March 9, 2004

Math 507: Functional Analysis (Spring, 2004)

D1 Suppose K is a compact subset of a Hilbert space, H. Prove that there is a closed separable subspace H_1 of H with $K \subseteq H_1$.

Comment I needed this result in the course. It was clear at the time. Is it still?

D2 (Holomorphic polynomial examples) a) Prove that there is a sequence of polynomials $\{P_n(z)\}$ so that $\lim_{n\to\infty} P_n(z)$ exists for all $z \in \mathbb{C}$ and is 1 if $z = 0$ and 0 if $z \neq 0$.

b) Prove that there is a sequence of polynomials $\{P_n(z)\}$ so that $\lim_{n\to\infty} P_n(z)$ exists for all $z \in \mathbb{C}$ and is 1 if $\operatorname{Re} z \geq 0$ and 0 if $\operatorname{Re} z < 0$.

D3 Suppose that $K(s,t)$ is a continuous real function on $[0,1] \times [0,1]$, and $T(g)(t) = \int_0^1 K(s,t)g(s)\,ds$. If $g \in C([0,1])$, then $f = T(g)$ is also continuous on $[0,1]$, and the eigenfunction expansion associated to T and K for f converges absolutely and uniformly to f in $[0,1]$ (not just in L^2).

D4 Suppose $L = D^2 + xD$ where $D = \frac{d}{dx}$ on $[0,1]$.

a) Find w so that wL is self-adjoint.

b) Find boundary conditions on $[0,1]$ so that L with these boundary conditions is a regular Sturm-Liouville problem for which 0 is not an eigenvalue.

D5 Suppose that L is a Banach limit on ℓ^∞. Show that there are sequences X and Y in ℓ^∞ so that $L(XY) \neq L(X)L(Y)$. Here the product XY is defined "pointwise" or "coordinatewise": $(XY)_n = X_n Y_n$. (This problem is found in many texts.)

Definition Two norms $\|\ \|_1$ and $\|\ \|_2$ on a vector space V are said to be *equivalent* if there is a $c > 0$ so that $c\|v\|_1 \leq \|v\|_2 \leq \frac{1}{c}\|v\|_1$ for all $v \in V$.

D6 a) Prove that all norms on $\mathbb{R}^n$ are equivalent.

b) Give (and verify!) an example of a V and two norms which are *not* equivalent.

D7 If V is complete with respect to both $\|\ \|_1$ and $\|\ \|_2$, and if $\|v\|_1 \leq C\|v\|_2$ holds for all $v \in V$, then $\|\ \|_1$ and $\|\ \|_2$ are equivalent. Can the hypotheses be weakened in this?

D8 Show that c^* is isometrically isomorphic to ℓ^1. Are c and c_0 isometrically isomorphic?

Remark This is problem 4 of section 3.6 in Conway's *A Course in Functional Analysis*.

D9 Show that if $X \in \ell^\infty$ $\|X\|_\infty \leq 1$, then there is a sequence $\{X_n\}$, X_n in ℓ^∞ such that $\|X_n\|_\infty \leq 1$, $\|X_n - X\|_\infty \to 0$, and each X_n takes on only a finite number of values.

Remark This is problem 3 of section 3.7 in Conway's *A Course in Functional Analysis*.