640:403:05 **Answers to the Second Exam** **4/17/2001**

(12) 1. Let C be any simple closed curve described in the positive sense in the z plane, and write $g(w) = \int_C \frac{z^3-5z}{z-w}\,dz$.

a) Show that $g(w) = 2\pi i(w^3 - 5w)$ if w is inside C.

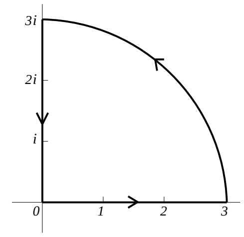
Answer $z^3 - 5z$ is analytic everywhere (it is a polynomial!). The CIF implies that if w is inside a simple closed curve, C, then $\frac{1}{2\pi i}g(w) = w^3 - 5w$ so the result requested is true.

b) Show that $g(w) = 0$ when w is outside C.

Answer If w is outside C, then the integrand (the function being integrated!) is analytic on and inside the simple closed curve, C. Therefore by Cauchy's Theorem, $g(w) = 0$.

(12) 2. Compute $\int_B \frac{z^4}{(z-(1+i))^3}\,dz$ where B is the simple closed curve shown: the line segment from 0 to 3, followed by the quarter-circular arc centered at 0 from 3 to $3i$, followed by the line segment from $3i$ to 0.

Answer -24π



Answer The CIF for derivatives states that $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ if f is analytic on and inside a simple closed curve C and z_0 is inside C. Here $f(z) = z^4$, $n = 2$, $z_0 = 1+i$, and C is the curve B. Then $f^{(2)}(1+i) = 4\cdot 3(1+i)^2 = 24i$. The answer we want is $24i$ divided by $2! = 2$ and multiplied by $2\pi i$, so the answer is -24π. The answer can also be obtained using the Residue Theorem, or by computing the appropriate Laurent series and integrating directly.

(12) 3. Suppose f is the function defined by $f(z) = \frac{z+1}{z(z-1)}$.

Find the Laurent series representing $f(z)$ in the annulus $0 < |z| < 1$. Be sure to explain *why* the series you write is valid in that annulus. Find explicit values of the coefficients of z^{10} and z^{-10} in the series.

(Partial) **Answer** The coefficients are -2 and 0.

Answer If $\frac{z+1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$ then $z+1 = A(z-1) + Bz$. $z = 0$ shows that $A = -1$ and $z = 1$ shows that $B = 2$. Then $f(z) = \frac{-1}{z} + \frac{2}{z-1}$. When $|z| < 1$, the geometric series with ratio z gives $\frac{2}{z-1} = -2\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} -2z^n$. Since we're dividing by z, we also need to know $0 < |z|$. So the Laurent series for f in the annulus $0 < |z| < 1$ is $-\frac{1}{z} + \sum_{n=0}^{\infty} -2z^n$. The coefficient of z^{10} is -2. The coefficient of z^{-10} is 0.

Another method to get the **Answer** $\frac{z+1}{z(z-1)} = -\frac{z+1}{z}\cdot\frac{1}{1-z} = -\frac{z+1}{z}\cdot\sum_{n=0}^{\infty} z^n$. There is some "overlapping" of powers because of the multiplication by $z+1$. This answer can be rewritten to be identical to the previous result.

(14) 4. a) Suppose that f is an entire function and there is a positive constant K so that $|f(z)| > K$ for all z. Prove that f must be a constant function.

Hint what can you do with something that is *not* 0?

Answer f can never be 0 since $|f(z)|$ is always positive. Therefore the function g defined by $g(z) = \frac{1}{f(z)}$ is defined for all z and is analytic: g is entire. Also, $|g(z)| < \frac{1}{K}$ for all z, so g is *bounded and entire*. Liouville's Theorem applies to show that g is constant and therefore so is f.

b) The exponential function is never 0 and is an entire function. Briefly explain why the exponential function does not contradict the assertion in part a).

Answer Every non-zero complex (and therefore real) number is a value of the exponential function (which is why log has values at every non-zero complex number). If $K > 0$, we can find z_0 with $e^{z_0} = \frac{K}{2}$ so any inequality of the form $|e^z| > K$ must be false for some z's.

(14) 5. If $F(z) = \frac{1}{z(z^2-1)(z^2+6)}$ compute the integral of F over the circle of radius 2 centered at 0, oriented counterclockwise as usual. Note that $\sqrt{6} > 2$.

Answer $-\frac{\pi i}{21}$

Answer F has isolated singularities at 0, ± 1, and $\pm\sqrt{6}i$. The Residue Theorem could be applied but the solution here will use techniques from earlier in the course. In the picture below, the singularities are

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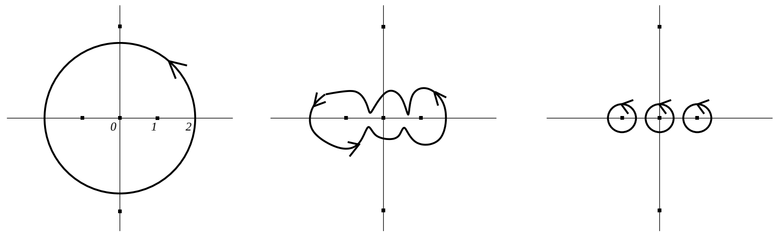
indicated by $\cdot$. Here is *one* approach to getting the answer. The important corollary to Cauchy's Theorem says that simple closed curves can be deformed without changing the value of the integral if the integrand is analytic in the region of the deformation. Deform the original circle into "lumps" around the singularities



at 0 and ± 1. Then "pinch off" the contour into three pieces, one around each of 0 and ± 1. I've drawn three circles for simplicity, but the important property is that each piece is a simple closed curve with exactly one singularity of f inside. We'll use the CIF on each circle. The left-hand circle has as integrand $\frac{g_1(z)}{z+1}$ where $g_1(z) = \frac{1}{z(z-1)(z^2+6)}$ is analytic on and inside this circle. So the CIF implies that the integral is $2\pi i \cdot g_1(-1) = 2\pi i \cdot \left(\frac{1}{(-1)(-2)((-1)^2+6)}\right) = 2\pi i \cdot \frac{1}{14} = \frac{\pi i}{7}$. The CIF on the middle circle shows that the integral is $2\pi i \cdot g_2(0)$ where $g_2(z) = \frac{1}{(z^2-1)(z^2+6)}$ (analytic on and inside this circle) and the integrand is seen as $\frac{g_2(z)}{z}$. Since $g_2(0) = -\frac{1}{6}$ the middle circle's integral is $-\frac{\pi i}{3}$. Finally, the integrand in the right-hand circle is $\frac{g_3(z)}{z-1}$ where $g_3(z) = \frac{1}{z(z+1)(z^2+6)}$. Again the CIF shows that the value of the integral is $2\pi i \cdot g_3(0) = 2\pi i \cdot \left(\frac{1}{14}\right) = \frac{\pi i}{7}$. The original integral is the sum of $\frac{\pi i}{7} - \frac{\pi i}{3} + \frac{\pi i}{7} = -\frac{\pi i}{21}$.

Comment (about my answer) Clever students tried other ways to do this problem, sometimes successfully. Some students "decomposed" F into five pieces with partial fractions, and then integrated each piece. Other students tried to find the Laurent expansion for F valid in the annulus containing the circle of radius 2. Only the coefficient of the $\frac{1}{z}$ term would matter, of course! This gives the correct answer also but is intricate.

(12) 6. Suppose the following is known about the coefficients of a power series $\sum_{n=0}^{\infty} a_n z^n$: All a_n's are complex numbers with $|a_n| \leq 12$. Explain why the power series converges for all z with $|z| \leq \frac{1}{10}$. Also verify that for those z's the sum of the series is always within a closed disc of radius 17 centered at 0.

Comment 17 is an overestimate!

Answer An absolutely convergent series must converge. The triangle inequality (extended to infinite sums) implies that the sum with | |'s is an overestimate of the true sum. That is, $|\sum_{n=0}^{\infty} a_n z^n| \leq \sum_{n=0}^{\infty} |a_n||z|^n$. The conditions here imply the last sum is $\leq \sum_{n=0}^{\infty} 12 \left(\frac{1}{10}\right)^n = 12\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{120}{9}$, certainly less than $17 = \frac{153}{9}$.

(12) 7. The function $g(z) = (1+3z)e^{(z^2)}$ is analytic near 0.

a) Use results about power series of familiar functions to find terms up to and including degree 4 in the Taylor series of g centered at 0.

Answer $e^{(z^2)} = 1 + z^2 + \frac{1}{2}z^4 +$HIGHER ORDER TERMS, using the power series for exp centered at 0 which converges for all z. Multiply by $1+3z$ and discard all terms with degree > 4: $(1+3z)(1+z^2+\frac{1}{2}z^4) = 1 + z^2 + \frac{1}{2}z^4 + 3z + 3z^3 + \frac{3}{2}z^5$ "=" $1 + 3z + z^2 + 3z^3 + \frac{1}{2}z^4$.

b) Use your answer to a) to compute $g^{(4)}(0)$.

Answer 12

Answer $g^{(4)}(0)$ is the coefficient of z^4 multiplied by 4!, so it is $\frac{1}{2} \cdot 4! = \frac{1}{2} \cdot 24 = 12$.

(12) 8. a) Use results about power series of familiar functions to find an exact value of L: $\lim_{z \to 0} \frac{(\cos z) - 1 + \frac{z^2}{2!}}{z^4} = L$.

Answer The Taylor series at 0 for cosine is $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$, converging for all z. $1 - \frac{z^2}{2!}$ begins this series. Therefore the denominator in the problem statement has a common factor of z^4, and the result for $z \neq 0$ is: $\frac{(\cos z) - 1 + \frac{z^2}{2!}}{z^4} = \sum_{n=2}^{\infty}(-1)^n \frac{z^{2n-4}}{(2n)!}$. When $z = 0$ all terms but the first "drop out", so $L = \frac{1}{4!} = \frac{1}{24}$.

b) Suppose the function h is defined by $h(z) = \begin{cases} \frac{(\cos z) - 1 + \frac{z^2}{2!}}{z^4} & \text{if } z \neq 0 \\ L & \text{if } z = 0 \end{cases}$ where L is the number found in a). Explain carefully why h is entire (analytic in all of the complex numbers).

Answer The function h has convergent power series $\sum_{n=2}^{\infty}(-1)^n \frac{z^{2n-4}}{(2n)!}$, valid for all z. Such series converge to analytic functions.