640:411 **Answers to the First Exam** 10/19/2008

(13) 1. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are real sequences, and that for all positive integers, n, $x_n \le y_n \le z_n$. If both $\{x_n\}$ and $\{z_n\}$ converge and have the same limit, L, prove that $\{y_n\}$ converges and its limit is L. **Answer** Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = L$ there is $N_x \in \mathbb{N}$ so that if $n \ge N_x$, then $|x_n - L| < \varepsilon$. Therefore for such n, $L - \varepsilon < x_n < L + \varepsilon$. Similarly, there is $N_z \in \mathbb{N}$ so that if $n \ge N_z$, then $L - \varepsilon < z_n < L + \varepsilon$. Let $N = \max(N_x, N_z)$. If $n \ge N$, then $L - \varepsilon < x_n \le y_n \le z_n < L + \varepsilon$, so that $|y_n - L| < \varepsilon$. Thus $\lim_{n\to\infty} y_n = L$.

(13) 2. Suppose (X, d) is a metric space. If P and Q are connected subsets of X with $P \cap Q \ne \emptyset$, prove that $P \cup Q$ is connected. **Answer** Suppose there is a separation, A and B of $P \cup Q$. Then $A \cup B = P \cup Q$, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$, and neither A nor B is empty. Since $P \cap Q \ne \emptyset$, there is $x \in P \cup Q$ so that $x \in A$ or $x \in B$. We address the first case (the second is similar). Surely $A \cap P$ and $B \cap P$ is a separation of P since $\overline{A \cap P} \subset \overline{A} \cap P \subset \overline{A}$ and $B \cap P \subset B$ and therefore $\left(\overline{A \cap P}\right) \cap (B \cap P) = \emptyset$ (and similarly reversing the roles of A and B). But $x \in A$, so since P is connected, $P \subset A$. Similarly, if $x \in Q$, $Q \subset A$. Therefore $B = \emptyset$, a contradiction: no separation exists, and $P \cup Q$ is connected.

(15) 3. Suppose (X, d) is a metric space.

a) If A and B are subsets of X, prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. **Answer** If $x \in \overline{A}$, then given $r > 0$, $N_r(x) \cap A \ne \emptyset$. That is, either $x \in A$ or x is a limit point of A. Since $A \subset A \cup B$, if $x \in \overline{A}$, then given $r > 0$, $N_r(x) \cap (A \cup B) \ne \emptyset$ and thus $x \in \overline{A \cup B}$. The case $x \in \overline{B}$ is similar. Now if $x \in \overline{A \cup B}$, consider $N_r(x) \cap (A \cup B) = (N_r(x) \cap A) \cup (N_r(x) \cap B)$. This is not empty because x is an element of the closure of $A \cup B$. If there is $r > 0$ so that $N_r(x) \cap A = \emptyset$, there is always $b \in B$ with $b \in N_r(x)$. Also, if $0 < s < r$, there must be $b \in B$ with $b \in N_s(x)$ or else $N_s(x) \cap (A \cup B) = \emptyset$. Therefore $x \in \overline{B}$. The situation if $N_r(x) \cap B = \emptyset$ is similar, so either x is in $\overline{A}$ or in $\overline{B}$.

b) Give an example to show that the closure of the union of a *countable* number of subsets of X need not be equal to the union of the closures of each of the sets. **Answer** Take $X = \mathbb{R}$ with the usual metric, and $A_j = \left\{\frac{1}{j}\right\}$ for positive integer j. Here $\overline{A_j} = A_j$ but $0 \in \overline{\bigcup_{j=1}^{\infty} A_j}$. So the union of the closures is not the same as the closure of the union.

c) Give an example to show that $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ need not be equal. Here A and B are subsets of X. **Answer** Take $X = \mathbb{R}$ with the usual metric. Suppose $A = [0, 1)$ and $B = [1, 2]$ so that $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$. Therefore $A \cap B = \emptyset$ so the closure is empty, but $\overline{A} \cap \overline{B} = \{1\}$

(15) 4. Suppose (X, d) is a metric space.

a) If A is a subset of X, prove that $\operatorname{diam}(A) = \operatorname{diam}\left(\overline{A}\right)$. **Comment** $\operatorname{diam}(S) = \sup\{d(x, y) \colon x, y \in S\}$ if $S \subset X$. **Answer** Since $A \subset \overline{A}$, the sup for $\overline{A}$ is taken over more real numbers, and therefore $\operatorname{diam}(A) \le \operatorname{diam}\left(\overline{A}\right)$. If $\operatorname{diam}(A) < \operatorname{diam}\left(\overline{A}\right)$, then there is $\delta > 0$ so that $\operatorname{diam}(A) + \delta < \operatorname{diam}\left(\overline{A}\right)$ and therefore $d(x, y) + \delta < \operatorname{diam}\left(\overline{A}\right)$ for all x and y in A. But the diameter of the closure is a sup, so there must be z and w in $\overline{A}$ so that $d(x, y) + \frac{\delta}{2} < d(z, w)$ for all x and y in A. Since $z \in \overline{A}$ and $w \in \overline{A}$, there are elements $\tilde{z}$ and $\tilde{w}$ in A with $d(z, \tilde{z}) < \frac{\delta}{4}$ and $d(w, \tilde{w}) < \frac{\delta}{4}$. Estimate: $d(z, w) \overset{\Delta}{\le} d(\tilde{z}, z) + d(\tilde{z}, \tilde{w}) + d(\tilde{w}, w) < d(\tilde{z}, \tilde{w}) + 2\left(\frac{\delta}{4}\right) = d(\tilde{z}, \tilde{w}) + \frac{\delta}{2}$. This contradicts a

previous assertion (with $\tilde{z}$ as x and $\tilde{w}$ as y) so the diameters must be equal. (The text's proof is more economical.)
b) Give an example of a subset A of X with $\text{diam}(A) \neq \text{diam}(A^{\circ})$ and $A^{\circ} \neq \emptyset$. (A° is the interior of A.) **Answer** Take $X = \mathbb{R}$ with the usual metric. If $A = [0,1] \cup \{2\}$, then $A^{\circ} = (0,1)$, $\text{diam}(A) = 2$, and $\text{diam}(A^{\circ}) = 1$.

(15) 5. a) Suppose (X,d) is a metric space, K is a compact subset of X, U is an open subset of X, and $K \subset U$. Prove that there is $r > 0$ so that $\bigcup_{k \in K} N_r(k) \subset U$. **Answer** Suppose $k \in K$. Since U is open, there is $r_k > 0$ with $N_{\mathbf{2}r_k}(k) \subset U$. Then $\{N_{r_k}(k)\}_{k \in K}$ is an open cover of K (with no **2** here!). K is compact so there is a finite subcover, $\left\{N_{r_{k_j}}(k_j)\right\}_{1 \le j \le n}$. Define $r = \min\{r_{k_j} : 1 \le j \le n\}$. s is a positive real number since it is the minimum of a finite set of positive real numbers. If $k \in K$, then there is k_j with $d(k, k_j) < r$ (cover!). But $N_r(k) \subset N_{2r}(k_j)$ (triangle inequality) and $N_{2r}(k_j) \subset N_{2r_{k_j}}(k_j) \subset U$. So we have proved $\bigcup_{k \in K} N_r(k) \subset U$.
Alternative proof Suppose no such r exists. Then for any positive integer n we can find $k_n \in K$ and $v_n \notin U$ with $d(k_n, v_n) < \frac{1}{n}$. Since K is compact, the sequence $\{k_n\}$ has a subsequence which converges to q in K. But $q \in U$ so there's $\delta > 0$ with $N_\delta(q) \subset U$. Find n so that $\frac{1}{n} < \frac{\delta}{2}$ and $d(k_n, q) < \frac{\delta}{2}$, possible since q is a subsequential limit of $\{k_n\}$. Then (by $\Delta \le$) $v_n \in N_{\frac{1}{n}}(k_n) \subset N_{\frac{\delta}{2}}(k_n) \subset N_\delta(q) \subset U$. But this contradicts $v_n \notin U$.
b) Give an example to show that there can be a closed subset C of X and an open subset U of X with $C \subset U$ so that there is no $r > 0$ with $\bigcup_{x \in C} N_r(x) \subset U$. **Answer** Take $\mathbb{R}$ with the usual metric and let C be the positive integers and U be the open set $\bigcup_{n \in N} \left(n - \frac{1}{n}, n + \frac{1}{n}\right)$. The Archimedean property implies there is no positive r with $r < \frac{1}{n}$ for all $n \in \mathbb{N}$, so this C is as desired. It is not difficult to find examples of *connected* C's and U's satisfying this question in R^2.

(14) 6. a) Prove directly from the definition of compactness that the half-open interval $(0,1] \subset \mathbb{R}$ is not compact. ($\mathbb{R}$ has the usual topology.) **Answer** Take $U_n = \left(\frac{1}{n}, 1\right]$. Then $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of $(0,1]$ and $U_{n+1} \supset U_n$. It is a cover by the Archimedean property. The cover "nests" since $\frac{1}{n+1} < \frac{1}{n}$. If $\left\{U_{n_j}\right\}_{1 \le j \le N}$ is a finite subcover, $\bigcup_{1 \le j \le N} U_{n_j} = U_M$ where $M = \max\{n_j : 1 \le j \le N\}$. But $U_M = \left(\frac{1}{M}, 1\right] \neq (0,1]$ by the Archimedean property.
b) Prove that a Cauchy sequence in a metric space is bounded.
Answer Proved in class and in the text.

(15) 7. Suppose the following is known about three sequences:

If n is a positive integer, then $|x_n - 2| < \frac{5}{n}$, $|y_n - 6| < \frac{20}{\sqrt{n}}$, and $|z_n - 5| < \frac{6}{n^2}$.

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge, and their respective limits are 2, 6, and 5. The sequence whose n^{th} term is $x_n y_n - z_n$ converges and its limit is $2 \cdot 6 - 5 = 7$. Do not prove this, but find and verify a specific n so that $|(x_n y_n - z_n) - 7| < \frac{1}{1{,}000}$. This need not be a "best possible" n but you must supply a specific n and a proof of your estimate.
Answer $|(x_n y_n - z_n) - 7| = |(x_n y_n - z_n) - (2 \cdot 6 - 5)| \le |x_n y_n - 2y_n + 2y_n - 2 \cdot 6| + |z_n - 5| \le |x_n - 2|\,|y_n| + 2|y_n - 6| + |z_n - 5|$. Suppose $n \ge (100)^2$. Then $|y_n - 6| < \frac{20}{100} = \frac{1}{5}$ so that $|y_n| \le |y_n - 6| + |6| < 7$ as well as $|y_n - 6| < \frac{20}{\sqrt{n}}$. Further, we know $|x_n - 2|\,|y_n| < \frac{5}{n} \cdot 7 = \frac{35}{n}$. Therefore $|(x_n y_n - z_n) - 7| < \frac{35}{n} + \frac{20}{\sqrt{n}} + \frac{6}{n^2}$. Wow! Now take $n = 10^{10}$ so $\frac{35}{n} = \frac{35}{10^{10}} < \frac{1}{3{,}000}$, $\frac{20}{\sqrt{n}} = \frac{20}{10^5} < \frac{1}{3{,}000}$, and $\frac{6}{n^2} = \frac{6}{10^{20}} < \frac{1}{3{,}000}$. The total will be less than $\frac{1}{1{,}000}$.