Here are answers that would earn full credit. Other methods may also be valid.

(10) 1. Suppose $f(x)$ is a differentiable function with $f(8) = 5$, $f'(8) = 3$ and $f''(8) = -2$. If $F(x) = f\big(x^3\big)$, compute $F(2)$, $F'(2)$, and $F''(2)$. **Answer** $F(2) = f(2^3) = f(8) = 5$. Since $F(x) = f\big(x^3\big)$, the Chain Rule states that $F'(x) = f'\big(x^3\big)(3x^2)$ and $F'(2) = f'\big(2^3\big)(3\cdot 2^2) = (3)(3\cdot 4) = 36$. Since $F'(x) = f'\big(x^3\big)(3x^2)$ we know (Product Rule and Chain Rule) $F''(x) = f''\big(x^3\big)(3x^2)(3x^2) + f\big(x^3\big)(6x)$ so that $F''(2) = f''\big(2^3\big)(3\cdot 2^2)(3\cdot 2^2) + f'\big(2^3\big)(6\cdot 2) = (-2)(3\cdot 4)^2 + 5(12)$: a fine answer, which, if you must, equals -180.

(12) 2. Suppose $f(x) = \frac{\ln(x^2+5)}{x+1}$.

a) What is the domain of $f(x)$? Give some justification for your answer. **Answer** Since $x^2 + 5 \geq 5$ no restriction comes from the "ln". $x + 1 = 0$ when $x = -1$. The domain is all numbers *not* equal to -1.

b) Compute $\lim\limits_{x\to+\infty} f(x)$.

Answer Consider the limit of $\frac{\ln(x^2+5)}{x+1}$ as $x \to +\infty$. Both the top and the bottom of the fraction defining $f(x)$ approach ∞ as $x \to \infty$, so the limit is eligible for l'Hopital's rule: $\lim\limits_{x\to+\infty} \frac{\ln(x^2+5)}{x+1} \overset{\text{l'H}}{=} \lim\limits_{x\to+\infty} \frac{\frac{2x}{x^2+1}}{1} = \lim\limits_{x\to+\infty} \frac{2x}{x^2+1}$. This is still eligible for l'H (this fraction also behaves like $\frac{\infty}{\infty}$ as $x \to \infty$), so $\lim\limits_{x\to+\infty} \frac{2x}{x^2+1} \overset{\text{l'H}}{=} \lim\limits_{x\to+\infty} \frac{2}{2x} = 0$.

c) $y = f(x)$ has a horizontal asymptote. Use the answer to b) to write <u>the equation</u> of a horizontal asymptote to the curve $y = f(x)$. **Answer** $y = 0$.

d) Compute $\lim\limits_{x\to-1^1} f(x)$.

Answer As $x \to -1^-$, x is close to -1 and less than -1. So $x + 1$ is small and *negative*. Certainly $x^2 + 5$ is close to $(-1)^2 + 5 = 6$. Since $6 > e$, $\ln(x^2+5) > \ln(e) = 1$. Therefore $\frac{\ln(x^2+5)}{x+1} \approx \frac{\ln(6)}{\text{SMALL NEGATIVE NUMBER}}$. This is a large negative number. So $\lim\limits_{x\to-1^1} f(x) = -\infty$.

e) $y = f(x)$ has a vertical asymptote. Use the answer to c) to write <u>the equation</u> of a vertical asymptote to the curve $y = f(x)$. **Answer** $x = -1$.

(16) 3. Suppose $f(x) = (x^2 - 3)e^x$.

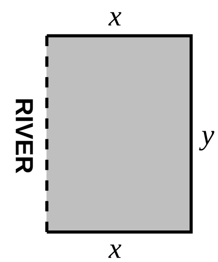
a) Find the first coordinates (the x values) of all relative maximums and minimums of the function $f(x) = (x^2-3)e^x$. Briefly explain your answers using calculus. **Answer** The relative extreme values occur at critical numbers. Since this function is differentiable, we compute $f'(x) = 2x(e^x) + (x^2-3)e^x = (x^2+2x-3)e^x$. But e^x is never 0, so that the only candidates for critical points are solutions of $x^2 + 2x - 3 = 0$. Since $x^2 + 2x - 3 = (x+3)(x-1)$, function's critical numbers are -3 and 1.

First Derivative Test Certainly $f'(0) = -3$ and $f'(100) > 0$ and $f'(-100) > 0$ also. So the function is increasing to the left of -3 and decreasing to the right of -3. Thus $f(-3) = 6e^{-3}$ is a relative maximum value of the function. Also the function is decreasing to the left of 1 and increasing to the right of 1. Thus $f(1) = -2e^{-1}$ is a relative minimum value of the function.

Second Derivative Test Since $f'(x) = (x^2+2x-3)e^x$ we see that $f''(x) = (2x+2)e^x + (x^2+2x-3)e^x = (x^2+4x-1)e^x$. But $f''(-3) = -4e^{-3} < 0$ which means that $f(x)$ is concave down, and $f(-3) = 6e^{-3}$ is a relative maximum value. And $f''(1) = 4e > 0$, therefore $f(x)$ is concave up, and $f(1) = -2e^{-1}$ is a relative minimum value.

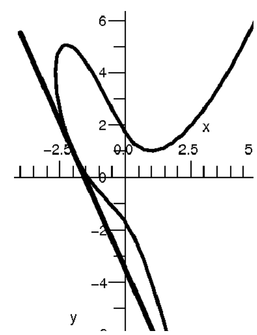
b) Find the first coordinates (the x values) of all inflection points of the function $f(x) = (x^2-3)e^x$. Briefly explain your answers using calculus. **Answer** This function is differentiable, and the inflection points must occur where $f''(x) = 0$. $f''(x)$ is computed above: $f''(x) = (x^2+4x-1)e^x$. The exponential function is never 0, and the roots of $x^2 + 4x - 1 = 0$ are $\frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)} = -2 \pm \sqrt{5}$. But we must still show that the concavity changes at these numbers. Since $f''(0) = -1$, $f(x)$ is concave down at 0. The sign of both numbers $f(\pm 1000)$ is positive (since the exponential function's values are positive and the x^2 overwhelms the other terms). Therefore to the left of $-2-\sqrt{5}$, $f(x)$ is concave up. The concavity switches and between the two roots of $f''(x) = 0$, the graph of $f(x)$ is concave down. And to the right of $-2+\sqrt{5}$, $f(x)$ is concave up. The numbers $-2 \pm \sqrt{5}$ are both first coordinates of points of inflection of $f(x)$.

(16) 4. A farmer has 400 feet of fencing and wants to enclose a rectangular pasture. One side of the pasture is along a river, and does not need fencing. What should the dimensions of the rectangular pasture be, so as to maximize the enclosed area?



Answer If the dimensions of the pasture (length measured in feet) are x and y with the side parallel to the river having length y, then the area A is given by $A = xy$ OBJECTIVE FUNCTION and the sides are restricted by $x \geq 0$, $y \geq 0$ and $2x + y = 400$ CONSTRAINT. Since $y = 400 - 2x$ we have $A = x(400 - 2x)$. The domain for this is $x \geq 0$ (this pasture is ridiculous, just a fence along the river!) and (with $y = 0$ so the pasture is just two fences perpendicular to the river) $2x \leq 400$: the domain is $[0, 200]$. If $x = 0$, $A = 0$, and if $x = 200$, $A = 0$.* Since $A = x(400 - 2x)$, $A' = (400 - 2x) + x(-2)$. The only critical number occurs when $400 - 4x = 0$ and $x = 100$. Then $y = 400 - 2(100) = 200$ and $A = 200(100) > 0$, certainly bigger than the endpoint values. These dimensions "maximize the enclosed area." (This is the first problem in chapter 7.)

(14) 5. The program `Maple` displays the image shown to the right when asked to graph the equation $y^2 = x^3 - 3xy + 3$.



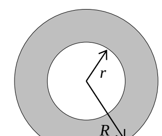
a) Verify by substitution that the point $P = (-2, 1)$ is on the graph of the equation. **Answer** $1^2 = 1$ and $(-2)^3 - 3(-2)1 + 3 = -8 + 6 + 3 = 1$, so the equation is correct.

b) Find $\frac{dy}{dx}$ in terms of y and x. **Answer** d/dx the equation: $2y\frac{dy}{dx} = 3x^2 - 3y - 3x\frac{dy}{dx}$ so that $(2y + 3x)\frac{dy}{dx} = 3x^2 - 3y$ and $\frac{dy}{dx} = \frac{3x^2 - 3y}{2y + 3x}$.

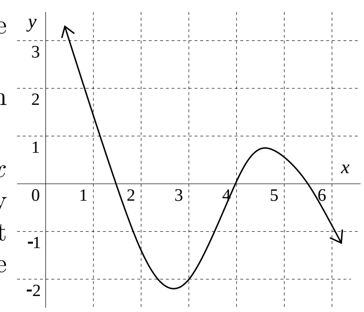
c) Find an equation for the line tangent to the graph at the point $P = (-2, 1)$. **Answer** If $x = -2$ and $y = 1$, $\frac{3x^2 - 3y}{2y + 3x}$ becomes $\frac{9}{-4} = -\frac{9}{4}$. One equation for the line is therefore $y - 1 = -\frac{9}{4}(x + 2)$.

d) Sketch this tangent line in the appropriate place on the image displayed. (Part of the picture is shown.)

(12) 6. Two circles have the same center. The inner circle has radius r which is increasing at the rate of 3 inches per second. The outer circle has radius R which is increasing at the rate of 2 inches per second. Suppose that A is the area of the region *between* the circles. At a certain time, r is 7 inches and R is 10 inches. What is A at that time? How fast is A changing at that time? Is A increasing or decreasing at that time? **Answer** $A = \pi R^2 - \pi r^2$. If $R = 10$ and $r = 7$, $A = \pi(10^2) - \pi(7^2) = 51\pi$. If we d/dt the equation, the result is $A' = 2\pi R\frac{dR}{dt} - 2\pi r\frac{dr}{dt} = 2\pi(10)(2) - 2\pi(7)(3) = -2\pi$. The area is changing at 2π inches per second and it is *decreasing*.



(20) 7. The graph of $y = f'(x)$, the *derivative* of the function $f(x)$, is shown to the right. Use the graph to answer the questions below.



The parts of this problem are *not* related but both parts use information from the graph of the derivative of $f'(x)$.

a) Use information from the graph of $f'(x)$ to find (as well as possible) the x where the *maximum value* of $f(x)$ in the interval $1 \leq x \leq 3$ will occur. Briefly explain using calculus why your answer is correct, including verification that the value of $f(x)$ you select is larger than $f(x)$ at *any* other number in the interval. **Answer** x is ≈ 1.6, the $x-$intercept of $y = f'(x)$ between 1 and 2. Call this x^*. $f'(x) < 0$ between x^* and 3, so the function decreases from x^* to 3: $f(x^*) > f(x)$ if $x^* < x \leq 3$. Also, $f'(x) > 0$ for x between 1 and x^*, so $f(x)$ is increasing in $[1, x^*]$. Therefore $f(x^*) > f(x)$ if $1 \leq x < x^*$.

b) Suppose that $f(3) = 5$. Use information from the graph and the tangent line approximation for $f(x)$ to find an approximate value of $f(3.04)$. Briefly explain using calculus and information from the graph why your approximate value for $f(3.04)$ is greater than or less than the exact value of $f(3.04)$. **Answer** Linear approximation gives $f(3.04) \approx f(3) + f'(3)(.04)$. The graph supplies $f'(3) = -2$ so $f(3.04) \approx 5 + (-2)(.04) = 4.92$. The tangent line to $y = f'(x)$ at $x = 3$ has slope > 0 (the graph of $y = f'(x)$ is *increasing* near $x = 3$) so the derivative of $f'(x)$ is positive there: $f''(x) > 0$ near $x = 3$, and $y = f(x)$ is concave up near $x = 3$. The approximate value is *less than* the exact value since the tangent line will lie below the graph of $y = f(x)$.

* Perhaps a more realistic domain is the *open* interval $0 < x < 200$ so the pasture is "real". Then we can investigate the edges of the domain with $\lim_{x \to 0^+} A = 0$ and $\lim_{x \to 200^-} A = 0$. The result is the same.