Colorful Mathematics!

By Lea Jesiolowski

A Little Bit of History …

The four color conjecture can be traced to Francis Guthrie during the year 1852. Studying law at the University College London, Guthrie one day decided to color a map of England. In the process, he noted that he only needed to use four colors, and that with only four colors, no two regions sharing a boundary would have to be shaded the same color [2]. Finding this discovery interesting, Francis went to his brother Frederick, a prominent physicist who would later found the Physical Society of London [8]. The question of whether or not for colors would suffice for any map also piqued the interest of Frederick and led him to pass the conjecture along to DeMorgan, who both of the brothers had studied under at the university. Augustus DeMorgan was an accomplished mathematician, logician, and teacher, and eventually founded the London Mathematical Society [4]. None of the men could formulate any answer, and on October 23, 1852, the exact same day the problem was presented, DeMorgan wrote to Hamilton:

> *A student of mine asked me today to give him a reason for a fact which I did not know was a fact - and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured - four colours may be wanted, but not more - the following is the case in which four colours are wanted. Query cannot a necessity for five or more be invented. ...... If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did....*

> Here, DeMorgan's citation of the Sphinx is a reference to ancient mythology. The Sphinx, a mythological creature, sat upon a large rock the town of Thebes. After learning a riddle from the muses, the Sphinx would pose this riddle to any traveler walking the road next to the rock. If the passerby did not know the answer, the Sphinx would strangle him. The catch was, the muses gave the Sphinx a riddle which no one could seem to answer. Indeed, DeMorgan would do no such thing to a person who could not solve the four color conjecture. However, he is implying that it appears to be a riddle which may be inherently unanswerable [6].

And what did Hamilton reply?

> *I am not likely to attempt your quaternion of colour very soon.* [7]

So for about a decade interest in the conjecture seemed to die down because no singular mathematician continued any in-depth research. Then, in 1860, the great logician Charles Sanders Peirce, son of Benjamin Peirce who was a graduate of Harvard later pursuing mathematics and astronomy, took a shot at it [2] [4]. Although he did not succeed, he at least maintained lifelong interest in the problem.

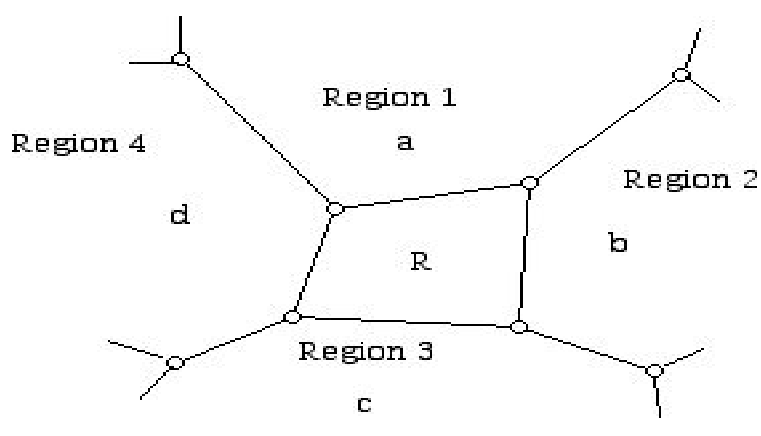
The first printed reference of the four color conjecture can be attributed to the distinguished mathematician Arthur Cayley, dating back to 1878. In a paper sent to the Royal Geographical Society entitled "On the Coloring of Maps" Cayley basically stated that this conjecture was intuitively easy, but extremely difficult to prove [2]. So, more or less, the idea had been kicked around for almost three decades with nothing to show for it but publishings as to why it was tough to prove in the first place!

Do We Have A Winner?

Then, on July 17, 1879, the mathematical community rejoiced as Alfred Bray Kempe, announced that he had proven the four color conjecture!! Kempe had studied mathematics under Cayley at Cambridge and was at the time occupied as a London barrister [7]. After a lifelong devotion and interest in mathematics, it appeared that Kempe had finally been rewarded with a proof to a seemingly improvable conjecture. What an exciting day that must have been. He submitted the proof to the American Journal of Mathematics and his proof was accepted by the mathematic community. He was even knighted in 1912, and was made into a Fellow of the Royal Society [2].

Kempe's Chains

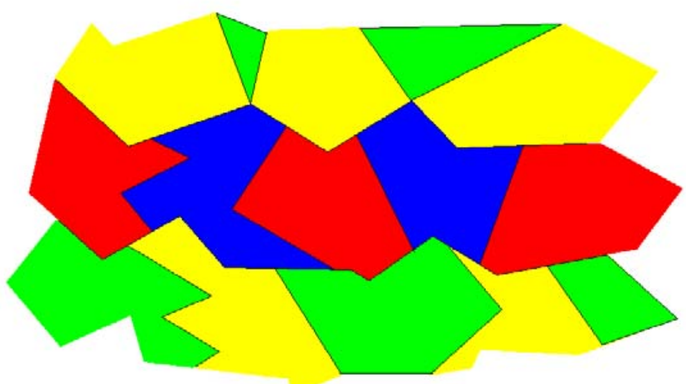
So how did Kempe appear to have proven the four color conjecture, making it into what he thought was an honest to goodness proof? The method used was entitled Kempe's Chains. Here is an example:



Suppose that a, b, c, and d are all different colors. In this diagram, regions 1 through 4 have all been colored with either a, b, c, or d. However, these four regions enclose region

R, which has yet to be colored. This inevitably raises a problem, because region R cannot be colored with either a, b, c, or d because that would disprove the four color conjecture right there. So Kempe, being the ever perfectionist, decided that he would recolor the map so that the coloring rule could be met and that R could be colored as well. So, here is how Kempe made his chains: Take the color of region 2: b. Move this color over one space to the left, and use it to fill in region R. Then, simply take color d, the one filling in region 4, and use that to fill in region 2. Using this method, Kempe demonstrated that these chains in a sense form barriers. There is a 'string' of two alternating colors. On the opposite sides of this string, you can place alternating strings of the two remaining unused colors. In this manner, a barrier is formed, and the two sets of similar alternating colors are always separated by a set in the middle [1].

Here is an example:



In this diagram, there are two chain schemes occurring: the yellow and green, and the red and blue. The alternating blue and red chain guards both of the yellow and green chains from ever touching each other. This avoids two of the same colors ever touching, thus satisfying the requirements of the four color conjecture.

So What's The Problem?

Well, Kempe decided to extend his argument to include a 5-gon, or a five-sided region surrounded by five regions which used all five colors. While Kempe successfully demonstrated this using the 4-gon (as shown above), his argument broke down in the case of a 5-gon. By simply moving one color into the empty region in the middle, and making a chain out of that color, and the color to the left of it, two of the same colors would inevitably touch. Unfortunately enough for Kempe, this error was overlooked when his proof was originally submitted. He enjoyed temporary acceptance, only to be later proven wrong by Percy John Heawood in 1890. It can be noted that Kempe's proof was accepted by the mathematical community for over a decade, despite this problem, once again demonstrating the continuing difficulty of actually proving the conjecture [1].

The Good Stuff Left Over From Kempe

Although the poor guy got the validity of his proof destroyed, good truly did come out of his work. Heawood was able to prove that all plane maps can be colored with five colors. He also made further contributions to the conjecture, and in 1898, he proved that if the number of regions surrounding any given region, R, was divisible by 3, then all of the regions themselves are four-colorable. This means that the region enclosed, and the regions surrounding it could all be colored with four colors and still satisfy the requirements of the conjecture. Heawood continued to extrapolate from

Kempe's work and began raising questions about the colorings on more general surfaces. If a graph G is embedded on a surface with the genus g > 0 then:

$$\chi(G) \leq \left\lfloor (7 + \sqrt{1+ 24g}\ )/2 \right\rfloor$$

A surface with genus 1 can be thought of as a donut with one hole, a surface with genus 1 to be a donut with two holes, etc [1] [5]. Along with adding such contributions as this to the four color problem, Heawood is also recognized for his devotion to the conjecture throughout his life, and for the overseeing of the restoration of Durham Castle in England, which had fallen into disrepair [4].

So What Next?

People sought to prove the four color conjecture over the next several decades. Some progress was made. Franklin proved that a 25 region map is four-colorable. Reynolds one-upped him in 1928, proving that any 28 region map was four-colorable. Franklin bumped that number up to 32 in 1938. Winn made it 36 in 1940. Then Ore and Stemple followed in 1970, proving that a map with 40 regions was four-colorable. Finally, Mayer beat them all, and in 1976 proved that any 90 region map was four-colorable [2]. But alas, who could prove that ***every*** map was four-colorable?

We Have Our Answer, Or Do We?

The four color theorem was proved in a way never done before. Two men, Wolfgang Haken and Kenneth Appel would change the methodology of proving conjectures forever. Appel was a programmer, and Haken, a professor at the University of Illinois, was schooled in the ways of Kempe methodology [4]. In 1976, the two men created a computer program, which by brute computational force (1200 hours to be exact) surveyed 1500 examples of regions, and colored them with four colors. The computational time was mostly eaten away by verifications and countless configurations [2]. Appel and Haken verified that by checking these examples, the problem was solved. They stated that there was only a finite set of examples because there were only a certain number of possible regional patterns. So, by checking one example of each pattern, they could prove it to be true for all maps. The results were in, and the computer worked. The four color conjecture was now the four color theorem.

The Aftermath

But sadly enough, Appel and Haken were not met with instantaneous glory at the feat of finally proving a more than century-old problem. Instead, the mathematical was fraught with controversy. Why you ask? The reasons [3]:

1. Aesthetics – Math and physics, although maybe not to the sane person's eye, prides itself on the beauty of its proofs. Following the theorem should be fluid and thrilling, as the numbers amazingly make a once stunning problem true. Instead, Appel and Haken gave the world a bunch of 1s and 0s, shoved them into a big computer, and after a while, out came a yes. G.H. Hardy famously said: "There is no permanent place in the world for ugly mathematics." The Appel and Haken proof was even said to be "like a bulldozer trying to create a Japanese rock garden" - Diaz.
2. Usefulness – Ok, so the computer says all of this stuff is true. But it gave absolutely no insight as to ***why*** it was true. Proofs in the past had

left the reader with that a-ha, eureka, I get it! But in this case, the reader simply had to look a huge indecipherable jumble of computer jargon. No one could see why four colors were all you needed; just that it really was all that was necessary.

3. Philosophical – Was this really a proof? Proofs are supposed to follow the strands of logic. They can be expressed in formal language. Proofs should be processed by the human mind, able to be rummaged through and searched for errors[1]. The Haken-Appel proof made no such thing possible. Who can scrounge around the insides of a computer anyway?

My Encounter With the GREAT Doron Zeilberger

So, two camps developed on the matter: the skeptics and the believers. The skeptics shunned the use of computers for proving theorems, arguing that it stole all the thunder of the beauty of the human method. But then there are the believers. Doron Zeilberger is a believer.

In an attempt to understand the viewpoint of a believer, I decided to speak with the great Dr. Zeilberger himself. Quite terrified (I find professors scary!) I tiptoed into the wonderful office of 704 Hill (I had to take two elevators to get there!). In the presence of Dr. Z, I learned quite a few things. Firstly, and definitely most true, mathematicians are extremely dedicated to defending their opinions. Secondly, computers aren't that bad according to Zeilberger. He surprisingly answered all of my questions with extreme sense. He says, why be afraid of computer errors? People always make errors, and you can design a good

[1] This is ironic because the human mind was unable to identify the error in Kempe's proof for 11 years.

enough computer program so that it won't make any. After all, humans are the people who tell the computer what to do. When asked about computers sucking the 'beauty' out of math, Dr. Z showed me a very interesting example. He showed me how humans proved an isosceles triangle to be identical when cut in half. Then, he showed me how the computer did it, and it indeed was a much 'prettier' way of going about it. In the end, it really did seem like we had no choice, computers are here to stay, and Dr. Z loves them.

What To Do Now?

Well, after examining the four-color theorem until my eyes bled, I have drawn a few conclusions. The fact that it is extremely difficult to prove by hand gives us something to work for. I was initially a skeptic, but I see how computers can help. However, I found certain holes in Dr. Z's argument. The computer proof he showed was beautiful only because it was easy enough to still write out on his chalk board. But when the proof gets more complicated, he'll never be able to explain to me the gazillions of bits, and why they are beautiful. So the beauty really will be gone. Finally, I think this issue only matters to about .00000000001% of the people in this world. I think everyone else really doesn't care who is proving what or how, just as long as they have TV and food. As for me, I'm glad I was given a project which I could at least understand, and am determined to solve this thing without a computer (I'm going to do it, I swear!).

Works Cited

[1] American Mathematical Society. Colorful Mathematics. http://www.ams.org/new-in-math/cover/coloring5.html

[2] Calude, Andrea S. "The Journey of the Four Color Theorem Through Time". June 5, 2001

[3] Diaz, Rony V. "Four Colors are Enough". The Manila Times. Life & Times. http://www.manilatimes.net/national/2003/jun/24/life/20030624lif3.html

[4] Encyclopedia.com

[5] Eric Weisstein's World of Mathematics, MathWorld.Wolfram.com. "Four Color Theorem" http://mathworld.wolfram.com/Four-ColorTheorem.html

[6] Mythology. The Sphinx of Greek Mythology. http://www.nmia.com/~sphinx/greek_sphinx.html

[7] O'Connor, J J and Robertson, E F. "The Four Color Theorem". http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/The_four_colour_theorem.html

[8] Physics Web. Frederick Guthrie: A Man of Action. November, 1999. http://physicsweb.org/article/world/12/11/10