Math 403:01 **Second exam** April 14, 2008

Due at the beginning of class, 5 PM, on Monday, April 19.

Rules

1. You may not discuss or communicate with any other person about this exam until after the end of class, 6:20 PM, on Monday, April 19.

2. You may use any other sources for help *except* for <u>other people</u>. For example, you may use textbooks, notes, "the Internet", and computers or calculators. Any answer must be supported by your own reasoning. You may *not* consult with any other person in any way.

3. If you have any questions about these rules, please let me know. The problems are understandable so no "hints" will be given (it is an exam!).

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(16) 1. If $f(z) = \dfrac{\sin(z^2)}{(\sin z)^2}$, find and classify *all* isolated singularities of f. If the isolated singularity is a pole, give the order of the pole and the residue of f at the pole.

(16) 2. Use complex analysis to show that $\int_0^\infty \frac{\sqrt{x}}{(1+x^2)^2}\,dx = \frac{\pi\sqrt{2}}{8}$. Show clearly any contour of integration and any residue computation. Explain why the limiting value of certain integrals is 0.

(16) 3. Suppose $f(z)$ is entire and $\sum\limits_{n=0}^{\infty} a_n z^n$ is its Taylor series centered at 0. Let $g(z) = f\left(\frac{1}{z}\right)$.

a) Write $g(z)$'s Laurent series in the annulus $0 < |z| < \infty$ using what's given about $f(z)$.

b) Suppose we know that if $\{z_n\}$ is *any* sequence in $\mathbb{C}$ with $\lim\limits_{n\to\infty} z_n = \infty$, then $\lim\limits_{n\to\infty} f(z_n) = \infty$.* Classify the isolated singularity of $g(z)$ at 0 with some explanation. What must then be true about the Laurent coefficients of $g(z)$?

c) Suppose that an entire function $f(z)$ has the property that if $\{z_n\}$ is *any* sequence in $\mathbb{C}$ with $\lim\limits_{n\to\infty} z_n = \infty$, then $\lim\limits_{n\to\infty} f(z_n) = \infty$. Show that $f(z)$ must be a non-constant polynomial.

(18) 4. Suppose $\{f_n(z)\}$ is a sequence of analytic functions in the unit disc (z's with $|z| < 1$). Also suppose there is a function $f(z)$ defined on the unit disc so that if n is a positive integer and if $|z| < 1$ then $|f_n(z) - f(z)| < \frac{1}{n}$.

a) Prove that $f(z)$ is analytic in the unit disc.

b) Prove that the sequence $\{f_n'(0)\}$ converges and that its limit is $f'(0)$.

c) Give an example of a sequence of differentiable functions $\{g_n(x)\}$ in *real calculus* defined for $-1 < x < 1$ with $\lim\limits_{n\to\infty} g_n(x) = 0$ for every x in $(-1, 1)$ but the sequence $\{g_n'(0)\}$ does *not* converge.

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* If $\{w_n\}$ is a sequence of complex numbers, $\lim\limits_{n\to\infty} w_n = \infty$ means this: Given *any* $M > 0$, there is a positive integer N (which may depend on M) so that if $n > N$, then $|w_n| > M$.

(18) 5. In “real calculus”, $\arctan'(x) = \frac{1}{1+x^2}$ and $\arctan(0) = 0$. Define $A(z)$ for complex z as follows:

If z is not $\pm i$, then $A(z) = \displaystyle\int_C \frac{1}{1+w^2}\,dw$ where C is a curve (*any curve*: C need not be simple!) from 0 to z not passing through $\pm i$.

a) If z is not equal to $\pm i$, explain briefly why there must be such a curve.

Comment Pictures accompanied by some written discussion (with complete sentences!) will be sufficient.

b) Will the result of the integration always be the same, no matter which eligible curve C is chosen? Explain why or why not.

Comment First discuss any values of $A(1)$ as completely as possible. This then implies results about $A(z)$ for all $z \neq \pm i$ and those results should be presented.

c) The answer to b) will show that $A(z)$ is “multivalued”. Explain how the domain of $A(z)$ can be restricted so that this defect is repaired: find a <u>largest connected open set, U, containing $\mathbb{R}$</u> so $A(z)$ is a (single-valued!) function whose derivative is $\frac{1}{1+z^2}$ and also $A(z)$ is the usual $\arctan(x)$ when $z = x$ is a real number.

Comment Many different U's are valid solutions to this problem. Explain why the suggested U verifies the requirements.

(16) 6. Suppose $u(x, y)$ is a *harmonic function*** defined for all of $\mathbb{R}^2$, and that $u(x, y)$ is bounded above: there is a real number M so that $u(x, y) \leq M$ for all (x, y) in $\mathbb{R}^2$. Prove that $u(x, y)$ must be constant.

Hint Math 403 mostly investigates *analytic functions*. We discussed creating an analytic function function $f(z)$ with $\mathrm{Re}\,(f(z)) = u(x, y)$ under certain circumstances. Show that there is such an *entire* function in this case. Then explain why this $f(z)$ must be constant.

** A harmonic function is a real-valued function which has continuous second derivatives of both variables (xx, xy, and yy) satisfying $u_{xx} + u_{yy} = 0$.