, \Lambda)$ be the collection of sets $A \subseteq \mathbb{R}$ such that for some $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{N}, \Lambda)$, there are

1. an $\mathcal{R}$-cardinal κ,
2. $\mathcal{M} \trianglelefteq \mathcal{R}$ witnessing that $\mathcal{R}$ is super Σ-closed at κ, and
3. $\alpha < \lambda^{\mathcal{S}_\kappa^{\mathcal{M}}} - 1$

such that

$$A \leq_w Code(\Sigma_{\mathcal{S}_\kappa^{\mathcal{M}}(\alpha), \mathcal{U}})$$

where $\mathcal{U}$ is the comparison tree on $\mathcal{P}$ with last model $\mathcal{S}_\kappa^{\mathcal{M}}$.

Definition 7.3 *We then say that Λ is a fullness preserving iteration strategy for $\mathcal{N}$ if for every $\mathcal{N}$-cardinal η, letting Λ^η be the fragment of Λ that acts on stacks above η, $\Gamma(\mathcal{N}, \Lambda^\eta) = \Gamma(\mathcal{P}, \Sigma)$.*

The next definition introduces our method of finding branches for trees in the generic extension. We recall the definition of $s(\vec{\mathcal{T}}, \xi)$ (see Definition 6.7 of [4]). Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is an almost non-dropping stack on $\mathcal{P}$. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}}, b}(\mathcal{P}^b)$. Then $\mathcal{Q}$ is an lsp. For $\xi + 1 \leq \lambda^{\mathcal{Q}}$, we let

$$s(\vec{\mathcal{T}}, \xi) = \{\alpha : \exists a \in (\delta_\xi^{\mathcal{Q}})^{<\omega} \exists f \in \mathcal{P}^b(\alpha = \pi^{\vec{\mathcal{T}}, b}(f)(a))\} \cap \delta_{\xi+1}^{\mathcal{Q}}$$

Definition 7.4 (Successful coiteration in Σ-closed mice) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair. Suppose $\mathcal{N}$ is a Σ-closed mouse, g is an $\mathcal{N}$-generic and $\mathcal{R} \in \mathcal{N}[g]$ is a hod premouse. Let κ be an $\mathcal{N}$-cardinal such that g is a $< \kappa$-generic and let $\mathcal{M}$ be as in Definition 7.2. Let*

$$((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\vec{\mathcal{T}}_\gamma : \gamma < \eta))$$

be the output of $(\mathcal{P}, \Sigma^{\mathcal{M}})$-construction of $\mathcal{M}$ where extenders used have critical point $> \kappa$. Let $\mathcal{S}_\kappa^{\mathcal{M}} = \mathcal{N}_\eta$ be the Σ-iterate of $\mathcal{P}$, $\mathcal{U}_\kappa^{\mathcal{M}} = \vec{\mathcal{T}}_{\eta-1}$. Let

$$\pi_\kappa^{\mathcal{M}} = \begin{cases} \pi^{\mathcal{U}_\kappa^{\mathcal{M}}} : \mathcal{P} \to \mathcal{S}_\kappa^{\mathcal{M}} & : \Sigma \text{ is a strategy} \\ \pi^{\mathcal{U}_\kappa^{\mathcal{M}}, b} : \mathcal{P} \to \mathcal{S}_\kappa^{\mathcal{M}} & : \text{otherwise.} \end{cases}$$

Notice that $\pi_\kappa^{\mathcal{M}} \in \mathcal{N}$.

For each $\xi < \eta$, *we let* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}$ *be the tree on* $\mathcal{R}$ *that is constructed by comparing* $\mathcal{R}$ *with the construction producing* $\mathcal{N}_\xi$. *More precisely,* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}$ *is constructed as follows. Suppose we have constructed* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma$. *We assume, part of the inductive construction, that no predicate appearing in a model producing* $\mathcal{N}_\xi$ *has been part of a disagreement. We proceed as follows.*

1. *Suppose* $\gamma = \beta + 1$. *If there is a disagreement on the* $\mathcal{N}_\xi$ *side then we stop the construction. More precisely, letting* $\mathcal{Q}$ *be the last model of* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma$, *the following holds:*
   (a) *If there is* η *such that* $\mathcal{Q}|\eta = \mathcal{N}_\xi|\eta$, $\mathcal{N}_\xi|\eta \neq \mathcal{N}_\xi||\eta$ *and* $\mathcal{Q}||\eta \neq \mathcal{N}_\xi||\eta$, *then we stop the construction.*
   (b) *If there is* η *such that* $\mathcal{Q}|\eta = \mathcal{N}_\xi|\eta$, $\mathcal{N}_\xi|\eta = \mathcal{N}_\xi||\eta$, $\mathcal{Q}||\eta \neq \mathcal{Q}|\eta$ *and* $\eta \notin dom(E^{\mathcal{Q}})$ *then we stop the construction.*
   (c) *If there is* η *such that* $\mathcal{Q}|\eta = \mathcal{N}_\xi|\eta$, $\mathcal{N}_\xi|\eta = \mathcal{N}_\xi||\eta$, $\mathcal{Q}||\eta \neq \mathcal{Q}|\eta$ *and* $\eta \in dom(E^{\mathcal{Q}})$ *then let* $E^{\mathcal{Q}}_\eta$ *be the next extender used in* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma$.
2. *Suppose* γ *is limit. Suppose further that* $\mathcal{N}_\xi \vDash$ "$\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma)$ *is not Woodin" and there is a cofinal well-founded branch* b *of* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma$ *such that* $\mathcal{Q}(b, \mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma)$*-exists and* $\mathcal{Q}(b, \mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma) \trianglelefteq \mathcal{N}_\xi$. *Then* II *continues by playing* b. *If there is no such* b *then stop the construction.*
3. *Suppose then* γ *is limit but* $\mathcal{N}_\xi \vDash$ "$\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma)$ *is Woodin". Let* μ *be such that* $\delta(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma) = \delta^{\mathcal{N}_\xi}_\mu$. *Suppose there is a cofinal well-founded branch* b *of* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma$ *such that for some* $\beta \in b$, $s(\mathcal{T}_\xi, \mu) \subseteq rng(\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi} \restriction \gamma}_{\beta,b})$. *Then* II *plays* b. *If there is no such* b *then we stop the construction.*

Suppose now that $(\mathcal{P}, \Sigma)$ *is a hod pair. Then we say* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}$ *is successful if it has a last model* $\mathcal{Q}$ *such that* $\mathcal{Q} \trianglelefteq \mathcal{N}_\xi$. *We also say that* $(\mathcal{R}, \mathcal{M}_\xi)$ *coiteration is successful.*

Next, suppose that $(\mathcal{P}, \Sigma)$ *is an sts hod pair. Then we say* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}$ *is successful if either*

1. $\mathcal{R}$ *is of lsa type,* $\mathcal{M}^+(\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}) = \mathcal{M}^+(\mathcal{S}^{\mathcal{M}}_\kappa | \delta^{\mathcal{S}^{\mathcal{M}}_\kappa})$ *and* $\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}, b}$ *exists or*
2. $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa,\xi}$ *has a last model* $\mathcal{Q}$ *such that* $\mathcal{Q} \trianglelefteq \mathcal{N}_\xi$.

Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{N}$ is a Σ-premouse which is Σ-closed. Suppose further that κ is an $\mathcal{N}$-cardinal, g is $< \kappa$-generic over $\mathcal{N}$, $\mathcal{R} \in \mathcal{N}|\kappa[g]$ is a hod premouse and $\mathcal{M}$ witnesses Σ-closure of $\mathcal{N}$ at κ.

Definition 7.5 (Good hod premice) *We say $\mathcal{R}$ is $(\mathcal{M},\kappa)$-good if $(\mathcal{R},\mathcal{S}^{\mathcal{M}}_{\kappa})$ coiteration is successful. If $\mathcal{R}$ is $(\mathcal{M},\kappa)$-good then we let $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa}$ be the comparison tree on $\mathcal{R}$ constructed via $(\mathcal{R},\mathcal{S}^{\mathcal{M}}_{\kappa})$ coiteration process. We also let $\pi^{\mathcal{M}}_{\mathcal{R},\kappa}=\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\kappa}}$. We say $\mathcal{R}$ is good if for all sufficiently large κ there is $\mathcal{M}$ such that $\mathcal{R}$ is $(\mathcal{M},\kappa)$-good.*

Notice that goodness is relative to $\mathcal{N}$ which will always be clear from context.

Fix now some good $\mathcal{R}\in\mathcal{N}|\kappa[g]$ and suppose $\mathcal{T}\in\mathcal{N}[g]$ is a tree on $\mathcal{R}$ without fatal drops. Suppose $\mathcal{T}$ *irreducible* (i.e., doesn't have cutpoints) and for some $\beta+1\leq\lambda^{\mathcal{R}}$, $\mathcal{T}$ is based on the top window of $\mathcal{R}(\beta+1)$.

Definition 7.6 (Correctly guided) *We say $\mathcal{T}$ is correctly guided if whenever λ is an $\mathcal{N}$ cardinal such that $\mathcal{T}\in\mathcal{N}|\lambda[g]$, there is $\mathcal{M}$ witnessing Σ-closure of $\mathcal{N}$ at λ such that whenever $\gamma<lh(\mathcal{T})$ is limit, letting b be the branch of $\mathcal{T}\restriction\gamma$ chosen by $\mathcal{T}$, the following holds:*

1. *$\mathcal{R}(\beta+1)$ is of limit type and one of the following holds.*
   - *(a) If $\mathcal{M}^{+}(\mathcal{T}\restriction\gamma)\vDash$ "$\delta(\mathcal{T}\restriction\gamma)$ isn't Woodin" then $\mathcal{Q}(b,\mathcal{T}\restriction\gamma)\trianglelefteq\mathcal{M}^{+}(\mathcal{T}\restriction\gamma)$.*
   - *(b) If $\mathcal{M}^{+}(\mathcal{T}\restriction\gamma)\vDash$ "$\delta(\mathcal{T}\restriction\gamma)$ is Woodin" then $(\mathcal{M}^{+}(\mathcal{T}\restriction\gamma),\mathcal{S}^{\mathcal{M}}_{\lambda})$ coiteration is successful and if $\mathcal{Q}\trianglelefteq\mathcal{M}_{\xi}$ is the last model of $\mathcal{U}^{\mathcal{M}}_{\mathcal{M}^{+}(\mathcal{T}\restriction\gamma),\lambda}$ then letting E be extender derived from $\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{M}^{+}(\mathcal{T}\restriction\gamma),\lambda}}$, $Ult(\mathcal{Q}(b,\mathcal{T}\restriction\gamma),E)\trianglelefteq\mathcal{S}^{\mathcal{M}}_{\lambda}$.*
2. *$\mathcal{R}(\beta+1)$ is not of limit type, $\mathcal{Q}(b,\mathcal{T}\restriction\gamma)$ holds and $(\mathcal{Q}(b,\mathcal{T}\restriction\gamma),\mathcal{S}^{\mathcal{M}}_{\lambda})$ coiteration is successful.*

Definition 7.7 (Certified stacks) *Continuing with the notation of Definition 7.4, suppose Σ is an iteration strategy (rather than an sts strategy) and $\vec{\mathcal{T}}\in\mathcal{N}[g]$ is a stack on $\mathcal{P}$ with no fatal drops. Let λ be an $\mathcal{N}$-cardinal such that $\vec{\mathcal{T}}\in\mathcal{N}|\lambda[g]$ and let $\mathcal{M}$ witness Σ-closure of $\mathcal{N}$ at λ. We say $\vec{\mathcal{T}}$ is $\mathcal{M}$-certified if the following conditions hold.*

1. *Suppose $\mathcal{R}$ is a cutpoint of $\vec{\mathcal{T}}$. Then for some ξ, $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\lambda,\xi}$ is successful. Moreover, if $\mathcal{R}$ is such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}}}$-exists then $\mathcal{R}$ is good. Also, if $E=E_{\vec{\mathcal{T}}}$ is the undropping extender of $\vec{\mathcal{T}}_{\leq\mathcal{R}}$ then $Ult(\mathcal{P},E)$ is good. We let*

$$\pi_{\mathcal{R}}=\begin{cases}\pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\lambda}} & :\pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}}}\text{-exists}\\ \pi^{\mathcal{U}^{\mathcal{M}}_{Ult(\mathcal{P},E),\lambda,\xi}} & :\text{otherwise.}\end{cases}$$

   Also, let

$$\sigma_{\mathcal{P},\mathcal{R}} = \begin{cases} \pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}}} & : \pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}}}\text{-}exists \\ \pi_E & : otherwise. \end{cases}$$

Then $\pi^{\mathcal{M}}_{\mathcal{P},\lambda} = \pi_{\mathcal{R}} \circ \sigma_{\mathcal{P},\mathcal{R}}$.

2. *Suppose* $\mathcal{R}$ *is a cutpoint of* $\vec{\mathcal{T}}$ *and* $\mathcal{T}$ *is the longest irreducible initial segment of* $\vec{\mathcal{T}}$ *which is based on* $\mathcal{R}$*. Then* $\mathcal{T}$ *is correctly guided.*

If $\vec{\mathcal{T}} \in \mathcal{N}[g]$ *is a certified stack such that its last vn-component is of limit length and* b *is a branch of* $\vec{\mathcal{T}}$ *then we say* b *is certified if* $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\mathcal{T}}_b\}$ *is certified.*

Continuing with the notation of Definition 7.4, we let Φ^* be the strategy of $\mathcal{P}$ that chooses certified branches for stacks with no fatal drop (this means that the continuation of the stack via the branch is certified). Assuming that $\Phi^* = \Sigma \restriction \mathcal{N}[g]$, we can extend Σ to act on stacks with fatal drops. To do this suppose that $\vec{\mathcal{T}} \in N[g]$ is a stack on $\mathcal{P}$ according to Φ^* and with last model $\mathcal{Q}$. Suppose $\eta < o(\mathcal{Q})$. We need to describe a strategy for $\mathcal{O}^{\mathcal{Q}}_{\eta,\eta}$.

Suppose first that for some $\beta < \lambda^{\mathcal{Q}}$, $\eta \in (o(\mathcal{Q}(\beta)), \delta^{\mathcal{Q}}_{\beta+1})$. We can then use super fullness preservation of Σ to dfine the strategy of $\mathcal{O}^{\mathcal{Q}}_{\eta,\eta}$ over $\mathcal{N}[g]$ (see the discussion after Definition 7.1).

Next, suppose for some $\beta + 1 \leq \lambda^{\mathcal{Q}}$, $\eta \in (\delta^{\mathcal{Q}}_{\beta}, \delta^{\mathcal{Q}}_{\beta+1})$ and $\mathcal{Q}(\beta+1)$ is of limit type. Let $\gamma \in dom(\vec{E})^{\mathcal{Q}}$ be the least such that $crit(E^{\mathcal{Q}}_{\gamma}) = \delta^{\mathcal{Q}}_{\beta}$ and $\gamma > \eta$. Then $\mathcal{O}^{\mathcal{Q}}_{\eta,\eta} = \mathcal{O}^{Ult(\mathcal{Q},E^{\mathcal{Q}}_{\gamma})}_{\eta,\eta}$. We can now use the super fullness preservation of $\Sigma_{Ult(\mathcal{Q},E^{\mathcal{Q}}_{\gamma})(\beta+1)}$ to define the strategy of $\mathcal{O}^{\mathcal{Q}}_{\eta,\eta}$ over $\mathcal{N}[g]$. We let Φ be this strategy of $\mathcal{P}$.

Definition 7.8 (Sts stacks with certified branches) *Suppose* $(\mathcal{P}, \Sigma)$ *is an sts hod pair and* $\mathcal{N}$ *is a super* Σ*-closed mouse,* g *is an* $\mathcal{N}$*-generic and* $\mathcal{R} \in \mathcal{N}[g]$ *is a hod premouse. Let* κ *be an* $\mathcal{N}$*-cardinal such that* g *is a* $<\kappa$*-generic and let* $\mathcal{M}$ *be as in Definition 7.2. Suppose* $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq m < \omega) \in \mathcal{N}[g]$ *is a stack on* $\mathcal{P}$*. Let* λ *be an* $\mathcal{N}$ *cardinal such that* $\vec{\mathcal{T}} \in \mathcal{N}|\lambda[g]$ *and let* $\mathcal{M}$ *witness* Σ*-closure of* $\mathcal{N}$ *at* λ*. We say* $\vec{\mathcal{T}}$ *has* $\mathcal{M}$*-certified branches if the following conditions hold:*

1. *Suppose* $\mathcal{R}$ *is a cutpoint of* $\vec{\mathcal{T}}$*. Then for some* ξ*,* $\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\lambda,\xi}$ *is successful. Moreover, if* $\mathcal{R}$ *is such that* $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}},b}$*-exists then* $\mathcal{R}^b$ *is good. Also, if* $E = E_{\vec{\mathcal{T}}}$ *is the undropping extender of* $\vec{\mathcal{T}}_{\leq\mathcal{R}}$ *then* $Ult(\mathcal{P}^b, E)$ *is good. We let*

$$\pi_{\mathcal{R}} = \begin{cases} \pi^{\mathcal{U}^{\mathcal{M}}_{\mathcal{R},\lambda},b} & : \pi^{\vec{\mathcal{T}}_{\leq\mathcal{R}},b}\text{-}exists \\ \pi^{\mathcal{U}^{\mathcal{M}}_{Ult(\mathcal{P}^b,E),\lambda,\xi}} & : otherwise. \end{cases}$$

Also, let

$$\sigma_{\mathcal{P},\mathcal{R}} = \begin{cases} \pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b} & : \pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}\textit{-exists} \\ \pi_E \restriction \mathcal{P}^b & : \textit{otherwise.} \end{cases}$$

Then $\pi_{\mathcal{P}}^{\mathcal{S}_\lambda^{\mathcal{M}}, b} = \pi_{\mathcal{R}} \circ \sigma_{\mathcal{P},\mathcal{R}}$.

2. *Suppose* $\mathcal{R}$ *is a cutpoint of* $\vec{\mathcal{T}}$ *and* $\mathcal{T}$ *is the longest irreducible initial segment of* $\vec{\mathcal{T}}$ *which is based on* $\mathcal{R}$. *Then* $\mathcal{T}$ *is correctly guided.*

Continuing with the terminology of Definition 7.8, if $\vec{\mathcal{T}} \in \mathcal{N}[g]$ and $b \in \mathcal{N}[g]$ is a branch of $\vec{\mathcal{T}}$ then we say b is certified if $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\mathcal{T}}\}$ is certified. We let Φ^* be the strategy of $\mathcal{P}$ that chooses certified branches. Assuming that $\Phi^* = \Sigma \restriction \mathcal{N}[g]$, we can extend Σ to act on stacks with fatal drops just like in Definition 7.7. We leave the details to the reader and let Φ be this strategy of $\mathcal{P}$.

It remains to show that $\Phi^* = \Sigma \restriction \mathcal{N}[g]$ which is the content of the next lemma.

Lemma 7.9 *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* Σ *is a strategy with branch condensation and super fullness preservation. Suppose* $\mathcal{N}$ *is a model of* $ZFC -$ *Replacement which is super* Σ*-closed. Let* g *be* $\mathcal{N}$*-generic. Then* $\mathcal{N}[g]$ *is* Σ*-closed.*

Proof. We show that $\Phi^* = \Sigma \restriction \mathcal{N}[g]$. To show this we need to show that

1. if $\vec{\mathcal{T}} \in \mathcal{N}[g]$ is according to Φ^* then it is according to Σ, and
2. if $\vec{\mathcal{T}} \in \mathcal{N}[g]$ is a stack on $\mathcal{P}$ according to Φ^* such that its last normal component has a limit length then $\Phi^*(\vec{\mathcal{T}})$ is defined.

We start with clause 1. To see it, fix some $\vec{\mathcal{T}} \in \mathcal{N}[g]$ which is according to Φ^* and Σ and let $b = \Phi^*(\vec{\mathcal{T}})$. We need to show that $b = \Sigma(\vec{\mathcal{T}})$. We can assume that there is no club $C \subseteq tn(\vec{\mathcal{T}})$ as otherwise $\Sigma(\vec{\mathcal{T}}) = b_C = \Phi^*(\vec{\mathcal{T}})$. Let then $\mathcal{R}$ be a terminal node of $\vec{\mathcal{T}}$ such that $\mathcal{U} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{R}}$ is an irreducible normal tree on $\mathcal{R}$. Recall that because $\vec{\mathcal{T}}$ is according to Φ^*, $\vec{\mathcal{T}}$ doesn't have fatal drops. Because $\vec{\mathcal{T}}$ is according to Σ and Φ^*, we have that $\mathcal{R}$ is good. We now have two cases.

Case 1. $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}}$**-exists.** We then have again have two sub cases. Suppose β is the least such that $\mathcal{U}$ is based on $\mathcal{R}(\beta)$. We then have the following cases.

Case 1.1. Suppose $\beta = \gamma + 1$ and $\mathcal{R}(\beta)$ is not of limit type. It follows that $\mathcal{U}$ is based on the window $(\delta_\gamma^{\mathcal{R}}, \delta_\beta^{\mathcal{R}})$. Fix then λ such that $\mathcal{R}, \vec{\mathcal{T}} \in \mathcal{N}|\lambda[g]$ and let $\mathcal{M}$ witness that $\mathcal{N}$ is Σ-closed at λ. We then have that $\mathcal{U}^\frown \mathcal{M}_b^{\mathcal{U}}$ is correctly guided. We want to see that this implies that $b = \Sigma(\vec{\mathcal{T}})$.

Suppose first that $\mathcal{Q}(b,\mathcal{U})$-exists. Let $\mathcal{Q} = \mathcal{Q}(b,\mathcal{U})$. We then have that for some ξ, $\mathcal{U}^{\mathcal{M}}_{\mathcal{Q},\lambda,\xi}$ is successful. Let $\mathcal{S}$ be the last model of $\mathcal{U}^{\mathcal{M}}_{\mathcal{Q},\lambda,\xi}$ and let $\tau : \mathcal{Q} \to \mathcal{S}$. Because $\mathcal{S}$ is iterable, it follows that $\mathcal{Q}$ is iterable above $\delta(\mathcal{U})$. We need to see that it is iterable as a $\Sigma_{\mathcal{R}(\beta)}$-mouse. Notice that we have that $\pi_{\mathcal{R}} \restriction \mathcal{R}(\beta) = \tau(\mathcal{R}(\beta))$. The desired conclusion then is an easy consequence of branch condensation.

Suppose next $\mathcal{Q}(b,\mathcal{U})$ doesn't exist. Let then $\mathcal{S} \trianglelefteq \mathcal{S}_\lambda^{\mathcal{M}}$ be the iterate of $\mathcal{Q} = \mathcal{M}_b^{\mathcal{U}}$. We then have that

$$\pi^{\mathcal{M}}_{\mathcal{P},\lambda} = \pi_{\mathcal{Q}} \circ \pi_b^{\vec{\mathcal{T}}}.$$

It again follows from branch condensation of Σ that $\Sigma(\vec{\mathcal{T}}) = b$.

Case 1.2. Suppose $\beta = \gamma + 1$ and $\mathcal{R}(\beta)$ is of limit type. It again follows that $\mathcal{U}$ is based on the top window of $\mathcal{R}$. We again have two cases and they both are similar two the cases considered above. First suppose that $\mathcal{Q}(b,\mathcal{U})$ doesn't exist. Then we finish as above.

Suppose then $\mathcal{Q}(b,\mathcal{U})$-exists. Let $\mathcal{Q} = \mathcal{Q}(b,\mathcal{U})$. If $\mathcal{Q} \trianglelefteq \mathcal{M}^+(\mathcal{M}(\mathcal{U}))$ then we must have that $\Sigma(\mathcal{U}) = b$. Otherwise, we have that for some ξ, $\mathcal{U}^{\mathcal{M}}_{\mathcal{M}^+(\mathcal{U}),\lambda,\xi}$ is successful. Let $\mathcal{S}^*$ be the last model of $\mathcal{U}^{\mathcal{M}}_{\mathcal{M}^+(\mathcal{U}),\lambda,\xi}$ and let $\tau : \mathcal{M}^+(\mathcal{U}) \to \mathcal{S}^*$. Let $\mathcal{S} = Ult(\mathcal{Q}, E)$ where E is the $(\delta(\mathcal{U}), \tau(\delta(\mathcal{U})))$-extender derived from τ. Because $\mathcal{S}$ is iterable, it follows that $\mathcal{Q}$ is iterable above $\delta(\mathcal{U})$. We need to see that it is iterable as a $\Sigma^{sts}_{\mathcal{M}^+(\mathcal{U})}$-mouse. Again, the desired conclusion is an easy consequence of branch condensation.

Case 2. $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}}$ **doesn't exist.** Suppose β is least such that $\mathcal{U}$ is based on $\mathcal{R}(\beta)$. Suppose first that $\beta + 1 < \lambda^{\mathcal{R}}$. Then the argument of case 1 applied to $Ult(\mathcal{P}, E_{\vec{\mathcal{T}}_{\leq \mathcal{R}}})$ shows that $\Sigma(\mathcal{U}) = b$. We thus assume that $\mathcal{R}$ is of limit type and $\beta = \lambda^{\mathcal{R}}$. Notice that we cannot have that $\mathcal{R}$ is of lsa type. Therefore, we must have that $\mathcal{Q}(b,\mathcal{U})$-exists. The argument from case 1.2 now shows that $\Sigma(\mathcal{U}) = b$.

This finishes the proof of clause 1. We now prove clause 2. Fix $\vec{\mathcal{T}} \in \mathcal{N}[g]$ according to Φ^* such that its last normal component has a limit length. We need to show that $\Phi^*(\vec{\mathcal{T}})$ is defined. First suppose that there is a club $C \subseteq tn(\vec{\mathcal{T}})$. Then b_C is according to Φ^*. Suppose then that there is no such club. Fix λ such that $\vec{\mathcal{T}} \in \mathcal{N}|\lambda[g]$ and let $\mathcal{M}$ witness that $\mathcal{N}$ is Σ-closed at λ. Let $\mathcal{R}$ be a cutpoint of $\vec{\mathcal{T}}$

such that $\mathcal{U} = \vec{\mathcal{T}}_{\geq \mathcal{R}}$ is a normal tree on $\mathcal{R}$. We again have two cases.

Case 1. $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}}$**-exists.** Let β be the least such that $\mathcal{U}$ is a tree on $\mathcal{R}(\beta)$. We then have the following cases.

Case 1.1. Suppose $\beta = \gamma + 1$ and $\mathcal{R}(\beta)$ is not of limit type. It follows that $\mathcal{U}$ is based on the window $(\delta^{\mathcal{R}}_{\gamma}, \delta^{\mathcal{R}}_{\beta})$. Fix then λ such that $\mathcal{R}, \vec{\mathcal{T}} \in \mathcal{N}|\lambda[g]$ and let $\mathcal{M}$ witness that $\mathcal{N}$ is Σ-closed at λ. It follows from the Dodd-Jensen property of $\Sigma_{\mathcal{R}(\beta)}$ that $(\mathcal{M}(\mathcal{U}), \mathcal{S}^{\mathcal{M}}_{\lambda})$ coiteration is successful. Let $\mathcal{W}$ be the tree on $\mathcal{M}(\mathcal{U})$ with last model $\mathcal{S}$ such that $\mathcal{S} \trianglelefteq \mathcal{S}^{\mathcal{M}}_{\lambda}$. Let $\tau : \mathcal{M}(\mathcal{U}) \rightarrow \mathcal{S}$ be the iteration embedding.

Suppose now that $\delta(\mathcal{W})$ is not Woodin in $\mathcal{S}^{\mathcal{M}}_{\lambda}$. Let $b = \Sigma(\vec{\mathcal{T}})$. Notice that b is the unique branch of $\mathcal{U}$ such that $\mathcal{Q}(b, \mathcal{U})$-exists and $Ult(\mathcal{Q}(b, \mathcal{U}), E_{\tau}) \trianglelefteq \mathcal{S}^{\mathcal{M}}_{\lambda}$. It follows that $b \in \mathcal{N}[g]$ and $\Phi^*(\mathcal{U}) = b$.

Suppose then $\delta(\mathcal{W})$ is Woodin in $\mathcal{S}^{\mathcal{M}}_{\lambda}$. Let $b = \Sigma(\vec{\mathcal{T}})$. Let $c = \Sigma(\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}^\frown\mathcal{W})$. Notice that c is the unique branch of $\mathcal{W}$ such that $s(\mathcal{K}, \xi) \subseteq \pi^{\mathcal{W}}_c[\delta(\mathcal{U})]$ where $\mathcal{K}$ is the normal tree on $\mathcal{P}$ with last model $\mathcal{S}^{\mathcal{M}}_{\lambda}$ and ξ is such that $\delta(\mathcal{W}) = \delta^{\mathcal{S}^{\mathcal{M}}_{\lambda}}_{\xi}$. It then follows that $c \in \mathcal{N}[g]$. We then again have that b is the unique branch of $\vec{\mathcal{T}}$ such that there is $(\pi^{\mathcal{W}}_c)^{-1}(s(\mathcal{K}, \xi)) \subseteq rng(\pi^{\mathcal{U}}_b)$. Hence, $b \in \mathcal{N}[g]$. It is now not hard to check that indeed $\Phi^*(\vec{\mathcal{T}}) = b$.

Case 1.2. Suppose $\beta = \gamma + 1$ and $\mathcal{R}(\beta)$ is of limit type. If $\beta < \lambda^{\mathcal{R}}$ then we must have that $\mathcal{Q}(b, \mathcal{U})$ exists where $b = \Sigma(\vec{\mathcal{T}}^\frown\mathcal{U})$. Then the argument from case 1.1 shows that $b \in \mathcal{N}[g]$ and $\Phi^*(\mathcal{U}) = b$. We then assume that $\beta = \lambda^{\mathcal{R}}$. Again, because of the same argument we can assume that $\mathcal{R}$ is of lsa type and if $b = \Sigma(\vec{\mathcal{T}}^\frown\mathcal{U})$ then $\mathcal{Q}(b, \mathcal{U})$ doesn't exist. In this case, b is the unique branch such that there is an embedding $\sigma : \mathcal{M}^{\mathcal{U}}_b \rightarrow \mathcal{S}^{\mathcal{M}}_{\lambda}$ such that $\pi^{\mathcal{M}}_{\lambda} = \sigma \circ \pi^{\vec{\mathcal{T}}}_b$. It then follows that $b \in \mathcal{N}[g]$ and that $\Phi^*(\vec{\mathcal{T}}) = b$.

This finishes the proof of the lemma in the case Σ is an iteration strategy. □

The proof of Lemma 7.9 can be used to prove an equivalent lemma for sts hod pairs. However, we have to require that stacks have only one main round. To prove the more general result we will require more closure properties.

Lemma 7.10 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is an sts strategy which is fullness preserving and has branch condensation. Suppose $\mathcal{N}$ is Σ-closed, Λ is a fullness preserving iteration strategy for $\mathcal{N}$ and g is $\mathcal{N}$-generic. Let Φ^1 be the fragment of Φ^* that acts on stacks with a single main round. Then $\Phi^1 \restriction b(\Phi^1) = \Sigma \restriction (\mathcal{N}[g] \cap b(\Sigma))$.*

Proof. The proof is very similar to the proof of Lemma 7.9. Because of this, we only indicate the parts that are different. The part that is different is to show that if $\vec{\mathcal{T}}$ is a stack according to Φ^1 and $\vec{\mathcal{T}} \in b(\Sigma)$ then $\Sigma(\vec{\mathcal{T}}) \in \mathcal{N}[g]$ and $\Phi(\vec{\mathcal{T}}) = b$. The reason that the proof here requires more care is because we do not that $\pi^{\mathcal{S}_\lambda^{\mathcal{M}}}$ exists for various $(\mathcal{M}, \lambda)$.

Fix then $\vec{\mathcal{T}}$ as above. We can assume that there is a cutpoint $\mathcal{R}$ such that $\mathcal{U} = \vec{\mathcal{T}}_{\geq \mathcal{R}}$ is a normal tree on $\mathcal{R}$. Fix β such that $\mathcal{U}$ is based on $\mathcal{R}(\beta)$. Suppose first that $\mathcal{R}(\beta) \in pB(\mathcal{P}, \Sigma)$. It follows from the proof of Lemma 7.9 that it is enough to show that there is $(\mathcal{M}, \lambda)$ such that $\mathcal{R}(\beta)$ is $(\mathcal{M}, \lambda)$-good.

Let ν be such that $\vec{\mathcal{T}} \in \mathcal{N}|\nu[g]$. We can then find $(\vec{\mathcal{W}}, \mathcal{S}) \in I(\mathcal{N}, \Lambda)$ such that $\vec{\mathcal{W}}$ is above ν and for some $\mathcal{S}$-cardinal λ, $\mathcal{M} \trianglelefteq \mathcal{S}$ and α, $Code(\Sigma_{\mathcal{S}_\lambda^{\mathcal{M}}(\alpha)}) =_w Code(\Sigma_{\mathcal{R}(\beta)})$. It then follows that in $\mathcal{S}$, $\mathcal{R}$ is $(\mathcal{M}, \lambda)$-good. By elementarity, such an $(\mathcal{M}, \lambda)$ exists in $\mathcal{N}$.

Suppose then $\mathcal{R}(\beta) \in pI(\mathcal{P}, \Sigma)$. The proof is very similar. It is enough to find $(\mathcal{M}, \lambda)$ such that $\mathcal{M}^+(\mathcal{U})$ is $(\mathcal{M}, \lambda)$-good. We can find such an $(\mathcal{M}, \lambda)$ using the above argument and using the fact that if $\vec{\mathcal{T}} \in b(\Sigma)$. □

To generalize Lemma 7.9 to stacks with arbitrary many main rounds we need to know that $\mathcal{N}$ satisfies a certain fullness condition defined below. The reason is that we need to be able to certify models not just branches.

Definition 7.11 (Fullness condition) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is an sts strategy which is fullness preserving and has branch condensation. Suppose $\mathcal{N}$ is Σ-closed. We say $\mathcal{N}$ satisfies Σ-fullness condition if whenever g is $\mathcal{N}$-generic, $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq m) \in \mathcal{N}[g]$ is a stack on $\mathcal{P}$ according to Σ such that $\vec{\mathcal{T}} \in m(\Sigma)$ and κ is an $\mathcal{N}$-cardinal such that $\vec{\mathcal{T}} \in \mathcal{N}|\kappa[g]$, letting $\mathcal{Q} = \Sigma(\vec{\mathcal{T}})$, $\mathcal{Q} \in \mathcal{N}$ and for any $\mathcal{S}$ such that $\mathcal{M}^+(\mathcal{Q}|\delta^{\mathcal{Q}}) \trianglelefteq \mathcal{S} \trianglelefteq \mathcal{Q}$ and $\rho(\mathcal{S}) = \delta^{\mathcal{Q}}$, there is $\mathcal{M} \trianglelefteq \mathcal{N}$ such that*

1. *$\mathcal{M}$ witnesses Σ-closure of $\mathcal{N}$ at κ and*
2. *$\mathcal{M} \vDash ZFC - Powerset +$ "there are infinitely many Woodin cardinals $\langle \delta_n : n < \omega \rangle$"*
3. *$\mathcal{S} \in \mathcal{M}|\delta_0$, and $\mathcal{S}$ has an iteration strategy $\Lambda \in \mathcal{M}$ acting on trees that are above $\delta^{\mathcal{Q}}$ and such that if $h \subseteq Coll(\omega, < \delta_\omega)$-generic and D is the derived model of $\mathcal{M}$ computed via h then Λ has an extension $\Lambda^* \in D$ such that whenever $\mathcal{S}^*$ is a Λ^*-iterate of $\mathcal{S}$ and $\vec{\mathcal{U}} \in dom(\Sigma^{\mathcal{S}^*})$ then $\mathcal{M}[h] \vDash$ "$\vec{\mathcal{T}}^\frown \vec{\mathcal{U}}$ is branch certified stack (see Definition 7.8)".*

Lemma 7.12 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is an sts strategy which is fullness preserving and has branch condensation. Suppose $\mathcal{N}$ is Σ-closed, Λ is a $\Gamma(\mathcal{P}, \Sigma)$-fullness preserving strategy for $\mathcal{N}$ and whenever $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{N}, \Lambda)$, $\mathcal{S}$ has fullness condition. Suppose g is $\mathcal{N}$-generic. Then $\Phi^* = \Sigma \upharpoonright \mathcal{N}[g]$.*

Proof. The proof is just like the proof of Lemma 7.10 except now we can certify models by using the fullness condition. Namely, suppose $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq m) \in \mathcal{N}[g]$ is a stack according to both Φ^* and Σ. Suppose that there is no club $C \subseteq tn(\vec{\mathcal{T}})$. Let $\mathcal{R}$ be a cutpoint such that $\vec{\mathcal{T}}_{\geq \mathcal{R}}$ is a normal irreducible tree on $\mathcal{R}$ of limit length. Suppose that the branch certification process doesn't yield a branch for $\vec{\mathcal{T}}_{\geq \mathcal{R}}$. Then we let $\Phi^*(\vec{\mathcal{T}})$ be the union of $\mathcal{S}$ such that $\mathcal{M}^+(\vec{\mathcal{T}}_{\geq \mathcal{R}}) \trianglelefteq \mathcal{S}$, $\rho(\mathcal{S}) = \delta(\vec{\mathcal{T}}_{\geq \mathcal{R}})$ and $\mathcal{S}$ has an iteration strategy as in clause 3 of Definition 7.11. The proof of Lemma 7.10 then shows that $\Phi^*(\vec{\mathcal{T}}) = \Sigma(\vec{\mathcal{T}})$. □

Lemma 7.12 left open whether there can exist Σ-closed mice which satisfy Σ-fullness condition. Our source of such $\mathcal{N}$ is the universality of background constructions which will give us mouse capturing for short tree strategies.

Definition 7.13 (MC for sts strategies) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair such that Σ has branch condensation and is fullness preservation. We let $MC(\Sigma)$ stand for the following statement: for every transitive $a \in HC$ such that $\mathcal{P} \in a$,*

$$\wp(a) \cap \mathrm{HOD}_{\Sigma, a \cup \{a\}} = Lp^{\Sigma}(a).$$

Lemma 7.14 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is an sts strategy which is fullness preserving and has branch condensation. Suppose further that $MC(\Sigma)$ holds. Let $\mathcal{N}$ be a Σ-closed mouse which has a fullness preserving iteration strategy and let $(\delta_i : i < \omega)$ be a sequence of Woodin cardinals of $\mathcal{N}$. Then $\mathcal{N}|\delta_\omega$ satisfies Σ-fullness condition.*

Proof. To see this, let $\vec{\mathcal{T}} \in \mathcal{N}[g]$ be a maximal stack according to Σ and let $\mathcal{Q}$ be its last model. We have that $\mathcal{M}^+(\mathcal{Q}|\delta^{\mathcal{Q}}) \in \mathcal{N}[g]$ and need to show that $\mathcal{Q} \in \mathcal{N}[g]$ and clause 2 of Definition 7.11 is satisfied. Notice that $\mathcal{Q}$ is $OD_{\Sigma, \mathcal{Q}|\delta^{\mathcal{Q}}}$. It the follows from $MC(\Sigma)$ that $\mathcal{Q} \in \mathcal{N}|\delta_\omega[g]$.

To see clause 2 of Definition 7.11, fix $\mathcal{S}$ such that $\mathcal{Q}|\delta^{\mathcal{Q}} \trianglelefteq \mathcal{S} \trianglelefteq \mathcal{Q}$ such that $\rho(\mathcal{S}) = \delta^{\mathcal{Q}}$. Let Ψ be the strategy of $\mathcal{S}$ which acts on trees above $\delta^{\mathcal{Q}}$. Because Ψ is $OD_{\Sigma, \mathcal{S}}$ and $\mathcal{S} \in \mathcal{N}$, we have that letting $\lambda = (\delta_\omega^+)^{\mathcal{N}}$, $\Psi \upharpoonright \mathcal{N}|\lambda \in \mathcal{N}$. It then follows from the proof of Lemma 7.9, that letting $\Lambda = \Psi \upharpoonright \mathcal{N}|\lambda$, Λ witnesses clause 2 for $\mathcal{M} = \mathcal{N}|\delta_\omega$. Fixing now $\kappa < \delta_\omega$, using condensation, we have that there must be $\mathcal{M} \trianglelefteq \mathcal{N}|(\kappa^+)^{\mathcal{N}}$ with the desired property. □

8 $\Gamma(\mathcal{P},\Sigma)$ when $\lambda^{\mathcal{P}}$ is a successor

In this section, we translate the results of Section 5.6 of [3] to our current context. Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor and Σ is super fullness preserving and has branch condensation. Suppose first that $\mathcal{P}$ isn't of lsa type. We then let

$$\Gamma_\Sigma = (\Sigma^2_1(Code(\Sigma_{\mathcal{P}^-})))^{L(Mice_{\Sigma_{\mathcal{P}^-}},\mathbb{R})}.$$

Notice that Γ_Σ is a lightface good pointclass. Also $Mice_{\Sigma_{\mathcal{P}^-}}$ belongs to Γ_Σ and is a universal Γ_Σ set. We let

$$\Gamma(\mathcal{P},\Sigma) = \{A : \text{for cone of } x \in \mathbb{R},\ A \cap C_{\Gamma_\Sigma}(x) \in C_{\Gamma_\Sigma}(C_{\Gamma_\Sigma}(x))\} = Env(\Gamma_\Sigma).$$

Notice that if $(\mathcal{Q},\Lambda)$ is a tail of $(\mathcal{P},\Sigma)$ then $\Gamma(\mathcal{Q},\Lambda) = \Gamma(\mathcal{P},\Sigma)$.

The above definition defines $\Gamma(\mathcal{P},\Sigma)$ when $\lambda^{\mathcal{P}}$ is a successor but $\mathcal{P}$ isn't of lsa type. In particular, $\Gamma(\mathcal{P},\Sigma)$ is not an LSA pointclass. The difficulty with generating LSA pointclasses as $\Gamma(\mathcal{P},\Sigma)$ is the following: Suppose Γ is an LSA pointclass, i.e., $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma,\mathbb{R})$ and $L(\Gamma,\mathbb{R}) \vDash AD^+ + LSA$. Let α be such that $\alpha + 1 = \Omega^\Gamma$ and set $\Gamma^b = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$[6]. The difficulty is that the pair that generates Γ^b is the same as the pair that generates Γ.

In what follows we will use the following notation: if $(\mathcal{P},\Sigma)$ is a hod pair such that $\mathcal{P}$ is of lsa type then we let $\Gamma(\mathcal{P},\Sigma) = \{A \subseteq \mathbb{R} : \exists \mathcal{Q} \in pB(\mathcal{P},\Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q}}))\}$. We reserve the notation $\Gamma^u(\mathcal{P},\Sigma)$ for the upper part of the lsa pointclass. In what follows we will describe a way of defining it.

Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that Σ is fullness preserving and $\mathcal{P}$ is of lsa type. We then let

$$\Gamma^u(\mathcal{P},\Sigma) = \{A : \text{for cone of } x \in \mathbb{R},\ A \cap Lp^{\Sigma^{sts}}(x) \in Lp_2^{\Sigma^{sts}}(x)\}.$$

It is not immediately clear that $L(\Gamma^u(\mathcal{P},\Sigma)) \cap \wp(\mathbb{R}) = \Gamma^u(\mathcal{P},\Sigma)$. The next lemma shows that this is indeed true. Notice that if $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ then $\Gamma^u(\mathcal{Q},\Sigma_{\mathcal{Q}}) = \Gamma^u(\mathcal{P},\Sigma)$.

Lemma 8.1 *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\mathcal{P}$ is of lsa type and Σ is fullness preserving. Then*

$$L(\Gamma^u(\mathcal{P},\Sigma)) \cap \wp(\mathbb{R}) = \Gamma^u(\mathcal{P},\Sigma)$$

[6] The superscript "b" stands for bottom.

and the set $\{(x, y) : y$ *codes an* Σ^{sts}*-mouse over* $x\}$ *cannot be uniformized in* $L(\Gamma^u(\mathcal{P}, \Sigma))$. *Hence,*

$$L(\Gamma^u(\mathcal{P}, \Sigma)) \vDash AD^+ + LSA.$$

Proof. We only outline the argument. The proof is very much like the proof of Lemma 5.16 of [3]. Let Γ be any good pointclass such that $Code(\Sigma) \in \underset{\sim}{\Delta}_\Gamma$. Let F be as in Theorem 2.25 of [3] for Γ. Fix x such that if $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ then $Code(\Sigma)$ is Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$. We then have that the fully backgrounded hod pair construction of $\mathcal{N}_x^*|\delta_x$ reaches a tail of $(\mathcal{P}, \Sigma)$ (see Theorem 1.7 above or Corollary 18.12 of [4]). Let $(\mathcal{Q}, \Lambda)$ be this tail. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}, \Lambda^{sts}})^{\mathcal{N}_x^*|\delta_x}$.

Because Σ is fullness preserving we have that no level of $\mathcal{N}$ projects acres $\delta^{\mathcal{Q}}$. We also have that the least strong cardinal of $\mathcal{N}$ is a limit of Woodin cardinals (the proof of this is just like the proof of the similar fact in the proof of Lemma 5.16 of [3]). It then follows that $L(\Gamma^u(\mathcal{P}, \Sigma))$ can be realized as a derived model of $\mathcal{N}|\lambda$ via Ψ, where letting κ be the least strong of $\mathcal{N}$, $\lambda = (\kappa^+)^{\mathcal{N}}$ and Ψ is the strategy of $\mathcal{N}|\lambda$ induced by the background construction.

To carry out the above outline, we use several results. First we use the proof of Theorem 18.3 to conclude that Ψ is fullness preserving. Next we use Lemma 7.14 to conclude that $\mathcal{N}|\lambda$ satisfies Σ-fullness condition. Lemma 7.12 can the be used to show that $L(\Gamma^u(\mathcal{P}, \Sigma))$ can indeed be realized as a derived model of $\mathcal{N}|\lambda$ via Ψ. □

9 B-iterability

In this section, we import B-iterability technology to our current context. Most of what we will need was laid out in [3]. Here we will only sketch the necessary arguments.

Definition 9.1 (Suitable pair) $(\mathcal{P}, \Sigma)$ *is a* suitable pair *if*

1. $\mathcal{P}$ *is a hod premouse,* $\lambda^{\mathcal{P}}$ *is a successor ordinal and* $\mathcal{P}$ *is not of limit type,*
2. $(\mathcal{P}(\lambda^{\mathcal{P}} - 1), \Sigma)$ *is a hod pair such that* Σ *has branch condensation and is fullness preserving*
3. $\mathcal{P}$ *is a* $\Sigma_{\mathcal{P}(\lambda^{\mathcal{P}}-1)}$*-mouse above* $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$,
4. *for any* $\mathcal{P}$*-cardinal* $\eta > \delta^{\mathcal{P}}_{\lambda-1}$, *if* η *is a strong cutpoint then* $\mathcal{P}|(\eta^+)^{\mathcal{P}} = Lp^{\Sigma}(\mathcal{P}|\eta)$

Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are such that Σ and Λ have branch condensation and are fullness preserving. We then let $(\mathcal{P}, \Sigma) \leq_{DJ} (\mathcal{Q}, \Lambda)$ if and only if $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$. Notice that $\leq_{DJ}$ is a well-founded relation. We then let $\alpha(\mathcal{P}, \Sigma) = |(\mathcal{P}, \Sigma)|_{\leq_{DJ}}$, and we let $[\mathcal{P}, \Sigma]$ be the $=_{DJ}$ equivalence class of $(\mathcal{P}, \Sigma)$, i.e.,

$(\mathcal{Q}, \Lambda) \in [\mathcal{P}, \Sigma]$ iff $(\mathcal{Q}, \Lambda)$ is a hod pair such that Λ has branch condensation and is super fullness preserving and $\alpha(\mathcal{Q}, \Lambda) = \alpha(\mathcal{P}, \Sigma)$.

Notice that $[\mathcal{P}, \Sigma]$ is independent of $(\mathcal{P}, \Sigma)$. We let

$$\mathbb{B}(\mathcal{P}, \Sigma) = \{B \subseteq [\mathcal{P}, \Sigma] \times \mathbb{R} : B \text{ is } OD\}.$$

Note that $\mathbb{B}(\mathcal{P}, \Sigma)$ is defined for hod pairs not suitable pairs.

The following standard lemma features prominently in our computations of HOD. The proof is very much like the proof of Lemma 4.16 of [3]. Given a hod premouse $\mathcal{P}$ such that $\lambda^{\mathcal{P}}$ is successor and $\mathcal{P}$ is not of limit type, we let $\mathcal{P}^- = \mathcal{P}(\lambda^{\mathcal{P}} - 1)$.

Lemma 9.2 *Assume SMC and suppose $(\mathcal{P}, \Sigma)$ is a suitable pair. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ and $\kappa > \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is a $\mathcal{P}$-cardinal. Then there is $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ such that $(\mathcal{P}, \tau)$ locally term captures $B_{(\mathcal{P},\Sigma)}$ at κ for comeager set of $\mathcal{P}$-generics.*

If B is locally term captured for comeager many set generics over a suitable pair $(\mathcal{P}, \Sigma)$ then we let $\tau^{\mathcal{P},\Sigma}_{B,\kappa}$ be the invariant term in $\mathcal{P}$ locally term capturing B at κ for comeager many set generics. One way to get term capturing for all generics is to show that a suitable pair can be extended to a structure that has one more Woodin.

Definition 9.3 (n-Suitable pair) *$(\mathcal{P}, \Sigma)$ is an n-*suitable pair *if there is δ such that*

1. *$(\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}, \Sigma)$ is suitable,*
2. *$\mathcal{P} \vDash ZFC - Replacement +$ "there are n Woodin cardinals, $\eta_0 < \eta_1 < ... < \eta_{n-1}$ above δ",*
3. *$o(\mathcal{P}) = \sup_{i<\omega}(\eta_n^{+i})^{\mathcal{P}}$,*
4. *$\mathcal{P}$ is a Σ-mouse over $\mathcal{P}|\delta$,*
5. *for any $\mathcal{P}$-cardinal $\eta > \delta$, if η is a strong cutpoint then $\mathcal{P}|(\eta^+)^{\mathcal{P}} = Lp^{\Sigma}(\mathcal{P}|\eta)$.*

If $(\mathcal{P}, \Sigma)$ is n-Suitable then we let $\delta^{\mathcal{P}}$ be the δ of Definition 9.3 and

$$\mathcal{P}^- = ((\mathcal{P}|((\delta^{\mathcal{P}})^{+\omega})^{\mathcal{P}})^-.$$

We let $\lambda^{\mathcal{P}} = \lambda^{\mathcal{P}^-} + 1$. Clearly 0-suitable pair is just a suitable pair. The following are easy consequences of Lemma 9.2.

Lemma 9.4 *Assume SMC and suppose $(\mathcal{P}, \Sigma)$ is a n-suitable pair. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ and $\kappa > \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is a $\mathcal{P}$-cardinal. Then there is $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ such that $(\mathcal{P}, \tau)$ locally term captures $B_{(\mathcal{P},\Sigma)}$ at κ for comeager set of $\mathcal{P}$-generics.*

Corollary 9.5 *Assume SMC and suppose $(\mathcal{P}, \Sigma)$ is a n-suitable pair such that $n > 0$. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$, $\delta = \delta^{\mathcal{P}}$, $\nu = (\delta^{+\omega})^{\mathcal{P}}$ and $\kappa \in (\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \nu)$ is a $\mathcal{P}$-cardinal. Then $(\mathcal{P}|\nu, \tau^{\mathcal{P},\Sigma}_{B,\kappa})$ locally term captures $B_{(\mathcal{P},\Sigma)}$ at κ.*

Corollary 9.5 is our main method of showing that various B are term captured over the hod mice we will construct.

Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair. It is now a trivial matter to import the terminology of Section 4.1 of [3] to our current context. We will have that $S(\Sigma)$ consist of those $\mathcal{Q}$ such that $\mathcal{Q}^- \in pI(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-})$ is a suitable pair. Given $\mathcal{Q} \in S(\Sigma)$, we let $f_B(\mathcal{Q}) = \oplus_{\kappa < o(\mathcal{Q})} \tau^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}_{B,\kappa}$. Then the rest of the notions are defined for $F = \{f_B : B \in \mathbb{B}(\mathcal{P}, \Sigma)\}$. Therefore, in the sequel, we will freely use the terminology of Section 4.1 of [3].

Before we move on, we remark that the same notions make sense for lsa pairs as well. Here by suitable we mean just an lsa pair $(\mathcal{P}, \Sigma)$ such that Σ is a fullness preserving iteration strategy with branch condensation. Given a suitable lsa pair $(\mathcal{P}, \Sigma)$, we let $M(\mathcal{P}, \Sigma)$ be the set of pairs $(\mathcal{M}^+(\mathcal{Q}|\delta^{\mathcal{Q}}), \Sigma_{\mathcal{M}^+(\mathcal{Q}|\delta^{\mathcal{Q}})})$ such that there is a stack $\vec{\mathcal{T}}$ according to Σ and with last model $\mathcal{Q}$ such that I can start a new main round on $\mathcal{Q}$. We then let

$$\mathbb{B}(\mathcal{P}, \Sigma) = \{B \subseteq pM(\mathcal{P}, \Sigma) \times \mathbb{R} : B \text{ is } OD\}.$$

The rest of the concepts cary over word by word. We leave it to the reader.

10 Getting $\vec{B}$-guided pairs

In this section, we would like to prove that there are hod pairs $(\mathcal{P}, \Sigma)$ such that Σ is $\vec{B}$-guided for some $\vec{B}$. The non lsa case is exactly as in [3] (see Theorem 5.20). To prove such a result for lsa hod pairs, we need to construct a model which satisfies the hypothesis of Lemma 7.14. Full backgrounded constructions do produce such models. The following lemma shows just that. Suppose $(\mathcal{P}, \Sigma)$ is an lsa hod pair. We then let

$$Mice^{sts}_{\Sigma} = \{(a, \mathcal{M}) : a \in HC \wedge \mathcal{P} \in a \wedge \mathcal{M} \trianglelefteq Lp^{\Sigma^{sts}}(a) \wedge \rho(\mathcal{M}) = a\}.$$

Recall that if M is a model of some fragment of set theory, λ is a limit of Woodin cardinals of M and $g \subseteq Coll(\omega, <\lambda)$ is M-generic then $D(M, \lambda, g)$ is the derived model of M at λ as computed by g. More precisely, letting $\mathbb{R}^*$ be the symmetric reals, we set

$$\Gamma = \{A \subseteq \mathbb{R}^* : A \in M(\mathbb{R}^*) \text{ and } L(A, \mathbb{R}^*) \vDash AD^+\}.$$

Then $D(M, \lambda, g) = L(\Gamma, \mathbb{R}^*)$.

Lemma 10.1 *Suppose $(\mathcal{P}, \Sigma)$ a hod pair such that $\mathcal{P}$ is lsa type and Σ is fullness preserving and has strong branch condensation. Suppose further that $MC(\Sigma^{sts})$ holds and that there is a good pointclass Γ such that $Code(\Sigma) \in \undertilde{\Delta}_{\Gamma}$. Let F be as in Theorem 2.25 of [3] for Γ, and let $x \in dom(F)$ be such that $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Mice_{\Gamma}, Mice^{sts}_{\Sigma}$ and $(\mathcal{P}, \Sigma)$. Then the hod pair construction of $\mathcal{N}^*_x|\delta_x$ reaches a tail of $(\mathcal{P}, \Sigma)$.*

Moreover, if $\mathcal{Q}$ is a tail of $(\mathcal{P}, \Sigma)$ reached by some hod pair construction of $\mathcal{N}^*_x|\delta_x$, $\Lambda = \Sigma_{\mathcal{Q}}$ and $\mathcal{N}^*$ is the output of $(\mathcal{J}^{\vec{E}, \Lambda^{sts}}[\mathcal{Q}])^{\mathcal{N}^*_x|\delta_x}$, κ is the least $<\delta_x$-strong cardinal of $\mathcal{N}^*$ and Ψ is the strategy of $\mathcal{N} =_{def} \mathcal{N}^*|(\kappa^{+\omega})^{\mathcal{N}^*}$ then Ψ is fullness preserving (see Definition 7.3 and whenever $\mathcal{M}$ is a Ψ iterate of $\mathcal{N}$ such that the iteration embedding $\pi : \mathcal{N} \to \mathcal{M}$ exists then $\mathcal{M}$ has Λ^{sts}-fullness condition (see Definition 7.11) and whenever $g \subseteq Coll(\omega, <\pi(\kappa))$ is $\mathcal{M}$-generic then

$$\Lambda^{sts} \restriction D(\mathcal{M}, \pi(\kappa), g) \in D(\mathcal{M}, \pi(\kappa), g).$$

Proof. By an absoluteness argument, it is enough to show that the claim holds for $\mathcal{N}$. We need to show that

(i) Ψ is fullness preserving (as a Λ^{sts}-mouse),
(ii) iterates of $\mathcal{N}$ according to Ψ have the Λ^{sts}-fullness condition and
(iii) $\mathcal{N} \vDash$ "there are proper class of Woodin cardinals".

Given (i)-(iii), the last clause of the theorem follows from Lemma 7.12. We start with (iii).

Claim 1. $\mathcal{N}|\kappa \vDash$ "there are proper class of Woodin cardinals".
Proof. Because $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Mice^{sts}_{\Sigma}$, we have that

$$Lp^{\Sigma^{sts}}(\mathcal{N}^*) \in \mathcal{N}^*_x$$

and therefore, $Lp^{\Sigma^{sts}}(\mathcal{N}^*) \vDash$ "δ_x is Woodin". Let $(T, S) \in \mathcal{N}_x^*$ witness that $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Mice_\Sigma^{sts}$. Let $\pi : H \to \mathcal{N}_x^*$ be such that $crit(\pi) = \eta$, $\mathcal{N}_x^*|\eta = H|\eta$, $|H| = \eta$ and $(T, S)^\# \in rng(\pi)$. Then we have that $Lp^{\Sigma^{sts}}(\mathcal{N}^*|\eta) \in H$ implying that $Lp^{\Sigma^{sts}}(\mathcal{N}^*|\eta) \vDash$ "η is Woodin". By universality of $\mathcal{N}^*$, we have that $Lp^{\Sigma^{sts}}(\mathcal{N}^*|\eta) \trianglelefteq \mathcal{N}^*$, implying that $\mathcal{N}^*|\kappa \vDash$ "for proper class of η, $\mathcal{N}^*|\eta \vDash$ "η is Woodin"". □

Next we show (i). We show (i) for $\mathcal{N}^*$. The general result then follows by using the proof given below and absoluteness.

Claim 2. $\mathcal{N}^*$ is a Λ^{sts}-closed mouse.

Proof. Fix some $\mathcal{N}^*$-cardinal ν. We need to show that there is $\mathcal{M} \trianglelefteq \mathcal{N}^*$ such that $\mathcal{N}^*||\nu \trianglelefteq \mathcal{M}$ and some $\Sigma^{\mathcal{M}}$-hod pair construction of $\mathcal{M}$ in which extenders used have critical points $> \nu$ reaches $\mathcal{R} \in pI(\mathcal{Q}, \Lambda^{sts})$.

By universality, it is enough to show that for every ν, the $\Sigma^{\mathcal{N}^*}$-hod pair construction converges (in which extenders used have critical point $> \nu$). Towards contradiction suppose not. It then follows that there is a $\mathcal{T} \in \mathcal{N}^*$ such that $\mathcal{T}$ appears in a $\Sigma^{\mathcal{N}^*}$-hod pair construction of $\mathcal{N}^*$, $\mathcal{T} \in b(\Lambda^{sts})$ and $\mathcal{T} \notin dom(\Sigma^{\mathcal{N}^*})$. This means that while performing the construction producing $\mathcal{N}^*$, we have never encountered a reason for indexing the branch of $\mathcal{T}$ in $\mathcal{N}^*$. In particular, we must have that $\mathcal{M}^+(\mathcal{T}) \vDash$ "$\delta(\mathcal{T})$ is a Woodin cardinal" and $\mathcal{Q}(b, \mathcal{T})$ is an $\Lambda^{sts}_{\mathcal{M}^+(\mathcal{T})}$-mouse over $\mathcal{M}^+(\mathcal{T})$.

Working in $L[\mathcal{N}^*]$, let Σ_1 be the sts strategy of $\mathcal{M}^+(\mathcal{T})$ induced by $\Sigma^{\mathcal{N}^*}$. The stacks are according to Σ_1 if they are $\mathcal{N}^*$-certified in the sense of Definition 7.8. Let then $\mathcal{N}_1 = (\mathcal{J}^{\vec{E}, \Sigma_1})^{\mathcal{N}^*}$. We must then have that $\mathcal{Q}(b, \mathcal{T}) \ntrianglelefteq \mathcal{N}_1$. Because $\mathcal{N}_1$ is universal, $\mathcal{Q}(b, \mathcal{T})$ cannot win the comparison with $\mathcal{N}_1$. It follows that there must be $\mathcal{T}_1 \in \mathcal{N}_1$ such that $\mathcal{T}_1$ is according to $\Sigma^{\mathcal{N}_1}$ but $\Lambda_{\mathcal{M}^+(\mathcal{T})}(\mathcal{T}_1)$ has not been indexed in $\mathcal{N}_1$. By repeating this process, we build $(\mathcal{Q}_i, \mathcal{T}_i, \mathcal{N}_i, \Sigma_i : i < \omega)$ such that the following conditions hold.

1. $\mathcal{Q}_0 = \mathcal{Q}$, $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{N}_0 = \mathcal{N}^*$ and $\mathcal{T}_0 \in \mathcal{N}_0$.
2. $\mathcal{T}_i \in \mathcal{N}_i$ is a normal tree on $\mathcal{Q}_i$ according to $\Lambda^{sts}_{\mathcal{Q}_i}$ and $\mathcal{Q}_{i+1} = \mathcal{M}^+(\mathcal{T}_i)$.
3. $\Sigma_i \in L[\mathcal{N}_i]$ is the strategy of $\mathcal{Q}_{i+1}$ induced by $\Sigma^{\mathcal{N}_i}$ in the sense that stacks according to Σ_i are $\mathcal{N}_i$-certified.
4. $\mathcal{N}_{i+1} = (\mathcal{J}^{\vec{E}, \Sigma_i}[\mathcal{Q}_i])^{\mathcal{N}_i}$.
5. For each i, $\mathcal{T}_i \in b(\Lambda^{sts}_{\mathcal{Q}_i})$.

Let then $b_i = \Lambda^{sts}_{\mathcal{Q}_i}(\mathcal{T}_i)$. We then have that the direct limit of $\oplus_{i<\omega}\mathcal{T}_i$ given by b_i is ill founded, contradiction. □

The proof of $\mathcal{N}^*$ gives much more. It shows that various constructions of $\mathcal{N}^*$ reach $\Lambda_{\mathcal{Q}}$-iterate of $\mathcal{Q}$ via $\mathcal{T}$ such that $\mathcal{T} \in m(\Lambda^{sts}_{\mathcal{Q}})$. It then follows from absoluteness and universality that Ψ is fullness preserving. This completes the proof of (i).

Next we show (ii). We will do this only for $\mathcal{N}$. The general result follows from absoluteness and the proof given below.

Claim 3. $\mathcal{N}$ satisfies Λ^{sts}-fullness condition.

Proof. Fix an $\mathcal{N}$-generic h and a transitive $a \in \mathcal{N}[h]$ such that $\mathcal{Q} \in a$. Let $\mathcal{S} \trianglelefteq Lp^{\Sigma^{sts}}(a)$ be such that $\rho(\mathcal{S}) = a$. Fix η such that $a \in \mathcal{N}|\eta[h]$. We want to show that there is $\mathcal{M} \trianglelefteq \mathcal{N}$ such that

1. $\mathcal{N}||\eta \trianglelefteq \mathcal{M}$,
2. $\mathcal{M} \vDash$ "ZFC-Powerset+"there are infinitely many Woodin cardinals $(\delta_n : n < \omega)$",
3. $\mathcal{S} \in \mathcal{M}|\delta_0$ and $\mathcal{S}$ has an iteration strategy $\Phi \in \mathcal{M}$ such that if $k \subseteq Coll(\omega, <\delta_\omega)$-generic and D is the derived model of $\mathcal{M}$ computed via k then Φ has an extension $\Phi^* \in D$ such that whenever $\mathcal{S}^*$ is a Φ^*-iterate of $\mathcal{S}$ and $\vec{\mathcal{U}} \in dom(\Sigma^{\mathcal{S}^*})$ then $\mathcal{M}[h] \vDash$ "$\vec{\mathcal{U}}$ is branch certified stack (see Definition 7.8)".

Let $\mathcal{R}$ be the output of $(\mathcal{J}^{\vec{E},\Lambda^{sts}}[a])^{\mathcal{N}^*}$ where the extenders used have critical point $> (\kappa^{+\omega})^{\mathcal{N}^*}$. Because $\mathcal{R}$ is universal we have that $\mathcal{S} \trianglelefteq \mathcal{R}$. The fact that $\mathcal{S}$ has an iteration strategy of the desired form follows from the proof of Claim 2 above and the following observation. Let $\mathcal{W}$ be a model appearing in the construction producing $\mathcal{R}$ and such that $\mathcal{C}(\mathcal{W}) = \mathcal{S}$. It is then enough to show that $\mathcal{W}$ is iterable in $\mathcal{N}^*$ via a desired iteration strategy. To do this, we observe that it is enough to show that

(1) for every $\mathcal{N}^*$-regular cardinal $\eta > \delta^{\mathcal{Q}}$, if Γ is the fragment of the iteration strategy of $\mathcal{N}^*|\eta$ that acts on non-dropping trees that are above $\delta^{\mathcal{Q}}$ then Γ is Suslin, co-Suslin captured by $(\mathcal{N}^*, \delta_x, \Gamma^*)$ where Γ^* is the strategy of $\mathcal{N}^*$.

We omit the proof of (1) as it is just like the proof of Lemma 3.9 of [3]. □

□

The next theorem can now be proved using Lemma 2.5, Lemma 7.14, Lemma 10.1 and the proof of Theorem 5.20 of [3].

Theorem 10.2 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor ordinal and Σ has a branch condensation and is fullness preserving. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$. There is then $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $\vec{B} = \langle B_i : i < \omega \rangle \subseteq \mathbb{B}(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ such that $\vec{B}$ strongly guides $\Sigma_{\mathcal{Q}}$.*

11 The computation of HOD

Throughout this section we assume $AD^+ + SMC$ and let $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ be the Solovay sequence. Our goal is to compute $V_{\theta_\alpha}^{\mathrm{HOD}}$ for $\alpha \leq \Omega$. We will do it under some additional hypothesis described below. In the next section, we prove that our additional hypothesis essentially follows from $AD^+ +$ "No initial segment of the Solovay sequence satisfies LSA".

Suppose $(\mathcal{P}, \Sigma)$ is an sts pair such that Σ is fullness preserving and has branch condensation. We then let $\alpha(\mathcal{P}, \Sigma) = w(\Gamma(\mathcal{P}, \Sigma))$. Also, we let $\mathcal{P}^- = \mathcal{P}$. We then also say that $(\mathcal{P}, \Sigma)$ is suitable. If $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is of lsa type then, just for convenience, we let $\Sigma_{\mathcal{P}^-} = \Sigma^{sts}$.

Suppose first that $\alpha + 1 = \Omega$. We then let $\mathcal{I} = \{(\mathcal{Q}, \Lambda, B_0, ..., B_i) :$

1. $(\mathcal{Q}, \Lambda)$ is suitable, Λ is fullness preserving and has branch condensation, and $\alpha(\mathcal{Q}^-, \Lambda) = \alpha$,

2. for every $i < n$, $B_i \in \mathbb{B}(\mathcal{Q}^-, \Lambda)$, and

3. $(\mathcal{Q}, \Lambda)$ is strongly $\vec{B}$-iterable $\}$.

Define $\preceq$ on $\mathcal{I}$ by

$$(\mathcal{P}, \Sigma, \vec{B}) \preceq (\mathcal{Q}, \Lambda, \vec{C}) \leftrightarrow \vec{B} \subseteq \vec{C} \text{ and } (\mathcal{Q}, \Lambda, \vec{B}) \text{ is a } \vec{B}\text{-tail of } (\mathcal{P}, \Sigma, \vec{B}).$$

When $(\mathcal{R}, \Psi, \vec{B}) \preceq (\mathcal{Q}, \Lambda, \vec{C})$, there is a canonical map

$$\pi : H_{\vec{B}}^{\mathcal{R}, \Psi} \rightarrow H_{\vec{B}}^{\mathcal{Q}, \Lambda},$$

which is independent of $\vec{B}$-iterable branches. We let $\pi_{(\mathcal{R}, \Psi, \vec{B}), (\mathcal{Q}, \Lambda, \vec{B})}$ be this map. We then have that $(\mathcal{I}, \preceq)$ is a directed. Let

$$\mathcal{F} = \{H_{\vec{B}}^{\mathcal{Q}, \Lambda} : (\mathcal{Q}, \Lambda, \vec{B}) \in \mathcal{I}\}.$$

and we let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ under the iteration maps $\pi_{(\mathcal{R},\Psi,\vec{B}),(\mathcal{Q},\Lambda,\vec{B})}$. Let $\delta_\infty = \delta^{\mathcal{M}_\infty}$. For $(\mathcal{Q},\Lambda,B) \in \mathcal{I}$, we let $\pi_{(\mathcal{Q},\Lambda,B),\infty} : H_B^{\mathcal{Q},\Lambda} \to \mathcal{M}_\infty$. Standard arguments show that $\mathcal{M}_\infty$ is well-founded.

Following Section 4.4 of [3], we let ϕ be the following sentence: for every $\alpha+1 < \Omega$, letting $\Gamma_\alpha = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$, there is a hod pair $(\mathcal{P},\Sigma)$ such that

1. $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$,
2. Σ is fullness preserving and has branch condensation,
3. for any $\mathcal{Q} \in pI(\mathcal{P},\Sigma) \cup pB(\mathcal{P},\Sigma)$, if $\lambda^{\mathcal{Q}}$ is a successor ordinal then
   (a) there is a sequence $\langle B_i : i < \omega \rangle \subseteq \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ which guides $\Sigma_{\mathcal{Q}}$ and
   (b) for any $B \in \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ there is $\mathcal{R} \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ such that $\Sigma_{\mathcal{R}}$ respects B.
4. $L(\Gamma_\alpha, \mathbb{R}) \vDash LSA$ if and only if $\mathcal{P}$ is of lsa type.

Our additional hypothesis, ψ, is a conjunction of ϕ with the following statement: If $\Omega = \alpha+1$ then there is a suitable $(\mathcal{P},\Sigma)$ which is $\emptyset$-iterable, $\lambda^{\mathcal{P}}$ is a successor and such that

1. $(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ is either a hod pair or an sts pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$ and $\Sigma_{\mathcal{P}^-}$ is fullness preserving and has branch condensation,
2. for any $B \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ there is an $\emptyset$-iterate $(\mathcal{Q},\Phi)$ of $(\mathcal{P},\Sigma)$ such that $(\mathcal{Q},\Phi)$ is strongly B-iterable.
3. $\mathcal{M}_\infty$ is well-founded and $\delta_\infty = \Theta = \theta_{\alpha+1}$.
4. $V \vDash LSA$ if and only if $\mathcal{P}$ is of lsa type.

We will use the following lemma to establish ψ. It can be proved exactly the same way as Lemma 4.23 of [3].

Lemma 11.1 *Suppose $\Gamma \subseteq \wp(\mathbb{R})$ is such that $L(\Gamma,\mathbb{R}) \vDash AD^+ + SMC + \Omega = \alpha+1$ and $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma,\mathbb{R})$. Suppose $\Gamma^* \subseteq \wp(\mathbb{R})$ is such that $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \vDash AD^+$ and there is a hod a pair $(\mathcal{P},\Sigma) \in \Gamma^*$ such that the following holds.*

1. *Σ has branch condensation and is Γ-fullness preserving.*
2. *$L(\Gamma,\mathbb{R}) \vDash LSA$ if and only if $\mathcal{P}$ is of lsa type.*

3. *$\lambda^{\mathcal{P}}$ is a successor ordinal, $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$ and $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$".*

4. *There is a sequence $\langle B_i : i < \omega\rangle \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ guiding Σ.*

5. *For any $B \in (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{R}}$ respects B.*

Then $L(\Gamma, \mathcal{R}) \vDash \psi$ and $\mathcal{M}_\infty^{L(\Gamma,\mathbb{R})} = \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$[7].

The next theorem is the adaptation of Theorem 2.24 of [3] to our current context. It can be proved via exactly the same proof. Because of this, we omit the proof.

Theorem 11.2 (Computation of HOD**)** *Assume AD^+. Suppose $\Gamma \subseteq \wp(\mathbb{R})$ is such that $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Then the following holds:*

1. *Suppose $L(\Gamma, \mathbb{R}) \vDash \phi$. Suppose $\beta + 1 < \Omega^\Gamma$. Let $(\mathcal{P}, \Sigma)$ witness ϕ for β. Then letting $\mathcal{M} = \mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$, $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda = \Sigma^{\mathcal{M}}$, for every $\alpha \leq \beta$*

$$\delta_\alpha^{\mathcal{M}} = \theta_\alpha^\Gamma \text{ and } \mathcal{M}|\theta_\alpha^\Gamma = (V_{\theta_\alpha^\Gamma}^{\mathrm{HOD}^\Gamma}, \vec{E} \restriction \theta_\alpha^\Gamma, \Lambda \restriction V_{\theta_\alpha^\Gamma}^{\mathrm{HOD}^\Gamma}, \in).$$

2. *If $L(\Gamma, \mathbb{R}) \vDash \psi$ then letting $\mathcal{M} = \mathcal{M}_\infty^{L(\Gamma,\mathbb{R})}$ $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda = \Sigma^{\mathcal{M}}$, for every $\alpha \leq \Omega^\Gamma$*

$$\delta_\alpha^{\mathcal{M}} = \theta_\alpha^\Gamma \text{ and } \mathcal{M}|\theta_\alpha^\Gamma = (V_{\theta_\alpha^\Gamma}^{\mathrm{HOD}^\Gamma}, \vec{E} \restriction \theta_\alpha^\Gamma, \Lambda \restriction V_{\theta_\alpha^\Gamma}^{\mathrm{HOD}^\Gamma}, \in).$$

3. *Suppose $\Gamma^* \subseteq \wp(\mathbb{R})$ is such that $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*, \mathbb{R}) \vDash AD^+$ and there is a hod a pair $(\mathcal{P}, \Sigma) \in \Gamma^*$ such that the following holds:*
   - *(a) Σ has branch condensation and is Γ-fullness preserving,*
   - *(b) $\lambda^{\mathcal{P}}$ is a successor ordinal, $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$ and $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Lambda_{\mathcal{P}^-}) = \alpha$",*
   - *(c) there is a sequence $\langle B_i : i < \omega\rangle \subseteq (\mathbb{B}(\mathcal{P}^-, \Lambda_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ guiding Σ,*
   - *(d) for any $B \in (\mathbb{B}(\mathcal{P}^-, \Lambda_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{R}}$ respects B.*

   Then $L(\Gamma, \mathcal{R}) \vDash \psi$ and $\mathcal{M}_\infty^{L(\Gamma,\mathbb{R})} = \mathcal{M}_\infty^+(\mathcal{P}, \Lambda)$.

Thus, working in a model of AD^+, if $\alpha < \Omega$ then to compute $\mathrm{HOD}|\theta_\alpha$ we only need to produce a hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. We will show that this is true in any model of AD^+ provided that there is no transitive class inner model containing the reals and satisfying $AD^+ + LSA$.

[7] Recall that $\mathcal{M}_\infty^+(\mathcal{P}, \Sigma)$ is the direct limit of all Σ-iterates of $\mathcal{P}$

12 The generation of the mouse full pointclasses

In this section, our goal is to show that if SMC holds and Γ is a mouse full pointclass such that $\Gamma \neq \wp(\mathbb{R})$ and there is a good pointclass Γ^* such that $\Gamma \subseteq \Gamma^*$ then there is $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma$. Recall that we let

$\#_{lsa}$: There is α such that $\theta_{\alpha+2} \leq \Theta$ and $L(\Gamma_{\alpha+1}) \vDash LSA$.

As in Section 6.1 of [3], we will construct $(\mathcal{P}, \Sigma)$ as above via a hod pair construction of some sufficiently strong background universe. However, the background universes given by Theorem 2.25 of [3] will not be sufficient for us. Below we describe a stronger background universe which we will use for such constructions.

Theorem 12.1 *Assume AD^+ and suppose Γ is a good pointclass and is not the last good pointclass. Let (N, Ψ) Suslin, co-Suslin capture Γ. There is then a function F defined on $\mathbb{R}$ such that for a Turing cone of x, $F(x) = \langle \mathcal{N}_x^*, \mathcal{M}_x, \delta_x^0, \delta_x^1, \Sigma_x, \Lambda_x \rangle$ such that*

1. *x codes N,*
2. *$\mathcal{N}_x^* | \delta_x^0 = \mathcal{M}_x | \delta_x^0$,*
3. *$\mathcal{M}_x$ is a Ψ-mouse and $\mathcal{M}_1^{\#,\Psi}(\mathcal{M}_x) \vDash$ "δ_x^0 is a Woodin cardinal",*
4. *for all $\eta < \delta_x^0$, $\mathcal{M}_1^{\#,\Psi}(\mathcal{M}_x | \eta) \vDash$ "δ_x^0 isn't a Woodin cardinal"*
5. *$\mathcal{N}_x^* \vDash$ "$\delta_x^0 < \delta_x^1$ are the only Woodin cardinals",*
6. *Σ_x is the unique iteration strategy of $\mathcal{M}_x$ (induced by the strategy of $\mathcal{M}_1^{\#,\Psi}(\mathcal{M}_x)$),*
7. *$\mathcal{N}_x^* | \delta_x^1$ is a (Ψ, Σ_x)-mouse over $\mathcal{M}_x$ and $\mathcal{N}_x^* = L^\Psi[\mathcal{N}_x^* | \delta_x^1]$,*
8. *Λ_x is a strategy of $\mathcal{N}_x^*$ and $(\mathcal{N}_x^*, \delta_x^1, \Lambda_x)$ Suslin, co-Suslin captures $Code(\Psi)$ and hence, $(\mathcal{N}_x^*, \delta_x^1, \Sigma_x)$ Suslin, co-Suslin captures Γ,*
9. *for any $\alpha < \delta_x^0$ and for any $\mathcal{N}_x^*$-generic $g \subseteq Coll(\omega, \alpha)$, $(\mathcal{N}_x^*[g], \Sigma_x)$ Suslin, co-Suslin captures $Code((\Sigma_x)_{\mathcal{M}_x \restriction \alpha})$ and its complement at $(\delta_x^0)^+$.*

The proof of Theorem 12.1 is very much like the proof of Theorem 2.25 of [3]. It is unfortunately unpublished and is also beyond the scope of this paper. Here is our theorem on generation of pointclasses.

Theorem 12.2 (The generation of the mouse full pointclasses) *Assume AD^+ and $\neg\#_{lsa}$. Suppose $\Gamma \neq \wp(\mathbb{R})$ is a mouse full pointclass such that $\Gamma \vDash SMC$. Then the following holds:*

1. *Suppose first that Γ is completely mouse full and let $A \subseteq \mathbb{R}$ witness it. Then the following holds:*
   (a) *Suppose $L(A,\mathbb{R}) \vDash \neg LSA$. Then there is $(\mathcal{P},\Sigma) \in L(A,\mathbb{R})$ such that $L(A,\mathbb{R}) \vDash$ "Σ has branch condensation and is fullness preserving" and $\Gamma(\mathcal{P},\Sigma) = \Gamma$.*
   (b) *Suppose $L(A,\mathbb{R}) \vDash LSA$. Then there is an sts pair $(\mathcal{P},\Sigma) \in L(A,\mathbb{R})$ such that $L(A,\mathbb{R}) \vDash$ "Σ has branch condensation and is fullness preserving" and $\Gamma(\mathcal{P},\Sigma) = \Gamma$".*
2. *If Γ is mouse full but not completely mouse full then there is a hod pair or an anomalous hod pair $(\mathcal{P},\Sigma)$ such that Σ has branch condensation and $\Gamma(\mathcal{P},\Sigma) = \Gamma$.*

Proof. Our proof has the same structure as the proof of Theorem 6.1 of [3]. The proof is by induction. Suppose $\Gamma \neq \wp(\mathbb{R})$ is a mouse full pointclass such that whenever Γ^* is properly contained in Γ and is a mouse full pointclass then there is a hod pair $(\mathcal{P},\Sigma)$ as in 1 or 2. We want to show that the claim holds for Γ. We examine several cases.

Case 1. θ^Γ isn't the largest Suslin cardinal.

Let $A \subseteq \mathbb{R}$ be such that $w(A) = \Gamma$. Recall that A_Γ is the set of reals σ which code a pair $\langle \sigma_0, \sigma_1 \rangle$ of continuous functions such that σ_0^{-1}"A is a code for a set in HP_Γ and σ_1^{-1}"A is a code for a quadruple $(\alpha, \Lambda, \mathcal{M}, \Psi)$ such that $(a, \Lambda, \mathcal{M}) \in Mice^\Gamma$ and Ψ is the unique strategy of $\mathcal{M}$. We let $B = A_\Gamma$.

For each hod pair $(\mathcal{P},\Sigma) \in \Gamma$, there is a sjs $\langle A_i : i < \omega \rangle$ such that $A_i \in \Gamma$ for every i and $Mice^\Gamma_\Sigma = A_0$. We then let C be the set of reals σ coding a continuous function such that $\sigma^{-1}[A]$ codes

1. a hod pair $(\mathcal{P},\Sigma)$ such that $Code(\Sigma) \in \Gamma$,
2. a sjs $\langle A_i : i < \omega \rangle$ such that $A_i \in \Gamma$ for all i and $Mice^\Gamma_\Sigma = A_0$.

Let Γ^* be a good pointclass such that $A, B, C \in \Delta_{\Gamma^*}$ and let F and (N, Ψ) be as in Theorem 12.1. Let x be such that $(\mathcal{N}^*_x, \delta^1_x, \Lambda_x)$ Suslin, co-Suslin captures A, B, C.

We claim that some model of Γ-hod pair construction of $\mathcal{N}_x^*|\delta_x^0$ is as desired. Here the proof is somewhat different than the proof of Theorem 6.1 of [3]. There the contradictory assumption that such construction do not reach Γ lead to a construction of a hod pair $(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{P}} = \delta^{\mathcal{P}}$ and $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is regular". This meant that a pointclass satisfying $AD_{\mathbb{R}}$ + "Θ is regular" had been reached giving the desired contradiction. In our current situation, if the constructions never stops then we will end up with an lsa type hod premouse $\mathcal{P}$ of height δ_x^0. The following claim shows how to finish the argument from there.

Claim 1. Suppose the Γ-hod pair construction of $\mathcal{N}_x^*|\delta_x^0$ doesn't stop and never reaches Γ. Then there is a pointclass satisfying $\#_{lsa}$.

Proof. Let $(\mathcal{P}^*, \Lambda)$ be the output of the hod pair construction of $\mathcal{N}_x^*|\delta_x^0$. Thus, we have that $o(\mathcal{P}^*) = \delta_x^0$. Let $\mathcal{P} = Lp_\omega^{\Lambda^{sts}}(\mathcal{M}^+(\mathcal{P}^*))$.

Subclaim 1. $\mathcal{P} \in \mathcal{N}_x^*$.

Proof. To see this, notice first that we have that

$$\Gamma(\mathcal{M}^+(\mathcal{P}^*), \Lambda) \subseteq \Gamma \text{ and } \Gamma(\mathcal{M}^+(\mathcal{P}^*), \Lambda) \neq \Gamma.$$

It then follows that there is $(\mathcal{Q}, \Phi) \in \Gamma$ such that $\Gamma(\mathcal{Q}, \Phi) = \Gamma(\mathcal{M}^+(\mathcal{P}^*), \Lambda)$ implying that, because of comparison, that $(\mathcal{M}^+(\mathcal{P}^*), \Lambda) \in \Gamma$. We can then fix z such that z codes a Wadge reduction of $Mice^{\Gamma}_{\Lambda^{sts}}$ to $Code(\Psi)$.

Let $\mathcal{P}^{**} = (\mathcal{J}^{\vec{E}, \Lambda^{sts}})^{\mathcal{N}_x^*|\delta_x^1}$ and let $\mathcal{R} = \mathcal{P}^{**}|((\delta_x^0)^{+\omega})^{\mathcal{P}^{**}}$. We claim that $\mathcal{P} \trianglelefteq \mathcal{P}^*$. Assume towards a contradiction that $\mathcal{P} \not\trianglelefteq \mathcal{P}^*$ and let $\mathcal{Q} \trianglelefteq \mathcal{P}$ be the least such that $\rho(\mathcal{Q}) = \delta_x^0$ and $\mathcal{Q} \not\trianglelefteq \mathcal{P}^*$. Let Φ be the strategy of $\mathcal{Q}$ which witnesses that $\mathcal{Q}$ is a Λ^{sts}-mouse. Fix w which codes a Wadge reduction of $Code(\Phi)$ to $Code(\Psi)$.

Let now Γ^{**} be a good pointclass such that $\Gamma^* < \Gamma^{**}$ and $Code(\Phi) \in \utilde{\Delta}_{\Gamma^{**}}$. Let F^* be as in Theorem 2.5 of [3] and let $y \in dom(F^*)$ be such that if $F^*(y) = (\mathcal{K}_y^*, \mathcal{W}_y, \lambda_y, \Phi_y)$ then $(\mathcal{K}_y^*, \delta_y, \Phi_y)$ Suslin, co-Suslin captures F, x, z, w and $Code(\Phi)$.

We then let $\mathcal{M}$ be the iterate of $\mathcal{N}_x^*$ according to Λ_x which is above δ_x^0 and is constructed via a (Ψ, Σ_x)-construction of $\mathcal{K}_x^*|\lambda_y$ done over $\mathcal{N}_x^*|\delta_x^0$. Let

$$\mathcal{N} = (\mathcal{J}^{\vec{E}, \Lambda^{sts}})^{\mathcal{M}|\pi_{\mathcal{N}_x^*, \mathcal{M}}(\delta_x^1)}.$$

Let $\mathcal{T}$ be the comparison tree of $\mathcal{N}$ and $\mathcal{Q}$ (we have that $\mathcal{N}$ doesn't move in this comparison) without its last branch. It then follows that $\mathcal{T} \in \mathcal{M}[w, \mathcal{Q}]$ and if $b = \Phi(\mathcal{T})$ then $b \in \mathcal{M}[w, \mathcal{Q}]$. But now universality of background constructions implies that in fact $\mathcal{N}$ reaches a level satisfying "there is a superstrong cardinal". This contradiction completes the proof. □

It then follows that $\mathcal{P} \in \mathcal{N}_x^*$ and hence $(\mathcal{P}, \Lambda)$ is an lsa type hod pair. However, at this point we do not know if Λ is fullness preserving mainly because we do not have enough absoluteness between $\mathcal{N}_x^*$ and V. Next, we produce an lsa type hod pair which does have fullness preservation.

Subclaim 2. There is an lsa hod pair $(\mathcal{Q}, \Phi)$ such that Φ is fullness preserving.

Proof. Fix a good pointclass Γ^{**} such that $\Gamma^* < \Gamma^{**}$ and that $F \in \utilde{\Delta}_{\Gamma^{**}}$. Let F^* be as in Theorem 2.5 of [3] and let $y \in dom(F^*)$ be such that if $F^*(y) = (\mathcal{K}_y^*, \mathcal{W}_y, \lambda_y, \Phi_y)$ then F, x are Suslin, co-Suslin captured by $(\mathcal{K}_y^*, \lambda_y, \Phi_y)$. Let $\mathcal{M}$ be the iterate Λ_x-iterate of $\mathcal{N}_x^*$ constructed via $\mathcal{J}^{\vec{E},\Psi}$-construction of $\mathcal{K}_y^*|\lambda_y$ and let $\mathcal{Q} = \pi_{\mathcal{N}_x^*,\mathcal{M}}(\mathcal{P})$. Let Φ be the strategy of $\mathcal{Q}$ which it inherits from Φ_y. It follows from the proof of Subclaim 1 that $\mathcal{Q}$ is full. It then follows from Theorem 18.3 of [4] that Φ is fullness preserving. □

The next subclaim finishes the proof of the claim.

Subclaim 3. $L(\Phi, \mathbb{R}) \vDash$ there is Γ' such that $L(\Gamma', \mathbb{R}) \vDash \#_{lsa}$.

Proof. It follows from Lemma 8.1 that $L(\Gamma(\mathcal{Q}, \Phi)) \vDash LSA$. Moreover, it follows from SMC that the set $\{(x, y) : y \notin Lp^{\Phi^{sts}}(x)\}$ cannot be uniformized by any function which is $OD_{\Phi^{sts},z}$ for some real z. Because $\Phi \notin \Gamma(\mathcal{Q}, \Phi))$, the claim follows. □

□

Thus we have that Γ-hod pair constructions of $\mathcal{N}_x^*|\delta_x^0$ have to stop. Next we show that they cannot stop because they break down implying that they stop because they reach Γ. An important remark is that it follows from the proof of the claim that Γ-hod pair construction cannot reach an lsa type hod pair $\mathcal{P}$ before reaching Γ as otherwise, letting Σ be its strategy, as in Subclaim 3, $(\mathcal{P}, \Sigma)$ produces a model satisfying $\#_{lsa}$. Thus, all hod mice reached by the Γ-hod pair construction before reaching Γ must either be not lsa type or the corresponding strategy must be not fullness preserving.

The proof that the construction doesn't break down is very much like the proof of Theorem 6.1 of [3] with one wrinkle. Suppose $\mathcal{P}$ is a model appearing in the Γ-hod pair construction of $\mathcal{N}_x^*|\delta_x^0$. Suppose $\lambda^{\mathcal{P}}$ is a successor and $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is a measurable cardinal. We have to show that adding a new extender to $\mathcal{P}$ doesn't project to or across δ. Below we only show how to handle this case, the rest is left to the reader (as it is very similar to the proof of Theorem 6.1 of [3]).

Suppose then $\mathcal{P}^*$ is a model of Γ-hod pair construction and Σ^* is its strategy. Suppose further that $\lambda^{\mathcal{P}^*}$ is a successor ordinal and if $\delta = \delta^{\mathcal{P}^*}_{\lambda^{\mathcal{P}^*}-1}$ then δ is a measurable cardinal in $\mathcal{P}^*$. Suppose that for some extender E the next model of the Γ-hod pair construction is $\mathcal{P} = (\mathcal{P}^*, E)$. Let Σ be the strategy of $\mathcal{P}$. If $\rho(\mathcal{P}) \geq ((\delta^{\mathcal{P}})^+)^{\mathcal{P}}$ then we can continue the construction.

Suppose then that $\rho(\mathcal{P}) \leq \delta$. If $\Gamma(\mathcal{P}, \Sigma) = \Gamma$ then we are done. Hence we assume that $\Gamma(\mathcal{P}, \Sigma) \neq \Gamma$ implying that $Code(\Sigma) \in \Gamma$ (using the comparison argument).

The argument now follows the same line of thought as the proof of Theorem 6.1 of [3]. Suppose first that $\rho(\mathcal{P}) < \delta$. Fix $\alpha < \Omega^\Gamma$ such that $w(Code(\Sigma)) < \theta^\Gamma_\alpha$. Let $\Gamma^* = \{A : w(A) < \theta^\Gamma_\alpha\}$. Notice that whenever $(\mathcal{Q}, \Lambda) \in \Gamma$ is such that

$$\Gamma(\mathcal{Q}, \Lambda) = \Gamma(\mathcal{P}, \Sigma) \text{ and } \mathcal{M}_\infty(\mathcal{P}, \Sigma) = \mathcal{M}_\infty(\mathcal{Q}, \Lambda).$$

It then follows that $\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \mathrm{HOD}^{L(\Gamma^*, \mathbb{R})}$. Let β be such that $\theta_\beta = \sup(\pi^\Sigma_{\mathcal{P},\infty} \restriction \delta^{\mathcal{P}})$. It then follows that $\rho(\mathcal{M}_\infty) < \theta_\beta$, contradiction!

We must then have that $\rho(\mathcal{P}) = \delta$. Let $\mathcal{Q} = \mathcal{P}^b$. Let F be the first extender on the sequence of $\mathcal{P}$ with critical point $\delta^{\mathcal{P}}$. Let $n \in \omega$ be largest such that that $\rho_n(\mathcal{P}) > \delta^{\mathcal{P}}$ and let $\mathcal{S} = Ult_n(\mathcal{P}, F)$. Notice that $\rho(\mathcal{S}) = \delta^{\mathcal{P}}$. Moreover, it follows from Γ-fullness that $\mathcal{S}^b = \mathcal{Q}$.

Let now $A \subseteq \delta^{\mathcal{P}}$ be the set coding $\mathcal{P}$. Notice that A is also the new subset defined over $\mathcal{S}$. Moreover, its not hard to see that comparison implies that whenever $(\mathcal{W}, \Phi)$ is a hod pair such $\Gamma(\mathcal{W}, \Phi) = \Gamma(\mathcal{P}, \Sigma)$, $\mathcal{Q} = \mathcal{W}^b$ and $\Phi_{\mathcal{Q}} = \Sigma_{\mathcal{Q}}$ then A is the new set defined over $\mathcal{W}$. It then follows that letting Γ^* be as in the previous paragraph then $L(\Gamma^*, \mathbb{R}) \vDash A \in OD_{Code(\Sigma_{\mathcal{Q}})}$. It then follows from SMC and the fullness of $\mathcal{Q}$ that in fact $A \in \mathcal{Q}$. But $\mathcal{Q} \trianglelefteq \mathcal{P}$ implying that $A \in \mathcal{P}$, contradiction.

Case 2. θ^Γ is the largest Suslin cardinal.

The difference between Case 1 and Case 2 is that we now cannot choose a good pointclass Γ^* such that $\Gamma \subseteq \Gamma^*$. The proof then is by reflection. Suppose that there is no lst hod pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$. The non existence of such a pair can be reflected.

Let then $A \subseteq \mathbb{R}$ be such that for some α, $L_\alpha(A, \mathbb{R}) \vDash ZF + AD^+ + LSA$ and that there is no sts pair $(\mathcal{P}, \Sigma) \in L_\alpha(A, \mathbb{R})$ such that in $L_\alpha(A, \mathbb{R})$ is Σ is fullness preserving and has branch condensation and

$$\Gamma(\mathcal{P}, \Sigma) = \{B \subseteq \mathbb{R} : w(B) < w(A)\}.$$

Let $\Gamma^* = \{B \subseteq \mathbb{R} : w(B) < w(A)\}$. It follows from Case 1 that there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\mathcal{P}$ is of lsa type and $\Gamma(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \Gamma^*$. We claim that $\Sigma^{sts} \in L_\alpha(\mathbb{R})$ and spend the rest of the argument showing it.

Let $\pi^* : \mathcal{P} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the iteration embedding. Let $\pi = \pi^* \restriction \mathcal{P}^b$. It follows from Theorem 11.2 that $\pi, \pi^*(\mathcal{P}) \in L_\alpha(A, \mathbb{R})$. We intend to use π to define Σ^{sts} over $L_\alpha(A, \mathbb{R})$. The proof is very much like our results proven in Section 7. Let $\mathcal{M} = \pi^*(\mathcal{P})$.

Suppose $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq m < \omega)$ is a finite stack on $\mathcal{P}$. We say the the branches of $\vec{\mathcal{T}}$ are π-certified if there is a sequence of embeddings $(\sigma_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$ and a sequence of hod pairs $(\mathcal{Q}_{\mathcal{R}}, \Lambda_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$, such that whenever $\mathcal{R} \in tn(\vec{\mathcal{T}})$, the following conditions hold:

1. if $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}$ exists then $\sigma_{\mathcal{R}} : \mathcal{R}^b \to \mathcal{M}$ and $\pi = \sigma \circ \pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}$,

2. if $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}$ doesn't exists then letting E be the undropping extender of $\vec{\mathcal{T}}_{\leq \mathcal{R}}$, then $\sigma : Ult(\mathcal{P}^b, E) \to \mathcal{M}$ and $\pi = \sigma \circ \pi_E$,

3. $(\mathcal{Q}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$ is a hod pair in $L_\alpha(A, \mathbb{R})$ such that $\Lambda_{\mathcal{R}}$ is fullness preserving and has branch condensation and letting $\tau_{\mathcal{R}} : \mathcal{Q}_{\mathcal{R}} \to \mathcal{M}_\infty(\mathcal{Q}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$, $\sigma_{\mathcal{R}} | \delta^{\mathcal{R}} \subseteq rng(\tau_{\mathcal{R}})$,

4. if $\alpha + 1 < \lambda^{\mathcal{R}}$ and $\mathcal{T}$ is the longest irreducible component of $\mathcal{T}$ based on $\mathcal{R}(\alpha)$ then letting $k : \mathcal{R}(\alpha) \to \mathcal{Q}_{\mathcal{R}}(\beta)$ be given by $k(x) = \tau_{\mathcal{R}}^{-1}(\sigma_{\mathcal{R}}(x))$, $\mathcal{T}$ is according to k-pullback of $\Lambda_{\mathcal{Q}_{\mathcal{R}}(\beta)}$ and $\sigma_{\mathcal{R}} \restriction \mathcal{R}(\alpha)$ is the iteration embedding according to the k-pullback of $(\Lambda_{\mathcal{R}})_{\mathcal{Q}(\beta)}$.

Suppose $\vec{\mathcal{T}}$ is a finite stack on $\mathcal{P}$ is such that for some cutpoint $\mathcal{R}$, $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}$ exists and $\vec{\mathcal{T}}_{\geq \mathcal{R}}$ is a normal irreducible tree on $\mathcal{R}$ based on the top window of $\mathcal{R}$. Suppose $\mathcal{Q}$ is an sts mouse such that $\mathcal{M}(\vec{\mathcal{T}}_{\geq \mathcal{R}}) \trianglelefteq \mathcal{Q}$ and $\mathcal{J}_1(\mathcal{Q}) \vDash$ "$\delta(\vec{\mathcal{T}}_{\geq \mathcal{R}})$ isn't a Woodin cardinal". We say $\mathcal{Q}$ is π-certified if

1. if $\mathcal{M}^+(\mathcal{M}(\vec{\mathcal{T}})) \vDash$ "$\delta(\vec{\mathcal{T}}_{\geq \mathcal{R}})$ isn't a Woodin cardinal" then $\mathcal{Q} \trianglelefteq \mathcal{M}^+(\mathcal{M}(\vec{\mathcal{T}}))$,

2. if $\mathcal{M}^+(\mathcal{M}(\vec{\mathcal{T}})) \vDash$ "$\delta(\vec{\mathcal{T}}_{\geq \mathcal{R}})$ is a Woodin cardinal" then $\mathcal{Q}$ has an iteration strategy Λ such that whenever $\mathcal{S}$ is a Λ-iterate of $\mathcal{Q}$ and $\vec{\mathcal{U}} \in \mathcal{S}$ is according to $\Sigma^{\mathcal{S}}$ then the branches of $\vec{\mathcal{T}}^\frown\{\mathcal{M}^+(\mathcal{M}(\vec{\mathcal{T}}))\}^\frown\vec{\mathcal{U}}$ are π-certified.

Suppose then $\vec{\mathcal{T}}$ is a finite stack on $\mathcal{P}$. We then say that $\vec{\mathcal{T}}$ is π-certified if

1. the branches of $\vec{\mathcal{T}}$ are π-certified and

2. whenever $\mathcal{R} \in tn(\vec{\mathcal{T}})$ is such that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}}, b}$ exists then letting $\mathcal{T}$ be the longest irreducible initial segment of $\vec{\mathcal{T}}$ that is based on the top window of $\mathcal{R}$, if $\alpha < lh(\mathcal{T})$ and $\mathcal{M}_\alpha^{\mathcal{T}} \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ isn't a Woodin cardinal" then letting $\mathcal{Q} \trianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$ be the longest such that $\mathcal{Q} \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ is a Woodin cardinal" then $\mathcal{Q}$ is π-certified.

It can now straightforward to check that the following are equivalent:

1. $\vec{\mathcal{T}}$ is according to Σ^{sts}.

2. $\vec{\mathcal{T}}$ is π-certified.

We leave the details to the reader. We comment that clause 4 in the definition of stacks that have π-certified branches guarantees that the embedding $\sigma_{\mathcal{R}} \restriction \mathcal{R}(\alpha)$ is according to Σ.

It then easily follows that $\Sigma^{sts} \in L_\alpha(A, \mathbb{R})$, contradiction. □

13 LSA pointclass from a Woodin limit of Woodins

Theorem 13.1 *Suppose there is a Woodin cardinal that is a limit of Woodin cardinal. Then there is an inner model satisfying $AD^+ LSA$.*

Proof. Woodin showed that, under our current hypothesis, there is an inner model that has divergent models of AD^+, i.e., there are sets of reals $A, B \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \vDash AD^+$, $L(B, \mathbb{R}) \vDash AD^+$, $A \notin L(B, \mathbb{R})$ and $B \notin L(A, \mathbb{R})$. Moreover, his constructions shows that we can assume that both $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ satisfy $MC + \Theta = \theta_0$. Thus, we assume that such a pair of models exists.

We let $\Gamma = L(A, \mathbb{R}) \cap L(A, \mathbb{R}) \cap \wp(\mathbb{R})$. We now assume that there is no inner model satisfying $AD^+ + LSA$. It follows from the proof of Subclaim 2 in the proof of Lemma 12.2 that

(1) there is no inner model M containing the ordinals and reals such that $M \vDash AD^+ +$ "there is an lst pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation".

Using Theorem 12.2 we get that there is $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ and $(\mathcal{Q}, \Lambda) \in L(B, \mathbb{R})$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) = \Gamma$. As in [3], we intend to compare $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ in V.

Before we do that, however, notice that because $L(\Gamma, \mathbb{R}) \vDash AD_{\mathbb{R}}$ (this is an unpublished result of Woodin), it follows that both $\mathcal{P}$ and $\mathcal{Q}$ are of limit type. The proof of Theorem 6.26 implies that in fact both $\mathcal{P}$ and $\mathcal{Q}$ have to be non-meek. The proof of Theorem 3.3 and Theorem 6.26 of [3] imply that there are $(\mathcal{R}, \Psi)$ and $(\mathcal{S}, \Phi)$ such that $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$, $\mathcal{S} \in pI(\mathcal{Q}, \Lambda)$, $\Psi = \Sigma_{\mathcal{R}}$, $\Phi = \Lambda_{\mathcal{S}}$, $\mathcal{R}^b = \mathcal{S}^b$, $\Psi_{\mathcal{R}^b} = \Phi_{\mathcal{S}^b}$ and whenever $\mathcal{W} \in pB(\mathcal{R}, \Psi) \cap pB(\mathcal{S}, \Phi)$, $\Psi_{\mathcal{W}} = \Phi_{\mathcal{W}}$. We then have that the further comparison of $\mathcal{R}$ and $\mathcal{S}$ doesn't use low level disagreements. It follows from Lemma 8.5 of [4] that the comparison of $\mathcal{R}$ and $\mathcal{S}$ is just an extender comparison.

Let now $(\mathcal{T}, \mathcal{U})$ be the trees on $\mathcal{R}$ and $\mathcal{S}$ respectively that are constructed using the extender comparison of $\mathcal{R}$ and $\mathcal{S}$ until we reach models $\mathcal{R}^*$ and $\mathcal{S}^*$ such that

$$\delta^{\mathcal{R}^*} = \delta^{\mathcal{S}^*} =_{def} \delta \text{ and } \mathcal{R}^*|\delta = \mathcal{S}^*|\delta.$$

Suppose first that we can reach such a stage in $< \omega_1$ many steps. Suppose next that $\mathcal{M}^+(\mathcal{R}^*|\delta) \vDash$ "δ isn't a Woodin cardinal". It then follows that $\mathcal{R}^* = \mathcal{S}^*$ and the comparison halts, contradiction.

It then must be the case that $\mathcal{M}^+(\mathcal{R}^*|\delta) \vDash$ "δ is a Woodin cardinal". Let $\mathcal{M} = \mathcal{M}^+(\mathcal{R}|\delta)$. Notice that we have that $\Psi^{sts}_{\mathcal{M}} = \Phi^{sts}_{\mathcal{M}}$. We also must have that $\Gamma(\mathcal{M}, \Psi^{sts}_{\mathcal{M}}) = \Gamma(\mathcal{M}, \Phi^{sts}_{\mathcal{M}}) = \Gamma$. (Otherwise, we have that $Code(\Psi^{sts}_{\mathcal{M}}) \in \Gamma$ and hence, using (1), we get that $\mathcal{R}^* \trianglelefteq Lp^{L(\Gamma,\mathbb{R}),\Psi^{sts}_{\mathcal{M}}}(\mathcal{M})$ and $\mathcal{S}^* \trianglelefteq Lp^{L(\Gamma,\mathbb{R}),\Psi^{sts}_{\mathcal{M}}}(\mathcal{M})$ implying that $\mathcal{R}^* \trianglelefteq \mathcal{S}^*$ or vice versa. It then follows that the comparison actually halts giving us contradiction.) But now we have that $Code(\Phi^{sts}_{\mathcal{M}}) \in L(\Gamma, \mathbb{R})$ implying that $\Gamma = \Gamma(\mathcal{M}, \Phi^{sts}_{\mathcal{M}}) \subset \Gamma$, contradiction.

We must then have that the construction of $(\mathcal{T}, \mathcal{U})$ lasts ω_1-steps. However, notice that in this case, as both sides use the same $\mathcal{Q}$-structures (this follows from (1)), the extender comparison of $\mathcal{R}$ and $\mathcal{S}$ can be done in both $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$. It then follows that, as ω_1 is measurable in both $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$, that $\mathcal{T}$ and $\mathcal{U}$ have branch which gives us a contradiction. □

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