Answers to Exam #1

(14) 1. Use the method of partial fractions to verify that $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{4}\ln 2 + \frac{1}{8}\pi$.

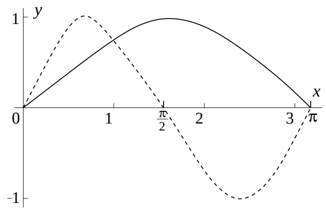
Answer Write $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)}{(x+1)(x^2+1)}$, so $1 = A(x^2+1)+(Bx+C)(x+1)$. When $x = -1$ we get $A = \frac{1}{2}$. Comparing x^2 coefficients of both sides, we see $B = -\frac{1}{2}$. Finally, comparing constant coefficients on both sides, we see that $1 = A + C$ so $C = \frac{1}{2}$. Compute: $\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\,dx = \frac{1}{2}\ln(x+1) + -\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan(x)\Big]_0^1 = \frac{1}{2}\ln(2) + -\frac{1}{4}\ln(2) + \frac{1}{2}\arctan(1) - \Big(\frac{1}{2}\ln(1) + -\frac{1}{4}\ln(1) + \frac{1}{2}\arctan(0)\Big) = \frac{1}{4}\ln(2) + \frac{1}{2}\frac{\pi}{4} = \frac{1}{4}\ln 2 + \frac{\pi}{8}$. Whew! All done.

(8) 2. Suppose $\mathcal{R}$ is the region bounded by $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$.

a) Find the volume of the solid that results from rotating $\mathcal{R}$ around the x-axis.
Answer $\pi\int_1^2 \pi\left(\frac{1}{x}\right)^2\,dx = \pi\cdot\left(\frac{-1}{x}\right)\Big]_1^2 = \frac{\pi}{2}$.

b) Find the volume of the solid that results from rotating $\mathcal{R}$ around the y-axis.
Answer $2\pi\int_1^2 x\left(\frac{1}{x}\right)\,dx = 2\pi$.

(12) 3. a) Carefully sketch $y = \sin x$ and $y = \sin 2x$ on the axes given for $0 \le x \le \pi$. Be sure to label each of the curves.
Answer The dashed curve is $y = \sin 2x$ and the solid curve is $y = \sin x$.



b) Find **all** points of intersection of the curves sketched in part a).
Answer The sketch has three intersection points. Since $\sin 2x = 2\sin x\cos x$, these occur when $2\sin x\cos x = \sin x$: when $\sin x = 0$, which occurs $x = 0$ or $x = \pi$ in the sketch, or when $\cos x = \frac{1}{2}$ which happens when $x = \frac{\pi}{3}$.

c) Find the area enclosed between the curves sketched in part a). Give an exact answer in terms of mathematical constants such as π and e.
Answer In $[0, \frac{\pi}{3}]$, the top curve is $\sin 2x$. The enclosed area there is $\int_0^{\frac{\pi}{3}} \sin 2x - \sin x\,dx = -\frac{1}{2}\cos 2x + \cos x\Big]_0^{\frac{\pi}{3}} = \frac{1}{4}$. In $[\frac{\pi}{3}, \pi]$, the top curve is $\sin x$. The enclosed area there is $\int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x\,dx = -\cos x + \frac{1}{2}\cos 2x\Big]_{\frac{\pi}{3}}^{\pi} = \frac{9}{4}$. So the total area is $\frac{1}{4} + \frac{9}{4} = \frac{5}{2}$.

(10) 4. a) Suppose w is a positive number. Define $A(w)$ to be the average value of $(\cos x)^2$ on the interval $0 \le x \le w$. Compute $A(w)$, and show how the integral is calculated. **Answer** $\int_0^w (\cos x)^2 dx = \int_0^w \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x)\Big]_0^w = \frac{1}{2}(w + \frac{1}{2}\sin 2w) - \frac{1}{2}(0 + \sin 0)$. The average value is the integral divided by the length of the interval, which is w, so $A(w) = \frac{1}{2} + \frac{1}{4}\frac{\sin 2w}{w}$.

b) What is the limit of $A(w)$ as $w \to 0^+$? **Answer** The limit of $\frac{\sin 2w}{w}$ can be computed using L'Hospital's rule. It is $\lim\limits_{w\to 0^+} \frac{2\cos 2w}{1} = 2$. So the limit requested is $\frac{1}{2} + \frac{1}{4}\cdot 2 = 1$. The limit can also be computed by noticing that $\sin 2w = 2\sin w\cos w$, and $\lim\limits_{w\to 0} \frac{\sin w}{w} = 1$, familiar from earlier calculus study.

(12) 5. a) Explain why $\displaystyle\int_1^\infty \frac{1}{x^2 + e^{2x}}\,dx$ converges. **Answer** $\frac{1}{x^2+e^{2x}}$ is positive on the interval $[1, \infty)$, and is certainly less than either $\frac{1}{x^2}$ or $\frac{1}{e^{2x}}$. Either of those integrals converges on that interval, so by comparison, the integral given converges.

b) Explain why the value of the integral in a) is less than $\frac{1}{2}$. **Answer** In a) either integral is "good enough" to show convergence. Here, however, since $\int_1^\infty \frac{1}{x^2}\,dx = \lim\limits_{A\to\infty} -\frac{1}{x}\Big]_1^A = 1$ and $\int_1^\infty \frac{1}{e^{2x}}\,dx = \lim\limits_{A\to\infty} -\frac{1}{2}e^{-2x}\Big]_1^A = \frac{1}{2e^2}$ we'd better use the second, exponential integral (decaying exponentials $\to 0$ much more rapidly than *any* inverse power of x). So the value of the integral given is less than $\frac{1}{2e^2}$ which is certainly less than $\frac{1}{2}$.

(12) 6. Calculate the following integrals, showing your work. a) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ **Answer** If $x = \tan\theta$, then $dx = (\sec\theta)^2\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$. So $\int \frac{dx}{\sqrt{1+x^2}} = \int \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta) + C = \ln(\sqrt{1+x^2} + x) + C$.

b) $\displaystyle\int \frac{dx}{2e^x+1}$ **Answer** If $u = e^x$, then $du = e^x\,dx$ so dx is $\frac{du}{u}$. The integral given becomes $\int \frac{1}{u(2u+1)}\,du$. Rewrite the integrand using partial fractions: $\frac{1}{u(2u+1)} = \frac{A}{u} + \frac{B}{2u+1}$ giving $1 = A(2u+1) + Bu$. When $u = 0$, we get $A = 1$. When $u = -\frac{1}{2}$ we get $B = -2$. Then antidifferentiating: $\int \frac{1}{u} + \frac{-2}{2u+1}\,du = \ln u - \ln(2u+1) + C$. Converting back to x's, the answer becomes $\ln(e^x) - \ln(2e^x+1) + C$. (Yes, $\ln(e^x)$ is x.)

(10) 7. Use a substitution followed by integration by parts to verify that $\displaystyle\int_0^1 e^{\sqrt{x}}\,dx = 2$.

Answer Try $w = \sqrt{x}$ and therefore $w^2 = x$ with $2w\,dw = dx$. The integral changes: $\int e^{\sqrt{x}}\,dx = \int e^w 2w\,dw$. This integral is a well-known candidate for integration by parts. Let's do it (I'll save the 2 until later):

$$\underbrace{\int e^w w\,dw}_{\int u\,dv} \underbrace{=}_{=} \underbrace{we^w}_{uv} \underbrace{-}_{-} \underbrace{\int e^w\,dw}_{\int v\,du} = we^w - e^w + C \quad \text{with} \quad \left.\begin{array}{c} u = w \\ dv = e^w\,dw \end{array}\right\}\left\{\begin{array}{c} du = dw \\ v = e^w \end{array}\right.$$

. We can reverse the substitution so the indefinite integral becomes (with the 2!) $2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C$. Therefore, $\int_0^1 e^{\sqrt{x}}\,dx = 2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right)\Big]_0^1 = (2(e-e)) - (2(0-1)) = 2$.

(12) 8. a) Suppose m and n are positive integers. Find a reduction formula for $\displaystyle\int x^m (\ln x)^n\,dx$. (Here the aim is to have an equation with the integral given here on one side and a similar integral with reduced n on the other side, since if we can push n to 0 we'll just have a polynomial to integrate, which is easy.)

Answer Again, integrate by parts.
$$\underbrace{\int x^m (\ln x)^n\,dx}_{\int u\,dv} \underbrace{=}_{=} \underbrace{\left(\frac{x^{m+1}}{m+1}\right)\cdot(\ln x)^n}_{uv} \underbrace{-}_{-} \underbrace{\int n(\ln x)^{n-1}\cdot\left(\frac{1}{x}\right)\left(\frac{x^{m+1}}{m+1}\right)dx}_{\int v\,du} + C$$

with $\left.\begin{array}{c} u = (\ln x)^n \\ dv = x^m\,dx \end{array}\right\}\left\{\begin{array}{c} du = n(\ln x)^{n-1}dx \\ v = \dfrac{x^{m+1}}{m+1} \end{array}\right.$. I'll clean this up a bit since the formula will be used in part b):
$$\int x^m(\ln x)^n dx = \left(\frac{x^{m+1}}{m+1}\right)(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx + C.$$

b) Use the formula obtained in a) to compute $\displaystyle\int x^{20}(\ln x)^2\,dx$.

Answer Since $n = 2$ we need the formula above twice.
$$\int x^{20}(\ln x)^2\,dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\int x^{20}\ln x\,dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20}\,dx\right) =$$
$$\frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20}\,dx\right) = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\cdot\frac{x^{21}}{21}\right) + C$$
I doubt whether I would "simplify" anything here except under duress.

(10) 9. The integral $\displaystyle\int_0^2 \left(x^3+1\right)^{7/2}\,dx$ is approximated using the Trapezoidal Rule by dividing $[0,2]$ into n segments of equal length. How large should n be in order to guarantee that the error is at most 10^{-6}?

Note You must give some reason explaining why any overestimates of derivatives you make are valid on the entire interval.

Answer We will need to overestimate $|f''(x)|$ for all x's in $[0,2]$ and for $f(x) = \left(x^3+1\right)^{7/2}$. So $f'(x) = \frac{7}{2}\left(x^3+1\right)^{5/2}(3x^2)$ and $f''(x) = \frac{7}{2}\cdot\frac{5}{2}\cdot\left(x^3+1\right)^{3/2}\left(3x^2\right)^2 + \frac{7}{2}\left(x^3+1\right)^{5/2}6x$. Notice that x^3+1 is increasing in $[0,2]$ since its derivative, $3x^2$, is positive there. So the value of x^3+1 for any x in $[0,2]$ is less than the value of x^3+1 at the right-hand endpoint, which is 9. The other pieces of f'' are also increasing and positive, so $f''(2)$ will be an appropriate overestimate of $|f''(x)|$ for all x in $[0,2]$.

Now $f''(2) = \frac{7}{2}\cdot\frac{5}{2}\cdot\left(2^3+1\right)^{3/2}\left(3\cdot 2^2\right)^2 + \frac{7}{2}\left(2^3+1\right)^{5/2}(6\cdot 2)$. This is 44,226, which I will "plug into" $M_2(b-a)^3/12n^2$ in place of M_2. Of course $a = 0$ and $b = 2$, so the Trapezoid Rule error will be less than $44,226\cdot 8/12n^2$. This is less than 10^{-6} when $n \geq 172,000$ (the approximate result my calculator reports). The number is large and, to me, rather unimpressive.

Detailed partial credit information is available on the web: look at the `Exam material` page, please.