Harmonic Analysis on Compact Symmetric Spaces: the Legacy of Élie Cartan and Hermann Weyl

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1 Introduction

In his lecture *Relativity theory as a stimulus in mathematical research* [Wey4], Hermann Weyl says that "Frobenius and Issai Schur's spadework on finite and compact groups and Cartan's early work on semi-simple Lie groups and their representations had nothing to do with it [relativity theory]. But for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups, and my experience in this regard is probably not unique."

Weyl's first encounter with Lie groups and representation theory as a tool to understand relativity theory occurred in connection with the Helmholtz-Lie space problem and the problem of decomposing the tensor product $\otimes^k \mathbb{C}^n$ under the mutually commuting actions of the general linear group $\mathrm{GL}(n, \mathbb{C})$ (on each copy of $\mathbb{C}^n$) and the symmetric group $\mathfrak{S}_k$ (in permuting the k copies of $\mathbb{C}^n$).[1] He later described the tensor decomposition problem in general terms [Wey3] as "an epistemological principle basic for all theoretical science, that of projecting the actual upon the background of the possible." Mathematically, the issue was to find subspaces of tensor space that are invariant and irreducible under all transformations that commute with $\mathfrak{S}_k$. This had already been done by Frobenius and Schur around 1900, but apparently Weyl first became aware of these results in the early 1920's. The subspaces in question, which are the ranges of minimal projections in the group algebra of $\mathfrak{S}_k$, are exactly the irreducible (polynomial) representations of $\mathrm{GL}(n, \mathbb{C})$, and all irreducible representations arise this way for varying k by including multiplication by integral powers of $\det(g)$ in the action. It seems clear from his correspondence with Schur at this time that these results were Weyl's starting point for his later work in representation theory and invariant theory.

[1] see [Haw, §11.2-3]

Near the end of his monumental paper on representations of semisimple Lie groups [Wey1, Kap. IV, §4], Weyl considers the problem of constructing all the irreducible representations of a simply-connected simple Lie group G such as $\mathrm{SL}(n, \mathbb{C})$. This had been done on a case-by-case basis by Cartan [Car1], starting with the defining representations for the classical groups (or the adjoint representation for the exceptional groups) and building up a general irreducible representation by forming tensor products. By contrast, Weyl, following the example of Frobenius for finite groups, says that "the correct starting point for building representations does not lie in the adjoint group, but rather in the *regular representation*, which through its reduction yields *in one blow* all irreducible representations." He introduces the infinite-dimensional space $C(U)$ of all continuous functions on the compact real form U of G ($U = \mathrm{SU}(n)$ when $G = \mathrm{SL}(n, \mathbb{C})$) and the right translation representation of U on $C(U)$. He then obtains the irreducible representations of U and their characters by using the eigenspaces of compact integral operators given by left convolution with positive-definite functions in $C(U)$, in analogy with the decomposition of tensor spaces for $\mathrm{GL}(n, \mathbb{C})$ using elements of the group algebra of $\mathfrak{S}_k$. The details are spelled out in the famous Peter–Weyl paper [Pe-We], which proves that the normalized matrix entries of the irreducible unitary representations of U furnish an orthonormal basis for $L^2(U)$, and that every continuous function on U is a uniform limit of linear combinations of these matrix entries.

In the introduction to [Car2], É. Cartan says that his paper was inspired by the paper of Peter and Weyl, but he points out that for a compact Lie group their use of integral equations "gives a transcendental solution to a problem of an algebraic nature" (namely, the completeness of the set of finite-dimensional irreducible representations of the group). Cartan's goal is "to give an algebraic solution to a problem of a transcendental nature, more general than that treated by Weyl." Namely, to find an explicit decomposition of the space of all L^2 functions on a homogeneous space into an orthogonal direct sum of group-invariant irreducible subspaces.

Cartan's paper [Car2] then stimulated Weyl [Wey2] to treat the same problem again and write "the systematic exposition by which I should like to replace the two papers Peter–Weyl [Pe-We] and Cartan [Car2]." In his characteristic style of finding the core of a problem through generalization, Weyl takes the finite-dimensional irreducible subspaces of functions (which he calls the *harmonic sets* by analogy with the case of spherical harmonics) on the compact homogeneous space X as his starting point.[2] Using the invariant measure on the homogenous space, he constructs integral operators that intertwine the representation of the compact group U on $C(X)$ with the left regular representation on $C(U)$.

In this paper we approach the Weyl–Cartan results by way of algebraic groups. The *finite* functions on a homogeneous space for a compact connected Lie group (that is, the functions whose translates span a finite-dimensional subspace) can be viewed as *regular* functions on the complexified group (a complex reductive alge-

[2] Weyl's emphasis on function spaces, rather than the underlying homogeneous space, is in the spirit of the recent development of *quantum groups*; his immediate purpose was to make his theory sufficiently general to include also J. von Neumann's theory of almost-periodic functions on groups, in which the functions determine a compactification of the underlying group.

braic group). Irreducible subspaces of functions under the action of the compact group correspond to irreducible subspaces of regular functions on the complex reductive group—this is Weyl's *unitarian trick*. We describe the algebraic group version of the Peter–Weyl decomposition and geometric criterion for simple spectrum of a homogeneous space (due to E. Vinberg and B. Kimelfeld). We present R. Richardson's algebraic group version of the Cartan embedding of a symmetric space, and the celebrated results of Cartan and S. Helgason concerning finite-dimensional spherical representations.

We then turn to more recent results of J.-L. Clerc [Cle] concerning the complexified Iwasawa decomposition and zonal spherical functions on a compact symmetric space, and S. Gindikin's construction ([Gin1], [Gin2], [Gin3]) of the *horospherical Cauchy–Radon transform*, which shows that compact symmetric spaces have canonical dual objects that are complex manifolds.

We make frequent citations to the extraordinary books of A. Borel [Bor] and T. Hawkins [Haw], which contain penetrating historical accounts of the contributions of Weyl and Cartan. Borel's book also describes the development of algebraic groups by C. Chevalley that is basic to our approach. For a survey of other developments in harmonic analysis on symmetric spaces from Cartan's paper to the mid 1980's see Helgason [Hel3]. Thanks go to the referee for pointing out some notational inconsistencies and making suggestions for improving the organization of this paper.

2 Algebraic Group Version of Peter–Weyl Theorem

2.1 Isotypic Decomposition of $\mathcal{O}[X]$

The paper [Pe-We] of Peter and Weyl considers compact Lie groups U; because the group is compact left convolution with a continuous function is a compact operator. Hence such an operator, if self-adjoint, has finite-dimensional eigenspaces that are invariant under right translation by elements of U. The finiteness of the invariant measure on U also guarantees that every finite-dimensional representation of U carries a U-invariant positive-definite inner product, and hence is *completely reducible* (decomposes as the direct sum of irreducible representations).[3]

Turning from Weyl's transcendental methods to the more algebraic and geometric viewpoint preferred by Cartan, we recall that a subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is an *algebraic group* if it is the zero set of a collection of polynomials in the matrix entries. The *regular functions* $\mathcal{O}[G]$ are the restrictions to G of polynomials in matrix entries and $\det^{-1}$. In particular, G is a complex Lie group and the regular functions on G are holomorphic. A finite-dimensional complex representation (π, V) of G is *rational* if the matrix entries of the representation are regular functions on G. The group G is *reductive* if every rational representation is completely reducible.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. From the work of Cartan, Weyl, and Chevalley, one knows the following:

[3] This is the Hurwitz "trick" (*kunstgriff*) that Weyl learned from I. Schur; see Hawkins [Haw, §12.2].

(1) There is a simply-connected complex linear algebraic group G with Lie algebra $\mathfrak{g}$.

(2) The finite-dimensional representations of $\mathfrak{g}$ correspond to rational representations of G.

(3) There is a real form $\mathfrak{u}$ of $\mathfrak{g}$ and a simply-connected compact Lie group $U \subset G$ with Lie algebra $\mathfrak{u}$.

(4) The finite-dimensional unitary representations of U extend uniquely to rational representations of G, and U-invariant subspaces correspond to G-invariant subspaces.[4]

(5) The irreducible rational representations of G are parameterized by the positive cone in a lattice of rank l (Cartan's theorem of the *highest weight*).[5]

The highest weight construction is carried out as follows: Fix a Borel subgroup $B = HN^+$ of G (a maximal connected solvable subgroup). Here $H \cong (\mathbb{C}^\times)^l$, with $l = \operatorname{rank}(G)$, is a maximal algebraic torus in G, and N^+ is the unipotent radical of B associated with a set of positive roots of H on $\mathfrak{g}$. Let $\bar{B} = HN^-$ be the opposite Borel subgroup. We can always arrange the embedding $G \subset \mathrm{GL}(n, \mathbb{C})$ so that H consists of the diagonal matrices in G, N^+ consists of the upper-triangular unipotent matrices in G, and N^- consists of the lower-triangular unipotent matrices in G. Let $\mathfrak{h}$ be the Lie algebra of H and $\Phi \subset \mathfrak{h}^*$ the roots of $\mathfrak{h}$ on $\mathfrak{g}$. Write $P(\Phi) \subset \mathfrak{h}^*$ for the *weight lattice* of H and $P_{++} \subset P(\Phi)$ for the *dominant weights*, relative to the system of positive roots determined by N^+. For $\lambda \in P(\Phi)$ we denote by $h \mapsto h^\lambda$ the corresponding character of H. It extends to a character of B by $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N^+$.

An irreducible rational representation (π, E) of G is then determined (up to equivalence) by its *highest weight*. The subspace E^{N^+} of N^+-fixed vectors in E is one-dimensional, and H acts on it by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}$. The subspace E^{N^-} of N^--fixed vectors in E is also one-dimensional, and H acts on it by the character $h \mapsto h^{-\lambda^*}$ where $\lambda^* = -w_0 \cdot \lambda$. Here w_0 is the element of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ that interchanges positive and negative roots.

For each $\lambda \in P_{++}$ we fix a model (π_λ, E_λ) for the irreducible rational representation with highest weight λ. Then $(\pi_{\lambda^*}, E_\lambda^*)$ is the contragredient representation. Fix a highest weight vector $e_\lambda \in E_\lambda$ and a lowest weight vector $f_{\lambda^*} \in E_\lambda^*$, normalized so that

$$\langle e_\lambda, f_{\lambda^*} \rangle = 1.$$

Here we are using $\langle v, v^* \rangle$ to denote the tautological duality pairing between a vector space and its dual (in particular, this pairing is complex linear in both arguments). For dealing with matrix entries as regular functions on the complex algebraic group G

[4] This is Weyl's *unitary trick*.

[5] The first algebraic proofs of this that did not use case-by-case considerations were found by Chevalley and Harish-Chandra in 1948; see [Bor, Ch. VII, §3.6-7].

this is more convenient than using a U-invariant inner product on E_λ and identifying E_λ^* with E_λ via a conjugate-linear map.

Let X be an irreducible affine algebraic G space. Denote the regular functions on X by $\mathcal{O}[X]$. There is a representation ρ of G on $\mathcal{O}[X]$:

$$\rho(g)f(x) = f(g^{-1}x) \quad \text{for } f \in \mathcal{O}[X] \text{ and } g \in G.$$

Because the G-action is algebraic, $\mathrm{Span}\{\rho(G)f\}$ is a finite-dimensional rational G-module for $f \in \mathcal{O}[X]$. There is a tautological G-intertwining map

$$E_\lambda \otimes \mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \to \mathcal{O}[X],$$

given by $v \otimes T \mapsto Tv$. For $\lambda \in P_{++}$ let

$$\mathcal{O}[X]^{N^+}(\lambda) = \{f \in \mathcal{O}[X] \,:\, \rho(hn)f = h^\lambda f \quad \text{for } h \in H \text{ and } n \in N^+\}. \tag{1}$$

The key point is that the choice of a highest weight vector e_λ gives an isomorphism

$$\mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \cong \mathcal{O}[X]^{N^+}(\lambda). \tag{2}$$

Here a G-intertwining map T applied to the highest weight vector gives the function $\varphi = Te_\lambda \in \mathcal{O}[X]^{N^+}(\lambda)$, and conversely every such function φ defines a unique intertwining map T by this formula.[6] From (2) we see that the highest weights of the G-irreducible subspaces of $\mathcal{O}[X]$ comprise the set

$$\mathrm{Spec}(X) = \{\lambda \in P_{++} \,:\, \mathcal{O}[X]^{N^+}(\lambda) \neq 0\} \quad \text{(the } G \textit{ spectrum} \text{ of } X\text{)}$$

Using the isomorphism (2) and the reductivity of G, we obtain the decomposition of $\mathcal{O}[X]$ under the action of G, as follows:

Theorem 2.1. *The isotypic subspace of type (π_λ, E_λ) in $\mathcal{O}[X]$ is the linear span of the G-translates of $\mathcal{O}[X]^{N^+}(\lambda)$. Furthermore,*

$$\mathcal{O}[X] \cong \bigoplus_{\lambda \in \mathrm{Spec}(X)} E_\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda) \quad \textit{(algebraic direct sum)} \tag{3}$$

as a G-module, with action $\pi_\lambda(g) \otimes 1$ on the λ summand.

The action of G on $\mathcal{O}[X]$ is not only linear; it also preserves the algebra structure. Since $\mathcal{O}[X]^{N^+}(\lambda) \cdot \mathcal{O}[X]^{N^+}(\mu) \subset \mathcal{O}[X]^{N^+}(\lambda + \mu)$ under pointwise multiplication and $\mathcal{O}[X]$ has no zero divisors (X is irreducible), it follows from (3) that

$$\mathrm{Spec}(X) \text{ is an additive subsemigroup of } P_{++}.$$

The multiplicity of π_λ in $\mathcal{O}[X]$ is $\dim \mathcal{O}[X]^{N^+}(\lambda)$ (which may be infinite). All of this was certainly known (perhaps in less precise form) by Cartan and Weyl at the time [Pe-We] appeared. We now consider Cartan's goal in [Car2] to determine the decomposition (3) when G acts transitively on X; especially, when X is a symmetric space. This requires determining the *spectrum* and the *multiplicities* in this decomposition.

[6] Weyl uses a similar construction in [Wey2], defining intertwining maps by integration over a compact homogeneous space.

2.2 Multiplicity Free Spaces

We say that an irreducible affine G-space X is *multiplicity free* if all the irreducible representations of G that occur in $\mathcal{O}[X]$ have multiplicity one. Thanks to the theorem of the highest weight, this property can be translated into a geometric statement (see [Vi-Ki]). For a subgroup $K \subset G$ and $x \in X$ write $K_x = \{k \in L \,:\, k \cdot x = x\}$ for the isotropy group at x.

Theorem 2.2 (Vinberg–Kimelfeld)**.** *Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X. Then X is multiplicity free. In this case, if $\lambda \in \mathrm{Spec}(X)$ then $h^\lambda = 1$ for all $h \in H_{x_0}$.*

Proof. If $B \cdot x_0$ is open in X, then it is Zariski dense in X (since X is irreducible). Hence $f \in \mathcal{O}[X]^{N^+}(\lambda)$ is determined by $f(x_0)$, since on the dense set $B \cdot x_0$ it satisfies $f(b \cdot x_0) = b^{-\lambda} f(x_0)$. In particular, if $f \neq 0$ then $f(x_0) \neq 0$, and hence $h^\lambda = 1$ for all $h \in H_{x_0}$. Thus

$$\dim \mathcal{O}[X]^{N^+}(\lambda) \leq 1 \quad \text{for all } \lambda \in P_{++}.$$

Now apply Theorem 2.1. ♦

Remark. The converse to Theorem 2.2 is true; this depends on some results of Rosenlicht [Ros] and is the starting point for the classification of multiplicity free spaces (see [Be-Ra]).

Example: Algebraic Peter–Weyl Decomposition

Theorem 2.2 implies the algebraic version of the Peter-Weyl decomposition of the regular representation of G. Consider the reductive group $G \times G$ acting on $X = G$ by left and right translations. Denote this representation by ρ:

$$\rho(y,z)f(x) = f(y^{-1}xz), \quad \text{for} \quad f \in \mathcal{O}[G] \text{ and } x, y, z \in G.$$

Take $H \times H$ as the Cartan subgroup and $\bar{B} \times B$ as the Borel subgroup of $G \times G$. Let $x_0 = I$ (the identity in G). The orbit of x_0 under the Borel subgroup is

$$(\bar{B} \times B) \cdot x_0 = N^- H N^+ \quad \text{(Gauss decomposition)} \tag{4}$$

This orbit is open in G since $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$. Hence G is multiplicity free as a $G \times G$ space. The $G \times G$ highest weights (relative to this choice of Borel subgroup) are pairs $(w_0\mu,\, \lambda)$, with $\lambda, \mu \in P_{++}$. The diagonal subgroup $\widetilde{H} = \{(h,h) \,:\, h \in H\}$ fixes x_0, so if $(w_0\mu,\, \lambda)$ occurs as a highest weight in $\mathcal{O}[X]$, then

$$h^{w_0\mu+\lambda} = 1 \quad \text{for all } h \in H.$$

This means that $\mu = -w_0\lambda = \lambda^*$; hence $E_\mu = E_{\lambda^*}$ is the contragredient representation of G.

Now set $\psi_\lambda(g) = \langle \pi_\lambda(g) e_\lambda,\, f_{\lambda^*} \rangle$. This function satisfies $\psi_\lambda(x_0) = 1$ and

$$\psi_\lambda(\bar{b}^{-1} g b) = \langle \pi_\lambda(g)\pi_\lambda(b) e_\lambda,\, \pi_{\lambda^*}(\bar{b}) f_{\lambda^*} \rangle = b^\lambda \bar{b}^{w_0\lambda^*} \psi_\lambda(g)$$

for $b \in B$ and $\bar{b} \in \bar{B}$. Hence ψ_λ is a $B \times \bar{B}$ highest weight vector for $G \times G$ of weight $(w_0\lambda^*, \lambda)$. This proves that $\operatorname{Spec}(X) = \{(w_0\lambda^*, \lambda) : \lambda \in P_{++}\}$.

Theorem 2.3. *For $\lambda \in P_{++}$ let $V_\lambda = \operatorname{Span}\{\rho(G \times G)\psi_\lambda\}$. Then $V_\lambda \cong E_{\lambda^*} \otimes E_\lambda$ as a $G \times G$ module. Furthermore,*

$$\mathcal{O}[G] = \bigoplus_{\lambda \in P_{++}} V_\lambda. \tag{5}$$

In particular, $\mathcal{O}[G]$ is multiplicity free as a $G \times G$ module, while under the action of $G \times 1$ it decomposes into the sum of $\dim E_\lambda$ copies of E_λ for all $\lambda \in P_{++}$.

The function ψ_λ in Theorem 2.3 is called the *generating function* [Žel] for the representation π_λ. Since $\psi_\lambda(n^- h n^+) = h^\lambda$ and $N^- H N^+$ is dense in G, it is clear that

$$\psi_\lambda(g)\psi_\mu(g) = \psi_{\lambda+\mu}(g). \tag{6}$$

The semigroup P_{++} of dominant integral weights is free with generators $\lambda_1, \ldots, \lambda_l$, called the *fundamental weights*.

Proposition 2.4. (Product Formula) *Set $\psi_i(g) = \psi_{\lambda_i}(g)$. Let $\lambda \in P_{++}$ and write $\lambda = m_1\lambda_1 + \cdots + m_l\lambda_l$ with $m_i \in \mathbb{N}$. Then*

$$\psi_\lambda(g) = \psi_1(g)^{m_1} \cdots \psi_l(g)^{m_l} \quad \textit{for } g \in G. \tag{7}$$

Remark. From the product formula it is evident that the existence of a rational representation with highest weight λ is equivalent to the property that the functions $n^- h n^+ \mapsto h^{\lambda_i}$ on $N^- H N^+$ extend to regular functions on G for $i = 1, \ldots, l$.

Example. Suppose $G = \mathrm{SL}(n, \mathbb{C})$. Take B as the group of upper-triangular matrices. We may identify P with $\mathbb{Z}^n$, where $\lambda = [\lambda_1, \ldots, \lambda_n]$ gives the character

$$h^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad h = \operatorname{diag}[x_1, \ldots, x_n].$$

Then P_{++} consists of the monotone decreasing n-tuples and is generated by

$$\lambda_i = [\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0] \quad \text{for } i = 1, \ldots, n-1.$$

The fundamental representations are the exterior powers $E_{\lambda_i} = \bigwedge^i \mathbb{C}^n$ of the defining representation, for $i = 1, \ldots, n-1$. The generating function $\psi_i(g)$ is the ith principal minor of g. The Gauss decomposition (4) is the familiar LDU matrix factorization from linear algebra, and

$$N^- H N^+ = \{g \in \mathrm{SL}(n, \mathbb{C}) : \psi_i(g) \neq 0 \quad \text{for } i = 1, \ldots, n-1\}.$$

Let $K \subset G$ be a subgroup and let $\mathcal{O}[G]^{R(K)}$ be the *right* K-invariant regular functions on G (those functions f such that $f(gk) = f(g)$ for all $k \in K$). This subspace of $\mathcal{O}[G]$ is invariant under left translations by G.

Corollary 2.5. *Let E_λ^K be the subspace of K-fixed vectors in E_λ. Then*

$$\mathcal{O}[G]^{R(K)} \cong \bigoplus_{\lambda \in P_{++}} E_\lambda \otimes E_{\lambda^*}^K \tag{8}$$

as a G module under left translations, with G acting by $\pi_\lambda \otimes 1$ on the λ-isotypic summand. Thus the multiplicity of π_λ in $\mathcal{O}[G]^{R(K)}$ is $\dim E_{\lambda^*}^K$.

For any closed subgroup K of G whose Lie algebra is a complex subspace of $\mathfrak{g}$, the coset space G/K is a complex manifold on which G acts holomorphically, and the elements of $\mathcal{O}[G]^{R(K)}$ are holomorphic functions on G/K. When K is a *reductive* algebraic subgroup, then the manifold G/K also has the structure of an affine algebraic G-space such that the regular functions are exactly the elements of $\mathcal{O}[G]^{R(K)}$ (a result of Matsushima [Mat]; see also Borel and Harish-Chandra [Bo-Ha]). Also, when K is reductive then $\dim E_{\lambda^*}^K = \dim E_\lambda^K$, since the identity representation is self-dual.

The pair (G, K) is called *spherical* if

$$\dim E_\lambda^K \leq 1 \quad \text{for all } \lambda \in P_{++}.$$

In this case, we refer to K as a *spherical subgroup* of G. When K is reductive, this property is equivalent to G/K being a multiplicity-free G-space, by Corollary 2.5.

3 Complexifications of Compact Symmetric Spaces

3.1 Algebraic Version of Cartan Embedding

Cartan's paper [Car2] studies the decomposition of $C(U/K_0)$, where U is a compact real form of the simply-connected complex semisimple group G and $K_0 = U^\theta$ is the fixed-point set of an involutive automorphism θ of U. The compact symmetric space $X = U/K_0$ is simply-connected and hence the group K_0 is connected.[7] The involution extends uniquely to an algebraic group automorphism of G that we continue to denote as θ. The algebraic subgroup group $K = G^\theta$ is connected and is the complexification of K_0 in G, hence reductive. By Matsushima's theorem G/K is an affine algebraic variety. It can be embedded into G as an affine algebraic subset as follows (see [Ric1], [Ric2]):

Define

$$g \star y = gy\theta(g)^{-1}, \quad \text{for } g, y \in G.$$

We have $(g \star (h \star y)) = (gh) \star y$ for $g, h, y \in G$, so this gives an action of G on itself which we will call the *θ-twisted conjugation* action. Let

$$Q = \{y \in G \,:\, \theta(y) = y^{-1}\}.$$

Then Q is an algebraic subset of G. Since $\theta(g \star y) = \theta(g)y^{-1}g^{-1} = (g \star y)^{-1}$, we have $G \star Q = Q$.

[7] This theorem of Cartan extends Weyl's results for compact semisimple groups–see Borel [Bor, Chap. IV, §2].

Theorem 3.1 (Richardson)**.** *The θ-twisted action of G is transitive on each irreducible component of Q. Hence Q is a finite union of Zariski-closed θ-twisted G-orbits.*

The proof consists of showing that the tangent space to a twisted G-orbit coincides with the tangent space to Q.

Corollary 3.2. *Let $P = G \star 1 = \{g\theta(g)^{-1} \, : \, g \in G\}$ be the orbit of the identity element under the θ-twisted conjugation action. Then P is a Zariski-closed irreducible subset of G isomorphic to G/K as an affine G-space (relative to the θ-twisted conjugation action of G).*

There is a θ-stable noncompact real form G_0 of G so that K_0 is a maximal compact subgroup of G_0. The symmetric space G_0/K_0 is the *noncompact dual* to U/K_0. The *Cartan embedding* is the map $G_0/K_0 \to P_0 \subset G_0$, where $P_0 = G_0 \star 1 = \exp \mathfrak{p}_0$ and $\mathfrak{p}_0$ is the -1 eigenspace of θ in $\mathfrak{g}_0$ (P_0 is Cartan's space $\mathcal{E}$–see Borel [Bor, Ch. IV, §2.4]).

3.2 Classical Examples

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a connected classical group whose Lie algebra is simple. The involutions and associated symmetric spaces G/K for G can be described in terms of the following three kinds of geometric structures on $\mathbb{C}^n$ (in the second and third type, G is the isometry group of the form and K is the subgroup preserving the indicated decomposition of $\mathbb{C}^n$):

(1) **nondegenerate bilinear forms** $G = \mathrm{SL}(n, \mathbb{C})$ and $K = \mathrm{SO}(n, \mathbb{C})$ or $\mathrm{Sp}(n, \mathbb{C})$

(2) **polarizations** $\mathbb{C}^n = V_+ \oplus V_-$ with $V_\pm$ totally isotropic subspaces for a bilinear form (zero or nondegenerate)

(3) **orthogonal decompositions** $\mathbb{C}^n = V_+ \oplus V_-$ with $V_\pm$ nondegenerate subspaces for a nondegenerate bilinear form

The proof that these structures give all the possible involutive automorphisms of the classical groups (up to inner automorphisms) can be obtained from following characterization of automorphisms of the classical groups:

Proposition 3.3. *Let σ be a regular automorphism of the classical group G.*

(1) *If $G = \mathrm{SL}(n, \mathbb{C})$ then there exists $s \in G$ so that σ is either $\sigma(g) = sgs^{-1}$ or $\sigma(g) = s(g^t)^{-1}s^{-1}$.*

(2) *If G is $\mathrm{Sp}(n, \mathbb{C})$ then there exists $s \in G$ so that $\sigma(g) = sgs^{-1}$.*

(3) *If G is $\mathrm{SO}(n, \mathbb{C})$ with $n \neq 2, 4$, then there exists $s \in \mathrm{O}(n, \mathbb{C})$ so that $\sigma(g) = sgs^{-1}$.*

Proof. The Weyl dimension formula implies that the defining representation (and its dual, in the case $G = \mathrm{SL}(n, \mathbb{C})$) is the unique representation of smallest dimension. So this representation is sent to an equivalent representation (or its dual) by σ. The existence of the element s follows from this equivalence (see [Go-Wa, §11.2.4] for details).[8] ♦

Example. Let $G = \mathrm{SL}(n, \mathbb{C})$ and $\theta(g) = (g^t)^{-1}$. Then

$$K = \mathrm{SO}(n, \mathbb{C}), \quad U = \mathrm{SU}(n), \quad K_0 = \mathrm{SO}(n), \quad G_0 = \mathrm{SL}(n, \mathbb{R}).$$

Also $g \star y = gyg^t$ and $Q = \{y \in G \, : \, y^t = y\} = P$, so there is one orbit. Hence the map $gK \mapsto gg^t$ gives the algebraic embedding

$$\mathrm{SL}(n, \mathbb{C})/\,\mathrm{SO}(n, \mathbb{C}) \cong \{y \in M_n(\mathbb{C}) \, : \, y = y^t, \, \det y = 1\}.$$

For the other classical examples, see Goodman–Wallach [Go-Wa, §11.2.5].

3.3 Complexified Iwasawa Decomposition

The real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ into $+1$ and -1 eigenspaces of the Cartan involution θ. The noncompact real group G_0 has an Iwasawa decomposition [9] $G_0 = K_0 A_0 N_0$. Here $A_0 = \exp \mathfrak{a}_0$ is a vector group with $\mathfrak{a}_0$ a maximal abelian subspace of $\mathfrak{p}_0$, and N_0 is a nilpotent subgroup normalized by A_0. Let A and N be the complexifications of A_0 and N_0 in G, respectively. Then A is a complex algebraic torus of rank l (the *rank* of G/K) and N is a unipotent subgroup. There is a θ-stable Cartan subgroup H of G such that $A \subset H$ and the following holds (see Vust [Vus] for the general case and Goodman-Wallach [Go-Wa, §12.3.1] for the classical groups):

(1) KAN is a Zariski-dense subset of G.

(2) The subgroup $M = \mathrm{Cent}_K(A)$ is reductive and normalizes N.

(3) Let $T = H \cap K$. Then $H = AT$ and $A \cap T$ is finite.

(4) There exists a Borel subgroup B with $HN \subset B \subset MAN$.

Thus MAN is a parabolic subgroup of G with reductive Levi component MA and unipotent radical N. We will give a more precise description of the set KAN in the next section.

[8] This type of result was one motivation for Weyl to learn Cartan's theory of representations of semisimple Lie groups–see Borel[Bor, Chap. III, §1] for more details.

[9] When $G_0 = \mathrm{SL}(n, \mathbb{R})$ this decomposition is the so-called QR factorization of a matrix obtained by the Gram-Schmidt orthogonalization algorithm.

4 Representations on Symmetric Spaces

4.1 Spherical Representations

We continue with the same setting and notation as in Section 3.3; in particular, P_{++} is the set of B-dominant weights. If $\lambda \in P_{++}$ and $E_\lambda^K \neq 0$ then λ will be called a K *spherical* highest weight and E_λ a K *spherical* representation.

Proposition 4.1. (i) *K is a spherical subgroup of G.*

(ii) *Let $T = H \cap K$. If $\lambda \in P_{++}$ is a K spherical highest weight, then*

$$t^\lambda = 1 \quad \textit{for all } t \in T. \tag{9}$$

Proof. Since B contains AN, the Iwasawa decomposition shows that BK is dense in G, so B has an open orbit on G/K. Hence K is a spherical subgroup by Theorem 2.2. Since T is the stabilizer in H of the point $K \in G/K$, condition (9) likewise holds. ♦

We say that λ is *θ-admissible* if it satisfies (9).

Example. Let $G = \mathrm{SL}(n, \mathbf{C})$ and $\theta(g) = (g^t)^{-1}$. Here $A = H$ (diagonal matrices in G), $N =$ all upper-triangular unipotent matrices, and $M = T \cong (\mathbf{Z}/2\mathbf{Z})^{n-1}$ consists of all matrices

$$t = \mathrm{diag}[\delta_1, \ldots, \delta_n], \quad \delta_i = \pm 1,\ \det(t) = 1.$$

Hence the θ-admissible highest weights $\lambda = [\lambda_1, \ldots, \lambda_{n-1}, 0]$ are those with λ_i *even* for all i. ♦

Remark. In general, the subgroup $F = T \cap A$ is finite and consists of elements of order 2, since $h = \theta(h) = h^{-1}$ for $h \in F$. Thus a θ-admissible highest weight λ is trivial on T and its restriction to A is *even*, in the sense that $h^\lambda = 1$ for $h \in F$.

Cartan [Car2] proved the implication (i) $\Longrightarrow$ (iii) in the following theorem and gave some indications for the proof of the converse (see Borel [Bor, Chap. IV §4.4-5]). Thus the following result is sometimes called the *Cartan–Helgason* theorem, although part (ii) and the first complete proof of the theorem is due to Helgason [Hel1].

Theorem 4.2. *Let (π_λ, E_λ) be an irreducible rational representation of G with highest weight λ (relative to B). The following are equivalent:*

(i) $E_\lambda^K \neq 0$.

(ii) *MN fixes the B-highest weight vector in E_λ.*

(iii) *λ is θ-admissible.*

Proof. The equivalence of (ii) and (iii) follows by a Lie algebra argument using $\mathfrak{sl}_2$ representation theory (see [Hel1] or [Go-Wa, §12.3.3]), and the implication (i) $\Longrightarrow$ (iii) comes from Proposition 4.1. We give Helgason's analytic proof that (iii) $\Longrightarrow$ (i).[10] Let λ be θ-admissible. Define

$$v_0 = \int_{K_0} \pi_\lambda(k) e_\lambda \, dk. \tag{10}$$

Then $v_0 \in (E_\lambda)^K$ by the unitarian trick, since K is connected. To show $v_0 \neq 0$, let ψ_λ be the generating function for π_λ. Then

$$\langle v_0,\, f_{\lambda^*}\rangle = \int_{K_0} \psi_\lambda(k)\, dk. \tag{11}$$

We use the following properties:

(1) Let σ be the complex conjugation of G whose fixed-point set is G_0. Then $\chi(\sigma(a)) = \overline{\chi(a)}$ for any regular character χ of A.

(2) If $h \in H \cap G_0 = (T \cap G_0)A_0$, then $h^\lambda > 0$ by (1), since $h = t\exp(x)$ with $t \in T$ and $x \in \mathfrak{a}_0$.

(3) $\psi_\lambda(g) \geq 0$ for $g \in G_0$ by (2) and the Gauss decomposition.

Since $\psi_\lambda(1) = 1$ and $K_0 \subset G_0$, property (3) shows that the integral (11) is nonzero. ♦

Example. Let $G = \mathrm{SL}(n, \mathbb{C})$ and $\theta(g) = (g^t)^{-1}$. Here A_0 consists of real diagonal matrices, $G_0 = \mathrm{SL}(n, \mathbb{R})$, and

$$\psi_\lambda(g) = \det{}_1(g)^{m_1} \cdots \det{}_{n-1}(g)^{m_{n-1}},$$

where $\det_i$ is the ith principal minor and $m_i = \lambda_i - \lambda_{i+1}$. Since λ is θ-admissible iff all λ_i are even, condition (3) in the proof of Theorem 4.2 obviously holds. For example, the highest weight $\lambda = [2, 0, \ldots, 0]$ is admissible, and the corresponding spherical representation $E_\lambda = S^2(\mathbb{C}^n)$. The K-fixed vector in E_λ is $\sum_i e_i \otimes e_i$, where $\{e_i\}$ is the standard basis for $\mathbb{C}^n$. ♦

The l *fundamental* K-spherical highest weights $\mu_1, \ldots, \mu_l$ (with $l = \dim A$ the rank of G/K) are linearly independent, and the general spherical highest weight is $\mu = m_1\mu_1 + \cdots + m_l\mu_l$ with $m_i \in \mathbb{N}$ (see [Hel2, Ch. V, §4]). Let $\Lambda \subset P_{++}$ be the subsemigroup of spherical highest weights. Since K is reductive and the identity representation is self-dual, $E_\lambda^K \neq 0$ if and only if $E_{\lambda^*}^K \neq 0$. Hence Λ is invariant under the map $\lambda \mapsto \lambda^*$ on P_{++}.

Corollary 4.3. *As a G-module,* $\mathcal{O}[G/K] \cong \bigoplus_{\mu \in \Lambda} E_\mu$.

[10] An algebraic-geometric proof was given later by Vust [Vus].

4.2 Zonal Spherical and Horospherical Functions

For each $\mu \in \Lambda$ choose a K-fixed *spherical vector* $e_\mu^K \in E_\mu$ and a MN-fixed *conical vector* $e_\mu \in E_\mu$, normalized so that

$$\langle e_\mu, e_{\mu^*}^K \rangle = 1, \qquad \langle e_\mu^K, e_{\mu^*}^K \rangle = 1. \tag{12}$$

The *zonal spherical function* $\varphi_\mu \in \mathcal{O}[G]$ is the representative function determined by pairing the K-fixed vectors in E_μ and E_{μ^*}:

$$\varphi_\mu(g) = \langle \pi_\mu(g) e_\mu^K, e_{\mu^*}^K \rangle.$$

From the definition it is clear that

$$\varphi_\mu(kgk') = \varphi_\mu(g) \quad \text{and} \quad \varphi_\mu(1) = 1$$

for $k, k' \in K$ and $g \in G$. Thus φ_μ is a regular function on G/K that is constant on the K-orbits.

The *zonal horospherical function* $\Delta^\mu \in \mathcal{O}[G]$ is the representative function determined by pairing the MN-fixed vector in E_μ with the K-fixed vector in E_{μ^*}:

$$\Delta^\mu(g) = \langle \pi_\mu(g) e_\mu, e_{\mu^*}^K \rangle.$$

From the definition it is clear that

$$\Delta^\mu(kgman) = a^\mu \Delta^\mu(g) \quad \text{and} \quad \Delta(1) = 1 \tag{13}$$

for $k \in K$, $g \in G$, and $man \in MAN$. Properties (13) with $g = 1$ determine Δ^μ uniquely, since KAN is dense in G. We can view Δ^μ as a holomorphic function on the affine symmetric space $K\backslash G$ that transforms by the character $man \mapsto a^\mu$ along the MAN orbits. The existence of a regular function on G with these transformation properties is equivalent to the existence of the K-spherical representation π_μ (just as for the generating functions ψ_λ in Section 2.2, which are the zonal horospherical functions associated with the diagonal embedding of G as a spherical subgroup of $G \times G$). Let μ and ν be K-spherical highest weights. From (13) and the density of the set $KMAN$ it follows that

$$\Delta^\mu(g)\Delta^\nu(g) = \Delta^{\mu+\nu}(g) \quad \text{for } g \in G. \tag{14}$$

Let $\mu_1, \ldots, \mu_l$ be the fundamental K-spherical highest weights, and define[11]

$$\Delta_j(g) = \Delta^{\mu_j}(g).$$

For a general K-spherical highest weight $\mu = m_1\mu_1 + \cdots + m_l\mu_l$ formula (14) implies the *product formula*

$$\Delta^\mu(g) = \Delta_1(g)^{m_1} \cdots \Delta_l(g)^{m_l}. \tag{15}$$

[11] Gindikin [Gin3] calls $\{\Delta_j\}$ the *Sylvester functions*; Theorem 4.4 shows they play the same role for the KAN decomposition as the generating functions $\{\psi_j\}$ for the N^-HN^+ decomposition.

Set

$$\Omega = \{g \in G \,:\, \Delta_j(g) \neq 0 \quad \text{for } j = 1, \ldots, l\}.$$

The weight μ is *regular* if $m_i \neq 0$ for $i = 1, \ldots, l$. If μ is regular, then we see from (15) that $\Omega = \{g \in G \,:\, \Delta^{\mu}(g) \neq 0\}$. Using techniques originating with Harish-Chandra [H-C], Clerc [Cle] obtained the following precise description of the complexified Iwasawa decomposition:

Theorem 4.4. *One has $\Omega = KAN$. Let $g = k(g)a(g)n(g)$ be the Iwasawa factorization in G_0.*

(i) *The function $g \mapsto n(g)$ extends holomorphically to a map from Ω to N.*

(ii) *The functions $g \mapsto k(g)$ and $g \mapsto a(g)$ extend to multivalent holomorphic functions on Ω, with values in K and A, respectively. The branches are related by elements of the finite subgroup $F = T \cap A$.*

(iii) *Let $g \mapsto \mathcal{H}(g)$ be the multivalent $\mathfrak{a}$-valued function on Ω such that $a(g) = \exp \mathcal{H}(g)$. Then*

$$\Delta^{\mu}(g) = e^{\langle \mathcal{H}(g),\, \mu\rangle} \quad \textit{for } g \in \Omega \textit{ and } \mu \in \Lambda.$$

Theorem 4.4 and (10) yield a formula analogous to Harish-Chandra's integral formula [H-C] for zonal spherical functions on the noncompact symmetric space G_0/K_0:

Corollary 4.5. *For $g \in G$ let $K_g = \{k \in K_0 \,:\, gk \in \Omega\}$. Then K_g is an open set in K_0 whose complement has measure zero. For $\mu \in \Lambda$ one has*

$$\varphi_\mu(g) = \int_{K_g} e^{\langle \mathcal{H}(gk),\, \mu\rangle}\, dk.$$

Clerc, elaborating on methods introduced by E. P. Van den Ban [VdBan], uses this integral representation and the method of complex stationary phase to determine the asymptotic behavior of $\varphi_\mu(u)$ as $\mu \to \infty$ in a suitable cone when u is a regular element of U; see [Cle, Théorème 3.4] for details.

4.3 Horospherical Cauchy–Radon Transform

By Theorem 4.2 the G-modules $\mathcal{O}[G]^{R(K)}$ and $\mathcal{O}[G]^{R(MN)}$ are multiplicity free and have the same spectrum (the set Λ of K-spherical highest weights). Using the normalized K-fixed vectors and MN-fixed highest weight vectors, we can thus define bijective G-intertwining maps

$$T : \bigoplus_{\mu\in\Lambda} E_\mu \xrightarrow{\cong} \mathcal{O}[G]^{R(K)}, \qquad \sum_{\mu\in\Lambda} v_\mu \mapsto \sum_{\mu\in\Lambda} d(\mu)\langle v_\mu,\, \pi_{\mu^*}(g)e^K_{\mu^*}\rangle$$

and

$$S : \bigoplus_{\mu\in\Lambda} E_\mu \xrightarrow{\cong} \mathcal{O}[G]^{R(MN)}, \qquad \sum_{\mu\in\Lambda} v_\mu \mapsto \sum_{\mu\in\Lambda} \langle v_\mu,\, \pi_{\mu^*}(g)e_{\mu^*}\rangle.$$

In both cases we assume that the components $v_\mu = 0$ for all but finitely many $\mu \in \Lambda$.

Let $f \in \mathcal{O}[G]^{R(K)}$. The (algebraic) Peter–Weyl expansion of f is

$$f(g) = \sum_{\mu\in\Lambda} d(\mu)\langle v_\mu,\, \pi_{\mu^*}(g)e^K_{\mu^*}\rangle \tag{16}$$

where $v_\mu \in E_\mu$ and $v_\mu = 0$ for all but finitely many μ. Here $d(\mu) = \dim E_\mu$. Following Gindikin [Gin2], we define the *horospherical Cauchy–Radon transform* $f \mapsto \hat{f}$ by

$$\hat{f}(g) = \sum_{\mu\in\Lambda} \langle v_\mu,\, \pi_{\mu^*}(g)e_{\mu^*}\rangle$$

Note that the dimension factor is removed, and the spherical vector is replaced by the conical vector in E_{μ^*}. It is easy to check that this definition does not depend on the choice of spherical and conical vectors, subject to the normalizations (12). We can express this transform in terms of the maps S and T just introduced as follows: If $v \in \bigoplus_\mu E_\mu$ and $f = Tv$, then $\hat{f} = Sv$. Since S and T are G-module isomorphisms, it follows that the map $f \mapsto \hat{f}$ gives a G-module isomorphism between the function spaces $\mathcal{O}[G]^{R(K)}$ and $\mathcal{O}[G]^{R(MN)}$. We now express this isomorphism in a more analytic form.

Theorem 4.6. *The horospherical Cauchy–Radon transform is given by the integral formula*

$$\hat{f}(g) = \sum_{\mu\in\Lambda} \int_U f(u)\Delta^\mu(u^{-1}g)\,du \qquad \textit{for } g \in G \tag{17}$$

(the integrals are zero for all but finitely many μ).

Remark. The integrands in (17) are invariant under $u \mapsto uk$ with $k \in K_0$, so the integrals can be viewed as taken over the compact symmetric space $X = U/K_0$.

Proof. Let f be given by (16) and let $\mu \in \Lambda$. Since f is right K-invariant and $E^K_\mu = \mathbb{C}e^K_\mu$, we have

$$\int_U f(u)\pi_\mu(u^{-1}g)e_\mu\,du = c_\mu(g)e^K_\mu \tag{18}$$

for some function $c_\mu(g)$ on G. From the Schur orthogonality relations and (12) we find that

$$c_\mu(g) = \langle v_\mu,\, \pi_{\mu^*}(g)e_{\mu^*}\rangle.$$

Evaluating both sides of (18) on the vector $e^K_{\mu^*}$, and summing on μ, we obtain (17). ♦

Remarks. 1. The horospherical Cauchy–Radon transform is the representation-theoretic expression of the double fibration

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ Z = G/K & & G/MN = \Xi \end{array}$$

This sets up a *correspondence* between Z and Ξ: a point gK of Z maps to the *pseudosphere* $gKMN \cong K/M$ in Ξ, and a point gMN in Ξ maps to the *horosphere* $gMNK \cong N$ in Z (see Gindikin [Gin1] for some examples).

2. Let $\bar{N} = \theta(N)$. Then $\bar{N}MAN$ is Zariski-dense in G (the generalized Gauss decomposition) and $A\bar{N}K$ is also Zariski-dense in G (the Iwasawa decomposition). Thus the solvable group $A\bar{N}$ has an open orbit in G/K and in G/MN, but the two homogeneous spaces are not isomorphic as complex manifolds, even though they have the same G spectrum and multiplicities.

An invariant (holomorphic) differential operator $P(D)$ on A has a polynomial symbol $P(\mu)$ such that

$$P(D)a^{\mu} = P(\mu)a^{\mu} \quad \text{for } \mu \in \Lambda.$$

If μ is a K-spherical highest weight, then the Weyl dimension formula asserts that

$$d(\mu) = \prod_{\alpha>0} \frac{(\mu+\delta, \alpha)}{(\delta, \alpha)} \qquad \text{where } \delta = \tfrac{1}{2}\textstyle\sum_{\alpha>0} \alpha.$$

Since $\mu = 0$ on $\mathfrak{t}$, we can view $\mu \mapsto d(\mu)$ as a polynomial function $W(\mu)$ on $\mathfrak{a}^*$. Following Gindikin [Gin3], we define the *Weyl operator* $W(D)$ to be the differential operator on A with symbol $W(\mu)$.

Since A normalizes MN, the space $\mathcal{O}[G]^{R(MN)}$ is stable under $R(A)$. The complex horospherical manifold Ξ is a fiber bundle over the compact flag manifold $F = G/MAN$ (a projective variety), with fiber A. The operator $W(D)$ acts by differentiation along the fibers.

Using the Weyl operator, Gindikin [Gin2] obtains the following inversion formula for the horospherical Cauchy–Radon transform:

Theorem 4.7. *Let $f \in \mathcal{O}[G]^{R(K)}$. Then*

$$f(g) = \int_{K_0} (W(D)\hat{f})(gk)\, dk \qquad \textit{for } g \in G \tag{19}$$

Remark. The integrand in (19) is invariant under right translations by M_0, so the integral is taken over the compact flag manifold $K_0/M_0 = G_0/M_0A_0N_0$ associated with the dual noncompact symmetric space.

Proof. It suffices to prove (19) when $f(g) = d(\mu)\langle v_\mu,\, \pi_{\mu^*}(g)e^K_{\mu^*}\rangle$ with $v_\mu \in E_\mu$. In this case,

$$\hat{f}(ga) = \langle v_\mu,\, \pi_{\mu^*}(ga)e_{\mu^*}\rangle = a^{\mu^*}\langle v_\mu,\, \pi_{\mu^*}(g)e_{\mu^*}\rangle.$$

Hence $W(D)\hat{f}(g) = d(\mu)\hat{f}(g)$ since $d(\mu) = d(\mu^*)$. Thus

$$\int_{K_0} (W(D)\hat{f})(gk)\, dk = d(\mu) \int_{K_0} \langle v_\mu,\, \pi_{\mu^*}(g)\pi_{\mu^*}(k)e_{\mu^*}\rangle\, dk = f(g),$$

since the integration of $\pi_{\mu^*}(k)e_{\mu^*}$ over K_0 yields $e^K_{\mu^*}$. ♦

4.4 Cauchy–Radon Transform as a Singular Integral

Denote by $Z = G/K$ the complex symmetric space with origin $x_0 = K$. Let $\zeta_0 = MN$ denote the origin in Ξ. For $z = g \cdot x_0 \in Z$ and $\zeta = y \cdot \zeta_0 \in \Xi$ we set $\Delta_j(z \mid \zeta) = \Delta_j(g^{-1}y)$. This is well-defined by the transformation properties (13), and we have

$$\Delta_j(z \mid \zeta a) = a^{\mu_j} \Delta_j(z \mid \zeta) \quad \text{for } a \in A.$$

Following Gindikin ([Gin2], [Gin3]), we define the *Cauchy–Radon kernel* on $Z \times \Xi$ by

$$K(z \mid \zeta) = \prod_{1 \le j \le l} \frac{1}{1 - \Delta_j(z \mid \zeta)} \,.$$

This function is meromorphic and invariant under the diagonal action of G, since $\Delta_j(g \cdot z \mid g \cdot \zeta) = \Delta_j(z \mid \zeta)$ for $g \in G$. The singular set of $K(z \mid \zeta)$ is the union of the manifolds $\{\Delta_j(z \mid \zeta) = 1\}$ in $Z \times \Xi$ for $j = 1, \dots, l$.

Recall that $X = U/K_0$ is the compact symmetric space corresponding to θ. Define

$$\Xi(0) = \{\zeta \in \Xi \,:\, |\Delta_j(x \mid \zeta)| < 1 \quad \text{for all } x \in X\}.$$

By definition, $U \cdot \Xi(0) = \Xi(0)$. Furthermore, the product formula (15) implies that

$$K(x \mid \zeta) = \sum_{\mu \in \Lambda} \Delta^\mu(u^{-1}g) \quad \text{(absolutely convergent series)} \tag{20}$$

for $x = u \cdot x_0 \in X$ and $z = g \cdot \zeta_0 \in \Xi(0)$. Since A normalizes the subgroup MN, the right multiplication action of A on G gives a right action of A on Ξ, denoted by $\zeta, a \mapsto \zeta \cdot a$. This action commutes with the left action of G on Ξ.

Lemma 4.8. (i) *The map $(U/M_0) \times A \to \Xi$ given by $(u, a) \mapsto u \cdot \zeta_0 \cdot a$ is regular and surjective.*

(ii) *Let $A_+ = \{a \in A \,:\, |a^{\mu_j}| < 1$ for $j = 1, \dots, l\}$. Then $U \cdot \zeta_0 \cdot A_+ \subset \Xi(0)$. Hence $\Xi(0)$ is a nonempty open subset of Ξ.*

Proof. Since U is a maximal compact subgroup of G, the Iwasawa decomposition of G shows that $G = UMAN$. This implies (i).

Clerc [Cle, Lemme 2.3], using a representation-theoretic argument originating with Harish-Chandra [H-C], shows that $|\Delta^\mu(u)| \le 1$ for $\mu \in \Lambda$ and $u \in U$. Let $a \in A$. Then Clerc's estimate implies that

$$|\Delta^\mu(ua)| = |\Delta^\mu(u)|\, |a^\mu| \le |a^\mu|. \tag{21}$$

If $a \in A_+$ then $|a^\mu| < 1$. Hence for $u, u' \in U$ we have

$$|\Delta^\mu(u \cdot x_0 \mid u'a \cdot \zeta_0)| = |\Delta^\mu(u^{-1}u'a)| < 1$$

by (21). This implies (ii). ♦

Using Lemma 4.8, we can obtain Gindikin's singular integral formula for the horospherical Cauchy-Radon transform. The noncompact real symmetric space G/U is the space of compact real forms of G, and by the Cartan decomposition of G it is a contractible manifold. For $\nu = gU \in G/U$ we define a compact totally-real cycle $X(\nu) = g \cdot X \subset Z$ and an open set $\Xi(\nu) = g \cdot \Xi(0) \subset \Xi$. This furnishes an open covering

$$\Xi = \bigcup_{\nu \in G/U} \Xi(\nu)$$

with a contractible parameter space.

Theorem 4.9 (Gindikin)**.** *For $f \in \mathcal{O}[Z]$ the horospherical Cauchy–Radon transform is given on each set of the covering $\{\Xi(\nu)\}$ by the Cauchy-type singular integral*

$$\hat{f}(\zeta) = \int_{X(\nu)} f(x) K(x \mid \zeta)\, dx \quad \textit{for} \quad \zeta \in \Xi(\nu) \tag{22}$$

(the integrand is continuous on $X(\nu)$).

Proof. Use formula (20) for $K(x \mid \zeta)$ when $\zeta \in \Xi(0)$, and then translate by $g \in G$ to get the formula in general. ♦

5 Concluding Remarks

In this paper we described the harmonic analysis of finitely-transforming functions on a compact symmetric space using algebraic group and Lie group methods, extending the fundamental results of Cartan and Weyl. Our presentation of the horospherical Cauchy-Radon transform has emphasized groups and homogeneous spaces as in [Gin2]; in fact, the integral formulas hold for all holomorphic functions (not just the G-finite functions) on X and Ξ, and also for hyperfunctions. Gindikin's point of view is that a compact symmetric space has a canonical dual object that is a complex manifold, and he develops this transform emphasizing complex analysis and integral geometry (see [Gin3]).

An analytic problem that we have not discussed is the holomorphic extension of real analytic functions on a compact symmetric space. These functions extend holomorphically to complex neighborhoods of the space. The geometric and analytic properties of these neighborhoods were studied by B. Beers and A. Dragt [Be-Dr], L. Frota-Mattos [Fr-Ma] and M. Lasalle [Las].

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