**Answers to vector calculus review problems for the final exam in Math 251:05–10**

1. Compute the following integrals. Use Green's Theorem, Stokes' Theorem or the Divergence Theorem wherever they are helpful.

a) $\iint_D xy\,dA$, where $D$ is the triangle in the $xy$-plane with vertices (0,0), (2,0), and (0,2).

**Answer** Convert to an iterated integral: $\int_0^2\int_0^{2-x} xy\,dy\,dx = \int_0^2 x\frac{(2-x)^2}{2}\,dx = \int_0^2 \frac{x^3}{2} - 2x^2 + 2x\,dx = \frac{2}{3}$

b) $\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$ and $C$ is the segment of the parabola $y = x^2$ beginning at $(-1,1)$ and ending at $(1,1)$.

**Answer** $\begin{cases} x = t \text{ so } \; dx = dt \\ y = t^2 \text{ so } dy = 2t\,dt \end{cases}$ and $\int_C x^2\,dx - xy\,dy = \int_{-1}^1 t^2\,dt - t^3\,2t\,dt = \int_{-1}^1 t^2 - 2t^4\,dt = -\frac{2}{15}$.

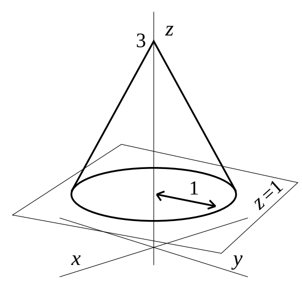
c) $\iint_S (x^3\mathbf{i} + y^3\mathbf{j} + \cos(xy)\mathbf{k})\cdot\mathbf{n}\,dS$, where $S$ is the unit sphere and $\mathbf{n}$ points inward.

**Answer** Use the Divergence Theorem. The normal points the "wrong" way, so this integral must equal $-\iiint_R \operatorname{div}(x^3\mathbf{i} + y^3\mathbf{j} + \cos(xy)\mathbf{k})\,dV = -\iiint_R 3x^2 + 3y^2\,dV$ where $R$ is the region inside the unit sphere. One way to evaluate the triple integral is with cylindrical coordinates (not spherical, because there is no $z^2$ in the integrand). In iterated cylindrical coordinates, this becomes $-\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} (3r^2)r\,dr\,d\theta\,dz$. The initial integral is $-\frac{3}{4}r^4\Big]_0^{\sqrt{1-z^2}} = -\frac{3}{4}(1-z^2)^2$. Multiply by $2\pi$ to compute the next integral, and expand for the last integral. So we have $-\int_{-1}^1 \frac{3\pi}{2}(1 - 2z^2 + z^4)dz = -\frac{3\pi}{2}(z - \frac{2}{3}z^3 + \frac{1}{5}z^5\Big]_{-1}^1 = -\frac{8\pi}{5}$.

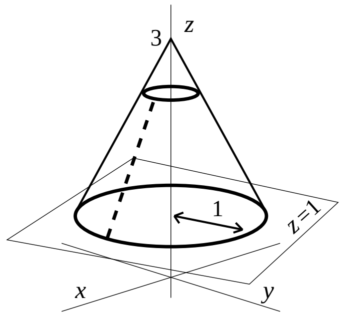
d) $\iint_S z^2\,dS$, where $S$ is the surface $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

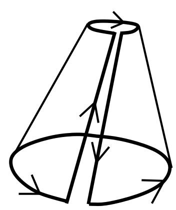
**Answer** Here $f(x,y) = \sqrt{x^2+y^2}$ and $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$ so $\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$ and $\iint_S z^2\,dS = \iint_{\text{The unit circle}} (x^2+y^2)\sqrt{2}\,dA_{xy} \overset{\text{Polar coordinates}}{=} \int_0^{2\pi}\int_0^1 \sqrt{2}(r^2)r\,dr\,d\theta = \frac{\pi}{\sqrt{2}}$.

e) $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz^2\mathbf{j} + z^3\mathbf{k}$ and $S$ is the lateral surface of the cone as shown, with $\mathbf{n}$ pointing outward.




**Answer** Use Stokes' Theorem. I'll be a bit cautious. I'd like to apply Stokes' Theorem to the boundary curve which is the circle at the bottom of the cone, where the circle is directed counterclockwise when viewed from high up on the $z$-axis. But the surface has a corner in the middle. We can't (or, rather, we shouldn't!) ignore questions of existence of derivatives. Simple examples in connection with Green's Theorem show that existence of the functions and suitable derivatives at *every* point inside a curve are important. I'll make a nice region for Stokes' Theorem by "drawing" a dashed line up the side of the cone, connecting a small circle around the top with the circle at the base. I think of what I've done as in the somewhat sloppier picture on the right: there I have "separated" the two sides of the dotted line, and I have drawn directions of integration along the edge curves. Integration along the two line segments which result from the original dotted line will cancel. We did arguments similar to this when using Green's Theorem. The line integral for the circle on top involves integrating three polynomials over a circle whose length $\to 0$ as the circle gets closer to $(0,0,3)$. Since the polynomials are continuous everywhere, the integral of the top circle will $\to 0$ because the length of the circle $\to 0$ and the integrands are bounded. Essentially this discussion is providing the foundation for the declaration that we *can* apply Stokes' Theorem to the whole cone with the bottom boundary circle. Stokes' Theorem tells me that $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot\mathbf{n}\,dS$, where $C$ is the circle at the base of the cone. But in fact I can apply Stokes' Theorem again to assert that the line integral over $C$ is the same as the flux integral over the base of the cone (this is the same logic as the second



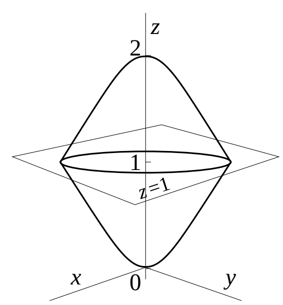
problem discussed in the course diary entry for 4/21/2006). So I compute the flux integral $\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where $D$ is the disc in the plane $z = 1$ inside the unit $(xy)$ circle in *that plane*. We compute $\nabla \times \mathbf{F}$: it is $\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz^2 & z^3 \end{pmatrix} = -2zx\mathbf{i} + y\mathbf{j} + (z^2 - z)\mathbf{k}$. On the plane $z = 1$, this is $-2x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$. The outward unit normal on the disk (the orientation needed for Stokes' Theorem) is $\mathbf{k}$ so the integrand in the flux computation over the disk, $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$, is just 0. The integral of that function over the disk $x^2 + y^2 \le 1$ is 0.
Another way to complete this problem is to compute the line integral over $C$. This is (remember that $z = 1$) the integral $\int_C \mathbf{F} \cdot \mathbf{n}\, dS = \int_C (y\mathbf{i} + x\mathbf{j} + \mathbf{k}) \cdot \mathbf{n}\, ds = \int_C -x\, dx + y\, dy$. Finally, that integral can be computed directly using the usual parameterization for the unit circle ($x = \cos\theta$ and $y = \sin\theta$ etc.) or it can be computed using a standard version of Green's Theorem. The result of either computation is 0 again.

2. Compute $\int_C e^x \sin z\, dx + y^2\, dy + e^x \cos z\, dz$, where $C$ is the oriented curve $\mathbf{x}(t) = (\cos t)^3\, \mathbf{i} + (\sin t)^3\, \mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$. First find a potential function.
**Answer** If $\mathbf{V} = e^x \sin z\mathbf{i} + y^2\mathbf{j} + e^x \cos z\mathbf{k}$ then $\nabla \times \mathbf{V} = 0$ is a direct and simple computation. Since the components of the vector field $\mathbf{V}$ are defined in all of $\mathbb{R}^3$, $\mathbf{V}$ must have a potential. We integrate to find it: $\int e^x \sin z\, dx = e^x \sin z + C_1(y, z)$ and $\int y^2\, dy = \frac{1}{3}y^3 + C_2(x, z)$ and $\int e^x \cos z\, dz = e^x \sin z + C_3(x, y)$. The results are three representations of the same function. The three functions $C_\star$ may contain functions only of the variables indicated. We compare the representations and declare that $f(x, y, z) = e^x \sin z + \frac{1}{3}y^3$ is one possible potential. Then $f(\text{THE END}) - f(\text{THE START})$ is the value of the integral over the curve $C$. When $t = 0$ we get $(1, 0, 0)$ as the position (THE START), and when $t = \pi/2$ we get $(0, 1, \pi/2)$ which is THE END. $f$'s value at THE START is 0 and at THE END it is $\frac{4}{3}$. So the integral's value is $\frac{4}{3}$.

3. A fluid has density 1500 and velocity field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the flow outward through the sphere $x^2 + y^2 + z^2 = 25$.
**Answer** Use the Divergence Theorem. The outward flux is given by the "other side" of the Divergence Theorem: $\int\int\int_V \operatorname{div} \mathbf{v}\, dV$, where $V$ is the region inside the sphere of radius 5 and center $(0, 0, 0)$. Here the divergence of $\mathbf{v}$ is $\frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial 2z}{\partial z} = 2$. So we need the integral of 2 over the sphere, and this is twice the volume of the sphere. Thus the flux is $2 \cdot \frac{4}{3}\pi 5^3 = \frac{1000\pi}{3}$. I think (note, please, that this problem was *not* written by me!) that we should multiply the flux by the density to get the total amount of fluid. Of course there are no units mentioned, which would help me, at least! If I am correct, then the answer is $1500 \cdot \frac{1000\pi}{3} = 500{,}000\pi$. Physically I do think this problem makes sense. If the flow were in gallons per minute or something, and if the density were in pounds per gallon, then we would get a product unit of pounds per minute: sensible.

4. Sketch the region $E$ contained between the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and let $S$ be the boundary of $E$.

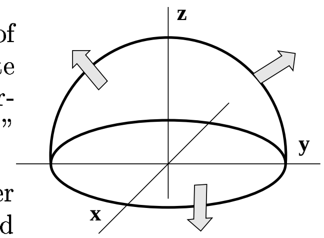
a) Find the volume of $E$.




**Answer** Use cylindrical coordinates. I've given a sketch of the surfaces. One is a paraboloid opening up, and the other is a paraboloid opening down. They intersect where $x^2 + y^2 = 2 - x^2 - y^2$, which occurs when $x^2 + y^2 = 1$, so $z$ must be 1. We can compute the volume of $E$ by taking the double integral of the height of the solid over the "base" of the solid. This solid is based over the unit circle in the $xy$=plane. The height over the point $(x, y)$ is $(2 - x^2 - y^2) - (x^2 + y^2) = 2 - 2x^2 - 2y^2$ (which is THE TOP−THE BOTTOM). I'll convert to polar coordinates: $\int_0^{2\pi} \int_0^1 (2 - 2r^2) r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2r - 2r^3)\, dr\, d\theta = \int_0^{2\pi} \left(r^2 - \frac{2}{4}r^4\right]_0^1 d\theta = \int_0^{2\pi} 1 - \frac{1}{2}\, d\theta = \pi$. The volume is $\pi$.

b) Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{n}$ is the outer normal to $S$.
**Answer** Use the Divergence Theorem. The divergence of $\mathbf{F}$ is $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$, a constant, so that the flux is three times the volume of $E$, or $3\pi$.

5. Find the total flux upward through the upper hemisphere ($z \ge 0$) of the sphere $x^2 + y^2 + z^2 = a^2$ of the vector field $\mathbf{T}(x, y, z) = \left(\frac{x^3}{3}\right)\mathbf{i} + \left(yz^2 + e^{\sqrt{zx}}\right)\mathbf{j} + \left(zy^2 + y + 2 + \sin\left(x^3\right)\right)\mathbf{k}$.




**Answer** I'll apply the Divergence Theorem to the upper half of the sphere of radius $a$ centered at the origin. There are two flux integrals. We want to calculate the flux over the upper hemisphere. First I'll compute the integral of the divergence of $\mathbf{T}$ over the upper half of the sphere, and then I'll compute the flux "out" through the disc of radius $a$ centered at the origin in the $xy$-plane.

**The divergence integral** The divergence of $\mathbf{T}$ is $x^2+z^2+y^2$. Both the upper hemisphere *and* the integrand $(x^2+z^2+y^2=\rho^2)$ hemisphere *and* the integrand $(x^2+z^2+y^2=\rho^2)$ are easy to describe in spherical coordinates. So the integral of the divergence of $\mathbf{T}$ in iterated spherical coordinates is $\int_0^{\pi/2}\int_0^{2\pi}\int_0^a (3r^2)\rho^2\left(\rho^2\sin(\phi)\right)\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\sin(\phi)\,d\phi\cdot\int_0^{2\pi}1\,d\theta\cdot\int_0^a\rho^4\,d\rho = 1\cdot 2\pi\cdot\frac{a^4}{4}=\frac{a^4\pi}{2}$.

**The flux "out" through the bottom disc** Here $\mathbf{n}$ is $-\mathbf{k}$. Also on the bottom disc, $\mathbf{T}(x,y,0)=\left(\frac{x^3}{3}\right)\mathbf{i}+1\mathbf{j}+\left(y+2+\sin\left(x^3\right)\right)\mathbf{k}$. So we need to integrate $\mathbf{T}\cdot\mathbf{n}=-\left(y+2+\sin\left(x^3\right)\right)$ over the disc of radius $a$ centered at the origin. The $y$ and $\sin\left(x^3\right)$ integrate to 0 since these functions are "odd" or antisymmetric and the region is symmetric both in $x$ and in $y$. The integral of $-2$ is minus twice the area: $-2\pi a^2$.

**Putting it together** The Divergence Theorem states that $\frac{a^4\pi}{2}=-2\pi a^2+$ THE FLUX WE WANT. Therefore the answer to the question is $\frac{a^4\pi}{2}+2\pi a^2$.

6. Suppose $\mathbf{F}=-2xz\mathbf{i}+y^2\mathbf{k}$. **Note** There is *no* $\mathbf{j}$ component in $\mathbf{F}$.

a) Compute curl $\mathbf{F}$.

**Answer** We compute $\nabla\times\mathbf{F}$: it is $\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -2xz & 0 & y^2\end{pmatrix}=2y\mathbf{i}-2x\mathbf{j}+0\mathbf{k}$.

b) Compute the outward unit normal $\mathbf{n}$ for the sphere $x^2+y^2+z^2=a^2$.

**Answer** If $f(x,y,z)=x^2+y^2+z^2$, an outward normal vector is $\nabla f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$, I hope you can see that it does point "out" in this case. We can *normalize* it by dividing by the correct constant: $n=\frac{x}{a}\mathbf{i}+\frac{y}{a}\mathbf{j}+\frac{z}{a}\mathbf{k}$,

c) If $R$ is any region on the sphere $x^2+y^2+z^2=a^2$, verify that $\iint_R(\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS=0$.

**Answer** $(2y\mathbf{i}-2x\mathbf{j}+0\mathbf{k})\cdot\left(\frac{x}{a}\mathbf{i}+\frac{y}{a}\mathbf{j}+\frac{z}{a}\mathbf{k}\right)=\frac{(xy-xy)}{a}=0$.

d) Suppose $C$ is a simple closed curve on the sphere $x^2+y^2+z^2=a^2$. Show that the line integral $\int_C -2xz\,dx+y^2\,dz=0$.

**Answer** By Stokes' Theorem, the line integral equals the surface integral over the region $R$ in the sphere whose boundary is $C$. We computed such integrals in c), and they are all 0.