Math 403:01 **A complex variables workshop!** April 7, 2010

Your goal here is to compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx$ using the Residue Theorem.

1 We may write $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx = \lim_{R\to\infty}\int_{I_R} f(z)\,dz$ where I_R is the interval from $-R$ to R on the real line. f is an analytic function with two isolated singularities. It is given by a formula:

$f(z) =$ ____________________.

2 The isolated singularities of f are at the complex numbers $A =$ ______________ in the upper half plane and $B =$ ______________ in the lower half plane.

The type of the isolated singularity at A is (circle one)

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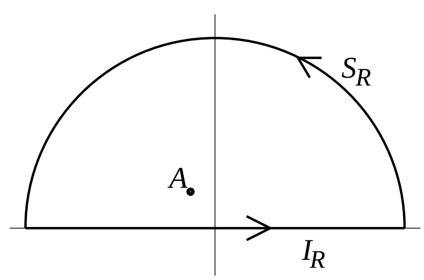
3 We can factor the denominator of f and rewrite its formula as follows:

$f(z) =$ ____________________.

4 The function f has an isolated singularity at A in the upper half plane. Since f has a ____ (insert here a useful and precise descriptive phrase about f's singularity at A), the residue of f at A is easy to compute.

That residue is ____________________.

5 Suppose S_R is the counterclockwise oriented semicircle of radius R in the upper half plane centered at 0. We will write $I_R + S_R$ to mean the simple closed curve obtained by following I_R by S_R. If R is large enough (as shown) so A is inside $I_R + S_R$, the Residue Theorem tells us that the value of $\displaystyle\int_{I_R+S_R} f(z)\,dz$ is ____________________.



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6 If $|z| = R$ where R is a large positive number, then the "reverse triangle inequality" applied to the original formula for f allows us to underestimate the denominator of $|f(z)|$ in terms of R (note that there will be several expressions that are subtracted):

So the denominator of $|f(z)| \geq$ ______________________.

7 The length of S_R is ______________. The ML inequality allows us to overestimate the modulus of $\int_{S_R} f(z)\,dz$ in terms of R.

$\left|\int_{S_R} f(z)\,dz\right| \leq$ ______________________.

8 We therefore conclude that $\lim_{R\to\infty} \int_{S_R} f(z)\,dz =$ ______________.

9 Combine the results of **8** and **5** to compute the value of $\lim_{R\to\infty} \int_{I_R} f(z)\,dz$.

The value of this limit is ______________________.

10 Putting it all together, we finally can write

$\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx =$ ______________________.

`Maple` reports that this quantity is approximately 3.62759 87284 68435 7012.