Formula sheet for the final exam in Math 291, fall 2002

Cauchy-Schwarz: $|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$. Triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

Distance from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$ is $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$.

Distance from $P_1(x_1, y_1, z_1)$ to plane $ax+by+cz=d$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\mathbf{n}=\langle a,b,c\rangle$

$\|\mathbf{a}\| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

$\|\mathbf{a}\cdot\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$ (If $=0$, then $\mathbf{a}\perp\mathbf{b}$.) $\quad \|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta$ (If $\mathbf{a}\|\mathbf{b}$ this $=0$.)

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \quad \text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}}$

Volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $\|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\|$

Arc length: $\int_a^b \|\mathbf{r}'(t)\|\,dt \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \quad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \overset{2\text{ dim}}{=} \frac{|y''(t)x'(t) - x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^3/2} \overset{y=f(x)}{=} \frac{|f"(x)|}{(1+(f'(x))^2)^{3/2}}$$

$$\tau = \frac{(\mathbf{r}'(t)\times\mathbf{r}'(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2}.$$ Frenet-Serret: $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}+\tau\mathbf{B}$, $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

Tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$: $z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

Linear approximation to $f(x,y)$ at (a,b): $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent plane to $F(x,y,z)=0$:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

If y implicitly defined by $y=f(x)$ in $F(x,y)=0$ then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If z implicitly defined by $z=f(x,y)$ in $F(x,y,z)=0$ then $z_x = -\dfrac{F_x}{F_z}$ and $z_y = -\dfrac{F_y}{F_z}$.

$\nabla f = \langle f_x, f_y, f_z\rangle = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} \quad D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z)\cdot\mathbf{u}$

Some chain rules:

If $z=f(x,y)$ and $x=x(t)$ and $y=y(t)$, then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dy}{dt}$.

If $z=f(x,y)$ and $x=g(s,t)$ and $y=h(s,t)$, then $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$.

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$. Let $H=H(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

a) If $H>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $H>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $H<0$, then $f(a,b)$ is not a local maximum or minimum (f has a saddle point).

A real-valued function $F(\mathbf{x})$ is continuous at $\mathbf{x_0}$ if, given any $\varepsilon>0$, there is a $\delta>0$ so that whenever $\|\mathbf{x}-\mathbf{x_0}\|<\delta$, then $|F(\mathbf{x})-F(\mathbf{x_0})|<\varepsilon$.

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Lagrange multipliers for one constraint

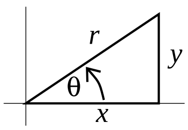
If $[G(\text{the variables}) = \text{a constant}]$ is the constraint and we want to extremize the objective function, $F(\text{the variables})$, then the extreme values can be found among F's values of the solutions of the system of equations $\nabla G = \lambda \nabla F$ (a vector abbreviation for the equations $\lambda \frac{\partial F}{\partial \star} = \frac{\partial G}{\partial \star}$ where $\star$ is each of the variables) **and** the constraint equation.

Polar coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad \theta = \arctan\left(\frac{y}{x}\right)$

$dA = r\,dr\,d\theta$

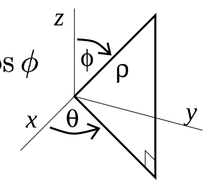


Spherical coordinates

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$



Change of variables in 2 dimensions

$\int\int_R f(x,y)\,dA = \int\int_{\tilde{R}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$; $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, the Jacobian.

Total mass of a mass distribution $\rho(x,y,z)$ over a region R of $\mathbb{R}^3$ is $\int\int\int_R \rho(x,y,z)\,dV$.
Area = integral of $1\,dA$; **arc length** = integral of $1\,ds$; **volume** = integral of $1\,dV$.

Line integral formulas

$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$

$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(x(t),y(t))x'(t)\,dt + Q(x(t),y(t))y'(t)\,dt$

Green's Theorem

$\int_C P\,dx + Q\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

These P, Q pairs will give R's area $\begin{cases} P=-y \text{ and } Q=0 \\ P=0 \text{ and } Q=x \\ P=-\frac{1}{2}y \text{ and } Q=\frac{1}{2}x \end{cases}$

In $\mathbb{R}^2$, a **conservative vector field** $\mathbf{V} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a **gradient vector field**: there's $f(x,y)$ with $\nabla f = \mathbf{V}$ so $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. f is a **potential** for $\mathbf{V}$. A conservative vector field is **path independent**. Work done by such a vector field over a **closed curve** is 0. For V conservative with potential f: $\int_C P\,dx + Q\,dy = f(\text{THE END}) - f(\text{THE START})$.
If $P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is conservative, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. If the region is **simply connected** (means **no holes**) then the converse is true, and f is both $\int P(x,y)\,dx$ and $\int Q(x,y)\,dy$.

Surfaces: If $\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}$, **flux** is $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$.

As a graph: $z = f(x,y)$ and $\mathbf{N} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ and $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\,dA_{xy}$.

Parametrically: $\mathbf{P}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ and $\mathbf{N} = \frac{\partial \mathbf{P}}{\partial u} \times \frac{\partial \mathbf{P}}{\partial v}$ and $dS = \left\|\frac{\partial \mathbf{P}}{\partial u} \times \frac{\partial \mathbf{P}}{\partial v}\right\| dA_{uv}$.

If $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\mathbf{F}$ is a vector field then $\begin{cases} \text{curl } F = \nabla\times\mathbf{F}, \text{ a vector field.} \\ \text{div } F = \nabla\cdot\mathbf{F}, \text{ a function.} \end{cases}$

Potentials in $\mathbb{R}^3$

If $\mathbf{F} = \nabla f$ and C is a curve, then $\int_C P\,dx + Q\,dy + R\,dz = f(\text{THE END}) - f(\text{THE START})$, path independence holds, the work over a closed curve is 0, and $\text{curl}(\nabla f) = 0$. Conversely, if F is defined in all of $\mathbb{R}^3$ with curl $F = 0$ (the cross-partials "match") then $\mathbf{F}$ has a potential, f, so $\nabla f = \mathbf{F}$. f is obtained by comparing partial integrals of the components of $\mathbf{F}$.

Stokes' Theorem (As you "walk" along C, S is to the left and $\mathbf{n}$ is up.)

$\left[\int\int_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS =\right] \quad \int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot d\mathbf{r} \quad \left[= \int_C P\,dx + Q\,dy + R\,dz\right]$

Divergence Theorem ($\mathbf{n}$ is unit *outward* normal to E, a region in $\mathbb{R}^3$ with boundary S.)

$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int\int_E \text{div } F\,dV \quad \left[= \int\int\int_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,dV\right]$