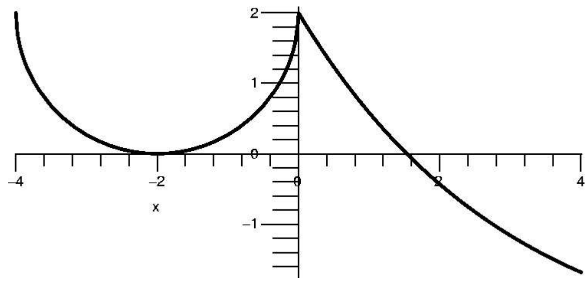
The function in 2009 AB 6

Information about the derivative, f', of a function f defined on the interval $-4 \le x \le 4$ is given. This derivative is continuous, and a graph of f' is shown, similar to what is displayed to the right. Additionally we are told that the graph of f' on $-4 \le x \le 0$ is a semicircle with an x-intercept at $x = -2$. We can use this information together with what we're told about f' for $x > 0$ to get a complete algebraic description.



Graph of f'

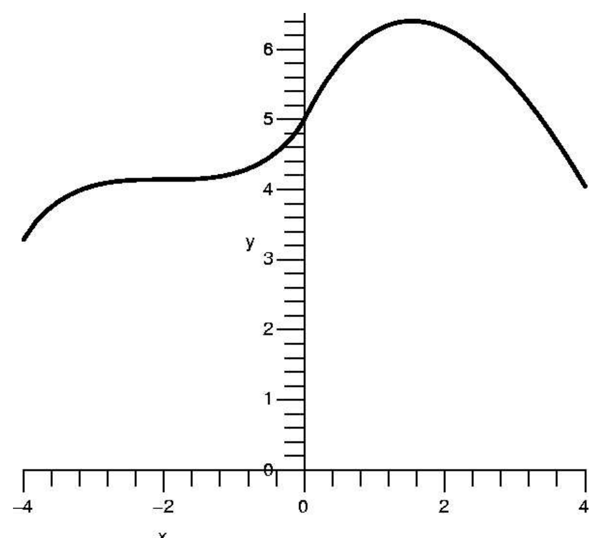
$$f'(x) = \begin{cases} 2 - \sqrt{4-(x+2)^2} & \text{for } -4 \le x \le 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \le 4 \end{cases}.$$

We can use this to get an algebraic description of f. If $x < 0$, then $\int_0^x f'(w)\,dw = \int_0^x 2 - \sqrt{4-(w+2)^2}\,dw = \int_0^x 2 - \sqrt{-w^2-4w}\,dw = \pi + 2x - \left(\frac{1}{2}x+1\right)\sqrt{-x^2-4x} - 2\arcsin\left(\frac{1}{2}x+1\right)$. The antidifferentiation can be obtained using a trigonometric substitution, which is <u>not</u> part of AB calculus*. If $x > 0$, then $\int_0^x f'(w)\,dw = \int_0^x 5e^{-w/3} - 3\,dw = 15 - 15e^{-x/3} - 3x$. Both antiderivatives have value 0 when $x = 0$ so since we know that $f(0) = 3$ we must add 3 to get a complete algebraic description of f:

$$f(x) = 3 + \begin{cases} \pi + 2x - \left(\frac{1}{2}x+1\right)\sqrt{-x^2-4x} - 2\arcsin\left(\frac{1}{2}x+1\right) & \text{for } -4 \le x \le 0 \\ 15 - 15e^{-x/3} - 3x & \text{for } 0 < x \le 4 \end{cases}.$$

A graph of f on $-4 \le x \le 4$ is shown to the right. We can "read off" values of f in answer to (b): $f(4) = 8 - 15e^{-4/3}$ and $f(-4) = -3 + 2\pi$. We can get the latter value by using area or accumulation reasoning, which seems much simpler than this explicit algebraic solution.

The graph seems to show that the inflection points are at $x = -2$ and $x = 0$. This agrees with supporting reasoning for (a). The absolute maximum of f on $-4 \le x \le 4$ appears to be at $x = 3\ln\left(\frac{5}{3}\right)$ which also needs justification.



Graph of f

* All of the computations and graphs here were done by `Maple`.