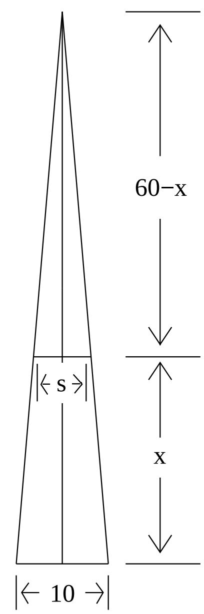
152:5–7 & 9–11 Some answers to *my* review questions for the first exam 2/15/2007

1. Suppose $\mathcal{R}$ is the region bounded by $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$.

a) Find the volume of the solid that results from rotating $\mathcal{R}$ around the x-axis.
Answer $\pi\int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \pi\cdot\left(\frac{-1}{x}\right)\Big]_1^2 = \frac{\pi}{2}$.

b) Find the volume of the solid that results from rotating $\mathcal{R}$ around the y-axis.
Answer $2\pi\int_1^2 x\left(\frac{1}{x}\right) dx = 2\pi$.

2. A flat-sided monolith* is 60 feet tall with a square base that is 10 feet on each side. What is the volume of the monolith?
Answer Our coordinate system's origin is at the center of the base of the monolith. We see a slice through the central axis of this solid. The height ranges from 0 to 60, and the width of the square cross-sections, from 10 to 0. If the height is x and the side of the cross-section is s then $\frac{60-x}{60} = \frac{s}{10}$ so $s = 10 - \frac{1}{6}x$. The volume is the sum of cross-section areas, $A(x) = s^2$, multiplied by a bit of height (dx) so the volume is $\int_0^{60} A(x)\,dx = \int_0^{60}\left(10 - \frac{1}{6}x\right)^2 dx$. This is $\left(\frac{-1}{-\frac{1}{6}}\right)\frac{1}{3}(10)^3 = 2{,}000$.



3. a) Suppose w is a positive number. Define $A(w)$ to be the average value of $(\cos x)^2$ on the interval $0 \le x \le w$. Compute $A(w)$, and show how the integral is calculated.
Answer $\int_0^w (\cos x)^2 dx = \int_0^w \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x)\Big]_0^w = \frac{1}{2}(w + \frac{1}{2}\sin 2w) - \frac{1}{2}(0+\sin 0)$. The average value is the integral divided by the length of the interval, which is w, so $A(w) = \frac{1}{2} + \frac{1}{4}\frac{\sin 2w}{w}$.

b) What is the limit of $A(w)$ as $w \to 0^+$?
Answer The limit of $\frac{\sin 2w}{w}$ can be computed using L'Hospital's rule. It is $\lim\limits_{w\to 0^+}\frac{2\cos 2w}{1} = 2$. So the limit requested is $\frac{1}{2} + \frac{1}{4}\cdot 2 = 1$. The limit can also be computed by noticing that $\sin 2w = 2\sin w\cos w$, and $\lim\limits_{w\to 0}\frac{\sin w}{w} = 1$, familiar from earlier calculus study.

4. Use the method of partial fractions to verify that $\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{4}\ln 2 + \frac{1}{8}\pi$.
Answer Write $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)}{(x+1)(x^2+1)}$, so $1 = A(x^2+1)+(Bx+C)(x+1)$. When $x = -1$ we get $A = \frac{1}{2}$. Comparing x^2 coefficients of both sides, we see $B = -\frac{1}{2}$. Finally, comparing constant coefficients on both sides, we see that $1 = A + C$ so $C = \frac{1}{2}$. Compute: $\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\,dx = \frac{1}{2}\ln(x+1) + -\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan(x)\Big]_0^1 = \frac{1}{2}\ln(2) + -\frac{1}{4}\ln(2) + \frac{1}{2}\arctan(1) - \left(\frac{1}{2}\ln(1) + -\frac{1}{4}\ln(1) + \frac{1}{2}\arctan(0)\right) = \frac{1}{4}\ln(2) + \frac{1}{2}\frac{\pi}{4} = \frac{1}{4}\ln 2 + \frac{\pi}{8}$. Whew! All done.

5. a) Here's a formula from the *Tables of Indefinite Integrals* by G. Petit Bois (1906):
$\int \frac{x^2}{x^3+5x^2+8x+4}\,dx = \log(x+1) + \frac{4}{x+2}$. Please verify this formula using the method of partial fractions.
Answer The formula *can* be verified by differentiation and algebraic manipulation, but you're asked to decompose the integrand using partial fractions. The formula, which has $x+1$, is a hint. I *guess* that -1 is a root of x^3+5x^2+8x+4. If we plug in $x = -1$, the value is $(-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0$. Divide x^3+5x^2+8x+4 by $x+1$. The quotient is $x^2+4x+4 = (x+2)^2$. The partial fraction expansion for $\frac{x^2}{x^3+5x^2+8x+4}$ is $\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ and this is $\frac{A(x+2)^2+Bx(x+2)+C(x+1)}{(x+1)(x+2)^2}$. We need A, B, and C so that $A(x+2)^2+Bx(x+2)+C(x+1) = x^2$. If $x = -1$, then $A = 1$. If $x = -2$, then $-C = 4$ so $C = -4$. Consider the x^2 coefficient: then $A + B = 1$ and since $A = 1$, B must be 0. Therefore $\frac{x^2}{x^3+5x^2+8x+4} = \frac{1}{x+1} + \frac{-4}{(x+2)^2}$. An antiderivative of $\frac{1}{x+1}$ is $\ln|x+1|$ and an antiderivative of $\frac{-4}{(x+2)^2}$ is $\frac{4}{x+2}$. We have verified the formula.

b) Here's another formula from the same text: $\int \frac{x^2}{x^3+x^2+x+1}\,dx = \frac{1}{2}\log\left((x+1)\sqrt{x^2+1}\right) - \frac{1}{2}\arctan x$. Again, please verify this formula using the method of partial fractions.

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* **monolith** 1. A large block of stone, especially one used in architecture or sculpture.
2. Something, such as a column or monument, made from one large block of stone.

Answer Again, the formula gives a clue: let $x = -1$ in $x^3 + x^2 + x + 1$ then $(-1)^3 + (-1)^2 + (-1) + 1 = 0$ so divide $x^3 + x^2 + x + 1$ by $x + 1$: the quotient is $x^2 + 1$. The form of the partial fraction expansion is $\frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+1)(x+1)}{(x+1)(x^2+1)}$. This is $\frac{x^2}{x^3+x^2+x+1}$ when $x^2 = A(x^2+1) + (Bx+C)(x+1)$. If $x = -1$ we see $A = \frac{1}{2}$. The x^2 coefficients give $A + B = 1$ so $B = \frac{1}{2}$. The constant terms give C: $0 = A + C$ so $C = -\frac{1}{2}$. Antidifferentiate $\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}$: $\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x$. Since $\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) = \frac{1}{2}\ln\left((x+1)\sqrt{x^2+1}\right)$, we're done. (The formulas in this problem are on page 12 of the Dover reprint.)

6. Calculate the following integrals, showing your work. a) $\int \frac{dx}{\sqrt{1+x^2}}$

Answer If $x = \tan\theta$, then $dx = (\sec\theta)^2\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$. So $\int \frac{dx}{\sqrt{1+x^2}} = \int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C = \ln(\sqrt{1+x^2} + x) + C$.

b) $\int \frac{dx}{2e^x+1}$

Answer If $u = e^x$, then $du = e^x\, dx$ so dx is $\frac{du}{u}$. The integral given becomes $\int \frac{1}{u(2u+1)}\, du$. Rewrite the integrand using partial fractions: $\frac{1}{u(2u+1)} = \frac{A}{u} + \frac{B}{2u+1}$ giving $1 = A(2u+1) + Bu$. When $u = 0$, we get $A = 1$. When $u = -\frac{1}{2}$ we get $B = -2$. Then antidifferentiating: $\int \frac{1}{u} + \frac{-2}{2u+1}\, du = \ln u - \ln(2u+1) + C$. Converting back to x's, the answer becomes $\ln(e^x) - \ln(2e^x+1) + C$. (Yes, $\ln(e^x)$ is x.)

7. a) Explain why $\int_1^\infty \frac{1}{x^2+e^{2x}}\, dx$ converges.

Answer $\frac{1}{x^2+e^{2x}}$ is positive on the interval $[1,\infty)$, and is certainly less than either $\frac{1}{x^2}$ or $\frac{1}{e^{2x}}$. Either of those integrals converges on that interval, so by comparison, the integral given converges.

b) Explain why the value of the integral in a) is less than $\frac{1}{2}$.

Answer In a) either integral is "good enough" to show convergence. Here, however, since $\int_1^\infty \frac{1}{x^2}\, dx = \lim_{A\to\infty} -\frac{1}{x}\Big]_1^A = 1$ and $\int_1^\infty \frac{1}{e^{2x}}\, dx = \lim_{A\to\infty} -\frac{1}{2}e^{-2x}\Big]_1^A = \frac{1}{2e^2}$ we'd better use the second, exponential integral (decaying exponentials $\to 0$ much more rapidly than *any* inverse power of x). So the value of the integral given is less than $\frac{1}{2e^2}$ which is certainly less than $\frac{1}{2}$.

8. In this problem, $f(x) = x(\ln x)^2$.

a) Verify that $\lim_{x\to 0^+} f(x) = 0$.

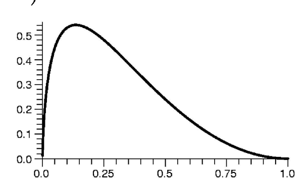
Hint Write the limit so you can apply L'H, but be sure to indicate *why* you need L'H *whenever* you use it.

Answer $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$. This is $\frac{\infty}{\infty}$, so we may apply L'Hopital's rule. We consider the limit of the derivative of the top over the derivative of the bottom: $\lim_{x\to 0^+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-2\ln x}{\frac{1}{x}}$. Again, $\frac{\infty}{\infty}$, so again L'H and then $\lim_{x\to 0^+} \frac{\frac{-2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} 2x = 0$.

b) Carefully compute the improper integral $\int_0^1 f(x)\, dx$. Indicate why the limits you need exist, and what these limits are.

Answer Integration by parts: $\int u\, dv = uv - \int v\, du$ becomes $\left.\begin{array}{c} u = (\ln x)^2 \\ dv = x\, dx \end{array}\right\} \left\{\begin{array}{c} du = 2(\ln x)\left(\frac{1}{x}\right) dx \\ v = \frac{1}{2}x^2 \end{array}\right.$ so the integral changes: $\int x(\ln x)^2\, dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\, dx$. But $\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$ with another integration by parts: $\left.\begin{array}{c} u = \ln x \\ dv = x\, dx \end{array}\right\} \left\{\begin{array}{c} du = \left(\frac{1}{x}\right) dx \\ v = \frac{1}{2}x^2 \end{array}\right.$. Thus $\int_0^1 x(\ln x)^2\, dx = \frac{1}{2}x^2(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big]_0^1 = \frac{1}{4}$ since $\lim_{x\to 0^+} x^2(\ln x)^2 = \lim_{x\to 0^+} x \cdot x(\ln x)^2 = 0$ as we saw and, similarly, $\lim_{x\to 0^+} x^2(\ln x) = 0$.

c) Here's a graph of $x(\ln x)^2$ drawn by `Maple` on the interval $[0,1]$. Does this graph (approximately) confirm your computation in b)? Why?



Answer Yes. It's almost a triangle of height $\frac{1}{2}$, base 1, and area $\frac{1}{4}$, which is b)'s answer.

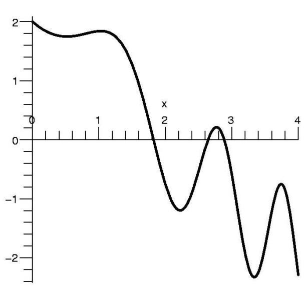
9. Use a substitution followed by integration by parts to verify that $\int_0^1 e^{\sqrt{x}}\,dx = 2$.
Answer Try $w = \sqrt{x}$ and therefore $w^2 = x$ with $2w\,dw = dx$. The integral changes: $\int e^{\sqrt{x}}\,dx = \int e^w 2w\,dw$. This integral is a well-known candidate for integration by parts. Let's do it (I'll save the 2 until later):

$$\begin{array}{llll} \int e^w w\,dw & = we^w & - \int e^w\,dw = we^w - e^w + C \\ \int u\,dv & = uv & - \int v\,du \end{array} \quad \text{with} \quad \left.\begin{array}{r} u = w \\ dv = e^w\,dw \end{array}\right\}\left\{\begin{array}{l} du = dw \\ v = e^w \end{array}\right.$$

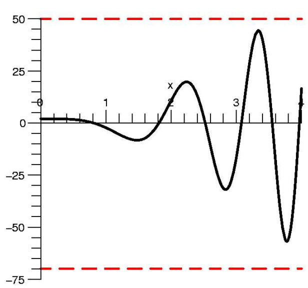
. We can reverse the substitution so the indefinite integral becomes (with the 2!) $2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C$. Therefore, $\int_0^1 e^{\sqrt{x}}\,dx = 2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right)\Big]_0^1 = (2(e-e)) - (2(0-1)) = 2.$

10. The integral $\int_0^2 \left(x^3+1\right)^{7/2}\,dx$ is approximated using the Trapezoidal Rule by dividing $[0,2]$ into n segments of equal length. How large should n be in order to guarantee that the error is at most 10^{-6}?
Note You must give some reason explaining why any overestimates of derivatives you make are valid on the entire interval.
Answer We will need to overestimate $|f''(x)|$ for all x's in $[0,2]$ and for $f(x) = \left(x^3+1\right)^{7/2}$. So $f'(x) = \frac{7}{2}\left(x^3+1\right)^{5/2}(3x^2)$ and $f''(x) = \frac{7}{2}\cdot\frac{5}{2}\cdot\left(x^3+1\right)^{3/2}\left(3x^2\right)^2 + \frac{7}{2}\left(x^3+1\right)^{5/2}6x$. Notice that x^3+1 is increasing in $[0,2]$ since its derivative, $3x^2$, is positive there. So the value of x^3+1 for any x in $[0,2]$ is less than the value of x^3+1 at the right-hand endpoint, which is 9. The other pieces of f'' are also increasing and positive, so $f''(2)$ will be an appropriate overestimate of $|f''(x)|$ for all x in $[0,2]$.
Now $f''(2) = \frac{7}{2}\cdot\frac{5}{2}\cdot\left(2^3+1\right)^{3/2}\left(3\cdot 2^2\right)^2 + \frac{7}{2}\left(2^3+1\right)^{5/2}(6\cdot 2)$. This is 44,226, which I will "plug into" $\frac{K(b-a)^3}{12n^2}$ in place of K. Of course $a=0$ and $b=2$, so the Trapezoidal Rule error will be less than $\frac{44{,}226\cdot 8}{12n^2}$. This is less than 10^{-6} when $\frac{44{,}226\cdot 8}{12n^2} < \frac{1}{1{,}000{,}000}$ so n is any integer greater than $\sqrt{\frac{44{,}226\cdot 8\cdot 1{,}000{,}000}{12}}$.
Note This is as far as I would go on an exam where there's no calculator use. Such an answer is totally acceptable to me. The n is about 172,000, rather unimpressive.

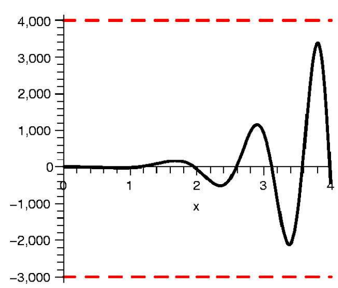
11. In this problem, $f(x) = 2 - x + \sin(x^2)$. Assume that the graphs of the functions below on the interval $[0,4]$ are correct. Information from these graphs may be used to answer the questions which follow.



The graph of $y = f(x)$



The graph of $y = f''(x)$, the second derivative of f



The graph of $y = f^{(4)}(x)$, the fourth derivative of f

a) How many subdivisions are needed to estimate $\int_0^4 f(x)\,dx$ with the Trapezoidal Rule to an accuracy of 10^{-10}? **Suggestion** Use the picture and the formula sheet.
Answer I think that you can take M_2 to be 70 here. Then $\frac{K(b-a)^3}{12n^2} = \frac{70(4-0)^3}{12n^2}$. This is less than 10^{-10} when n is an integer greater than $\left(\frac{70\cdot 4^3\cdot 5\cdot 10^{10}}{12}\right)^{1/2}$.
Note Without a calculator, leave this as is! Actually, it is ≈2,000,000.

b) How many subdivisions are needed to estimate $\int_0^4 f(x)\,dx$ with Simpson's Rule to an accuracy of 10^{-10}? **Suggestion** Use the picture and the formula sheet.
Answer I think that you can take M_4 to be 4,000 here. Then $\frac{K(b-a)^5}{180n^4} = \frac{4{,}000(4-0)^5}{180n^4}$. This is less than 10^{-10} when n is an integer greater than $\left(\frac{4{,}000\cdot 4^5\cdot 10^{10}}{180}\right)^{1/4}$.
Note Without a calculator, leave this as is! Actually, it is ≈4,000.