January 27, 2000

Math 504: Complex Variables (Spring, 2000)

B1 Prove that the only compactly supported real analytic function on $\mathbb{R}$ is the zero function. Prove that the only compactly supported real analytic function on $\mathbb{R}^2$ is the zero function.

B2 Suppose that a power series $\displaystyle\sum_{(j,k)=(0,0)}^{\infty} \nu_{j,k} x^j y^k$ converges in some neighborhood of $0 = (0,0) \in \mathbb{R}^2$, where the $\nu_{j,k}$ are complex constants. What conditions on these constants are needed to insure that the sum of the series is a holomorphic function?

$\mathbb{C}^*$ will denote $\mathbb{C} \setminus \{0\}$, and D will denote the open unit disc in $\mathbb{C}$: those z's with $|z| < 1$. $L^p(D)$ is the set of measurable functions f on D with $\int_D |f|^p \, dA$ finite (dA is area measure in the plane).

B3 For which $p \in [1, \infty]$ and $n \in \mathbb{Z}$ is $z^n \in L^p(D)$?

B4 For which $p \in [1, \infty]$ is $e^{\frac{1}{z}} \in L^p(D)$?

B5 Can you find a function with an essential singularity at 0 so that the answer to the previous question is different?

B6 What are the Cauchy transforms of dx on $[0,1]$, $d\theta$ on ∂D (both of these are 1-dimensional Lebesgue measures), and the unit mass (the delta function) at 0? Note that $\frac{1}{z^2} = O(|z|^{-1})$ as $z \to \infty$ and is holomorphic in $\mathbb{C}^*$. Is $\frac{1}{z^2}$ the Cauchy transform of some measure?

B7 Use facts about Laurent series representations in $\mathbb{C}^*$ to prove that $\dfrac{\partial f}{\partial \bar{z}} = g$ has a solution $f \in C^\infty(\mathbb{C}^*)$ for every $g \in C^\infty(\mathbb{C}^*)$.