Differential equations–separation of variables. Factor one side of the equation so that $dy/dx = f(x)g(y)$. Then $dy/g(y) = f(x)dx$. Integrate, not forgetting the constant C of integration. Use the initial condition $(x, y) = (x_0, y_0)$ to find C.
EX: For $dy/dx = x + xy$, with $y = 2$, when $x = 0$, then $dy/dx = x(1 + y)$, or $dy/(1 + y) = x\,dx$. Integrating, $\ln(1+y) = x^2/2+C$. So $1+y = e^{x^2/2+C}$, or $y = Ae^{x^2/2} - 1$, with $A = e^C$. Set $x = 0$, $y = 2$, $2 = A-1$ and $y = 3e^{x^2/2} - 1$.

Exponential population growth: $P = P_0e^{kt}$, where P_0 is the initial population and k is a constant.

$\lim_{n\to\infty} \frac{n^a}{b^n} = 0$, if $b > 1$; $\lim_{n\to\infty} \frac{a^n}{n!} = 0$; f, g poly., $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{an^j + \text{lower terms}}{bn^k + \text{lower terms}} = \begin{cases} a/b, & \text{if } j = k; \\ 0, & \text{if } j < k;\ \pm\infty, \text{ if } j > k. \end{cases}$

$\lim_{n\to\infty} c^{1/n} = 0$, if $c > 0$; $\lim_{n\to\infty} n^{1/n} = 0$; $\lim_{n\to\infty} (1 + (1/n))^n = e$. Use L'Hospital's Rule to verify these results.

Geometric series $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, if $|x| < 1$. $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$, diverges if $p \le 1$.

Divergence Test: The series $\sum a_n$ diverges if either $\lim_{n\to\infty} a_n$ does not exist or exists and is not 0.

Integral Test: Suppose that f is a positive, continuous, decreasing function. Let $a_n = f(n)$ for each positive integer n. Then the integral $\int_1^{\infty} f(x)\,dx$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together. Use for $\sum 1/n^p$, $\sum 1/(n(\ln n)^p)$.
EX: For $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$, find $\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{\ln 2}^{\infty} = \infty$, using $u = \ln x$, $du = \frac{dx}{x}$. The series diverges.

Comparison Test: Assume $0 \le a_n \le b_n$. If $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.
EX: Note that $\frac{1}{n^2 \ln n} \le \frac{1}{n^2}$, if $n \ge 3$, as $\ln n \ge 1$, if $n \ge 3$. Since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^2 \ln n}$.

Limit Comparison: If $a_n, b_n > 0$, $\lim_{n\to\infty}(a_n/b_n) = L$, with $L \ne 0, \infty$, $\sum a_n$ and $\sum b_n$ both converge or diverge.
EX: Compare $\sum \frac{\sqrt{n}}{n+1}$ to $\sum \frac{1}{\sqrt{n}}$, since $\left(\frac{\sqrt{n}}{n+1}\right)\Big/\left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n+1} \to 1$, as $n \to \infty$. Since $\sum \frac{1}{\sqrt{n}}$ diverges, both diverge.

Alternating Series: If $\sum_{n=0}^{\infty} (-1)^n a_n$ is a series with $a_n \ge 0$, $a_{n+1} \le a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, the series converges.

Absolute Convergence: The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. Then $\sum a_n$ converges also.
Conditional Convergence: The series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ratio Test: Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. If $L < 1$, $\sum a_n$ converges. If $L > 1$, $\sum a_n$ diverges. If $L = 1$, the test fails. Often useful for series involving a^n or $n!$. Root Test: Similar but with $\lim_{n\to\infty} |a_n|^{1/n} = L$. These tests can be used to find radius of convergence for power series.
EX: For $\sum \frac{3^n}{n!}$, $\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \to 0$, as $n \to \infty$. As $L = 0 < 1$, the series converges absolutely.

Series	Test
$\sum \frac{1}{n(lnn)^4}$	Integral
$\sum \frac{n^2}{(n+1)2^n}$	Ratio
$\sum \frac{(-1)^n n}{n^2+1}$	Alter. Ser.

Series	Test
$\sum \frac{(n!)^2}{(2n)!}$	Ratio
$\sum \frac{n^2+1}{n^2+n}$	Divergence
$\sum \frac{n^2+n}{n^4+1}$	Lim. Comp. to $\sum \frac{1}{n^2}$

Series	Test
$\sum \frac{1}{(\ln n)2^n}$	Comp: $\sum \frac{1}{(\ln n)2^n} \le \sum \frac{1}{2^n}$
$\sum \frac{\ln n}{n}$	Comp: $\sum \frac{1}{n} \le \sum \frac{\ln n}{n}$
$\sum \frac{\sin n}{n^2}$	Comp: $\sum \frac{\lvert\sin n\rvert}{n^2} \le \sum \frac{1}{n^2}$

Estimating Sums with the Integral Test: The error in approximating $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{N} a_n$ is at most $\int_N^{\infty} f(x)dx$.
EX: How large should N be to insure that $\sum_{n=1}^{N} ne^{-n^2}$ is within $1/10^{10}$ of $\sum_{n=1}^{\infty} ne^{-n^2}$? We need $\int_N^{\infty} xe^{-x^2}\,dx \le 1/10^{10}$. Integrating, $e^{-N^2}/2 \le 1/10^{10}$, or $10^{10}/2 \le e^{N^2}$, or $10\ln 10 + \ln 2 \le N^2$, or $\sqrt{10\ln 10 + \ln 2} \le N$, N an integer.
With Alternating Series: The error in approximating $\sum_{n=0}^{\infty} (-1)^n a_n$ with $\sum_{n=0}^{N} (-1)^n a_n$ has absolute value at most a_{n+1}.

Radius of convergence: For the power series, $\sum c_n(x-a)^n$, there is a number $R \ge 0$ so that if $|x-a| < R$, the series converges absolutely and if $|x-a| > R$, the series diverges. If $|x-a| = R$, the series may converge or diverge. Find R with the ratio test. The interval of convergence I is the set of points where the series converges. I contains the interval $(a-R, a+R)$ and may or may not contain the boundary points $a-R$ and $a+R$.
EX: For $\sum \frac{nx^{2n}}{3^n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x|^{2n+2}}{3^{n+1}} \cdot \frac{3^n}{n|x|^{2n}} = \left(\frac{n+1}{n}\right) \cdot \frac{|x|^2}{3} \to \frac{|x|^2}{3}$, as $n \to \infty$. The series converges absolutely if $|x|^2/3 < 1$, or $|x|^2 < 3$, or $|x| < \sqrt{3}$. If $|x| = \sqrt{3}$, the series is $\sum n$ and diverges. Then, $R = \sqrt{3}$ and $I = (-\sqrt{3}, \sqrt{3})$.

EX: Use power series integration to find the power series of $\arctan x$ at $a = 0$. Setting $x = -t^2$ in the geometric series: $\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$ and $\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

Taylor Polynomials: The n-th Taylor poly. of f is $T_n(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \cdots + f^n(a)(x-a)^n/n!$, assuming f has n derivatives at a. Also $f(x) = T_n(x) + R_n(x)$, $|R_n(x)| \le M|x-a|^{n+1}/(n+1)!$, with M an upper bound of $|f^{n+1}(t)|$ with t in the interval with endpoints a and x. EX: Find $T_2(x)$ of $f(x) = (1+x)^{1/2}$ at $a = 0$. Estimate $|R_2(x)|$ with x in $[0, .1]$. Here, $f(x) = (1+x)^{1/2}$, $f'(x) = (1/2)(1+x)^{-1/2}$, $f''(x) = (-1/4)(1+x)^{-3/2}$, $f'''(x) = (3/8)(1+x)^{-5/2}$. Then, $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = -1/4$. Thus, $T_2(x) = 1 + x/2 - x^2/8$. Also, $|R_2(x)| \le M|x|^3/3!$, with M an upper bound of $|f'''(t)| = (3/8)(1+t)^{-5/2}$, with t in $[0, .1]$. Since $1+t$ is increasing, $(3/8)(1+t)^{-5/2}$ is decreasing on $[0, .1]$ and its maximum value occurs at $t = 0$ and is $3/8$. Thus, $M = 3/8$, $|x| \le .1$, and $|R_2(x)| \le (3/8)(.1)^3/6 = (.1)^3/16$.

$e^x = \sum_{n=0}^{\infty} x^n/n!$; $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$; $\cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$; $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n x^{n+1}/(n+1)$

Parametric curves: If $x = f(t), y = g(t)$, then the slope of the tangent line is given by $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{g'(t)}{f'(t)}$. The arc length from $t = a$ to $t = b$ is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$. Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$. $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$. The area bounded by $y = f(\theta)$, $\theta = a$, $\theta = b$ is $\int_a^b \frac{1}{2}r^2\,d\theta = \int_a^b \frac{1}{2}(f(\theta))^2\,d\theta$.