An Algebraic Group Approach to Compact Symmetric Spaces *†

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Abstract

The most classical example of noncommutative harmonic analysis is the theory of spherical harmonics:

(a) Under the action of the special orthogonal group G in n variables, the space of square-integrable functions on the $(n-1)$-sphere decomposes as an orthogonal direct sum of mutually inequivalent irreducible subspaces.

(b) The algebra of polynomials in n variables decomposes as the tensor product of the *G-invariant polynomials* and the *harmonic polynomials* (separation of variables).

(c) The restrictions of the harmonic polynomials to the sphere gives all the irreducible G-invariant spaces in (a).

In this paper we describe how (a) and (b) carry over to harmonic analysis on a compact Riemannian symmetric space and its tangent space. Our approach is through complex algebraic groups; we embed the compact symmetric space in a complexification which is an affine algebraic subset of the complexified isometry group of the space. We describe this embedding in explicit matrix form for the symmetric spaces of *classical type*, which are given by three elementary linear algebra constructions. Property (a) generalizes to the context of *multiplicity-free* spaces, and we obtain Helgason's theorem characterizing the highest weights of the irreducible spherical representations. The harmonic analysis of polynomial functions on the tangent space to a symmetric space was carried out by Kostant and Rallis as a generalization of (b) and (c). We describe a new proof of their basic theorem and illustrate it in all the cases where the isotropy group is a classical group.

1 Introduction

In the introduction to [Car], É. Cartan says that his paper was inspired by the paper of F. Peter and H. Weyl on harmonic analysis on compact groups [Pe-We], but he points out that for a compact Lie group Weyl's use of integral equations 'gives a transcendental solution to

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†The results in this paper are part of the book *Representations and Invariants of the Classical Groups*, Cambridge U. Press (1997). The `*.dvi` files of this paper and the book are available by anonymous FTP to `math.rutgers.edu`. After logging in, change directory to `/pub/goodman` and see the `README` file for further information.

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a problem of an algebraic nature' (namely, the completeness of the set of finite-dimensional irreducible representations of the group). Cartan's goal is 'to give an algebraic solution to a problem of a transcendental nature, more general than that treated by Weyl' (namely, finding an explicit decomposition of the space of all L^2 functions on a homogeneous space into an orthogonal direct sum of group-invariant irreducible subspaces).

In this paper we have a similar goal. Recall that the space of functions on the circle with finite Fourier series can be identified with the algebra of finite Laurent polynomials $\mathbb{C}[z, z^{-1}]$ (where $z = e^{i\theta}$). In a similar way, the 'finite' functions on a homogeneous space for a compact connected Lie group (that is, the functions whose translates span a finite-dimensional subspace) can be viewed as polynomial ('regular') functions on the complexified group (a complex reductive algebraic group). By Weyl's 'unitarian trick' the irreducible subspaces of functions under the action of the compact group correspond to irreducible subspaces of regular functions on the complex reductive group. The advantage of this correspondence is that we can then apply algebraic group techniques to show that for a symmetric space the irreducible representations occur with multiplicity one. We also determine the highest weights of these representations using a mixture of algebraic and transcendental methods (this result was first obtained in complete generality by Helgason [Hel2]).

In the last sections of the paper we describe the corresponding results for the decomposition of the tangent-space ('isotropy') representation of a symmetric space. Here the group action on the underlying space is now linear, but the multiplicities of the irreducible spaces of polynomials are not one. Just as in the case of spherical harmonics there is a tensor product decomposition into invariant polynomials (the analogue of 'radial' functions) and functions on a homogeneous space for the isotropy group (the analogue of spherical harmonics). However, the homogeneous space for the isotropy group is generally not symmetric and not multiplicity-free, unlike the classical case of the sphere.

There is an interesting 'transcendental' problem that we do not discuss here. Just as in the case of functions on the circle, the functions on a compact symmetric space that are real-analytic but not 'finite' extend holomorphically to a complex neighborhood of the space. The geometric and analytic propreties of these neighborhoods have been studied by Beers-Dragt [Be-Dr], Frota-Mattos [Fr-Ma] and Lassalle [Las]. The explict matrix models for the complexifications of the classical symmetric spaces given in this paper were not used in the cited papers, however. It would be interesting to reexamine this question in the context of these matrix domains.

2 Representations on $\mathrm{Aff}(X)$

2.1 Isotypic Decomposition

Let G be a connected complex reductive algebraic group, and let X be an affine algebraic set on which G acts regularly. We write $\mathrm{Aff}(X)$ for the algebra of *regular functions* on X (the algebra often denoted by $\mathbb{C}[X]$) and we denote by ρ_X the associated representation of G on $\mathrm{Aff}(X)$, given by

$$\rho_X(g)f(x) = f(g^{-1}x), \quad \text{for } f \in \mathrm{Aff}(X).$$

For example, when X is a finite-dimensional vector space and G acts linearly, then $\mathrm{Aff}(X) = \mathcal{P}(X)$, the polynomial functions on X, and G preserves the spaces of homogeneous polynomials.

Fix a Borel subgroup $B = HN$ of G, with H a maximal torus in G and N the unipotent radical of B. Taking $G \subset \mathrm{GL}(n, \mathbb{C})$, we can always conjugate G so that H consists of the diagonal matrices in G and N consists of the upper-triangular unipotent matrices in G. Write $P(G) \subset \mathfrak{h}^*$ for the *weight lattice* of G and $P_{++}(G)$ for the *dominant weights*, relative to the system of positive roots determined by N (since the Borel subgroups in G are all conjugate, the notations $P(G)$ and $P_{++}(G)$ are unambiguous once B is fixed). For $\lambda \in P(G)$ we denote by $h \mapsto h^\lambda$ the corresponding character of H. We extend this to a character of B by setting $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N$.

An irreducible regular representation (π, V) of G is then determined (up to equivalence) by its *highest weight*. The subspace V^N of N-fixed vectors in V is one-dimensional, and H acts on it by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}(G)$. For each such λ we fix a model (π^λ, V^λ) for the irreducible representation with highest weight λ. Let $\mathrm{Aff}(X)^N$ be the space of N-fixed regular functions on X. For every character $b \mapsto b^\lambda$ of B, let $\mathrm{Aff}(X)^N(\lambda)$ be the N-fixed regular functions f of weight λ:

$$\rho_X(b)f = b^\lambda f \qquad \text{for } b \in B. \tag{1}$$

We can then describe the G-isotypic decomposition of $\mathrm{Aff}(X)$ as follows.

Theorem 2.1 *For $\lambda \in P_{++}(G)$, the isotypic subspace of type π^λ in $\mathrm{Aff}(X)$ is the span of $\rho_X(G)\mathrm{Aff}(X)^N(\lambda)$. This subspace is isomorphic to $V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$ as a G-module, with action $\pi^\lambda(g) \otimes 1$. Thus*

$$\mathrm{Aff}(X) \cong \bigoplus_{\lambda \in P_{++}(G)} V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$$

This theorem shows that the G-multiplicities in $\mathrm{Aff}(X)$ are the dimensions of the spaces $\mathrm{Aff}(X)^N(\lambda)$. We have $\mathrm{Aff}(X)^N(\lambda) \cdot \mathrm{Aff}(X)^N(\mu) \subset \mathrm{Aff}(X)^N(\lambda + \mu)$ under pointwise multiplication. Hence the set

$$\mathcal{S}(X) = \{\lambda \in P_{++}(G) \,:\, \mathrm{Aff}(X)^N(\lambda) \neq 0\} \quad \text{(the *spectrum* of } X\text{)}$$

is an additive semigroup. The theorem above shows that this semigroup completely determines the G-isotypic decomposition of $\mathrm{Aff}(X)$.

2.2 Function Models for Irreducible Representations

Let $\bar{N}$ be the unipotent group *opposite* to N; if we take a matrix form of G so that $G = G^t$, H is diagonalized and N is upper-triangular, then $\bar{N} = N^t$. We shall obtain all the irreducible representations of G as representations induced from characters of the Borel subgroup $\bar{B} = H\bar{N}$.

We begin with the representation of G on the function space

$$\mathcal{R}(\bar{N}\backslash G) = \{f \in \mathrm{Aff}(G) \,:\, f(\bar{n}g) = f(g) \text{ for } \bar{n} \in \bar{N}\}.$$

The G-action is by right translation. We decompose this space into irreducible subspaces as follows. For $\lambda \in P_{++}(G)$ let

$$\phi_\lambda : V^{\lambda^*} \otimes V^\lambda \to \mathrm{Aff}(G), \qquad \phi_\lambda(v^* \otimes v)(g) = \langle v^*, \pi^\lambda(g)v\rangle$$

Choose an N-fixed vector $v_\lambda \in V^\lambda$ and a $\bar{N}$-fixed vector $v_\lambda^* \in V^{\lambda^*}$, normalized so that

$$\langle v_\lambda^*, v_\lambda\rangle = 1.$$

This can be done, since v_λ^* has weight $-\lambda$ and so is orthogonal to all weight spaces in V^λ except $\mathbb{C}v_\lambda$.

Theorem 2.2 *The space $\mathcal{R}(\bar{N}\backslash G)$ contains every irreducible regular representation of G exactly once:*

$$\mathcal{R}(\bar{N}\backslash G) = \bigoplus_{\lambda \in P_{++}(G)} \phi_\lambda(v_\lambda^* \otimes V^\lambda). \tag{2}$$

From the decomposition of $\mathcal{R}(\bar{N}\backslash G)$ we obtain the following function models for the irreducible representations of G.

Theorem 2.3 (Borel-Weil) *Let $\lambda \in P_{++}(G)$. Let $\mathcal{R}_\lambda \subset \mathrm{Aff}(G)$ be the subspace of functions such that*

$$f(\bar{n}hg) = h^\lambda f(g), \quad \text{for } \bar{n} \in \bar{N},\ h \in H,\ g \in G. \tag{3}$$

Then $\mathcal{R}_\lambda = \phi_\lambda(v_\lambda^* \otimes V^\lambda)$. Hence $\mathcal{R}_\lambda$ is spanned by the right translates of the function

$$f_\lambda(g) = \langle v_\lambda^*, \pi^\lambda(g)v_\lambda\rangle,$$

and the restriction of the right regular representation R of G to $\mathcal{R}_\lambda$ is an irreducible representation with highest weight λ. The function f_λ is uniquely determined by the property $f(\bar{n}hn) = h^\lambda$ for $\bar{n} \in \bar{N}$, $h \in H$, $n \in N$.

We call the function $f_\lambda(g)$ in Theorem 2.3 the *generating function* for the representation with highest weight λ. It can be calculated from the fundamental representations of G as follows.

Corollary 2.4 *Let $\lambda_1, \ldots, \lambda_r$ be generators for the additive semigroup $P_{++}(G)$. Set $f_i(g) = f_{\lambda_i}(g)$. Let $\lambda \in P_{++}(G)$ and write $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r$ with $m_i \in \mathbb{N}$. Then*

$$f_\lambda(g) = f_1(g)^{m_1} \cdots f_r(g)^{m_r} \quad \text{for } g \in G. \tag{4}$$

Example

Suppose $G = \mathrm{GL}(n, \mathbb{C})$. Take B as the group of upper-triangular matrices. We may identify $P(G)$ with $\mathbb{Z}^n$, where $\lambda = [\lambda_1, \ldots, \lambda_n]$ gives the character

$$h^\lambda = x^{\lambda_1} \cdots x^{\lambda_n}, \quad h = \mathrm{diag}[x_1, \ldots, x_n].$$

Then $P_{++}(G)$ consists of the monotone decreasing n-tuples and is generated by

$$\lambda_i = [\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0] \quad \text{for } i = 1, \ldots, n \text{ and } \lambda_{n+1} = -\lambda_n.$$

We can take $V^{\lambda_i} = \bigwedge^i \mathbb{C}^n$. The generating function is

$$f_{\lambda_i}(g) = \ i\text{th principal minor of } g.$$

for $i = 1, \ldots, n$. Here $f_{\lambda_{n+1}}(g) = (\det g)^{-1}$, which is a regular function on $\mathrm{GL}(n, \mathbb{C})$.

2.3 Multiplicity-free Spaces

We say that X is *multiplicity-free* as a G-space if all the irreducible representations of G that occur in $\mathrm{Aff}(X)$ have multiplicity one. We now obtain a geometric condition for an affine G-space X to be multiplicity free.

For a subgroup $K \subset G$ and $x \in X$ we write $K_x = \{k \in K : k \cdot x = x\}$ for the isotropy group at x. Let $\mathfrak{k} = \mathrm{Lie}(K)$. Then the Lie algebra of K_x is $\mathfrak{k}_x = \{Y \in \mathfrak{k} : d\rho(Y)_x = 0\}$.

Theorem 2.5 (Vinberg-Kimelfeld) *Let X be an irreducible affine G-space. Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X. Equivalently, suppose*

$$\dim(\mathfrak{b}/\mathfrak{b}_{x_0}) = \dim X.$$

Then X is multiplicity-free. In this case, if the representation π^λ with highest weight λ occurs in $\mathrm{Aff}(X)$, then $h^\lambda = 1$ for all $h \in H_{x_0}$.

Proof. It suffices by Theorem 2.1 to show that

$$\dim \mathrm{Aff}(X)^N(\lambda) \leq 1 \quad \text{for all } \lambda \in P_{++}(G).$$

Suppose $B \cdot x_0$ is open in X (and hence dense in X, by the irreduciblity of X). Then $f \in \mathrm{Aff}(X)^N(\lambda)$ is determined by $f(x_0)$, since on the dense set $B \cdot x_0$ it satisfies $f(b \cdot x_0) = b^{-\lambda} f(x_0)$. □

Let $K \subset G$ be a reductive algebraic subgroup. Then G/K has the structure of an affine algebraic set. The pair (G, K) will be called *spherical* if

$$\dim(V^\lambda)^K \leq 1.$$

for every $\lambda \in P_{++}(G)$. By Frobenius reciprocity this is equivalent to the space G/K being multiplicity-free.

From the conjugacy of Borel subgroups in G and Theorem 2.5 we have the following criterion for spherical pairs.

Corollary 2.6 *Suppose there exists a connected solvable subgroup S of G so that $\mathfrak{s} + \mathfrak{k} = \mathfrak{g}$. Then (G, K) is spherical.*

When (G, K) is a spherical pair an irreducible representation V^λ of G will be called *K-spherical* if $(\dim V^\lambda)^K = 1$. These are precisely the representations that occur in the decomposition of $\mathrm{Aff}(G/K)$ into G-irreducible subspaces. Thus the semigroup $\mathcal{S}(G/K)$ consists of the highest weights of K-spherical representations of G.

3 Representations on Symmetric Spaces

3.1 Algebraic Models for Symmetric Spaces

Let G be a connected reductive algebraic group, and let θ be an involutive automorphism of G. The differential of θ at 1, which we continue to denote as θ, is then an automorphism of $\mathfrak{g}$ which satisfies $\theta^2 = I$. Let $K = G^\theta$. The space G/K can be embedded into G as an affine algebraic subset as follows.

Define

$$g \star y = gy\theta(g)^{-1}, \quad \text{for } g, y \in G.$$

We have $(g \star (h \star y)) = (gh) \star y$ for $g, h, y \in G$, so this gives an action of G on itself which we will call the *θ-twisted conjugation* action. Let

$$Q = \{y \in G : \theta(y) = y^{-1}\}.$$

Then Q is an algebraic subset of G. Since $\theta(g \star y) = \theta(g)y^{-1}g^{-1} = (g \star y)^{-1}$, we have $G \star Q = Q$.

Theorem 3.1 (Richardson) *The θ-twisted action of G is transitive on each irreducible component of Q. Hence Q is a finite union of closed θ-twisted G-orbits.*

The proof consists of showing that the tangent space to a twisted G-orbit coincides with the tangent space to Q.

Corollary 3.2 *Let $P = G \star 1 = \{g\theta(g)^{-1} : g \in G\}$ be the orbit of the identity element under the θ-twisted conjugation action. Then P is a closed irreducible subset of G isomorphic to G/K as a G-space (relative to the θ-twisted conjugation action of G).*

3.2 Classical Symmetric Spaces

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a connected classical group with $\mathrm{Lie}(G)$ a simple Lie algebra. The involutions θ and associated symmetric spaces G/K for G can be described in terms of three kinds of geometric structures on $\mathbb{C}^n$:

(1) nondegenerate bilinear forms (symmetric or skew symmetric);

(2) polarizations $\mathbb{C}^n = V_+ \oplus V_-$ with $V_\pm$ totally isotropic subspaces relative to a bilinear form (zero form or nondegenerate symmetric or skew-symmetric form);

(3) orthogonal decompositions $\mathbb{C}^n = V_+ \oplus V_-$ relative to a nondegenerate bilinear form (symmetric or skew-symmetric).

In case (1) G is $\mathrm{SL}(n, \mathbb{C})$ and K is the subgroup preserving the bilinear form (in the Cartan classification, these are called types AI and AII). For case (2) G is $\mathrm{SL}(n, \mathbb{C})$ (if the form is identically zero) or the group preserving the bilinear form on $\mathbb{C}^n$ (if the form is nondegenerate) and K is the subgroup preserving the given decomposition of $\mathbb{C}^n$ (Cartan types AIII, BDI and CII, respectively). For case (3) G is the group preserving the bilinear form and K is the subgroup preserving the given decomposition of $\mathbb{C}^n$ (Cartan types DIII and CI). Thus there are seven types of symmetric spaces for the classical groups that arise this way.

The proof that these seven types give all the possible involutive automorphisms of the classical groups (up to inner automorphisms) can be obtained from following characterization of automorphisms of the classical groups.

Proposition 3.3 *Let σ be a regular automorphism of the classical group G.*

(1) *If $G = \mathrm{SL}(n, \mathbb{C})$ then there exists $s \in G$ so that σ is either $\sigma(g) = sgs^{-1}$ or $\sigma(g) = s(g^t)^{-1}s^{-1}$.*

(2) *If G is $\mathrm{Sp}(n, \mathbb{C})$ then there exists $s \in G$ so that $\sigma(g) = sgs^{-1}$.*

(3) *If G is $\mathrm{SO}(n, \mathbb{C})$ with $n \neq 2, 4$, then there exists $s \in \mathrm{O}(n, \mathbb{C})$ so that $\sigma(g) = sgs^{-1}$.*

Proof. Let π be the defining representation of G on $\mathbb{C}^m$ (where $m = n$ in cases (1) and (3), and $m = 2n$ in case (2)). The representation

$$\pi^\sigma(g) = \pi(\sigma(g))$$

also acts irreducibly on $\mathbb{C}^m$. The Weyl dimension formula implies that the defining representation (and its dual, in the case $G = \mathrm{SL}(n, \mathbb{C})$) is the unique representation of dimension m. The proposition follows easily from this fact. □

We can now describe all the involutions of the classical groups.

Theorem 3.4 *Let θ be an involution of the classical group G. Assume* $\mathrm{Lie}(G)$ *is simple. Then θ is given as follows, up to conjugation by an element of G.*

(1) *If $G = \mathrm{SL}(n, \mathbb{C})$, then there are three possibilities:*

> (a) *$\theta(x) = T(x^t)^{-1}T^t$ for $x \in G$, where $T \in G$ satisfies $T^t = T$. The property $T^t = T$ determines θ uniquely up to conjugation in G. The corresponding bilinear form $B(u, v) = u^tTv$, for $u, v \in \mathbb{C}^n$, is symmetric and nondegenerate.*
>
> (b) *$\theta(x) = T(x^t)^{-1}T^t$ for $x \in G$, where $T \in G$ satisfies $T^t = -T$. The property $T^t = -T$ determines θ uniquely up to conjugation in G. The corresponding bilinear form $B(u, v) = u^tTv$, for $u, v \in \mathbb{C}^n$, is skew-symmetric and nondegenerate.*
>
> (c) *$\theta(x) = JxJ^{-1}$ for $x \in G$, where $J \in \mathrm{GL}(n, \mathbb{C})$ and $J^2 = I_n$. Let*
>
> $$V_\pm = \{v \in \mathbb{C}^n \,:\, Jv = \pm v\}.$$
>
> *Then $V = V_+ \oplus V_-$ and θ is determined (up to conjugation in G) by* $\dim V_+$.

(2) *If G is $\mathrm{SO}(V, \omega)$ or $\mathrm{Sp}(V, \omega)$, then there are two possibilities:*

> (a) *$\theta(x) = JxJ^{-1}$ for $x \in G$, where J preserves the form ω and $J^2 = I$. Let*
>
> $$V_\pm = \{v \in V \,:\, Jv = \pm v\}.$$
>
> *Then $V = V_+ \oplus V_-$, the restriction of ω to $V_\pm$ is nondegenerate, and θ is determined (up to conjugation in G) by* $\dim V_+$.
>
> (b) *$\theta(x) = JxJ^{-1}$ for $x \in G$, where J preserves the form ω and $J^2 = -I$. Let*
>
> $$V_{\pm i} = \{v \in V \,:\, Jv = \pm iv\}.$$
>
> *Then $V = V_i \oplus V_{-i}$, the restriction of ω to $V_{\pm i}$ is zero and V_i is dual to V_{-i} via the form ω. The automorphism θ is uniquely determined (up to conjugation in G).*

We proceed to describe the symmetric spaces for the classical groups in more detail. Given the group G and involution θ, we set

$$P = \{g\theta(g)^{-1} \,:\, g \in G\}, \qquad Q = \{y \in G \,:\, \theta(y) = y^{-1}\}.$$

We write $s_p = [\delta_{p+1-i-j}]$ for the $p \times p$ matrix with 1 on the anti-diagonal and 0 elsewhere. Let $\tau(g) = (\bar{g}^t)^{-1}$. In all cases we will take the matrix form of G and the involution θ so that the following holds.

(1) $\tau(G) = G$ and G^τ is a compact real form of G.

(2) The diagonal subgroup H in G is a maximal torus and $\theta(H) = H$.

(3) $\tau\theta = \theta\tau$

It follows from (3) that $\sigma = \theta\tau$ is also a conjugation on G.

3.2.1 Involutions Associated with Bilinear Forms

Symmetric Bilinear Form–Type AI:

Let $G = \mathrm{SL}(n, \mathbb{C})$ and define the involution $\theta(g) = (g^t)^{-1}$. Then $\theta(g) = g$ if and only if g preserves the symmetric bilinear form $B(u, v) = u^t v$ on $\mathbb{C}^n$. Thus $K = G^\theta = \mathrm{SO}(\mathbb{C}^n, B)$. The θ-twisted action is

$$g \star y = gyg^t,$$

and $Q = \{y \in G \,:\, y^t = y\}$. A matrix $y \in Q$ defines a symmetric bilinear form $B_y(u, v) = u^t y v$ on $\mathbb{C}^n$. The θ-twisted G-orbit of y corresponds to all the bilinear forms G-equivalent to B_y. Since B_y is non-singular, there exists $g \in \mathrm{GL}(n, \mathbb{C})$ so that $g \star y = I_n$. Since $\det y = 1$, we have $\det g = \pm 1$; multiplying g by $\mathrm{diag}[-1, 1, \ldots, 1]$ if necessary, we may take $\det g = 1$. Thus Q is a single G-orbit in this case, and hence $Q = P$. By Corollary 3.2 we conclude that

$$\mathrm{SL}(n, \mathbb{C})/\mathrm{SO}(\mathbb{C}^n, B) \cong \{y \in M_n(\mathbb{C}) \,:\, y = y^t,\, \det y = 1\}$$

as a G-variety, under the map $gK \mapsto gg^t$. In this case the conjugation $\sigma = \theta\tau$ is given by $\sigma(g) = \bar{g}$.

Skew-symmetric Bilinear Form–Type AII:

Let $G = \mathrm{SL}(2n, \mathbb{C})$. Take

$$\mu = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and let T_n be the $2n \times 2n$ skew-symmetric block-diagonal matrix

$$T_n = \mathrm{diag}[\underbrace{\mu, \ldots, \mu}_{n\,\text{blocks}}].$$

Then $T_n^2 = -I_{2n}$ and $T_n^{-1} = T_n^t$. Define the involution θ by

$$\theta(g) = T_n (g^t)^{-1} T_n^t.$$

Since $\tau(T_n) = T_n$, we have $\theta\tau = \tau\theta$. For $g \in G$, $\theta(g) = g$ if and only if $g^t T_n g = T_n$. This means that g preserves the non-degenerate skew-symmetric bilinear form $\omega(u, v) = u^t T_n v$ on $\mathbb{C}^{2n}$. Thus $K = G^\theta = \mathrm{Sp}(\mathbb{C}^{2n}, \omega)$.

In this case the θ-twisted action of G is

$$g \star y = g y T_n g^t T_n^t.$$

and $Q = \{y \in G : (yT_n)^t = -yT_n\}$. A matrix $y \in Q$ defines a non-singular skew-symmetric bilinear form

$$\omega_y(u, v) = u^t y T_n v, \qquad u, v \in \mathbb{C}^{2n},$$

and the θ-twisted G-orbit of y corresponds to all the bilinear forms equivalent to ω_y. Arguing as in Type AI, we see that Q is a single G-orbit and hence $Q = P$. By Corollary 3.2 we conclude that

$$\mathrm{SL}(2n, \mathbb{C})/\mathrm{Sp}(\omega) \cong \{y \in M_n(\mathbb{C}) : yT_n = -(yT_n)^t, \, \det y = 1\}$$

under the map $gK \mapsto gT_n g^t T_n^t$. In this case the conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = T_n \bar{g} T_n^t.$$

3.2.2 Involutions Associated with Polarizations

Zero Bilinear Form–Type AIII:

Let $G = \mathrm{SL}(p+q, \mathbb{C})$. For integers $p \le q$ with $p + q = n$ define

$$J_{p,q} = \begin{bmatrix} 0 & 0 & s_p \\ 0 & I_{q-p} & 0 \\ s_p & 0 & 0 \end{bmatrix}.$$

Then $J_{p,q}^2 = I_n$, so we can define an involution θ of G by

$$\theta(g) = J_{p,q} g J_{p,q}.$$

Since $\tau(J_{p,q}) = J_{p,q}$, we have $\theta\tau = \tau\theta$. The linear transformations $P_\pm = \frac{1}{2}(I_n \mp J_{p,q})$ are the projections onto the ± 1 eigenspaces $V_\pm$ of $J_{p,q}$, and

$$\mathbb{C}^n = V_+ \oplus V_-. \tag{5}$$

We have $\dim V_+ = \mathrm{tr}(P_+) = \frac{1}{2}(n - (q-p)) = p$. The subgroup $K = G^\theta$ consists of all $g \in G$ that commute with $J_{p,q}$. This means that g leaves invariant the decomposition (5), so we have

$$K \cong \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})),$$

the group of all block diagonal matrices $g = \mathrm{diag}[g_1, g_2]$ with $g_1 \in \mathrm{GL}(p, \mathbb{C})$, $g_2 \in \mathrm{GL}(q, \mathbb{C})$ and $\det g_1 \det g_2 = 1$.

In this case $Q = \{y \in G : (yJ_{p,q})^2 = I_n\}$ and the θ-twisted action is

$$g \star y = g y J_{p,q} g^{-1} J_{p,q}.$$

For $y \in Q$ the matrix $z = yJ_{p,q}$ is a non-singular idempotent. Thus it defines a decomposition

$$\mathbb{C}^n = V_+(y) \oplus V_-(y),$$

where z acts by ± 1 on $V_\pm(y)$. The θ-twisted G-orbit of y corresponds to the G-conjugacy class of z, under the map $g \star y \mapsto (g \star y)J_{p,q}$. Hence $G \star y$ is determined by $\dim V_+(y)$, which can be any integer between 0 and n. In particular, the θ-twisted G-orbit of I is

$$P = \{y \in \mathrm{GL}(p+q, \mathbb{C}) \,:\, (yJ_{p,q})^2 = I_n,\ \mathrm{tr}(yJ_{p,q}) = q - p\}.$$

By Corollary 3.2 we conclude that

$$\mathrm{SL}(p+q, \mathbb{C})/\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})) \cong P$$

under the map $gK \mapsto gJ_{p,q}g^{-1}J_{p,q}$. The conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = J_{p,q}(\bar{g}^t)^{-1}J_{p,q}.$$

Skew-symmetric Bilinear Form–Type CI:

Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$, where Ω is the skew-symmetric form $\Omega(u, v) = u^t J_n v$ with

$$J_n = \begin{bmatrix} 0 & s_n \\ -s_n & 0 \end{bmatrix}.$$

We have $J_n^t = J_n^{-1}$ and $J_n^2 = -I_{2n}$. Thus $J_n \in G$ and the map

$$\theta(g) = -J_n g J_n$$

is an involution on G. Since $\tau(J_n) = J_n$, we see that θ commutes with τ. We can decompose

$$\mathbb{C}^{2n} = V_+ \oplus V_-,$$

where J_n acts by $\pm i$ on $V_\pm$. The form Ω vanishes on the subspaces $V_\pm$. Indeed, the projections onto $V_\pm$ are $P_\pm = \frac{1}{2}(1 \mp iJ_n)$, and we have $P_+^t = P_-$ since $J_n^t = -J_n$. Thus $P_+^t J_n P_+ = J_n P_- P_+ = 0$ and so $\Omega(P_+u, P_+v) = 0$ (the same holds for P_-). Thus Ω gives a nonsingular pairing between V_- and V_+. In particular, $\dim V_\pm = n$.

The subgroup $K = G^\theta$ consists of all $g \in G$ that commute with J_n. Thus g leaves invariant $V_\pm$. Since g preserves Ω, the action of g on V_- is dual to its action on V_+. Thus

$$K \cong \mathrm{GL}(V_+) \cong \mathrm{GL}(n, \mathbb{C}).$$

The θ-twisted action is

$$g \star y = gyJ_n g^{-1} J_n^{-1}$$

and $Q = \{y \in G \,:\, (yJ_n)^2 = -I_{2n}\}$. Let $y \in Q$ and set $z = yJ_n$. Then $z^2 = -I_{2n}$, so we can decompose

$$\mathbb{C}^{2n} = V_+(y) \oplus V_-(y), \tag{6}$$

where z acts by $\pm i$ on $V_\pm(y)$. We claim that $\Omega = 0$ on $V_\pm(y)$. Indeed, the projections onto $V_\pm(y)$ are $P_\pm = \frac{1}{2}(1 \mp iz)$, and from the relation $y^t J_n = J_n y^{-1}$ we calculate that

$z^t J_n = -J_n z$, so this follows just as in the case $y = I_n$. The subspaces $V_\pm(y)$ are thus maximal isotropic for the form Ω, and Ω gives a non-singular pairing between $V_+(y)$ and $V_-(y)$. Since y is determined by the decomposition (6), it follows that Q is a single θ-twisted G-orbit. Thus

$$P = \{y \in \mathrm{Sp}(n, \mathbb{C}) \,:\, (yJ_n)^2 = -I_{2n}\}.$$

By Corollary 3.2 we conclude that

$$\mathrm{Sp}(\mathbb{C}^{2n}, \Omega)/\mathrm{GL}(n, \mathbb{C}) \cong P$$

under the map $gK \mapsto gJ_n g^{-1} J_n^t$. The conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = -J_n(\bar{g}^t)^{-1} J_n.$$

Symmetric Bilinear Form–Type DIII:

Let $G = \mathrm{SO}(\mathbb{C}^n, B)$ with $n = 2l$ even, where $B(u, v) = u^t s_n v$. We define $\Gamma \in \mathrm{GL}(n, \mathbb{C})$ as follows. Let

$$\gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $l = 2r$ even, define the block-diagonal matrix

$$\Gamma_n = i\,\mathrm{diag}[\underbrace{\gamma, \ldots, \gamma}_{r}, \underbrace{-\gamma, \ldots, -\gamma}_{r}].$$

For $l = 2r + 1$ odd, set

$$\Gamma_n = i\,\mathrm{diag}[\underbrace{\gamma, \ldots, \gamma}_{r}, 1, -1, \underbrace{-\gamma, \ldots, -\gamma}_{r}].$$

Then $\Gamma_n s_n = -s_n \Gamma_n$, $\Gamma_n^t = \Gamma_n$ and $\Gamma_n^2 = -I_n$. Thus $\Gamma_n \in \mathrm{O}(\mathbb{C}, B)$ and the map

$$\theta(g) = -\Gamma_n g \Gamma_n$$

is an involution on G. Since $\tau(\Gamma_n) = -\Gamma_n$, we see that θ commutes with τ.

We can decompose

$$\mathbb{C}^n = V_+ \oplus V_-$$

where $V_\pm$ are the $\pm i$ eigenspaces of Γ_n. Since $\Gamma_n s_n = -s_n \Gamma_n$, the form B vanishes on the subspaces $V_\pm$, by the same calculation as in Type CI. As in that case, we have

$$K \cong \mathrm{GL}(V_+) \cong \mathrm{GL}(l, \mathbb{C}).$$

For this case $Q = \{y \in G \,:\, (y\Gamma_n)^2 = -I_n\}$ and the θ-twisted action is

$$g \star y = gy\Gamma_n g^{-1} \Gamma_n.$$

For $y \in Q$ the matrix $z = y\Gamma_n$ satisfies $z^2 = -I_n$, so we can decompose

$$\mathbb{C}^n = V_+(y) \oplus V_-(y),$$

where z acts by $\pm i$ on $V_\pm(y)$. We have

$$z^t s_n = \Gamma_n y^t s_n = \Gamma_n s_n y^{-1} = -s_n \Gamma_n y^{-1} = -s_n z.$$

This implies that the subspaces $V_\pm$ are totally isotropic for the form B (by the same calculation as in Type CI). It follows that Q is a single θ-twisted G-orbit. Thus

$$P = \{y \in \mathrm{SO}(\mathbb{C}^n, B) \,:\, (y\Gamma_n)^2 = -I_n\}.$$

By Corollary 3.2 we conclude that

$$\mathrm{SO}(\mathbb{C}^n, B)/\mathrm{GL}(l, \mathbb{C}) \cong P$$

under the map $gK \mapsto g\Gamma_n g^{-1}\Gamma_n$. The conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = -\Gamma_n (\bar{g}^t)^{-1} \Gamma_n.$$

3.2.3 Involutions Associated with Orthogonal Decompositions

Symmetric Bilinear Form–Type BDI:

Let $G = \mathrm{SO}(\mathbb{C}^n, B)$, where B is the symmetric bilinear form $B(u, v) = u^t s_n v$ on $\mathbb{C}^n$. For integers $p \le q$ with $p + q = n$ define $J_{p,q}$ as in type AIII. We have $J_{p,q}^t = J_{p,q}^{-1} = J_{p,q}$ and $J_{p,q} s_n = s_n J_{p,q}$. Since $g \in G$ if and only if $s_n g^t s_n = g$, we see that $J_{p,q} \in \mathrm{O}(\mathbb{C}, B)$ Thus the map

$$\theta(g) = J_{p,q} g J_{p,q}$$

is an involution on G. Clearly θ commutes with τ. The projections $P_\pm$ onto the ± 1 eigenspaces $V_\pm$ of $J_{p,q}$ commute with s_n. Hence $V_+ \perp V_-$ (relative to the form B), since $P_+^t s_n P_- = s_n P_+ P_- = 0$. We have $\dim V_+ = \mathrm{tr}(P_+) = \frac{1}{2}(n - (q - p)) = p$ and $\dim V_- = q$. The subgroup $K = G^\theta$ consists of all $g \in G$ that commute with $J_{p,q}$. This means that g leaves invariant the decomposition (5). The restrictions $B_\pm$ of B to $V_\pm$ are non-degenerate, since $V_- \perp V_+$, so we have

$$K \cong \mathrm{S}(\mathrm{O}(V_+, B_+) \times \mathrm{O}(V_-, B_-)) \cong \mathrm{S}(\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})),$$

the group of all block diagonal matrices $g = \mathrm{diag}[g_1, g_2]$ with $g_1 \in \mathrm{O}(p, \mathbb{C})$, $g_2 \in \mathrm{O}(q, \mathbb{C})$, and $\det g_1 \det g_2 = 1$.

We have $Q = \{y \in G \,:\, (yJ_{p,q})^2 = I_n\}$ and the θ-twisted action is

$$g \star y = g y J_{p,q} g^{-1} J_{p,q}.$$

The G-orbits in Q for this action correspond to the G-similarity classes of idempotent matrices $yJ_{p,q}$, with $y \in Q$.

For $y \in Q$ the matrix $z = yJ_{p,q}$ satisfies $z^2 = I_n$, so it gives a decomposition

$$\mathbb{C}^n = V_+(y) \oplus V_-(y), \tag{7}$$

where z acts by ± 1 on $V_\pm(y)$. Since $y^t s_n = s_n y^{-1}$, $J_{p,q} y^{-1} = y J_{p,q}$ and $J_{p,q} s_n = s_n J_{p,q}$, we have

$$z^t s_n = J_{p,q} y^t s_n = J_{p,q} s_n y^{-1} = s_n J_{p,q} y^{-1} = s_n z.$$

Hence the same argument that we used when $y = I_n$ shows that the subspaces $V_\pm(y)$ are mutually orthogonal (relative to the form B). This implies that the restrictions of B to $V_\pm$ are non-singular. Since y is determined by the decomposition (7), it follows that the θ-twisted G-orbit of y is determined by the integer

$$\dim V_+(y) = \frac{1}{2}(n - \operatorname{tr}(yJ_{p,q})).$$

In particular, $P = \{y \in \mathrm{SO}(\mathbb{C}^n, B) \,:\, (yJ_{p,q})^2 = I_n,\, \operatorname{tr}(yJ_{p,q}) = q - p\}$. Corollary 3.2 now implies

$$\mathrm{SO}(\mathbb{C}^n, B)/\mathrm{S}(\mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})) \cong P$$

under the map $gK \mapsto gJ_{p,q}g^{-1}J_{p,q}$. The conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = J_{p,q}(\bar{g}^t)^{-1}J_{p,q}.$$

Skew-symmetric Bilinear Form–Type CII:

Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$, where Ω is the skew-symmetric bilinear form $\Omega(u,v) = u^t J_n v$ as in Type CI. For $0 < p \le q$ with $p + q = n$, let $J_{p,q} \in \mathrm{GL}(n, \mathbb{C})$ be as in Type AIII and define

$$K_{p,q} = \begin{bmatrix} J_{p,q} & 0 \\ 0 & J_{p,q} \end{bmatrix}.$$

Since $J_{p,q}^t = J_{p,q}^{-1} = J_{p,q}$ and $s_n J_{p,q} s_n = J_{p,q}$, we have $K_{p,q} \in G$ and $K_{p,q}^2 = I_{2n}$. Thus the map

$$\theta(g) = K_{p,q} g K_{p,q}$$

is an involution on G. Clearly θ commutes with τ. As in Type BDI, the ± 1 eigenspaces of $K_{p,q}$ give a decomposition

(8) $$\mathbb{C}^{2n} = V_+ \oplus V_-$$

which is orthogonal relative to the form Ω. The subgroup $K = G^\theta$ consists of all $g \in G$ that commute with $K_{p,q}$. Since the restrictions of Ω to $V_\pm$ are non-degenerate and $\dim V_+ = \operatorname{tr}(P_+) = \frac{1}{2}(2n - \operatorname{tr}(K_{p,q})) = 2p$, we have

$$K \cong \mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C}),$$

in complete analogy with Type BDI.

Here $Q = \{y \in G \,:\, (yK_{p,q})^2 = I_{2n}\}$ and the θ-twisted action is

$$g \star y = gyK_{p,q}g^{-1}K_{p,q}.$$

Let $y \in Q$ and set $z = yK_{p,q}$. Since $y^t J_n = J_n y^{-1}$ and $K_{p,q}J_n = J_n K_{p,q}$, we have $z^t J_n = J_n z$. Thus the ± 1 eigenspaces of z are mutually orthogonal (relative to Ω) and give a decomposition

(9) $$\mathbb{C}^{2n} = V_+(y) \oplus V_-(y).$$

The same proof as in Type BDI shows that the θ-twisted G-orbit of y is determined by the integer $\operatorname{tr}(yK_{p,q})$. In particular,

$$P = \{y \in \mathrm{Sp}(\mathbb{C}^{2n}, \Omega) \,:\, (yK_{p,q})^2 = I_{2n},\, \operatorname{tr}(yK_{p,q}) = 2(q-p)\}.$$

By Corollary 3.2 we conclude that

$$\mathrm{Sp}(\mathbb{C}^{2n}, \Omega)/(\mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C})) \cong P$$

under the map $gK \mapsto gJ_n g^{-1} J_n^{-1}$. The conjugation $\sigma = \theta\tau$ is given by

$$\sigma(g) = K_{p,q}(\bar{g}^t)^{-1} K_{p,q}.$$

3.3 Iwasawa Decompositions

Let G be a connected classical group and let θ be an involutive regular automorphism of G. Let

$$K = \{g \in G \, : \, \theta(g) = g\}.$$

Let τ be a conjugation in G such that the corresponding real form is compact. From the examples of the previous section we know that we may assume $\tau\theta = \theta\tau$. We write $\sigma = \tau\theta$; then σ is another conjugation of G. Let $G_0 = G^\sigma$ and $K_0 = G^\tau \cap K = G^\sigma \cap K$. Then G_0 is a (noncompact) real form of G and K_0 is a compact real form of K.

We shall prove that (G, K) is a spherical pair by finding a Borel subgroup B with KB dense in G. In fact, we shall construct a solvable subgroup AN^+ of G, a semidirect product of a torus A with a unipotent group N^+, such that $K \cap (AN^+)$ is finite and KAN^+ is dense in G. This gives the so-called (complexified) *Iwasawa decomposition* of G.

Take $G \subset \mathrm{GL}(n, \mathbb{C})$ and θ as in Section 3.2. Let $\mathfrak{g}$ be the Lie algebra of G. We will write θ for $d\theta$. As usual, we will consider $\mathfrak{k}$ =Lie(K) to be a Lie subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{k} = \{X \in \mathfrak{g} \, : \, \theta X = X\}.$$

Set $\mathfrak{p} = \{X \in \mathfrak{g} \, : \, \theta X = -X\}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as a K-module under $\mathrm{Ad}|_K$.

Let $\mathfrak{h} = \mathrm{Lie}(H)$ and let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the rootspace decomposition of $\mathfrak{g}$ relative to H. We can then describe the action of θ on $\mathfrak{g}$ as follows. Since H is θ-stable, it is clear that

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha^\theta}$$

where we write $\alpha^\theta = \alpha \circ \theta$.

Recall from Section 3.1 that G/K is embedded as the identity component of the algebraic set

$$Q = \{g \in G \, : \, \theta(g) = g^{-1}\}.$$

Since H is commutative, the set $H \cap Q$ is a commutative subgroup of G consisting of semisimple elements. Hence the identity component $A = (H \cap Q)^\circ$ is an (algebraic) torus. By definition,

$$\theta(a) = a^{-1} \quad \text{for all } a \in A.$$

An algebraic torus with this property is called θ-*anisotropic*. For the conjugation $\sigma = \theta\tau$ and $a \in A$ we have

$$\sigma(a) = \tau\theta(a) = \tau(a^{-1}) = \bar{a}.$$

If χ is a regular character of A then

$$\chi(\sigma(a)) = \overline{\chi(a)} \quad \text{(complex conjugate).}$$

We describe this property by saying that A is *σ-split*.

Set $T = H \cap K$. Then T° is a torus and there is a finite group C such that $T = T^\circ \times C$.

Lemma 3.5 *One has $H = AT^\circ$ and $A \cap T = \{a \in A : a^2 = 1\} \cong (\mathbb{Z}/2\mathbb{Z})^m$, where $m = \operatorname{rank}(A)$. Thus $C \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some r with $0 \le r \le m$.*

Set

$$\Phi_0 = \{\alpha \in \Phi : \langle \alpha, X \rangle = 0 \quad \text{for all } X \in \mathfrak{a}\}.$$

We shall verify case-by-case later in this Section that the following condition holds for the involutions of the classical groups:

$(*)$ For all $\alpha \in \Phi_0$, θ acts as the identity on $\mathfrak{g}_\alpha$.

For now, we assume this fact and prove the following consequence of $(*)$ (a Lie subalgebra of $\mathfrak{g}$ is called a *toral subalgebra* if it is commutative and consists of semisimple elements).

Lemma 3.6 *Assume $(*)$ is satisfied. Let $\mathfrak{l}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$. Hence if $Y \in \mathfrak{p}$ and $[Y, \mathfrak{a}] = 0$ then $Y \in \mathfrak{a}$. In particular, $\mathfrak{a}$ is a maximal toral subalgebra of $\mathfrak{p}$.*

Proof. Since $\theta(\mathfrak{l}) = \mathfrak{l}$, we have $\mathfrak{l} = \mathfrak{m} \oplus (\mathfrak{l} \cap \mathfrak{p})$. Let $X \in \mathfrak{l} \cap \mathfrak{p}$ and write

$$X = X_0 + X_1 + \sum_{\alpha \in \Phi} X_\alpha\,,$$

with $X_0 \in \mathfrak{t}$ and $X_1 \in \mathfrak{a}$. Since $[X, \mathfrak{a}] = 0$ and $[\mathfrak{a}, \mathfrak{t}] = 0$, we have

$$0 = [Y, X] = \sum_{\alpha \in \Phi} \alpha(Y) X_\alpha \quad \text{for all } Y \in \mathfrak{a}.$$

Hence $X_\alpha = 0$ for all $\alpha \in \Phi \setminus \Phi_0$. Thus using condition $(*)$ we can write

$$-X = \theta(X) = X_0 - X_1 + \sum_{\alpha \in \Phi_0} X_\alpha.$$

It follows that $X_0 = X_\alpha = 0$ and hence $X = X_1 \in \mathfrak{a}$. This implies that $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Since A is a torus, the elements of $\mathfrak{a}$ are semisimple, so $\mathfrak{a}$ is a maximal toral subalgebra of $\mathfrak{p}$. $\square$

Define $M = \operatorname{Cent}_K(A)$ (the centralizer of A in K). It is clear from $(*)$ that $\mathfrak{m} = \operatorname{Lie}(M)$ has the rootspace decomposition

$$\mathfrak{m} = \mathfrak{t} + \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha,$$

where Φ_0 is the set of roots vanishing on $\mathfrak{a}$, as above. Let $L = \operatorname{Cent}_G(A)$. Then L is connected [Bor, Cor. 11.12] and $\operatorname{Lie}(L) = \mathfrak{l}$.

Lemma 3.7 *One has $L = AM^\circ$ and $M = TM^\circ$. Hence M is connected if and only if T is connected.*

Proof. Let $x \in L$. Then the semisimple and unipotent components $x_s, x_u \in L$ since they commute with A. We can write

$$x_u = \exp Y,$$

where Y is a nilpotent element of $\mathfrak{l}$. Since $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$, $[\mathfrak{a}, \mathfrak{m}] = 0$ and the elements of $\mathfrak{a}$ are semisimple, it follows that $Y \in \mathfrak{m}$. Hence $x_u \in M^\circ$.

One knows from [Bor, Cor. 11.12] that there is a torus $S \subset L$ such that

$$A \cup \{x_s\} \subset S.$$

Let $\mathfrak{s} = \mathrm{Lie}(S) \supset \mathfrak{a}$. If $Z \in \mathfrak{s}$ and $X \in \mathfrak{a}$ then

$$0 = \theta([Z, X]) = [\theta(Z), -X].$$

Hence $Z - \theta(Z) \in \mathfrak{p}$ and commutes with $\mathfrak{a}$. Thus $Z - \theta(Z) \in \mathfrak{a}$ by Lemma 3.6. Hence $\theta(Z) \in \mathfrak{s}$, so we have $\theta(\mathfrak{s}) = \mathfrak{s}$. Thus

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus \mathfrak{a}$$

Since $S = \exp(\mathfrak{s})$, this shows that

$$S = A \cdot S_0,$$

where $S_0 = S \cap K = \exp(\mathfrak{s} \cap \mathfrak{k}) \subset M^\circ$. Thus we can factor $x_s = ab$ with $a \in A$ and $b \in S_0$. This proves that $x = abx_u$ and hence $L = AM^\circ$. This implies that $M = (A \cap T)M^\circ = TM$. $\square$

We are assuming that $G \subset \mathrm{GL}(n, \mathbb{C})$ and A is a subgroup of the diagonal matrices. Hence we can find a subset $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ such that the characters

$$a \mapsto \chi_j(a) = x_{i_j} \quad \text{for } a = \mathrm{diag}[x_1, \ldots, x_n]$$

freely generate $\mathcal{X}(A)$. We fix such a set of characters and we give $\mathcal{X}(A)$ the corresponding lexicographic order. Let the unipotent subgroups $N^\pm$ of G be defined relative to this order (so the weights of $\mathrm{Ad}(A)$ on N^+ are positive, and the weights on N^- are negative). Then we have

$$\theta(N^+) = N^-. \tag{10}$$

Indeed for $\chi \in \mathcal{X}(A)$ and $a \in A$,

$$\chi^\theta(a) = \chi(\theta(a)) = \chi(a)^{-1},$$

since A is θ-anisotropic. Thus θ gives an order-reversing automorphism of $\mathcal{X}(A)$. This implies (10).

A total ordering of $\mathcal{X}(H)$ will be called *compatible* with the chosen order on $\mathcal{X}(A)$ if $\mu|_A > \nu|_A$ implies that $\mu > \nu$ in $\mathcal{X}(H)$. We construct a compatible order on $\mathcal{X}(H)$ as follows. Let $\Sigma = \{\nu_1, \ldots, \nu_r\} \subset \mathcal{X}(A)$ be the weights of A on $\mathbb{C}^n$, enumerated so that

$$\nu_1 > \nu_2 > \cdots > \nu_r$$

relative to the order that we have fixed on $\mathcal{X}(A)$. Enumerate the standard basis for $\mathbb{C}^n$ as $\{e_{j_1}, \ldots, e_{j_n}\}$ so that e_{j_i} has weight ν_1 for $1 \leq i \leq m_1$, weight ν_2 for $m_1 + 1 \leq i \leq m_2$, and so forth. Each vector e_{j_i} transforms according to a weight μ_{j_i} of H. We give $\mathcal{X}(H)$ the lexicographic order in which $\mu_{j_1} > \mu_{j_2} > \ldots$. This order is clearly compatible with the order on $\mathcal{X}(A)$.

Let Φ^+ be the roots for H that are positive relative to the order just defined. Let B be the Borel subgroup of G defined by the positive system Φ^+. Write $\mathfrak{n}^\pm = \mathrm{Lie}(N^\pm)$ and set $\Phi_1^+ = \Phi^+ \setminus \Phi_0$. Then

$$\mathfrak{n}^+ = \sum_{\alpha \in \Phi_1^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \sum_{\alpha \in \Phi_1^+} \mathfrak{g}_{-\alpha}.$$

Thus $AN^+ \subset B$.

Lemma 3.8 *One has the vector-space direct sum decompositions*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

Hence N^-MAN^+ and KAN^+ are open Zariski-dense subsets of G and $K \cap (AN^+)$ is finite.

Proof. Since Φ is the disjoint union $\Phi_0 \cup \Phi_1^+ \cup (-\Phi_1^+)$, we can use the rootspace decomposition of $\mathfrak{g}$ to write $X \in \mathfrak{g}$ as

$$X = \sum_{\beta \in \Phi_1^+} X_{-\beta} + \{H_0 + \sum_{\alpha \in \Phi_0} X_\alpha\} + H_1 + \sum_{\beta \in \Phi_1^+} X_\beta,$$

where $H_0 \in \mathfrak{t}$, $H_1 \in \mathfrak{a}$ and $X_\alpha \in \mathfrak{g}_\alpha$. This gives the first decomposition of $\mathfrak{g}$. For the second decomposition, we write

$$X_{-\beta} = X_{-\beta} + \theta(X_{-\beta}) - \theta(X_{-\beta})$$

for $\beta \in \Phi_1^+$ and note that $X_{-\beta} + \theta(X_{-\beta}) \in \mathfrak{k}$ and $\theta(X_{-\beta}) \in \mathfrak{n}^+$ by (10).

Consider the maps $N^- \times M \times A \times N^+ \to G$ and $K \times A \times N^+ \to G$ given by multiplication in G of the elements from each factor. From the decompositions of $\mathfrak{g}$ we see that differentials of these maps are surjective at 1. Since G is connected, it follows that the images are Zariski-dense. □

Theorem 3.9 *K is a spherical subgroup of G. If λ is the B-highest weight of an irreducible K-spherical representation of G, then*

$$t^\lambda = 1 \quad \textit{for all } t \in T. \tag{11}$$

Proof. The Borel subgroup B contains AN^+, so KB is dense in G by Lemma 3.8. This proves that K is spherical. Since $T = K \cap B$, condition (11) is satisfied by the highest weight of a K-spherical representation by Theorem 2.5. □

We call $\lambda \in P_{++}(G)$ *θ-admissible* if it satisfies (11).

We now work out explicit Iwasawa decompositions for the seven types of classical symmetric spaces G/K associated with an involution θ, following the notation of Section 3.2. For each type we verify condition $(*)$ above, describe the maximal θ-anisotropic torus

$A = (H \cap Q)^\circ$ and the subgroup $T = H \cap K$. We give a total order on $\mathcal{X}(A)$ and a compatible order on $\mathcal{X}(H)$. We describe the weight decomposition

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r, \tag{12}$$

where $G \subset \mathrm{GL}(n, \mathbb{C})$ and A acts on V_i by the character μ_i. The enumeration is chosen so that $\mu_1 > \cdots > \mu_r$. The group M consists of the elements of K that preserve the decomposition (12) and N^+ consists of the elements $g \in G$ so that $I - g$ is strictly upper block-triangular relative to the decomposition (12). We give the system of positive roots Φ^+ for the compatible order on $\mathcal{X}(H)$, and we find the explicit form of the θ-admissibility condition (11) for the Φ^+-dominant weights. The information is summarized in the *Satake diagram*, which is obtained from the Dynkin diagram of $\mathfrak{g}$ by the following procedure.

(S1) If a simple root vanishes on $\mathfrak{a}$, then the corresponding node in the Dynkin diagram is marked by $\bullet$.

(S2) If two simple roots have the same nonzero restriction to $\mathfrak{a}$, then the corresponding nodes are marked by $\circ$ and are joined by a curved arrow.

(S3) The labels on the nodes $\circ$ (where m_i always denotes a nonnegative integer) are the coefficients of the corresponding fundamental weights in the θ-admissible Φ^+-dominant weights. Nodes joined by a curved arrow have the same coefficient and nodes marked by $\bullet$ have coefficient zero.

Notation: D_p is the group of invertible diagonal $p \times p$ matrices, $s_p = [\delta_{p+1-i-j}]$ is the $p \times p$ matrix with 1 on the anti-diagonal and 0 elsewhere. For $a = \mathrm{diag}[a_1, \ldots, a_p] \in D_p$ let

$$\varepsilon_i(a) = a_i, \qquad \check{a} = s_p a s_p = \mathrm{diag}[a_p, \ldots, a_1].$$

Bilinear Forms–Type AI

Here $G = \mathrm{SL}(n, \mathbb{C})$, θ is the involution $\theta(g) = (g^t)^{-1}$, and $K \cong \mathrm{SO}(n, \mathbb{C})$. The maximal torus H is θ-anisotropic. Hence $A = H$ and $T \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$ consists of all matrices

$$t = \mathrm{diag}[\delta_1, \ldots, \delta_n], \quad \delta_i = \pm 1, \, \det(t) = 1.$$

There are no roots that vanish on $\mathfrak{a}$, so condition $(*)$ is vacuously satisfied. Hence A is a maximal θ-anisotropic torus and $M = T$. Take the characters $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{n-1}$ as an ordered basis for $\mathcal{X}(A)$. The eigenspace decomposition (12) in this case is

$$\mathbb{C}^n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$$

and the associated system of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \, : \, 1 \le i < j \le n\}.$$

Let $\lambda = \sum \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{N}$ be a weight of H. Suppose $\lambda_1 \ge \cdots \ge \lambda_{n-1} \ge 0$ is Φ^+-dominant. Then $t^\lambda = 1$ for all $t \in T$ if and only if λ_i is even for all i. Thus λ is θ-admissible if and only if

$$\lambda = 2m_1\varpi_1 + \cdots + 2m_l\varpi_l, \quad m_i \in \mathbb{N},$$

where $\varpi_1, \ldots, \varpi_l$ (with $l = n - 1$) are the fundamental weights. The Satake diagram is shown in Figure 12.1.

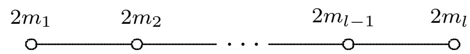


Figure 12.1: Satake Diagram of Type AI

Bilinear Forms–Type AII

In this case $G = \mathrm{SL}(2n, \mathbb{C})$, $\theta(g) = T_n(g^t)^{-1}T_n^{-1}$, where $T_n = \mathrm{diag}[\mu, \ldots, \mu]$ (n copies of $\mu = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$). We have $K \cong \mathrm{Sp}(n, \mathbb{C})$. The maximal torus $H = D_{2n} \cap G$ in G is θ-invariant, with

$$\theta(h) = \mathrm{diag}[x_2^{-1}, x_1^{-1}, \ldots, x_{2n}^{-1}, x_{2n-1}^{-1}]$$

for $h = \mathrm{diag}[x_1, \ldots, x_{2n}]$. Thus A consists of all matrices

$$a = \mathrm{diag}[x_1, x_1, \ldots, x_n, x_n], \qquad x_1 x_2 \cdots x_n = 1,$$

and has rank $n-1$. We take generators $\chi_1, \ldots, \chi_{n-1}$ for $\mathcal{X}(A)$ as $\chi_i(a) = \varepsilon_{2i-1}(a)$ and give $\mathcal{X}(A)$ the corresponding lexicographic order. The group T consists of all matrices

$$t = \mathrm{diag}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}], \quad x_i \in \mathbb{C}^\times$$

and is a torus of rank n. The roots vanishing on $\mathfrak{a}$ are

$$\Phi_0 = \{\pm(\varepsilon_1 - \varepsilon_2), \ldots, \pm(\varepsilon_{2n-1} - \varepsilon_{2n})\}$$

and a calculation shows that θ acts by 1 on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_0$. Thus condition $(*)$ is satisfied and hence A is a maximal θ-anisotropic torus.

The decomposition (12) in this case is

$$\mathbb{C}^{2n} = V_1 \oplus \cdots \oplus V_n, \qquad V_i = \mathbb{C}e_{2i-1} + \mathbb{C}e_{2i}.$$

Note that V_i is non-isotropic for the skew form defined by T_n. One calculates that M consists of the block diagonal matrices

$$m = \mathrm{diag}[g_1, \ldots, g_n], \quad g_i \in \mathrm{Sp}(V_i). \tag{13}$$

Thus $M \cong \times^n \mathrm{Sp}(1, \mathbb{C})$.

The ordered basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n-1}$ for $\mathcal{X}(H)$ is compatible with the order we have given to $\mathcal{X}(A)$. Let Φ^+ be the corresponding system of positive roots. Let $\lambda = \sum_{i=1}^{2n-1} \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \cdots \geq \lambda_{2n-1} \geq 0$ be a Φ^+-dominant weight. Then $t^\lambda = 1$ for all $t \in T$ if and only if $\lambda_{2i-1} = \lambda_{2i}$ for $i = 1, \ldots, n-1$ and $\lambda_{2n-1} = 0$. Thus λ is θ-admissible if and only if

$$\lambda = m_2\varpi_2 + \cdots + m_{l-1}\varpi_{l-1}, \quad m_i \in \mathbb{N}.$$

The Satake diagram is shown in Figure 12.2; note that $l = n - 1$ is odd.

Figure 12.2: Satake Diagram of Type AII

Polarizations–Type AIII

We have $G = \mathrm{SL}(n, \mathbb{C})$ and $\theta(g) = J_{p,q} g J_{p,q}$ with $0 < p \leq q$ and $p + q = n$. Here

$$J_{p,q} = \begin{bmatrix} 0 & 0 & s_p \\ 0 & I_{q-p} & 0 \\ s_p & 0 & 0 \end{bmatrix}.$$

We have $K \cong \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$. The maximal torus $H = D_n \cap G$ is θ-invariant. For $h \in H$ write $h = \mathrm{diag}[a, b, c]$, with $a, c \in D_p$ and $b \in D_{q-p}$. Then

$$\theta(h) = \mathrm{diag}[\check{c}, b, \check{a}].$$

Thus $A \cong D_p$ consists of all

$$h = \mathrm{diag}[a, I_{q-p}, \check{a}^{-1}], \quad a \in D_p.$$

We take generators $\chi_1, \ldots, \chi_p$ for $\mathcal{X}(A)$ as $\chi_i(h) = \varepsilon_i(a)$, and we give $\mathcal{X}(A)$ the corresponding lexicographic order. We have $h \in T$ provided

$$h = \mathrm{diag}[a, b_{q-p}, \check{a}], \quad a \in D_p,\, b \in D_{q-p},\, \det(h) = 1.$$

Thus $T \cong D_{q-1}$ is connected. The roots vanishing on $\mathfrak{a}$ are

$$\Phi_0 = \{\pm(\varepsilon_i - \varepsilon_j) \,:\, p + 1 \leq i < j \leq q\},$$

and it is obvious that θ acts by 1 on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_0$. Thus condition $(*)$ is satisfied and hence A is a maximal θ-anisotropic torus.

The decomposition (12) in this case is

$$\mathbb{C}^{2n} = V_1 \oplus \cdots \oplus V_p \oplus V_0 \oplus V_{-p} \oplus \cdots \oplus V_{-1}$$

where $V_i = \mathbb{C}e_i$, $V_0 = \mathbb{C}e_{p+1} + \cdots + \mathbb{C}e_q$ and $V_{-i} = \mathbb{C}e_{n+1-i}$ for $i = 1, \ldots, p$. Here A acts on $V_{\pm i}$ by the character $\chi_i^{\pm 1}$ and acts on V_0 by 1. Hence M consists of the block diagonal matrices

$$x = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \check{a} \end{bmatrix}, \qquad a \in D_p, \quad b \in \mathrm{GL}(q-p, \mathbb{C}), \quad \det x = 1.$$

The ordered basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ for $\mathcal{X}(H)$ is compatible with the order we have given to $\mathcal{X}(A)$. Let Φ^+ be the corresponding system of positive roots. Let $\lambda = \sum \lambda_i \varepsilon_i$, with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \cdots \geq \lambda_n$, be a Φ^+-dominant weight. Then $t^\lambda = 1$ for all $t \in T$ if and only if $\lambda_1 = -\lambda_n$, $\lambda_2 = -\lambda_{n-1}, \ldots, \lambda_p = -\lambda_{q+1}$, and $\lambda_j = 0$ for $p+1 \le j \le q$. Thus λ is θ-admissible if and only if

$$\lambda = [\lambda_1, \ldots, \lambda_p, \underbrace{0, \ldots, 0}_{q-p}, -\lambda_p, \ldots, -\lambda_1]$$

where $\lambda_1 \ge \cdots \lambda_p \ge 0$ are arbitrary integers. Thus λ is θ-admissible if and only if

$$\lambda = m_1(\varpi_1 + \varpi_l) + m_2(\varpi_2 + \varpi_{l-1}) \cdots + m_p(\varpi_p + \varpi_q), \quad m_i \in \mathbb{N}.$$

The Satake diagram is shown in Figure 12.3.

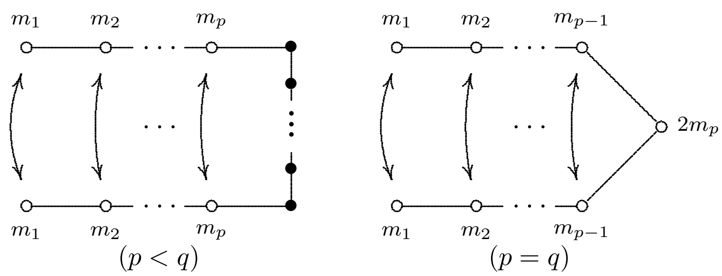


Figure 12.3: Satake Diagrams of Type AIII

Polarizations–Type CI

Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$ where Ω is the bilinear form with matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We take the involution $\theta(g) = -JgJ$. Here $K \cong \mathrm{GL}(n, \mathbb{C})$ and the maximal torus $H = D_{2n} \cap G$ is θ-anisotropic. Hence $A = H$ and $M = T \cong (\mathbb{Z}/2\mathbb{Z})^n$ consists of all matrices

$$t = \mathrm{diag}[\delta_1, \ldots, \delta_n, \delta_1, \ldots, \delta_n], \quad \delta_i = \pm 1.$$

Since $\Phi_0 = \emptyset$, condition $(*)$ is vacuously satisfied and hence A is a maximal θ-anisotropic torus. We define an order on $\mathcal{X}(A)$ using the characters

$$\chi_i(h) = \varepsilon_i(a), \qquad h = \mathrm{diag}[a, a^{-1}]$$

for $i = 1, \ldots, n$. Let Φ^+ be the corresponding system of positive roots. Let $\lambda = \sum_{i=1}^{n} \lambda\varepsilon_i$ with $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$ be a dominant weight. Then $t^\lambda = 1$ for all $t \in T$ if and only if λ_i is even for all i. Thus λ is θ-admissible if and only if

$$\lambda = 2m_1\varpi_1 + \cdots + 2m_l\varpi_l, \quad m_i \in \mathbb{N},$$

where $\varpi_1, \ldots, \varpi_l$ (with $l = n-1$) are the fundamental weights. The Satake diagram is shown in Figure 12.1.

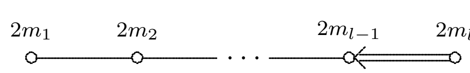


Figure 12.4: Satake Diagram of Type CI

Polarizations–Type DIII

Take $G = \mathrm{SO}(\mathbb{C}^n, B)$, with $n = 2l$ even and B the form with matrix s_n. We take the involution $\theta(g) = -\Gamma_n g \Gamma_n$ with Γ_n defined as in Section 3.2. As in Type CII, we have $K \cong \mathrm{GL}(l, \mathbb{C})$. The maximal torus $H = D_n \cap G$ is θ-stable. Write elements of H as

$$h = \mathrm{diag}[a, \check{a}^{-1}], \quad \text{with } a = [a_1, \ldots, a_l].$$

Then $\theta(h) = \mathrm{diag}[b, \check{b}^{-1}]$, where

$$b = \begin{cases} \mathrm{diag}[a_2, a_1, \ldots, a_{2p}, a_{2p-1}] & \text{(when } l = 2p) \\ \mathrm{diag}[a_2, a_1, \ldots, a_{2p}, a_{2p-1}, a_{2p+1}] & \text{(when } l = 2p+1) \end{cases}$$

We have $A \cong D_p$ consisting of all $h = \mathrm{diag}[a, \check{a}^{-1}]$ with

$$a = \begin{cases} \mathrm{diag}[x_1, x_1^{-1}, \ldots, x_p, x_p^{-1}] & \text{(when } l = 2p) \\ \mathrm{diag}[x_1, x_1^{-1}, \ldots, x_p, x_p^{-1}, 1] & \text{(when } l = 2p+1) \end{cases}$$

We take generators $\chi_1, \ldots, \chi_p$ for $\mathcal{X}(A)$ as $\chi_i(a) = \varepsilon_{2i-1}(a)$, where $p = [l/2]$, and we put the corresponding lexicographic order on $\mathcal{X}(A)$. The group T consists of all matrices $h = \mathrm{diag}[a, \check{a}^{-1}]$ with

$$a = \begin{cases} \mathrm{diag}[x_1, x_1^{-1}, \ldots, x_p, x_p^{-1}] & \text{(when } l = 2p) \\ \mathrm{diag}[x_1, x_1^{-1}, \ldots, x_p, x_p^{-1}, 1] & \text{(when } l = 2p+1) \end{cases}$$

Thus $T = T^\circ$ is a torus of rank p (when l is even) or rank $p+1$ (when l is odd). The roots vanishing on $\mathfrak{a}$ are

$$\pm(\varepsilon_1 + \varepsilon_2), \pm(\varepsilon_3 + \varepsilon_4), \ldots, \pm(\varepsilon_{2p-1} + \varepsilon_{2p}) \,.$$

We leave it as an exercise to check that θ acts by 1 on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_0$. Thus condition $(*)$ is satisfied and hence A is a maximal θ-anisotropic torus.

The χ_i eigenspace for A on $\mathbb{C}^n$ is

$$V_i = \mathbb{C}e_{2i-1} + \mathbb{C}e_{n-2i+1}, \quad i = 1, \ldots, p$$

and the χ_i^{-1} eigenspace is

$$V_{-i} = \mathbb{C}e_{2i} + \mathbb{C}e_{n-2i+2}, \quad i = 1, \ldots, p.$$

When $l = 2p+1$ there is also the space $V_0 = \mathbb{C}e_l + \mathbb{C}e_{l+1}$ where A acts by 1. The subspaces $V_{\pm 1}, \ldots, V_{\pm p}$ are B-isotropic, while B is non-degenerate on V_0 (when $l = 2p+1$). The space V_{-i} is dual to V_i relative to B. We have

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_p \oplus V_0 \oplus V_{-1} \oplus \cdots \oplus V_{-p}$$

(where we set $V_0 = 0$ when $l = 2p$ is even). From this decomposition we calculate that

$$M \cong \begin{cases} \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_p) & \text{when } l = 2p \\ \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_p) \times \mathrm{SO}(V_0) & \text{when } l = 2p+1 \end{cases} \tag{14}$$

(note that $\mathrm{SO}(V_0) \cong \mathrm{GL}(1, \mathbb{C})$ since $\dim V_0 = 2$).

The ordered basis

$$\begin{array}{ll} \varepsilon_1 > -\varepsilon_2 > \varepsilon_3 > -\varepsilon_4 > \cdots > \varepsilon_{2p-1} > -\varepsilon_{2p} & (\text{when } l = 2p), \\ \varepsilon_1 > -\varepsilon_2 > \varepsilon_3 > -\varepsilon_4 > \cdots > \varepsilon_{2p-1} > -\varepsilon_{2p} > \varepsilon_{2p+1} & (\text{when } l = 2p+1) \end{array}$$

for $\mathcal{X}(H)$ is compatible with the order on $\mathcal{X}(A)$. Let Φ^+ be corresponding system of positive roots. Let Φ^+ be the corresponding system of positive roots. We see that Φ^+ is obtained from the standard choice of positive roots by the action of the Weyl group element that tranforms the ordered basis $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{l-1} > \pm\varepsilon_l$ into the ordered basis above (the choice of $\pm$ depending on whether l is even or odd). It follows that when l is even, the simple roots in Φ^+ are

$$\alpha_1 = \varepsilon_1 + \varepsilon_2,\ \alpha_2 = -\varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-2} = -\varepsilon_{l-2} - \varepsilon_{l-1},$$

$$\alpha_{l-1} = \varepsilon_{l-1} + \varepsilon_l,\ \alpha_l = \varepsilon_{l-1} - \varepsilon_l.$$

If p is odd, then the simple roots in Φ^+ are

$$\alpha_1 = \varepsilon_1 + \varepsilon_2,\ \alpha_2 = -\varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-2} = \varepsilon_{l-2} + \varepsilon_{l-1},$$

$$\alpha_{l-1} = -\varepsilon_{l-1} - \varepsilon_l,\ \alpha_l = -\varepsilon_{l-1} + \varepsilon_l.$$

The simple roots vanishing on $\mathfrak{a}$ in this case are $\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}$. The roots α_{l-1} and α_l have the same restriction to $\mathfrak{a}$.

A weight $\lambda = \sum_{i=1}^{l} \lambda_i \varepsilon_i$ is Φ^+-dominant if and only if

$$\lambda_1 \geq -\lambda_2 \geq \lambda_3 \geq -\lambda_4 \geq \cdots \geq |\lambda_l|.$$

Let λ be a Φ^+-dominant weight. Then $t^\lambda = 1$ for all $t \in T$ if and only if

$$\lambda_1 = -\lambda_2,\ \lambda_3 = -\lambda_4,\ \cdots,\ \lambda_{2p-1} = -\lambda_{2p}$$

in the case $l = 2p$. When $l = 2p + 1$ there is the additional condition $\lambda_{2p+1} = 0$. Writing λ in terms of the fundamental weights, we find that it is θ-admissible if and only if

$$\lambda = \begin{cases} m_2\varpi_2 + \cdots + m_{l-2}\varpi_{l-2} + 2m_l\varpi_l & (l \text{ even}) \\ m_2\varpi_2 + \cdots + m_{l-3}\varpi_{l-3} + m_{l-1}(\varpi_{l-1} + \varpi_l) & (l \text{ odd}) \end{cases}$$

The Satake diagram is shown in Figure 12.5.

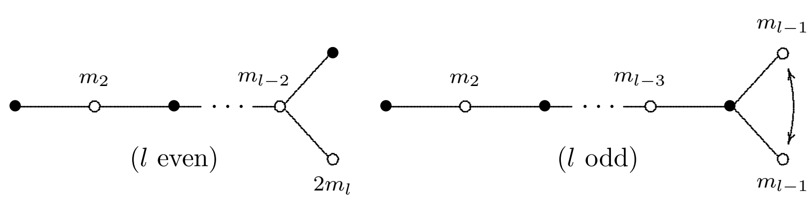


Figure 12.5: Satake Diagrams of Type DIII

Orthogonal Decompositions–Type BDI

Now $G = \mathrm{SO}(\mathbb{C}^n, B)$, the involution $\theta(g) = J_{p,q} g J_{p,q}$ with $1 \le p \le q$ and $p + q = n$ as in Type AIII, and $K \cong \mathrm{S}(\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C}))$. This is the only case in which K is not connected; we have

$$K^\circ \cong \mathrm{SO}(p, \mathbb{C}) \times \mathrm{SO}(q, \mathbb{C}).$$

The maximal torus $H = D_n \cap G$ is θ-stable. We can write $h \in H$ as $\mathrm{diag}[a, b, \check{a}^{-1}]$, with $a \in D_p$ arbitrary and b of the form

$$b = \begin{cases} \mathrm{diag}[c, \check{c}^{-1}] & \text{when } n = 2l \\ \mathrm{diag}[c, 1, \check{c}^{-1}] & \text{when } n = 2l + 1 \end{cases} \tag{15}$$

with $c \in D_{l-p}$. We calculate that

$$\theta(h) = \mathrm{diag}[a^{-1}, b, \check{a}].$$

Thus $A \cong D_p$ consists of all diagonal matrices

$$h = \mathrm{diag}[a, I_{q-p}, \check{a}^{-1}], \quad a \in D_p.$$

We take generators $\chi_1, \ldots, \chi_p$ for $\mathcal{X}(A)$ as $\chi_i(h) = \varepsilon_i(a)$ and we give $\mathcal{X}(A)$ the corresponding lexicographic order. The group T consists of all diagonal matrices $h = \mathrm{diag}[a, b, \check{a}^{-1}]$ where b is given by (15) and $a^2 = I_p$. Thus $T \cong (\mathbb{Z}/2\mathbb{Z})^p \times D_{l-p}$. The subgroup $T_0 = H \cap K^\circ$ consists of all such diagonal matrices that satisfy the additional condition $a_1 \cdots a_p = 1$.

The roots vanishing on $\mathfrak{a}$ are

$$\{\pm\varepsilon_i \pm \varepsilon_j \,:\, p + 1 \le i < j \le l\} \quad (\text{when } n = 2l)$$

$$\{\pm\varepsilon_i \pm \varepsilon_j \,:\, p + 1 \le i < j \le l\} \cup \{\varepsilon_i \,:\, p + 1 \le i \le l\} \quad (\text{when } n = 2l + 1)$$

It is clear that θ acts by 1 on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_0$, since the matrices in $\mathfrak{g}_\alpha$ are of block-diagonal form $\mathrm{diag}[0, x, 0]$, with $x \in M_{q-p}(\mathbb{C})$. Thus condition $(*)$ is satisfied and hence A is a maximal θ-anisotropic torus of rank p.

The χ_i eigenspace for A on $\mathbb{C}^n$ is

$$V_i = \mathbb{C}e_i, \quad i = 1, \ldots, p.$$

and the χ_i^{-1} eigenspace is

$$V_{-i} = \mathbb{C}e_{n+1-i}, \quad i = 1, \ldots, p.$$

The space $V_0 = \mathbb{C}e_{p+1} \oplus \cdots \oplus \mathbb{C}e_q$ is the 1-eigenspace for A. The subspaces $V_{\pm 1}, \ldots, V_{\pm p}$ are B-isotropic, while B is non-degenerate on V_0. The space V_{-i} is dual to V_i relative to B. We have

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_p \oplus V_0 \oplus V_{-1} \oplus \cdots \oplus V_{-p}. \tag{16}$$

From this decomposition we see that M consists of all matrices in block-diagonal form

$$m = \mathrm{diag}[a, b, a] \quad \text{with } a = [\pm 1, \ldots, \pm 1] \text{ and } b \in \mathrm{SO}(V_0). \tag{17}$$

The ordered basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l$ for $\mathcal{X}(H)$ is compatible with the order on $\mathcal{X}(A)$. Let Φ^+ be the corresponding system of positive roots. Let $\lambda = \sum_{i=1}^{l} \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{N}$ and suppose λ is Φ^+-dominant. Then $t^\lambda = 1$ for all $t \in T$ if and only if $\lambda_j = 0$ for $p+1 \le j \le l$ and λ_i is even for $i = 1, \dots, p$. Thus λ is θ-admissible if and only if

$$\lambda = [\lambda_1, \dots, \lambda_p, \underbrace{0, \dots, 0}_{l-p}]$$

where $\lambda_1 \ge \cdots \lambda_p \ge 0$ are arbitrary even integers. If we only require that $t^\lambda = 1$ for all $t \in H \cap K^\circ$, then the parity condition becomes

$$\lambda_i - \lambda_j \in 2\mathbb{Z} \quad \text{for all } 1 \le i < j \le p.$$

We shall say that λ is K°-*admissible* when this is satisfied. When we write λ in terms of the fundamental dominant weights the admissibility conditions become the following. First assume $n = 2l+1$ is odd (Type BI). Then λ is θ-admissible if and only if

$$\lambda = \begin{cases} 2m_1\varpi_1 + 2m_2\varpi_2 + \cdots + 2m_p\varpi_p & (p < l) \\ 2m_1\varpi_1 + 2m_2\varpi_2 + \cdots + 2m_{l-1}\varpi_{l-1} + 4m_l\varpi_l & (p = l) \end{cases}$$

where $m_i \in \mathbb{N}$. For λ to be K°-admissible, however, the coefficient of ϖ_p only has to be an integer (not necessarily even) when $p < l$, and the coefficient of ϖ_l only has to be even (not necessarily a multiple of 4) when $p = l$. The Satake diagram is shown in Figure 12.6, where the coefficients shown in parentheses apply to K°-admissible weights.

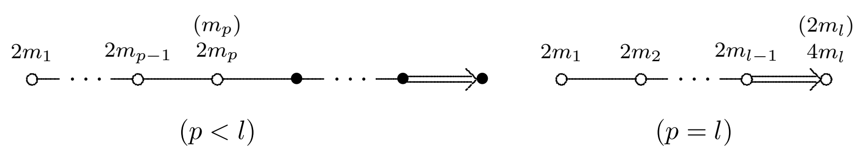


Figure 12.6: Satake Diagrams of Type BI

Now assume $n = 2l$ is even (Type DI). λ is θ-admissible if and only if

$$\lambda = \begin{cases} 2m_1\varpi_1 + 2m_2\varpi_2 + \cdots + 2m_p\varpi_p & (p < l-1) \\ 2m_1\varpi_1 + 2m_2\varpi_2 + \cdots + 2m_{l-2}\varpi_{l-2} + 2m_{l-1}(\varpi_{l-1} + \varpi_l) & (p = l-1) \\ 2m_1\varpi_1 + 2m_2\varpi_2 + \cdots + 2m_{l-2}\varpi_{l-2} + 2m_{l-1}\varpi_{l-1} + 2m_l\varpi_l & (p = l) \end{cases}$$

where $m_i \in \mathbb{N}$. For λ to be K°-admissible, however, the coefficient of ϖ_p only has to be an integer (not necessarily even) when $p < l-1$, while the coefficients of ϖ_{l-1} and ϖ_l have to be equal integers (not necessarily even) when $p = l-1$. When $p = l$ then K°-admissibility is the same as θ-admissibility. The Satake diagram is shown in Figure 12.7, where the coefficients shown in parentheses apply to K°-admissible weights.

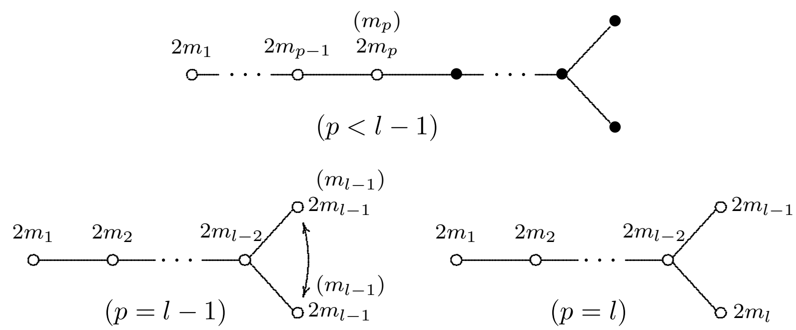


Figure 12.7: Satake Diagrams of Type DI

Orthogonal Decompositions–Type CII

In this case $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ where Ω is the bilinear form with matrix $J = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix}$. We take the involution $\theta(g) = K_{p,q} g K_{p,q}$, for $1 \le p \le q$ and $p + q = l$, with $K_{p,q}$ as in Section 3.2. We have $K \cong \mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C})$. The maximal torus $H = D_l \cap G$ is θ-stable. We write $h \in H$ as

(18) $$h = \mathrm{diag}[x, \check{x}^{-1}], \quad \text{where } x = \mathrm{diag}[a, b, c] \quad \text{with } a, c \in D_p,\ b \in D_{q-p}.$$

Then $\theta(h) = \mathrm{diag}[y, \check{y}^{-1}]$ with $y = \mathrm{diag}[\check{c}, b, \check{a}]$. Thus $A \cong D_p$ consists of all

$$h = \mathrm{diag}[x, x] \quad \text{with } x = \mathrm{diag}[a, I_{q-p}, \check{a}^{-1}], \quad a \in D_p.$$

We take generators $\chi_1, \ldots, \chi_p$ for $\mathcal{X}(A)$ as $\chi_i(h) = \varepsilon_i(a)$ and we give $\mathcal{X}(A)$ the corresponding lexicographic order. The group T consists of all

$$h = \mathrm{diag}[x, \check{x}^{-1}] \quad \text{with } x = \mathrm{diag}[a, b, \check{a}], \quad a \in D_p,\ b \in D_{q-p}.$$

Thus $T \cong D_q$ is connected.

The roots vanishing on $\mathfrak{a}$ are

$$\pm\varepsilon_i \pm \varepsilon_j, \quad p < i \le j \le q$$

$$\pm(\varepsilon_1 + \varepsilon_l), \pm(\varepsilon_2 + \varepsilon_{l-1}), \ldots, \pm(\varepsilon_p + \varepsilon_{q+1}).$$

A calculation similar to that done above in Type AIII shows that $\theta = 1$ on the corresponding root spaces. Thus condition $(*)$ is satisfied and hence A is maximal θ-anisotropic.

The χ_i eigenspace for A on $\mathbb{C}^{2l}$ is

$$V_i = \mathbb{C}e_i + \mathbb{C}e_{l+i}, \quad i = 1, \ldots, p$$

and the χ_i^{-1} eigenspace is

$$V_{-i} = \mathbb{C}e_{l+1-i} + \mathbb{C}e_{2l+1-i}, \quad i = 1, \ldots, p.$$

The 1-eigenspace of A is

$$V_0 = \mathbb{C}e_{p+1} + \cdots + \mathbb{C}e_q + \mathbb{C}e_{l+p+1} + \cdots + \mathbb{C}e_{l+q}.$$

The subspaces $V_{\pm 1}, \ldots, V_{\pm r}$ are Ω-isotropic, while Ω is non-degenerate on V_0. The space V_{-i} is dual to V_i relative to Ω. We have

$$\mathbb{C}^{2l} = V_1 \oplus \cdots \oplus V_p \oplus V_0 \oplus V_{-1} \oplus \cdots \oplus V_{-p}.$$

The elements of M leave invariant the spaces $V_{\pm i}$ and V_0, while the transformation $K_{p,q}$ acts by I on V_0 and interchanges V_i and V_{-i}. From this decomposition one calculates that

$$M \cong (\times^p \mathrm{Sp}(1, \mathbb{C})) \times \mathrm{Sp}(q-p, \mathbb{C}). \tag{19}$$

The weights of H on V_i are ε_i and $-\varepsilon_{l+1-i}$ for $i = 1, \ldots, p$ and the weights of H on V_0 are $\pm\varepsilon_i$ for $i = p+1, \ldots, q$. Hence the ordered basis

$$\varepsilon_1 > -\varepsilon_l > \varepsilon_2 > -\varepsilon_{l-1} > \cdots > \varepsilon_p > -\varepsilon_{q+1} > \varepsilon_{p+1} > \varepsilon_{p+2} > \cdots > \varepsilon_q$$

for $\mathcal{X}(H)$ is compatible with the order we have given to $\mathcal{X}(A)$. Let Φ^+ be the corresponding system of positive roots. Since Φ^+ is obtained from the usual set of positive roots by the action of the Weyl group element that tranforms the ordered basis $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_l$ into the ordered basis above, it follows that the simple roots in Φ^+ are

$$\alpha_1 = \varepsilon_1 + \varepsilon_l,\ \alpha_2 = -\varepsilon_l - \varepsilon_2, \ldots, \alpha_{2p-1} = \varepsilon_p + \varepsilon_{q+1},\ \alpha_{2p} = -\varepsilon_{q+1} - \varepsilon_{p+1},$$

$$\alpha_{2p+1} = \varepsilon_{p+1} - \varepsilon_{p+2}, \ldots, \alpha_{l-1} = \varepsilon_{q-1} - \varepsilon_q,\ \alpha_l = 2\varepsilon_q.$$

The simple roots vanishing on $\mathfrak{a}$ are thus $\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}$.

Thus $\lambda = \sum_{i=1}^{l} \lambda_i \varepsilon_i$ is Φ^+-dominant if and only if

$$\lambda_1 \geq -\lambda_l \geq \lambda_2 \cdots \geq \lambda_p \geq -\lambda_{q+1} \geq \lambda_{p+1} \geq \cdots \lambda_q \geq 0$$

We see that $t^\lambda = 1$ for all $t \in T$ if and only if

$$\lambda_1 = -\lambda_l,\ \lambda_2 = -\lambda_{l-1},\ \cdots\ \lambda_p = -\lambda_{q+1}$$

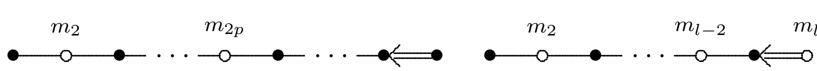
and $\lambda_j = 0$ for $j = p+1, \ldots, q$. Thus λ is θ-admissible if and only if

$$\lambda = [\lambda_1, \ldots, \lambda_p, \underbrace{0, \ldots, 0}_{q-p}, -\lambda_p, \ldots, -\lambda_1]$$

where $\lambda_1 \geq \cdots \lambda_p \geq 0$ are arbitrary integers. When we write λ in terms of the fundamental weights, then it is θ-admissible if and only if

$$\lambda = \begin{cases} m_2\varpi_2 + m_4\varpi_4 + \cdots + m_{2p}\varpi_{2p} & (2p < l) \\ m_2\varpi_2 + m_4\varpi_4 + \cdots + m_{l-2}\varpi_{l-2} + m_l\varpi_l & (2p = l) \end{cases}$$

where $m_i \in \mathbb{N}$. The Satake diagram is shown in Figure 12.8.



$(2p < l)$ $\qquad\qquad$ $(2p = l)$

Figure 12.8: Satake Diagrams of Type CII

3.4 Spherical Representations

We now determine the irreducible representations having a K-fixed vector. We follow the notation of the previous section: θ is an involution of the connected semisimple classical group G, H is a θ-stable Cartan subalgebra, $K = G^\theta$ or $K = (G^\theta)^\circ$ (in type BDI). We fix A, a maximal θ-anisotropic torus in H, and denote $M = \mathrm{Cent}_K(A)$, B a θ-admissible Borel subgroup of G, $N^+ \subset B$.

Theorem 3.10 (Helgason) *Let (π^λ, V^λ) be an irreducible regular representation of G with highest weight λ (relative to B). The following are equivalent:*
(1) *V^λ contains a nonzero K-fixed vector.*
(2) *V^λ contains a nonzero MN^+-fixed vector.*
(3) *$t^\lambda = 1$ for all $t \in T = H \cap K$.*

Proof. We have already shown that (1) $\Longrightarrow$ (3) in Theorem 3.9. We observe that (2) is equivalent to

$(2)'$ $\pi(M)v_\lambda = v_\lambda$, where v_λ is a nonzero B-extreme vector in V^λ.

This is clear, since MN^+ contains the unipotent radical of B, so $V^{MN^+} \subset \mathbb{C}v_\lambda$. Since $T \subset M$, clearly $(2)' \Longrightarrow (3)$.

Suppose (3) holds. We shall prove that $(2)'$ holds. We have $M = F \cdot M^\circ$ by Lemma 3.7 with $F \subset T$. Since T fixes v^λ, we only need to show that

$$d\pi^\lambda(\mathfrak{g}_\alpha)v_\lambda = 0 \quad \text{for all } \alpha \in \Phi_0 \tag{20}$$

Suppose $\alpha \in \Phi_0^+$. Then (20) is true since $\lambda + \alpha$ is not a weight of V^λ. Now by (3) we have $\lambda(\mathfrak{t}) = 0$. But for $\alpha \in \Phi_0$, the coroot $h_\alpha \in \mathfrak{t}$. Hence the reflection s_α fixes λ, since

$$s_\alpha(\lambda) = \lambda - \langle \lambda, h_\alpha \rangle \alpha = \lambda.$$

Thus $s_\alpha(\lambda - \alpha) = \lambda - s_\alpha(\alpha) = \lambda + \alpha$. If $\lambda - \alpha$ were a weight of V^λ, then $\lambda + \alpha$ would be a weight also by Weyl group symmetry, a contradiction. Hence $d\pi^\lambda(\mathfrak{g}_{-\alpha})v_\lambda = 0$ also.

Thus it only remains to prove that (3) $\Longrightarrow$ (1). For this we will need the following lemma. We take the conjugations τ and σ on G as in Section 3.2 and write $G_0 = G^\sigma$, $K_0 = K^\sigma = K^\tau$ and $A_0 = A^\sigma$ for the corresponding real forms. The group K_0 is compact, while G_0 and A_0 are noncompact.

Lemma 3.11 *Let $\lambda \in \mathcal{X}(H)$. Suppose $t^\lambda = 1$ for all $t \in T$. Then $a^\lambda > 0$ for all $a \in A_0$.*

Proof. We take G in the matrix form as in Section 3.2. For $a \in A$ we have $\sigma(a) = \bar{a}$. Thus $A_0 = A^\sigma$ consists of real matrices. In Section 3.3 we gave an isomorphism $\phi : A \cong D_p$ with A_0 corresponding to the real matrices in D_p. Let $\phi(a) = \mathrm{diag}[x_1, \ldots, x_p]$ for $a \in A$. Then

$$a^\lambda = x_1^{m_1} \cdots x_p^{m_p} \quad \text{with } m_i \in \mathbb{Z}.$$

By Lemma 3.7 we have $T \cap A = F = \{a \in A : a^2 = 1\}$. Under the isomorphism ϕ,

$$F \cong \{[\epsilon_1, \ldots, \epsilon_p] : \epsilon_i = \pm 1\}.$$

Thus $a^\lambda = 1$ for all $a \in F$ if and only if $m_i \in 2\mathbb{Z}$ for $i = 1, \ldots, p$. Obviously this implies that $a^\lambda > 0$ when $x_i \in \mathbb{R}$ for $i = 1, \ldots, p$. $\square$

Completion of proof of Theorem 3.10:

We assume (3) (and hence (2)$'$). Define

$$v_0 = \int_{K_0} \pi^\lambda(k) v_\lambda \, dk.$$

Then v_0 is invariant under K_0. Since K_0 is a compact real form of K, we also have v_0 invariant under K, by analytic continuation. To complete the proof, we only need to show that $v_0 \neq 0$ when condition (3) is satisfied.

Let v_λ^* be the lowest weight vector for the dual representation $(V^\lambda)^*$, normalized so that $\langle v_\lambda^*, v_\lambda \rangle = 1$. Let $f_\lambda(g) = \langle v_\lambda^*, \pi^\lambda(g) v_\lambda \rangle$ be the generating function for π^λ, as in Section 2.2. Then

$$\langle v_\lambda^*, v_0 \rangle = \int_{K_0} f_\lambda(k) \, dk. \tag{21}$$

We shall show that

$$f_\lambda(g) \geq 0 \quad \text{for } g \in G_0. \tag{22}$$

Since $f_\lambda(1) = 1$, this will imply that the integral (21) is positive, and hence $v_0 \neq 0$.

For $m \in M$, $a \in A$, and $n^\pm \in N^\pm$ we have

$$\begin{aligned} f_\lambda(n^- m a n^+) &= \langle \pi^{\lambda^*}(n^-)^{-1} v_\lambda^*, \pi^\lambda(a) \pi^\lambda(m n^+) v_\lambda \rangle \\ &= \langle v_\lambda^*, \pi^\lambda(a) v_\lambda \rangle = a^\lambda, \end{aligned}$$

since N^- fixes v_λ^* and MN^+ fixes v_λ (by (2)$'$). Hence if $a \in A_0$ then $f_\lambda(n^- m a n^+) > 0$ by Lemma 3.11. Since $N_0^- A_0 M_0 N_0^+$ is dense in G_0 (by the Gauss decomposition), this proves (22). $\square$

Corollary 3.12 *As a G-module,* $\mathrm{Aff}(G/K) \cong \bigoplus_\lambda V^\lambda$, *where λ runs over all θ-admissible dominant weights of H.*

Proof. This follows by Theorem 3.9 and Theorem 3.10. $\square$

From Corollary 3.12 and the Satake diagrams in Section 3.3, we conclude that the semigroups of highest weights for spherical representations have the following generators (where l is the rank of $\mathfrak{g}$ and $p = \dim \mathfrak{a}$ is the rank of G/K):

Type AI: $\{2\varpi_1, 2\varpi_2, \ldots, 2\varpi_l\}$ $\quad (p = l)$

Type AII: $\{\varpi_2, \varpi_4, \ldots, \varpi_l\}$ $\quad (p = l/2)$

Type AIII: $\{\varpi_1 + \varpi_l, \varpi_2 + \varpi_{l-1}, \ldots, \varpi_p + \varpi_{l+1-p}\}$ $\quad (2p \leq l + 1)$

Type CI: $\{2\varpi_1, 2\varpi_2, \ldots, 2\varpi_l\}$ $\quad (p = l)$

Type DIII: $\{\varpi_2, \varpi_4, \ldots, \varpi_{l-2}, 2\varpi_l\}$ $\quad (l \text{ even}, p = l/2)$

$\{\varpi_2, \varpi_4, \ldots, \varpi_{l-3}, \varpi_{l-1} + \varpi_l\}$ $\quad (l \text{ odd}, p = (l-1)/2)$

Type BI: $\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_p\}\quad (p < l)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-1},\, 4\varpi_l\}\quad (p = l)$

Type DI: $\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_p\}\quad (p < l-1)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-2},\, \varpi_{l-1} + \varpi_l\}\quad (p = l-1)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-2},\, 2\varpi_{l-1},\, 2\varpi_l\}\quad (p = l)$

Type CII: $\{\varpi_2, \varpi_4, \dots, \varpi_{2p}\}\quad (2p \le l)$

If we take $K = (G^\theta)^\circ$ in Type BDI (the only case in which G^θ is not connected), then the semigroup of highest weights of K-spherical representations has the following generators:

Type BI: $\{2\varpi_1, \dots, 2\varpi_{p-1},\, \varpi_p\}\quad (p < l)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-1},\, 2\varpi_l\}\quad (p = l)$

Type DI: $\{2\varpi_1, \dots, 2\varpi_{p-1},\, \varpi_p\}\quad (p < l-1)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-2},\, \varpi_{l-1} + \varpi_l\}\quad (p = l-1)$

$\{2\varpi_1,\, 2\varpi_2, \dots, 2\varpi_{l-2},\, 2\varpi_{l-1},\, 2\varpi_l\}\quad (p = l)$

4 Tangent-Space Representations

4.1 A Theorem of Kostant and Rallis

We now turn to the tangent space analysis. Let G be a connected, reductive, linear algebraic group. Let θ be an involutive regular automorphism of G. Let $\mathfrak{g}$ be the Lie algebra of G. We will write θ for $d\theta$. Let

$$K = \{g \in G \,:\, \theta(g) = g\}.$$

One can show that K is reductive (when G is a classical group we verified this by classification in Section 3.2).

As usual, we will consider $\mathfrak{k}$ =Lie(K) to be a Lie subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{k} = \{X \in \mathfrak{g} \,:\, \theta X = X\}.$$

Set $V = \{X \in \mathfrak{g} \,:\, \theta X = -X\}$ (this subspace was denoted by $\mathfrak{p}$ earlier). Then $\mathfrak{g} = \mathfrak{k} \oplus V$ as a K-module under $\mathrm{Ad}|_K$. Set $\sigma(k) = \mathrm{Ad}(k)|_V$ for $k \in K$. Then (σ, V) is a regular representation of K. Note that we may identify V with the tangent space to G/K at the coset K, with the action of K on V being the natural isotropy representation on the tangent space at a fixed point.

Let $\mathcal{P}(V)$ denote the polynomial functions on V and let $\mathcal{P}^j(V)$ denote the space of homogeneous polynomials on V of degree j. As usual, we have a representation μ of K on $\mathcal{P}(V)$ given by

$$\mu(k)f(v) = f(\sigma(k)^{-1}v)\quad \text{for } f \in \mathcal{P}(V),\, k \in K \text{ and } v \in V.$$

Let $\mathcal{P}(V)^K = \{f \in \mathcal{P}(V) \,:\, \mu(k)f = f \text{ for all } k \in K\}$. Then $\mathcal{P}(V)^K$ is graded by degree. Set

$$\mathcal{P}_+(V)^K = \{f \in \mathcal{P}(V)^K \,:\, f(0) = 0\}.$$

Let $U = \{g \in G : \tau(g) = g\}$. Let $K_0 = U \cap K$. Then K_0 is a compact form of K. Let $\mathfrak{u} = \mathrm{Lie}(U)$, thought of as a real subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap V).$$

Clearly $\mathfrak{u} \cap \mathfrak{k} = \mathrm{Lie}(K_0)$, which we denote by $\mathfrak{k}_0$. Set $V_0 = i(V \cap \mathfrak{u})$. Let K_0° be the identity component of K_0. Let $\mathfrak{a}_0 \subset V_0$ be a subspace that is maximal with respect to the condition that $[\mathfrak{a}_0, \mathfrak{a}_0] = 0$ and set

$$M = \{k \in K : \mathrm{Ad}(k)|_{\mathfrak{a}_0} = I\}.$$

Then clearly $\tau(M) = M$, where τ is the conjugation of G whose fixed-point set is a compact real form of G (we may assume τ commutes with θ). Thus M has a compact real form so M is reductive.

To state the Kostant-Rallis Theorem we need one more ingredient. We note that the subspace $\mathcal{P}^j(V) \cap (\mathcal{P}(V)\mathcal{P}_+(V)^K)$ is K-invariant. Thus there is a K-invariant subspace $\mathcal{H}^j$ in $\mathcal{P}^j(V)$ such that

$$\mathcal{P}^j(V) = \mathcal{H}^j \oplus \left\{\mathcal{P}^j(V) \cap (\mathcal{P}(V)\mathcal{P}_+(V)^K)\ \right\}.$$

Set $\mathcal{H} = \bigoplus_{j \geq 0} \mathcal{H}^j$.

Theorem 4.1 (Kostant-Rallis) *The map $\mathcal{H} \otimes \mathcal{P}(V)^K \longrightarrow \mathcal{P}(V)$ given by $h \otimes f \mapsto hf$ is a linear bijection. Furthermore $\mathcal{H}$ is equivalent with $\mathrm{Aff}(K/M) = \mathrm{Ind}_M^K(1)$ as a K-module. In particular, if (ρ, F) is an irreducible regular representation of K then $\mathrm{Hom}_K(F, \mathcal{P}(V))$ is a free $\mathcal{P}(V)^K$ module on $\dim F^M$ generators.*

The Kostant-Rallis Theorem generalizes a celebrated theorem of Kostant concerning the adjoint representation.

Theorem 4.2 (Kostant) *Let G be a connected, reductive, linear algebraic group. Let T be a maximal torus in G. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mu(g)f(X) = f(\mathrm{Ad}(g)^{-1}X)$ for $g \in G$, $f \in \mathcal{P}(\mathfrak{g})$, $X \in \mathfrak{g}$. Let*

$$\mathcal{P}(\mathfrak{g})^G = \{f \in \mathcal{P}(\mathfrak{g}) : \mu(g)f = f \text{ for all } g \in G\}.$$

Let $\mathcal{H}$ be a graded $\mu(G)$-invariant subspace of $\mathcal{P}(\mathfrak{g})$ such that

$$\mathcal{P}(\mathfrak{g}) = \mathcal{H} \oplus \left\{\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^G\right\}.$$

Then the map $\mathcal{H} \otimes \mathcal{P}(\mathfrak{g})^G \longrightarrow \mathcal{P}(\mathfrak{g})$ given by $h \otimes f \mapsto hf$ is a linear bijection and $(\mu, \mathcal{H})$ is equivalent with $\mathrm{Ind}_T^G(1)$ as a representation of G. In particular, if (ρ, F) is an irreducible regular representation of G then the space $\mathrm{Hom}_G(F, \mathcal{P}(\mathfrak{g}))$ of covariants of type ρ is a free $\mathcal{P}(\mathfrak{g})^G$ module on $\dim F^T$ generators, where F^T is the zero weight space in F.

To deduce this result from Theorem 4.1, take $G_1 = G \times G$ in place of G in Theorem 4.1 and let $\theta(g, h) = (h, g)$ for $(g, h) \in G_1$. Then $K = G_1^\theta \cong G$ (embedded diagonally). Let $\mathfrak{g}$ be the Lie algebra of G. Then (σ, V) is equivalent with $(\mathrm{Ad}, \mathfrak{g})$ as a representation of G. The complexification of $\mathfrak{a}_0$ is corresponds to the Lie algebra of a maximal torus of G. Hence M is a maximal torus in G.

4.2 Classical Examples

There are 16 pairs $(K, (\sigma, V))$ covered by the Kostant-Rallis Theorem, with $\mathfrak{g}$ simple and K a product of classical groups (7 pairs with $\mathfrak{g}$ classical and 9 with $\mathfrak{g}$ exceptional); see [Hel1, Ch. X, §6, Table V]. For the cases in which G is also a classical group, K and θ were determined in Sections 3.2 and M in Section 3.3. For that purpose the matrix forms of G and θ were chosen so that the diagonal subgroup H in G was a maximal torus and $A = H \cap Q$ was a maximal θ-anisotropic torus (where $\theta(g) = g^{-1}$ for $g \in Q$). In the following examples we have chosen the matrix form of G and the involution θ to facilitate the description of V as a K-module. The algebraically independent generating set for $\mathcal{P}(V)^K$ is obtained from the Chevalley Restriction Theorem and the classification of the invariants for finite reflection groups. Note that when M is a finite group the restricted root system coincides with the root system of $\mathfrak{h}$ on $\mathfrak{g}$.

In the following examples, s_n and $\check{x}$ have the same meaning as in Section 3.3.

1. (Type AI) Let $G = \mathrm{SL}(n, \mathbb{C})$ and $\theta(g) = (g^t)^{-1}$. Then $K = \mathrm{SO}(n, \mathbb{C})$ and V is the space of symmetric $n \times n$ matrices of trace 0. The action of K on V is $\sigma(k)X = kXk^{-1}$. Here we take $\mathfrak{a}$ to be the diagonal matrices in $\mathfrak{g}$. We have $W(\mathfrak{a}) = W_G = \mathfrak{S}_n$. The polynomials $u_i(X) = \mathrm{tr}(X^{i+1})$, for $i = 1, \ldots, n-1$, restrict on $\mathfrak{a}$ to generators for $\mathcal{P}(\mathfrak{a})^{W(\mathfrak{a})}$. Hence $\mathcal{P}(V)^K$ is the polynomial algebra with generators $u_1, \ldots, u_{n-1}$.

2. (Type AII) Let $G = \mathrm{SL}(2n, \mathbb{C})$ and $\theta(g) = -J(g^t)^{-1}J$ where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Then $K = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$, where Ω is the bilinear form with matrix J. The space V consists of all matrices ($n \times n$ blocks)

$$X = \begin{bmatrix} A & B \\ C & A^t \end{bmatrix} \qquad \text{with } \mathrm{tr}(A) = 0, \quad B^t = -B, \quad C^t = -C. \tag{23}$$

We take $\mathfrak{a} \subset V$ as the matrices

$$X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \qquad Z = \mathrm{diag}[z_1, \ldots, z_n], \quad \mathrm{tr}(Z) = 0.$$

The restricted root system in this case is of type A_{n-1}. The polynomial $u_i(X) = \mathrm{tr}(X^{i+1})$ restricts on $\mathfrak{a}$ to $2\mathrm{tr}(Z^{i+1})$. Hence $u_1, \ldots, u_{l-1}$ generate $\mathcal{P}(V)^K$, since their restrictions generate $\mathcal{P}(\mathfrak{a})^{W(\mathfrak{a})}$.

3. (Type AIII) Let $G = \mathrm{SL}(n, \mathbb{C})$. Take $q \geq p > 0$ with $p + q = n$ and define $\theta = \theta_{q,p}$, where $\theta_{q,p} = I_{q,p} g I_{q,p}$ and $I_{q,p} = \begin{bmatrix} I_q & 0 \\ 0 & -I_p \end{bmatrix}$. Then $K = \mathrm{S}(\mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C}))$ imbedded diagonally and V consists of all matrices in block form

$$v = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}, \qquad X \in M_{q,p}(\mathbb{C}), \quad Y \in M_{p,q}(\mathbb{C}). \tag{24}$$

As a K-module $V \cong F \oplus F^*$, where $F = M_{q,p}(\mathbb{C})$ with action

$$\rho(g_1, g_2)X = g_1 X g_2^{-1} \quad \text{for } g_1 \in \mathrm{GL}(q, \mathbb{C}),\ g_2 \in \mathrm{GL}(p, \mathbb{C}),\ \det(g_1)\det(g_2) = 1$$

(we can identify F^* with $M_{p,q}(\mathbb{C})$ with action $\rho^*(g_1, g_2)Y = g_2 Y g_1^{-1}$). The restriction of ρ to $\mathrm{SL}(q, \mathbb{C}) \times \mathrm{SL}(p, \mathbb{C})$ is irreducible and equivalent to the outer tensor product $\mathbb{C}^q \widehat{\otimes} \mathbb{C}^p$ of the defining representations.

In this matrix realization we take $\mathfrak{a} \subset V$ as the matrices v in (24) with

$$X = \begin{bmatrix} Z s_p \\ 0_{q-p} \end{bmatrix}, \quad Y = \begin{bmatrix} s_p Z & 0_{q-p} \end{bmatrix}, \qquad \text{where } Z = \mathrm{diag}[z_1, \ldots, z_p]. \tag{25}$$

The polynomials $u_i(v) = \mathrm{tr}((XY)^i)$ with v as in (24) are K-invariant. Since

$$(Z s_p)(s_p Z) = Z^2$$

for Z as above, we see that the restriction of u_i to $\mathfrak{a}$ is the $W(\mathfrak{a})$-invariant polynomial $Z \mapsto \mathrm{tr}(Z^{2i})$. These polynomials, for $i = 1, \ldots, p$, generate $\mathcal{P}(\mathfrak{a})^{W(\mathfrak{a})}$. Hence $\mathcal{P}(V)^K$ is the polynomial algebra generated by $u_1, \ldots, u_p$.

4. (Type CI) Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$, where Ω is the bilinear form with matrix J as in Example 2, and take $\theta(g) = JgJ^{-1}$. Then $K \cong \mathrm{GL}(n, \mathbb{C})$ consists of the matrices

$$k = \begin{bmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{bmatrix}, \qquad \text{with } g \in \mathrm{GL}(n, \mathbb{C}),$$

while V consists of the matrices ($n \times n$ blocks)

$$v = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \qquad \text{with } X^t = X, \quad Y^t = Y. \tag{26}$$

Let F be the space of $n \times n$ symmetric matrices, and let ρ be the representation of $\mathrm{GL}(n, \mathbb{C})$ on F given by $\rho(g)X = gXg^t$. Then $(\sigma, V) \cong (\rho \oplus \rho^*, F \oplus F^*)$. Here we can identify F^* with F as a vector space, with $g \in \mathrm{GL}(n, \mathbb{C})$ acting by $X \mapsto (g^t)^{-1} X g^{-1}$.

In this realization we take $\mathfrak{a} \subset V$ as the matrices

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \qquad \text{with } X = \mathrm{diag}[x_1, \ldots, x_n]. \tag{27}$$

This is a toral subalgebra of $\mathfrak{g}$ that is conjugate in G to the Lie algebra of the maximal anisotropic torus used in Section 3.3. The polynomials $u_i(v) = \mathrm{tr}((XY)^i)$, with v as in (26), are K-invariant. The restriction of u_i to $\mathfrak{a}$ is the polynomial $X \mapsto \mathrm{tr}(X^{2i})$. Since the restricted root system is of type C_n, these polynomials, for $i = 1, \ldots, n$, generate $\mathcal{P}(\mathfrak{a})^{W(\mathfrak{a})}$. It follows that $u_1, \ldots, u_n$ are algebraically independent generators of $\mathcal{P}(V)^K$.

5. (Type DIII) Let $G = \mathrm{SO}(\mathbb{C}^{2n}, B)$, where B is the bilinear form with matrix $\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$, and take $\theta(g) = JgJ^{-1}$ (J as in Example 2). Then K is the same as in Example 4 (Type CI), while V consists of the matrices ($n \times n$ blocks)

$$v = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \qquad \text{with } X^t = -X, \quad Y^t = -Y. \tag{28}$$

Let F be the space of $n \times n$ skew-symmetric matrices, and let ρ be the representation of $\mathrm{GL}(n,\mathbb{C})$ on F given by $\rho(g)X = gXg^t$. Then $(\sigma, V) \cong (\rho \oplus \rho^*,\ F \oplus F^*)$. Here we can identify F^* with F as a vector space, with $g \in \mathrm{GL}(n,\mathbb{C})$ acting by $X \mapsto (g^t)^{-1}Xg^{-1}$.

In this realization we take $\mathfrak{a} \subset V$ as the matrices

$$v = \begin{bmatrix} 0 & Xs_n \\ s_nX & 0 \end{bmatrix}, \qquad X = \begin{cases} \mathrm{diag}[Z, -\check{Z}] & \text{when } n = 2p \\ \mathrm{diag}[Z, 0, -\check{Z}] & \text{when } n = 2p+1. \end{cases} \tag{29}$$

Here $Z = \mathrm{diag}[z_1, \ldots, z_p]$. This is a toral subalgebra of $\mathfrak{g}$ that is conjugate in G to the Lie algebra of the maximal anisotropic torus used in Section 3.3. The polynomials $u_i(v) = \mathrm{tr}((XY)^i)$, with v as in (28), are K-invariant. The restriction of u_i to $\mathfrak{a}$ is the polynomial $Z \mapsto \mathrm{tr}(Z^{2i})$ (note that $Xs_n = -s_nX$ for X as in (29)). Since the restricted root system is of type C_p or BC_p, these polynomials, for $i = 1, \ldots, p$, generate $\mathcal{P}(\mathfrak{a})^{W(\mathfrak{a})}$. It follows that $u_1, \ldots, u_p$ are algebraically independent generators for $\mathcal{P}(V)^K$.

6. (Type BDI) Let $G = \mathrm{SO}(n,\mathbb{C})$ ($g^tg = I_n$ for $g \in G$). Take $p + q = n$, $q \geq p \geq 1$, $\theta = \theta_{q,p}$ as in Example 3. Then $K = \mathrm{S}(\mathrm{O}(q,\mathbb{C}) \times \mathrm{O}(p,\mathbb{C}))$, imbedded diagonally into G, while V consists of the matrices

$$v = \begin{bmatrix} 0 & X \\ -X^t & 0 \end{bmatrix} \qquad \text{with } X \in M_{q,p}(\mathbb{C}). \tag{30}$$

Here (σ, V) is the representation of K on $M_{q,p}(\mathbb{C})$ given by

$$\sigma(g_1, g_2)X = g_1Xg_2^{-1}.$$

Restricted to the subgroup $\mathrm{SO}(q,\mathbb{C}) \times \mathrm{SO}(p,\mathbb{C}) \subset K$ it is the irreducible representation $\mathbb{C}^q \widehat{\otimes} \mathbb{C}^p$ (outer tensor product of the defining representations) when $p \neq 2$ and $q \neq 2$. For $p = 2$ and $q > 2$ it is the sum of two irreducible representations (recall that $\mathrm{SO}(2,\mathbb{C}) \cong \mathrm{GL}(1,\mathbb{C})$).

In this realization we take $\mathfrak{a} \subset V$ as the matrices v in (30) with

$$X = \begin{bmatrix} Zs_p \\ 0 \end{bmatrix}, \qquad Z = \mathrm{diag}[z_1, \ldots, z_p].$$

This is a toral subalgebra of $\mathfrak{g}$ that is conjugate in G to the Lie algebra of the maximal anisotropic torus used in Section 3.3. The polynomials $u_i(v) = \mathrm{tr}((XX^t)^i)$, with v as in (30), are K-invariant. The restriction of u_i to $\mathfrak{a}$ is the polynomial $Z \mapsto \mathrm{tr}(Z^{2i})$. Suppose $p < q$. Then the restricted root system is of type B_p, so it follows that $u_1, \ldots, u_p$ are algebraically independent generators for $\mathcal{P}(V)^K$. Now suppose $p = q$. Then the restricted root system is of type D_p. In this case the *Pfaffian* polynomial $\mathrm{Pfaff}(v)$ is K-invariant and restricts to the $W(\mathfrak{a})$-invariant polynomial $Z \mapsto z_1 \cdots z_p$ on $\mathfrak{a}$. It follows that $\{u_1, \ldots, u_{p-1}, \mathrm{Pfaff}\}$ is a set of algebraically independent generators for $\mathcal{P}(V)^K$ when $p = q$.

7. (Type CII) Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \omega_n)$ where ω_n is the bilinear form with matrix $T_n = \mathrm{diag}[\mu, \ldots, \mu]$ (n copies of $\mu = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$). Take $q \geq p > 0$ with $p + q = n$ and let $\theta g = I_{2q,2p} g I_{2q,2p}$, as in Example 3. Then $K = \mathrm{Sp}(\mathbb{C}^{2q}, \omega_q) \times \mathrm{Sp}(\mathbb{C}^{2p}, \omega_p)$ embedded diagonally and V consists of all matrices

$$\text{(31)} \qquad v = \begin{bmatrix} 0 & X \\ T_p X^t T_q & 0 \end{bmatrix}, \qquad X \in M_{2q,2p}(\mathbb{C}).$$

Here $(k_1, k_2) \in K$ acts on $v \in V$ by $X \mapsto k_1 X k_2^{-1}$ for $k_1 \in K_1 = \mathrm{Sp}(\mathbb{C}^{2q}, \omega_q)$, $k_2 \in K_2 = \mathrm{Sp}(\mathbb{C}^{2p}, \omega_p)$ and $X \in M_{2q,2p}(\mathbb{C})$. Hence the representation (σ, V) is irreducible and equivalent to the outer tensor product $\mathbb{C}^{2q} \widehat{\otimes} \mathbb{C}^{2p}$ of the defining representations of K_1 and K_2.

We take $\mathfrak{a}$ to consist of all matrices (31) with

$$\text{(32)} \qquad X = \begin{bmatrix} Z \\ 0_{q-p} \end{bmatrix}, \qquad Z = \mathrm{diag}[z_1, z_1, \ldots, z_p, z_p] \in M_{2p}(\mathbb{C}).$$

This is a toral subalgebra of $\mathfrak{g}$ that is conjugate in G to the Lie algebra of the maximal anisotropic torus used in Section 3.3. The polynomials $u_i(v) = \mathrm{tr}((XX^t)^i)$, with v as in (31), are K-invariant. The restriction of u_i to $\mathfrak{a}$ is the polynomial $Z \mapsto \mathrm{tr}(Z^{2i})$. Since the restricted root system is of type BC_p (when $p < q$) or C_p (when $p = q$), it follows that $u_1, \ldots, u_p$ are algebraically independent generators for $\mathcal{P}(V)^K$.

8. (Type G) Let $K = (\mathrm{SL}(2) \times \mathrm{SL}(2))/\{(I, I), (-I, -I)\}$ and (σ, V) the representation on $V = \mathbb{C}^2 \widehat{\otimes} S^3(\mathbb{C}^2)$ (outer tensor product). Here M is isomorphic with $\times^2(\mathbb{Z}/2\mathbb{Z})$. One has

$$\mathcal{P}(V)^K = \mathbb{C}[u_1, u_2] \quad \text{with } \deg u_1 = 2 \text{ and } \deg u_2 = 6.$$

This example comes from the exceptional group G_2.

9. (Type FI) Let $K = (\mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(3, \mathbb{C}))/\{(I, I), (-I, I)\}$ and (σ, V) the representation on $\mathbb{C}^2 \widehat{\otimes} F$ (outer tensor product) with F the irreducible representation of $\mathrm{Sp}(3, \mathbb{C})$ having highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. M is isomorphic with $\times^4(\mathbb{Z}/2\mathbb{Z})$. One has

$$\mathcal{P}(V)^K = \mathbb{C}[u_1, u_2, u_2, u_4] \quad \text{with } \deg u_1 = 2, \deg u_2 = 6, \deg u_3 = 8, \text{ and } \deg u_4 = 12.$$

This example comes from the exceptional group F_4.

10. (Type FII) Let $K = \mathrm{Spin}(9, \mathbb{C})$ and (σ, V) the spin representation, $M \cong \mathrm{Spin}(7, \mathbb{C})$. The restricted root system is of type A_1 and hence $\mathcal{P}(V)^K = \mathbb{C}[u]$ with $\deg u = 2$. This example also comes from F_4.

11. (Type EI) Let $K = \mathrm{Sp}(4, \mathbb{C})$ and take the representation (σ, V) of $\mathrm{Sp}(4, \mathbb{C})$ on $\bigwedge^4 \mathbb{C}^8 / \bigwedge^2 \mathbb{C}^8$. Here we embed

$$\omega \wedge \textstyle\bigwedge^2 \mathbb{C}^8 \subset \bigwedge^4 \mathbb{C}^8$$

with ω a nonzero element of $(\bigwedge^2 \mathbb{C}^8)^K$ (this is the irreducible representation with highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$). M is isomorphic with $\times^6(\mathbb{Z}/2\mathbb{Z})$. $\mathcal{P}(V)^K$ is the polynomial algebra in 6 generators whose degrees are 2, 5, 6, 8, 9 and 12. This example comes from the exceptional group E_6.

12. (Type EII) Let $K = (\mathrm{SL}(2) \times \mathrm{SL}(6, \mathbb{C}))/\{(I, I), (-I, -I)\}$ and take (σ, V) to be the representation $\mathbb{C}^2 \widehat{\otimes} \bigwedge^3 \mathbb{C}^6$ (outer tensor product). M is locally isomorphic with $\mathrm{GL}(1, \mathbb{C}) \times$

$\mathrm{GL}(1,\mathbb{C})$ and the restricted root system is of type F_4. Hence $\mathcal{P}(V)^K$ is a polynomial algebra in four generators with degrees as in Example 9. This example also comes from the exceptional group E_6.

13. (Type EIII) Take $K = (\mathrm{GL}(1,\mathbb{C}) \times \mathrm{Spin}(10,\mathbb{C}))/\{(I,I),(-I,-I)\}$ (here the second $-I$ is the kernel of the covering $\mathrm{Spin}(10,\mathbb{C}) \to \mathrm{SO}(10,\mathbb{C})$. Let (σ, V) be the sum $(\rho_+, F^+) \oplus (\rho_-, F^-)$ of the two half spin representations of $\mathrm{Spin}(10,\mathbb{C})$ with $\rho_+(z,I) = zI$ and $\rho_-(z,I) = z^{-1}I$ for $z \in \mathrm{GL}(1,\mathbb{C})$. M is isomorphic with $\mathrm{GL}(4,\mathbb{C})$. The restricted root system is of type BC_2. Hence $\mathcal{P}(V)^K$ is the polynomial algebra in two generators, one of degree 2 and the other of degree 4. This example also comes from E_6.

14. (Type EV) Let $K = \mathrm{SL}(8,\mathbb{C})$ and take (σ, V) to be the representation of K on $\bigwedge^4 \mathbb{C}^8$. If we replace K by $\sigma(K)$ then M is isomorphic with $\times^7(\mathbb{Z}/2\mathbb{Z})$. $\mathcal{P}(V)^K$ is the polynomial algebra in seven generators whose degrees are 2, 6, 8, 10, 12, 14 and 18. This example comes from the exceptional group E_7.

15. (Type EVI) Let $K = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{Spin}(12,\mathbb{C}))/\{(I,I),(-I,-I)\}$ (the second $-I$ as in Example 13). (σ, V) is given by $\mathbb{C}^2 \widehat{\otimes} S$ (exterior tensor product) with S a half spin representation. M is locally isomorphic with $\times^3\mathrm{SL}(2,\mathbb{C})$. The restricted root system is of type F_4. Hence $\mathcal{P}(V)^K$ is a polynomial algebra in four generators whose degrees are as in Example 9. This example also comes from the exceptional group E_7.

16. (Type EVIII) Let $K = \mathrm{Spin}(16,\mathbb{C})$ and take (σ, V) to be a half spin representation. If we replace K by $\sigma(K)$ then M is isomorphic with $\times^8(\mathbb{Z}/2\mathbb{Z})$. $\mathcal{P}(V)^K$ is the polynomial algebra in eight generators whose degrees are 2, 8, 12, 14, 18, 20, 24 and 30. This example comes from the exceptional group E_8.

4.3 Comments on the Proof and Further Examples

Let K be a connected reductive linear algebraic group and let (σ, V) be a regular representation of K. We will now isolate the actual properties of the representations that we use to prove the Kostant-Rallis Theorem. We assume that there is a subspace $\mathfrak{a}$ in V such that

(1) The restriction $f \mapsto f|_{\mathfrak{a}}$ defines an isomorphism of $\mathcal{P}(V)^K$ onto a subalgebra $\mathcal{R}$ of $\mathcal{P}(\mathfrak{a})$.

(2) The subalgebra $\mathcal{R}$ of $\mathcal{P}(\mathfrak{a})$ is generated by algebraically independent homogeneous elements $u_1, \ldots, u_l$ with $l = \dim \mathfrak{a}$. Furthermore, there exists a graded subspace $\mathcal{A}$ of $\mathcal{P}(\mathfrak{a})$ such that the map $\mathcal{A} \otimes \mathcal{R} \to \mathcal{P}(\mathfrak{a})$ given by $a \otimes r \mapsto ar$ is a linear bijection.

(3) There exists $h \in \mathfrak{a}$ such that $|\sigma(K)h \cap \mathfrak{a}| \geq \dim \mathcal{A}$.

(4) Let h be as in (3) and set

$$\mathcal{X}_h = \{v \in V \,:\, f(v) = f(h) \text{ for all } f \in \mathcal{P}(V)^K\}.$$

If $v \in \mathcal{X}_h$ then $\dim Kv = \dim V - \dim \mathfrak{a}$.

Set $M = \{k \in K \,:\, \sigma(k)h = h\}$. Our proof of Theorem 4.1 actually proves the following result.

Theorem 4.3 *Assume that* (σ, V) *satisfies* (1)-(4). *Let* (σ, F) *be an irreducible regular representation of* K. *Then as a* $\mathcal{P}(V)^K$*-module, the space* $\operatorname{Hom}_K(F, \mathcal{P}(V))$ *is free on* $\dim F^M$ *generators.*

Here are some examples that are not isotropy representations for symmetric spaces but that nevertheless satisfy conditions (1)-(4).

1. Let $K = \mathrm{SL}(2,\mathbb{C})$ and let (σ, V) be the representation of K on $S^3(\mathbb{C}^2)$ (i.e. the irreducible four-dimensional representation). One can show that $\mathcal{P}(V)^K = \mathbb{C}[f]$ with f irreducible and homogeneous of degree 4. Let e_1, e_2 be the usual basis of $\mathbb{C}^2$ and let $h = e_1^3 + e_2^3$. If $u = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ then $\sigma(u)h = ih$. Set $\mathfrak{a} = \mathbb{C}h$. Thus $\sigma(K)h \cap \mathfrak{a} \supset \{h, -h, ih, -ih\}$. One has $f(h) \neq 0$ and

$$M = \left\{ \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} : \xi^3 = 1 \right\}.$$

We look upon $\mathcal{P}(\mathfrak{a})$ as $\mathbb{C}[t]$. Assuming that $f(h) = 1$, we then have $\operatorname{res}(\mathcal{P}(V)^K) = \mathbb{C}[t^4]$. Take

$$\mathcal{A} = \mathbb{C}1 \oplus \mathbb{C}t \oplus \mathbb{C}t^2 \oplus \mathbb{C}t^3.$$

Thus all conditions but (4) have been verified. Condition (4) follows since f is irreducible so $\mathcal{X}_h$ is irreducible. We can thus apply Theorem 4.3 and conclude that if F^k is the irreducible $(k+1)$-dimensional regular representation of K then $\operatorname{Hom}_K(F^k, \mathcal{P}(V))$ is a free $\mathbb{C}[f]$-module on d_k generators. Here

$$\begin{aligned} d_{6k+2j} &= 2k+1 \quad \text{for } j = 0,1,2 \text{ and } k = 0,1,2,\ldots, \\ d_{6k+3+2j} &= 2k+2 \quad \text{for } j = 0,1,2 \text{ and } k = 0,1,\ldots. \end{aligned}$$

2. Let $K = \mathrm{Sp}(3,\mathbb{C})$ and let $V \subset \bigwedge^3 \mathbb{C}^6$ be the irreducible K-submodule with highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Then $\mathcal{P}(V)^K = \mathbb{C}[f]$ with f an irreducible homogeneous polynomial of degree 4. Let $h = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$. Then $f(h) \neq 0$ so we may normalize f by $f(h) = 1$. Let $\mathfrak{a} = \mathbb{C}h$. Set

$$u = \begin{bmatrix} 0 & iI_3 \\ iI_3 & 0 \end{bmatrix}.$$

Then $\sigma(u)h = -ih$. Thus the conditions are satisfied as in Example 1. In this case M is the group of all matrices

$$k = \begin{bmatrix} b & 0 \\ 0 & (b^t)^{-1} \end{bmatrix}, \quad b \in \mathrm{SL}(3,\mathbb{C}).$$

3. Let $K = \mathrm{SL}(6,\mathbb{C})$ and let $V = \bigwedge^3 \mathbb{C}^6$. As in Examples 1 and 2, one has $\mathcal{P}(V)^K = \mathbb{C}[f]$ with f homogeneous of degree 4. We take h and u as in Example 2. Then the conditions (1)-(4) are satisfied and M is the group of all

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad b_1, b_2 \in \mathrm{SL}(3,\mathbb{C}).$$

5 Notes

Section 2.2. Theorem 2.3 is an algebraic version of the *Borel-Weil Theorem*; for the analytic version in terms of holomorphic vector bundles, see [Wal, Ch. 6, §6.3]. The functions in $\mathcal{R}_\lambda$ are uniquely determined by their restrictions to the unipotent group N. The finite-dimensional space $\mathcal{V}_\lambda \subset \mathrm{Aff}(N)$ of restrictions of functions in $\mathcal{R}_\lambda$ is characterized as the solution space of a system of differential equations (called an *indicator system*) in [Žel, Ch. XVI]. These equations, which express the nilpotence of the generators for $\mathfrak{n}$ on $\mathcal{V}_\lambda$, are also the basic tool in the algebraic construction of the finite-dimensional $\mathfrak{g}$-modules [Hum, §21.2].

Section 2.3. The open B-orbit condition in Theorem 2.5 is also a necessary condition for X to be multiplicity-free. This follows easily from the result of [Ros] that if B does not have an open orbit on X then there exists a non-constant B-invariant rational function on X (see [Vi-Ki]). The spherical pairs (G, K) with G connected and K reductive have been classified in [Krä]. The term *spherical subgroup* is also applied to any algebraic subgroup L (not necessarily reductive) such that L has an open orbit on G/B. This condition can be shown equivalent to $\mathrm{Ind}_K^G(\chi)$ being multiplicity-free for all regular characters χ of L [Vi-Ki, Theorem 1]. For example, any subgroup containing the nilradical of a Borel subgroup is spherical in this sense. (Such subgroups are called *horospherical.*) See [Bri] for a survey of results in this more general context.

Irreducible linear multiplicity-free actions were classified in [Kac]. The classification of reducible multiplicity-free linear actions was done (independently) in [Be-Ra] and [Lea]. For examples see [Ho-Um] and [How]).

Section 3.1. The results in this section are from [Ric1]. See also [Ric2].

Section 3.2. The symmetric spaces are labelled according to Cartan's classification (see [Hel1]).

Section 3.3. The Iwasawa decomposition for a general complex reductive group is obtained in [Vus]. See also [De-Pr].

Section 3.4. The problem of decomposing $\mathrm{Aff}(G/K)$ into irreducible subspaces was first treated by Cartan in [Car]. Theorem 3.10 was proved by Helgason [Hel2, Ch. V, §4.1]. For a proof using algebraic geometry see [Vus].

Section 4.1. Theorem 4.1 has many important applications to the representation theory of real reductive groups (see [Kos2]). Theorem 4.2 on the adjoint representation appears in [Kos1].

Section 4.2. The examples are labeled according to Cartan's classification of symmetric spaces [Hel1, Ch. X, §6 Table V]. For all the cases, in particular when G is exceptional, the Lie algebra $\mathfrak{m}$ of M and the restricted root system can be read off from the *Satake diagram* ; see [Ara] and [Hel1, Ch. X, Exercises Table VI]. The isotropy representations of K on V are obtained in all cases in [Wol, §8.11].

Section 4.3. Theorem 4.3 is related to a theorem of G. W. Schwarz [Sch]. The classification of representations with free modules of covariants is treated in [Pop, Ch. 5].

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