May 2, 2002

Review Problems for the final exam in section 1 of Math 403

What follows is basically the final exam I gave last year in Math 403. I'll *happily* discuss this exam with you on $\left\{\begin{array}{l}\text{Wednesday, May 8, 10 AM}\\ \text{Thursday, May 9, 3:30 PM}\end{array}\right.$ in Hill 525. I should also be available in my office (try Hill 304 first, then Hill 542) during much of the week, and accessible via e-mail most of the time. Please look over this semester's earlier exams and review sheets.

(20) 1. In this problem U will be the region in the complex plane defined by the inequalities $r = |z| > 1$ <u>and</u> $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{3\pi}{4}$.

a) Sketch the region U on the axes given. *[Axes omitted here.]*
Is U a connected open set? (Yes | No) Is U a simply connected open set? (Yes | No)

b) Suppose $F(z) = z^2$. If $z = re^{i\theta}$, write a formula for $F(z)$ in complex exponential form. If $V = F(U)$, the image of U under F, the collection of all values of F with domain restricted to U, sketch the region V on the axes given. *[Axes omitted here.]*
Is V a connected open set? (Yes | No) Is V a simply connected open set? (Yes | No)

c) Suppose $G(z) = \frac{1}{z}$. If $z = re^{i\theta}$, write a formula for $G(z)$ in complex exponential form. If $W = G(U)$, the image of U under G, the collection of all values of G with domain restricted to U. sketch the region W on the axes given. *[Axes omitted here.]*
Is W a connected open set? (Yes | No) Is W a simply connected open set? (Yes | No)

(10) 2. Find $\operatorname{Arg} z$ if $z = (1+i)^i$.

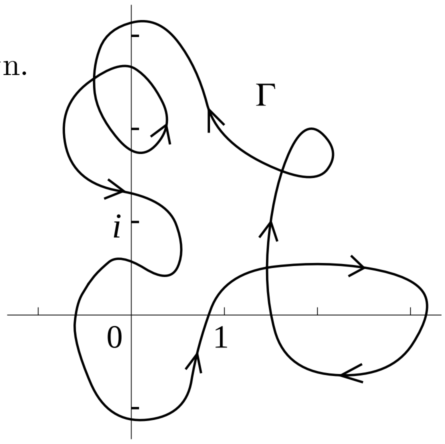
(12) 3. Describe all solutions of $z^3 = 2i$ algebraically in either rectangular or polar form. Sketch these solutions on the axes provided. *[Axes omitted here.]*

(18) 4. a) Suppose $c(x,y)$ is a harmonic function, so $\dfrac{\partial^2 c}{\partial x^2} + \dfrac{\partial^2 c}{\partial y^2} = 0$.

Prove that $f = \dfrac{\partial c}{\partial x} - i\dfrac{\partial c}{\partial y}$ is analytic.

b) Verify that $w(x,y) = \cos x \cosh y$ is harmonic, and find one harmonic conjugate.

(25) 5. Compute $\displaystyle\int_\Gamma \frac{z^2}{e^{2\pi i z} - 1}\,dz$ where Γ is the closed curve shown.



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(20) 6. a) If k is a real number with $-1 < k < 1$, derive the Laurent series representation

$$\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n} \qquad (|k| < |z| < \infty)\,.$$

b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula $\displaystyle\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$.

(25) 7. If A is real and $A > 1$, `Maple` reports that $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+A^2)} = \left(\frac{1}{2}\right)\left(\frac{\pi}{A+1}\right)$. Check this assertion using the method of residues. Show clearly the contour of integration and any residue computation. Explain why some integral tends to 0 in the limit.

(16) 8. What is the radius of convergence of the Taylor series expansion of $h(z) = \dfrac{e^z}{(z-1)(z+2)}$ when expanded around $z = i$? Give a numerical answer. Justify why the series must converge with at least that radius <u>and</u> why it can't have a larger radius.

Note Actual computation of the series is not practical.

(20) 9. Find the order of the pole of $H(z) = \dfrac{1}{(6\sin z + z^3 - 6z)^2}$ at $z = 0$.

(20) 10. Suppose T is the inside of the square with corners $1+i$, $-1+i$, $-1-i$, and $1-i$, and S is the inside of the square with corners $3+3i$, $-3+3i$, $-3-3i$, and $3-3i$. Suppose also that $f(z)$ is any entire function. Let M be the maximum of $|f''(z)|$ on T and let N be the maximum of $|f(z)|$ on S. Show that $M \le \frac{1}{2}N$.

Hint Begin by writing some complex variables formula connecting f'' and f.

(14) 11. Suppose $F(z) = z^3\left(\dfrac{1}{z+1} + \dfrac{1}{(z-1)^3}\right)$, and C is a simple closed curve which does *not* pass through 1 or -1. What are all possible values of $\displaystyle\int_C F(z)\,dz$ (and why)? Sketch examples of C's which will give each value you list.

(20) 12. Find complex numbers a, b, c, and d so that the linear fractional transformation $L(z) = \dfrac{az+b}{cz+d}$ takes $-i$ to 0, 0 to 1, and $2i$ to ∞.
Sketch the image of the unit circle $|z| = 1$ under the transformation L, and briefly explain why what was drawn is correct.