Unfair from Fair: Expectation

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The Solution

This was another expected payoff problem, where *E*, the expected number of flips, was the product of probability and flips summed from unity to infinity. To find the probability of successfully generating an unfair coin on the n^{th} flipping, we considered the following:

- On the first flipping, the probability of success is $\frac{3}{4}$.
- On the second flipping, the probability of success is $\frac{1}{4}\cdot\frac{3}{4}$, since this probability is the product of one failure and a subsequent success.
- Through induction, the probability of success on the n^{th} flipping is therefore $P(n) = \frac{1}{4^{n-1}}\cdot\frac{3}{4} = \frac{3}{4^n}$.

The payoff is consequently $D(n) = 2n$, since success on the n^{th} flipping requires 2n flips. The expected number of flips is therefore $E = 6\sum_{n=1}^{\infty}\frac{n}{4^n}$. This sum was evaluated using the following Maple command:

```
> 6*sum('n/4^n','n=1..infinity');
                              8/3
```

We decided to use Maple to generate this answer due to time and sleep constraints, and the assumption that this sum could be computed similar to the sum $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$. Our solution closely matches the approximations provided in the problem description, as well as a simulation we conducted with 1,000,000,000 trials, which yielded an *E* of approximately 2.666706216. The source for this poorly coded Java program can be found here.