Discrete Fourier and Wavelet Transforms: Mathematical Microscopes for Signal Processing

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Rutgers Math Club

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Abstract:

Signal processing has become an essential and ubiquitous part of contemporary scientific and technological activity, and the signals that need to be processed appear in most sectors of modern life. Signal processing is used for audio mp3 files, telecommunications (telephone and television), transmission and analysis of satellite images, and medical imaging (echograph, tomography, and nuclear magnetic resonance). All these applications involve the analysis, storage, transmission, and synthesis of complex time series.

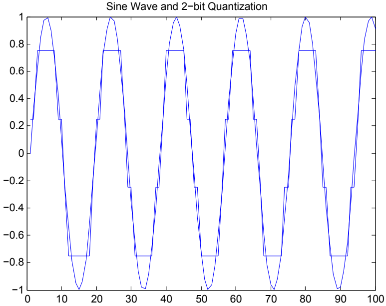
In this talk I will show how to use elementary linear algebra to create Fourier and wavelet transforms for manipulating digital signals (sounds and images). Much of the mathematics involved is of recent origin; for example, the new JPEG 2000 image algorithms use discrete wavelet transforms developed by Ingrid Daubechies and her collaborators in the 1990's.

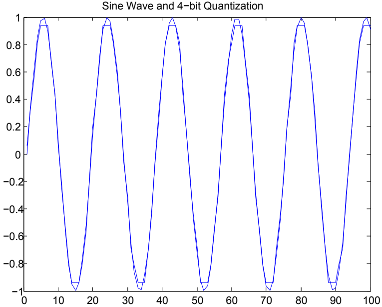
Continuous (calculus) to Discrete (linear algebra)

analog signal $f(t)$ $(a \le t \le b) \longrightarrow$ digital signal $\mathbf{v}$

$\mathbf{v} = Q[f(t_0), f(t_1), \ldots, f(t_{N-1})]^{\mathrm{T}}$ $\quad Q =$ quantize entries (# bits)

$t_j = a + j\Delta t$ with $\Delta t = (b-a)/N$ $\quad$ (sample at rate N)





more bits gives better sound but needs more memory space

Linear algebra model:

digital signal $\longleftrightarrow$ vector in complex vector space $\cong \mathbb{C}^N$

Periodic Discrete Signals and Transforms

Fix integer $N \geq 2$ $\quad \ell^2[\mathbb{Z}/N\mathbb{Z}] =$ vector space of N-periodic signals

$\phi : \mathbb{Z} \to \mathbb{C}$ with $\phi(k+N) = \phi(k)$ for all $k \in \mathbb{Z}$

inner product $\quad \langle \phi, \psi \rangle = \sum_{k=0}^{N-1} \overline{\phi(k)} \psi(k) \quad$ ($\overline{z}$ = complex conj.)

$\ell^2[\mathbb{Z}/N\mathbb{Z}] \longleftrightarrow \mathbb{C}^N$ by $\phi \longleftrightarrow x = [\phi(0), \cdots, \phi(N-1)]^{\mathrm{T}}$

Transform Method

Choose invertible matrix $A = [v_0, \cdots, v_{N-1}]$ $\quad$ (basis for $\mathbb{C}^N$)

Let $v_0^*, \ldots, v_{N-1}^*$ be rows of A^{-1} $\quad$ (dual basis)

A-transform $X = A^{-1}x = [c_0, \cdots, c_{N-1}]^{\mathrm{T}}$, entries $c_k = v_k^* x$

Inversion formula $\quad (\star) \quad x = AX = c_0 v_0 + \cdots + c_{N-1} v_{N-1}$

Example $\quad A^{-1} = \overline{A}^T$ (Orthonormal basis) $\quad \Longleftrightarrow v_k^* = \overline{v}_k^T$

$\Longleftrightarrow$ energy preserved: $(x, x) = (X, X) = |c_1|^2 + \cdots + |c_N|^2$

Transform Problem

Find the best transform for expanding a particular type of signal x

- Large percentage of the coefficients c_k in $(\star)$ are small
- Replace small c_k by 0 to get compressed signal $\tilde{x}$ with small percentage of nonzero coefficients (good approximation to x)

Shift Operator and Sampled Waves

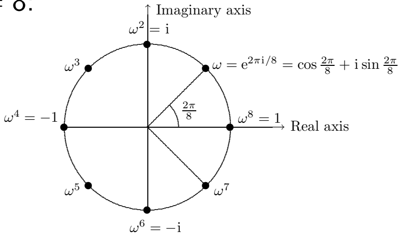
Shift operator For $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$ $\quad S\phi(k) = \phi(k-1) \quad (S^N = I)$

Problem Find **eigenvectors** for S

Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/N} \quad (\mathrm{i} = \sqrt{-1})$ $\quad$ Then $\omega^N = 1$ and

$1, \omega, \omega^2, \ldots, \omega^{N-1}$ are the solutions to $z^N = 1$ (roots of unity)

$N = 8$:



Set $\phi_p(k) = \omega^{kp}$ for $k, p = 0, \ldots, N-1$ $\quad$ Then $\phi_p \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$

ϕ_p = discrete sample of continuous wave f_p (frequency p):

$$f_p(t) = \mathrm{e}^{2\pi p t \mathrm{i}} = \cos(2\pi p t) + \mathrm{i}\sin(2\pi p t) \qquad (t = k/N)$$

$$\phi_p \longleftrightarrow \begin{cases} \text{low freq. wave when } p \approx 0 \pmod{N} & \omega^p \approx 1 \\ \text{high freq. wave when } p \approx N/2 \pmod{N} & \omega^p \approx -1 \end{cases}$$

Fourier Basis $\{\phi_0, \phi_1, \ldots, \phi_{N-1}\}$ for $\ell^2[\mathbb{Z}/N\mathbb{Z}]$

- Eigenvectors for S: $S\phi_p = \omega^{-p}\phi_p$ (eigenvalue $= \omega^{-p}$)
- Orthogonality: $\langle \phi_p, \phi_q \rangle = \begin{cases} N & \text{if } p \equiv q \mod N \\ 0 & \text{else} \end{cases}$

Discrete Fourier transform (DFT) of $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$

$$\widehat{\phi}(p) = \langle \phi_p, \phi \rangle = \sum_{k=0}^{N-1} \omega^{-kp}\,\phi(k) \quad \text{for } \phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$$

Fourier synthesis: $\phi = (1/N)\{\widehat{\phi}(0)\,\phi_0 + \cdots + \widehat{\phi}(N-1)\,\phi_{N-1}\}$

Matrix Description (calculated by FFT "Fast Fourier Transform")

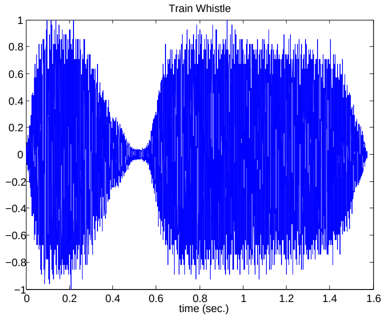
$\phi_p \longleftrightarrow E_p \in \mathbb{C}^N$ $(p = 0, \ldots, N-1)$ Fourier basis for $\mathbb{C}^N$

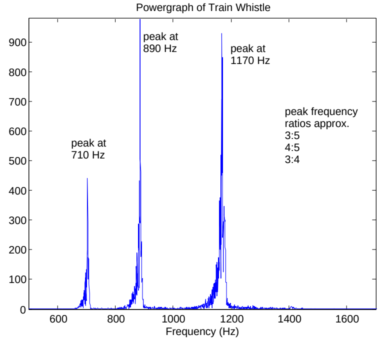
Fourier matrix $F_N = [\overline{E_0}, \cdots, \overline{E_{N-1}}]$ (j, k entry $\omega^{-(j-1)(k-1)}$)

$$N = 4\text{:}\quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}$$

If $\phi \longleftrightarrow y \in \mathbb{C}^N$ then $\widehat{\phi} \longleftrightarrow Y = F_N y$ and $y = (1/N)\overline{F}_N Y$

Frequency Analysis



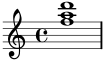


frequency peaks near 710Hz, 890Hz, 1170Hz $(N = 12,880)$

analog signal to synthesize peak frequencies

$f(t) = 0.45\sin(2\pi * 710t) + 0.95\sin(2\pi * 890t) + 0.93\sin(2\pi * 1170t)$

What is missing from this model?



(quarter note, rest, half note) d minor chord (whole note)

Need time & frequency information!

Wavelet Approach

Given: Digital signal $s_k = \begin{bmatrix} s_k(0) \\ s_k(1) \\ \vdots \\ s_k(N-1) \end{bmatrix}$ length $N = 2^k$ (level k)

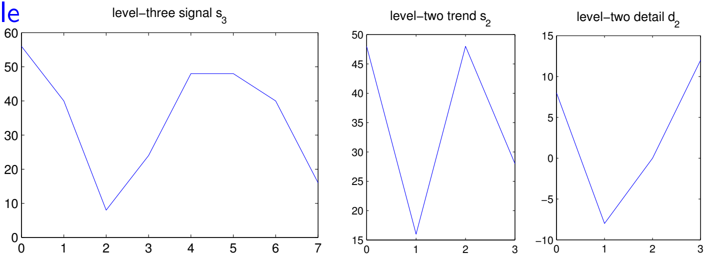
Haar Wavelet Transform (pyramid algorithm)

- Downsample $s_k \longrightarrow \begin{bmatrix} (s_k)_{\text{even}} \\ (s_k)_{\text{odd}} \end{bmatrix}$ with

$$(s_k)_{\text{even}} = \begin{bmatrix} s_k(0) \\ s_k(2) \\ \vdots \\ s_k(N-2) \end{bmatrix}, \ (s_k)_{\text{odd}} = \begin{bmatrix} s_k(1) \\ s_k(3) \\ \vdots \\ s_k(N-1) \end{bmatrix} \ (\text{length } N/2)$$

- Trend (level $k-1$) $s_{k-1} = \frac{1}{2}\left\{(s_k)_{\text{even}} + (s_k)_{\text{odd}}\right\}$
- Detail (level $k-1$) $d_{k-1} = \frac{1}{2}\left\{(s_k)_{\text{even}} - (s_k)_{\text{odd}}\right\}$
- Iterate on trend $s_{k-1} \longrightarrow [s_{k-2}, \ d_{k-2}], \quad s_{k-2} \longrightarrow \cdots$

Example

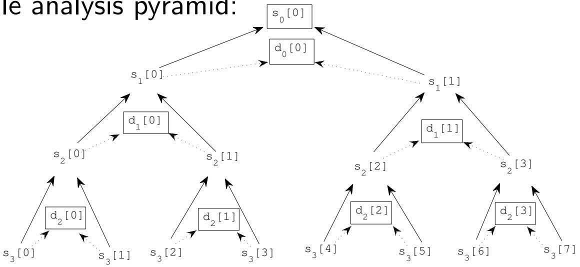


Three-scale analysis pyramid:

level 0

level 1

level 2



Three-scale multiresolution analysis $s_3 \longrightarrow [s_0, d_0, d_1, d_2]$

Multiresolution Synthesis and Haar Basis

Three-scale multiresolution synthesis $s_3 = W^{(3)} \begin{bmatrix} s_0 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$

Haar three-scale synthesis matrix (8×8):

$W^{(3)} = [u_0,\ v_0,\ v_1,\ S^4 v_1,\ v_2,\ S^2 v_2,\ S^4 v_2\ S^6 v_2]$ (S = shift)

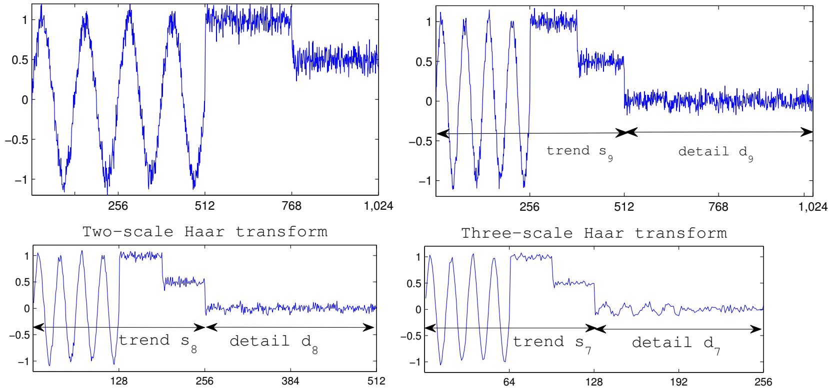
$$u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

trend coarse detail medium detail fine detail

Differences between detail vectors for Haar basis and Fourier basis:

- localized in time (multiple time scales)
- time shifted and rescaled samples of one function (wavelet)

Wave + Step Function + Noise   One-scale Haar transform

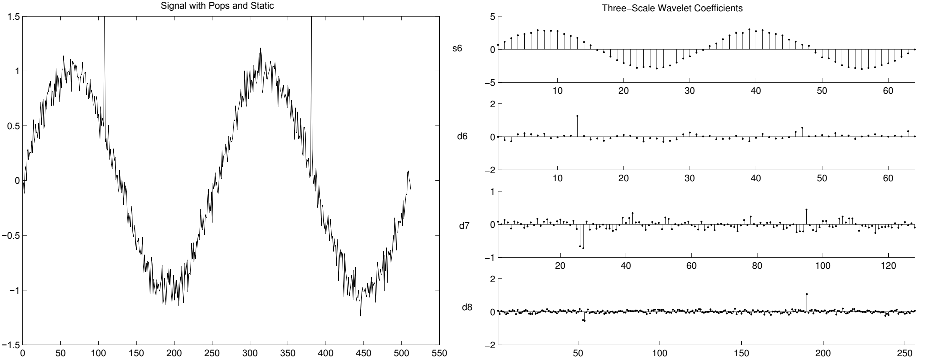


Pyramid algorithm: $x \longrightarrow [s_7, d_7, d_8, d_9]$ $\quad (N = 2^{10})$

There are many other mathematical microscopes!

Feature Extraction and Signal Compression

Three-scale LeGall wavelet transform
(used in JPEG 2000 lossless image compression algorithm)
signal $\longrightarrow [s_6, d_6, d_7, d_8] \quad (N = 2^9 = 512)$



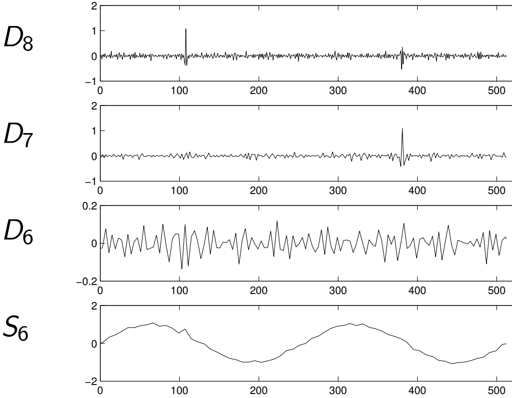
- Trend s_6 shows the low frequency signal content
- Time location of pops is clear at each level of detail
- Most of the detail coefficients are small (noise)

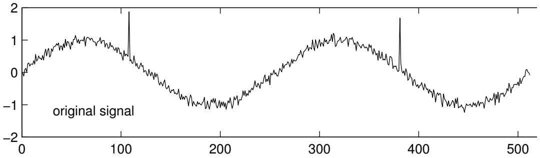
Three level wavelet synthesis of signal:

Let D_j = inverse wavelet transform of details d_j ($j = 8, 7, 6$)

S_6 = inverse wavelet transform of trend s_6

Then original signal $= D_8 + D_7 + D_6 + S_6$ (each term in $\mathbb{C}^{512}$)

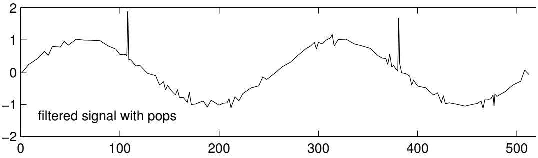




Threshold Compression

- 97% of detail coefficients in d_6, d_7 and d_8 are less than 0.2.
- Replace each such small coefficient by 0 to get $\tilde{d}_6$, $\tilde{d}_7$, $\tilde{d}_8$
- Calculate inverse wavelet transforms $\widetilde{D}_6$, $\widetilde{D}_7$, $\widetilde{D}_8$ and add to S_6

$S_6 + \widetilde{D}_6 + \widetilde{D}_7 + \widetilde{D}_8$:



Noise filtered out but pops still in (6 : 1 compression of signal)
(Not possible using Fourier basis)

W = one-scale wavelet analysis matrix

X = image matrix
(256 $\times$ 256 eight-bit matrix)

WXW^{T} = wavelet transform
(partitioned matrix)

Original Lena Image

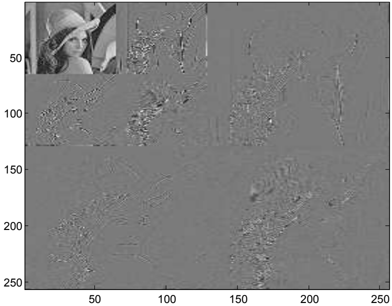
One-scale Wavelet Transform

trend 128×128	vertical details
horizontal details	diagonal details

Multiscale Image Transforms and Edge Detection

Pyramid algorithm for two-scale transform of image matrix

- Keep the 3 one-scale detail matrices
- Make a wavelet transform of the trend matrix



Two-scale wavelet transform

Inverse transform of details (omit level-two trend)

Famous Cartoon (mathematician to engineer after seeing machine)

"It works in practice, but does it work in theory?"

Problem How do we build wavelet transforms?

Linear Time-Invariant Filters

Given: signal ϕ and polynomial $f(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1}$
Form moving average of the signal (as for Haar trend and detail)

$$a_0\phi(k) + a_1\phi(k-1) + \cdots + a_{N-1}\phi(k-N+1) = (f(S)\phi)(k)$$

where $S =$ shift operator and $f(S) = a_0 + a_1 S + \cdots + a_{N-1} S^{N-1}$

Definition The linear transformation $f(S)$ on $\ell^2[\mathbb{Z}/N\mathbb{Z}]$ is called a filter, and $f(S)\phi$ is the filtered signal

Example On Fourier basis $S\phi_p = \omega^{-p}\phi_p$ $(\omega = \mathrm{e}^{2\pi \mathrm{i}/N})$ Hence

$$f(S)\phi_p = (a_0 + a_1\omega^{-p} + \cdots + a_{N-1}\omega^{-p(N-1)})\phi_p = f(\omega^{-p})\phi_p$$

- ϕ_p is an eigenvector for $f(S)$ with eigenvalue $\lambda = f(\omega^{-p})$
- If $\phi = \sum_{p=0}^{N-1} c_p \phi_p$ then $f(S)\phi = \sum_{p=0}^{N-1} f(\omega^{-p}) c_p \phi_p$
- Low pass filter: $f(S)$ attenuates high frequencies $(p \approx N/2)$
  $\iff f(-1) = 0 \iff f(z) = (z+1)^m g(z) \quad (m \geq 1)$
- High pass filter: $f(S)$ attenuates low frequencies $(p \approx 0)$
  $\iff f(1) = 0 \iff f(z) = (z-1)^m g(z) \quad (m \geq 1)$

Filter Banks and Discrete Wavelet Transforms

Construct a wavelet analysis transform of $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$ (N even) using lowpass and highpass filters followed by downsampling

- Choose a polynomial $h_0(z)$ with $h_0(-1) = 0$ to get a lowpass filter $H_0 = h_0(S)$ and calculate $H_0\phi$ (length N)
- Choose a polynomial $h_1(z)$ with $h_1(1) = 0$ to get a highpass filter $H_1 = h_1(S)$ and calculate $H_1\phi$ (length N)
- Introduce downsampling operator D on $\psi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$:
  $$(D\psi)(k) = \psi(2k) \quad (D\psi \text{ period } N/2), \quad \psi \longleftrightarrow v \in \mathbb{C}^N$$
  $$\ell^2[\mathbb{Z}/N\mathbb{Z}] \xrightarrow{D} \ell^2[\mathbb{Z}/(N/2)\mathbb{Z}], \quad D\psi \longleftrightarrow v_{\mathrm{even}} \in \mathbb{C}^{N/2}$$
- Calculate trend $= DH_0\phi$ and detail $= DH_1\phi$ (lengths $= N/2$)
- Define a wavelet transform $W\phi = \begin{bmatrix} \text{trend}(\phi) \\ \text{detail}(\phi) \end{bmatrix}$

Important: length(ϕ) = length($W\phi$), so $W \longleftrightarrow N \times N$ matrix

Perfect Reconstruction (PR) Problem   Is W invertible?

Energy Preservation Problem   Does W preserve the energy of a signal? (Important for signal compression)

Algebraic condition for PR

($\star\star$) $h_0(z)h_1(-z) - h_0(-z)h_1(z)$ is a nonzero monomial

Examples

- Haar transform

$$h_0(z) = 1 + z \quad \text{and} \quad h_1(z) = 1 - z$$

Here $h_0(z)h_1(-z) - h_0(-z)h_1(z) = (1+z)^2 - (1-z)^2 = 4z$

- LeGall transform (= CDF(2,2) transform)

$$h_0(z) = (1+z)^2(1 - 4z + z^2) \quad \text{and} \quad h_1(z) = (1-z)^2$$

- CDF(p,q) transforms (Ingrid Daubechies & collaborators, 1992)

Given positive integers p, q with $p + q$ even, there exits $g(z)$ so

$$h_0(z) = (1+z)^q g(z) \quad \text{and} \quad h_1(z) = (1-z)^p$$

satisfy ($\star\star$) (explicit formula for $g(z)$ with binomial coefficients)

- CDF(9,7) filters used in JPEG 2000 image compression algorithm

In these examples only the Haar transform preserves energy

Theorem

Suppose the polynomial $h_0(z)$ satisfies $h_0(-1) = 0$ and

$$(\sharp) \quad h_0(z)h_0(z^{-1}) + h_0(-z)h_0(-z^{-1}) = 2$$

Define $h_1(z) = zh_0(-z^{-1})$ and let $H_0 = h_0(S)$, $H_1 = h_1(S)$. Then H_0, H_1 are the low pass & high pass filters for an energy-preserving PR wavelet transform.

Remark Equation $(\sharp) \iff$ system of quadratic equations for the coefficients of $h_0(z)$

Examples

- Haar transform (normalized) Take $h_0(z) = (1+z)/\sqrt{2}$ Then $(\sharp) = \frac{1}{2}\{(1+z)(1+z^{-1}) + (1-z)(1-z^{-1})\} = 2$
- Daub4 transform Take $h_0(z) = (a + bz + cz^2 + dz^3)/\sqrt{32}$ where $a = 1+\sqrt{3}$, $b = 3+\sqrt{3}$, $c = 3-\sqrt{3}$, $d = 1-\sqrt{3}$ Then $h(-1) = a - b + c - d = 0$ and $(\sharp)$ is satisfied
- Daub2k transform (any positive integer k) Find complex roots of polynomial used in CDF(k,k) transform (No explicit formula)

See the Math 357 web page for a list of books about Fourier and wavelet transforms and links to web sites and expository articles. My course lecture notes (currently being revised) are also there.

http://www.math.rutgers.edu/courses/357

Thanks for looking and listening!