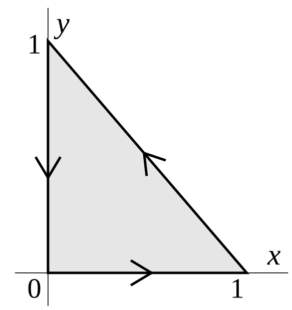
Math 291:01 **Solution of the second exam's bonus problem** 11/28/2002

Prove Green's Theorem ($\int_C P(x,y)\,dx + Q(x,y)\,dy = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$) if R is the triangular region in the xy-plane which is bounded by the positive x-axis, the positive y-axis, and line $x+y=1$, and if C is the oriented boundary of R: the line segment from $(0,0)$ to $(1,0)$, followed by the line segment from $(1,0)$ to $(0,1)$, and completed by the line segment from $(0,1)$ to $(0,0)$. $P(x,y)$ and $Q(x,y)$ are functions with continuous first partial derivatives on C and all of R.



Answer Begin with $\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$. This is $\int\int_R \frac{\partial Q}{\partial x}\,dA - \int\int_R \frac{\partial P}{\partial y}\,dA$. Now convert to iterated integrals, with a careful choice of order: $\int_0^1 \int_0^{1-y} \frac{\partial Q}{\partial x}\,dx\,dy - \int_0^1 \int_0^{1-x} \frac{\partial P}{\partial y}\,dy\,dx$. Apply the Fundamental Theorem of Calculus in the first integral with respect to x and in the second integral with respect to y. We get a sum of four integrals:

$\int_0^1 (Q(y,1-y) - Q((0,y))\,dy - \int_0^1 (P(x,1-x) - P(x,0))\,dx =$

$\underbrace{\int_0^1 Q(y,1-y)\,dy}_{\mathbf{A}} - \underbrace{\int_0^1 Q((0,y)\,dy}_{\mathbf{B}} - \underbrace{\int_0^1 P(x,1-x)\,dx}_{\mathbf{C}} + \underbrace{\int_0^1 P(x,0)\,dx}_{\mathbf{D}}.$

Now consider $\int_C P(x,y)\,dx + Q(x,y)\,dy$. Write $C = C_1 + C_2 + C_3$ where C_1 is the line segment from $(0,0)$ to $(1,0)$, C_2 is the line segment from $(1,0)$ to $(0,1)$, and C_3 is the line segment from $(0,1)$ to $(0,0)$.

The C_1 integral First parameterize the line segment from $(0,0)$ to $(1,0)$: $\begin{cases} x = t \\ y = 0 \end{cases}$ so that $\begin{cases} dx = dt \\ dy = 0\,dt \end{cases}$ with $0 \le t \le 1$. Then $\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_0^1 P(t,0)\,dt$. This is the value of the integral **D**.

The C_3 integral Parameterize the line segment from $(1,0)$ to $(0,0)$: $\begin{cases} x = 0 \\ y = 1 \end{cases}$ so that $\begin{cases} dx = 0\,dt \\ dy = dt \end{cases}$ with $0 \le t \le 1$ but with a minus sign, since the direction is "down". Then $\int_C P(x,y)\,dx + Q(x,y)\,dy = -\int_0^1 Q(0,t)\,dt$. This is the value of the integral **B** with its preceding minus sign.

The C_2 integral This is the most interesting case. Begin with $\begin{cases} x = t \\ y = 1-t \end{cases}$ so that $\begin{cases} dx = dt\,dt \\ dy = -dt \end{cases}$ with $0 \le t \le 1$, again prefixing the result with a minus sign because of the direction. Then $\int_C P(x,y)\,dx + Q(x,y)\,dy = -\int_0^1 (P(t,1-t) - Q(t,1-t))\,dt$. The value of this is the same as the sum of the integral **A** and the integral **C**, with **C**'s minus sign included.

We have verified the equality in Green's Theorem for this triangle.