Please write solutions to $N+1$ of these problems by Monday, November 6. These written solutions should be accompanied by explanations using complete English sentences. Any computer assistance should be appropriately documented. Students may work in groups: a group is N^* students. All students in each group should read all of the group's answers before the work is handed in. Thus all students in each group will therefore be responsible for all answers handed in by the group.

1. a) What is the maximum value of the function $f(x,y)=3x+5y$ subject to the constraint $x^2+y^2=1$, and where is it attained? Draw a picture of the constraint and the appropriate level set of the objective function.

b) Suppose n is a positive real number. What is the maximum value of the function $f(x,y)=3x+5y$ subject to the constraint $x^n+y^n=1$ and where is it attained? Your answers should all be functions of n.

c) What happens to the maximum value found in b) when $n \to \infty$? Try to draw a picture of the situation when n is large.

d) What happens to the maximum value found in b) when $n \to 0^+$? Try to draw a picture of the situation when n is small.

Comment Graphing programs don't seem to handle the extreme situations described in c) and d) very well. You may need to think about what appropriate pictures should be.

2. It is certainly possible for the set of critical points of a function defined in $\mathbb{R}^3$ to be a point (e.g., $x^2+y^2+z^2$) or a line (e.g., x^2+y^2) or a plane (e.g., x^2). Can you create a function $F:\mathbb{R}^3 \to \mathbb{R}$ whose set of critical points is the twisted cubic, $\mathbf{c}(t)=(t,t^2,t^3)$?

3. a) Find the second order Taylor polynomial of

$$z=\sqrt{6x+5y}$$

at the point $(x,y)=(1,2)$.

b) Suppose z is defined *implicitly* as a function of x and y near the point $(1,2,4)$ by the equation

$$5x+3xy-z^2+2xyz-3x^2z+1=0$$

Find the second order Taylor polynomial of z at the point $(1,2,4)$†.

4. Suppose $\mathcal{S}$ is the unit square where $0 \le x \le 1$ and $0 \le y \le 1$. Does

$$\int\!\!\int_{\mathcal{S}} \frac{1}{x+y}\,dA$$

OVER

* N here is 1 or 2 or 3.

† No, I don't think that the two functions of a) and b) are the same!

converge, and, if it does, what is its value? Include a simple sketch produced by `Maple` of the volume indicated in this integral.

5. Does either of these converge? If not, explain why. If yes, evaluate. Of course, explain your answers.

a) $\displaystyle\iint_{\mathbb{R}^2} e^{-|x|-|y|}\,dA$ b) $\displaystyle\iint_{\mathbb{R}^2} e^{-|x+y|}\,dA$

6. a) **The 1-dimensional wave equation** Suppose that $H(x,t)$ represents the height of a vibrating string over the point x of the real line at time t. Then for small vibrations and for homogeneous strings (think of a guitar string) H satisfies the partial differential equation $\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial t^2} = 0$. Verify that if $f(w)$ is any twice-differentiable function of one variable, then $H(x,t) = f(x-t)$ satisfies the wave equation. Comment on how the wave shape (the graph of f) travels along the string as t changes. If H_1 and H_2 satisfy the wave equation, verify that $H_1 + H_2$ and cH_1 (where c is any constant) also satisfy the wave equation. This is called "the principle of superposition".

b) **The Korteweg-de Vries equation** The following paragraph was written in 1844 by John Scott Russell, a Scottish engineer.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Quoted on `http://www.ma.hw.ac.uk/~chris/scott_russell.html`

This was the first published record of what is now called a soliton[α]. In 1895 the phenomenon was described with a partial differential equation named the *Korteweg-de Vries equation* (KdV): $u_t + u_{xxx} + 6uu_x = 0$, Korteweg and de Vries stated the equation with supporting reasoning. This equation is non-linear and does not satisfy the "principle of superposition". KdV and related equations have turned out to be very important both theoretically and in practical applications (fiber optics, transmission of nerve impulses, some chemical reactions).

i) If $K > 0$, show that $u_K = \frac{K}{2}\left(\operatorname{sech}\left(\frac{\sqrt{K}}{2}(x - Kt)\right)\right)^2$ is a solution of KdV[β].

ii) Verify that $7u_1$ and $u_2 + u_3$ are *not* solutions of KdV.

iii) Describe a connection between the speed of the soliton and its maximum height.[γ]

More information `http://math.cofc.edu/faculty/kasman/SOLITONPICS/index.html`

[α] Google lists more than 2,430,000 web pages in response to "soliton".

[β] Yes: sech means *hyperbolic secant*.

[γ] Russell wrote (1885), "The sound of a cannon travels faster than the command to fire it."