(20) 1. Show that $u(x, y) = xy + y$ is a harmonic function and find a harmonic conjugate $v(x, y)$ of $u(x, y)$. Write the resulting complex analytic function $u(x, y) + iv(x, y)$ in terms of the complex variable z.

(30) 2. In this problem $F(z) = z^2$.

a) Suppose R is the half-line $y = x$, $x \geq 0$, and L is the line $x = 2$. Carefully sketch the images of R and L under the mapping determined by F. State what the angle between the intersections of the images of R and L must be, and briefly explain why.

b) Suppose $S_0 = \{2\}$ and define $S_1 = F^{-1}(S_0)$, $S_2 = F^{-1}(S_1)$, $S_3 = F^{-1}(S_2)$, etc., so that $S_{n+1} = F^{-1}(S_n)$ if n is a non-negative integer. Here $F^{-1}(w)$ means *all* square roots of w. Sketch S_0, S_1, S_2, and S_3. How many points will there be in S_n if n is any positive integer? Suppose S is the union of *all* of the S_n's for *all* non-negative integers n. Identify any interior points of S. Identify any accumulation points of S.

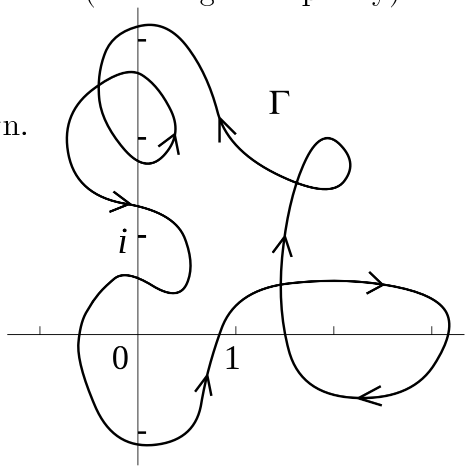
(20) 3. Find the order of the pole of

$$H(z) = \frac{1}{(6 \sin z + z^3 - 6z)^2}$$

at $z = 0$.

(20) 4. Consider the function $G(z) = z^5 + 5z^2 + e^z$. How many zeros (counting multiplicity) does G have in the annular region $1 < |z| < 2$?

(25) 5. Compute $\displaystyle\int_\Gamma \frac{z^2}{e^{2\pi i z} - 1}\,dz$ where Γ is the closed curve shown.



(20) 6. Suppose T is the inside of the square with corners 0, 1, i, and $1 + i$, and S is the inside of the square with corners $3 + 3i$, $-3 + 3i$, $-3 - 3i$, and $3 - 3i$. Suppose also that $f(z)$ is any entire function. Let M be the maximum of $|f''(z)|$ on T and let N be the maximum of $|f(z)|$ on S. Show that

$$M \leq \frac{1}{2} N\,.$$

Hint Begin by writing some complex variables formula connecting f'' and f.

(20) 7. a) If k is a real number with $-1 < k < 1$, derive the Laurent series representation

$$\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n} \qquad (|k| < |z| < \infty)\,.$$

b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula

$$\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}\,.$$

(25) 8. If A is real and $A > 1$, the program `Maple` reports that

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+A^2)} = \left(\frac{1}{2}\right)\left(\frac{\pi}{A+1}\right).$$

Check this assertion using the method of residues. Show clearly the contour of integration and any residue computation. Explain why some integral tends to 0 in the limit.

(20) 9. Find complex numbers a, b, c, and d so that the linear fractional transformation

$$L(z) = \frac{az+b}{cz+d}$$

takes $-i$ to 0, 0 to 1, and $2i$ to ∞.

Sketch the image of the unit circle $|z| = 1$ under the transformation L, and briefly explain why what was drawn is correct.

Final Exam for Math 403, section 1

May 5, 1999

NAME ____________________

Do all problems, in any order.
Show your work; an answer alone may not get full credit.
No notes or calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	30	
3	20	
4	20	
5	25	
6	20	
7	20	
8	25	
9	20	
Total Points Earned:		