SOME REMARKS ON COMPLEX CROWNS OF RIEMANN SYMMETRIC SPACES

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Abstract. We discuss several results and problems in which complex geometrical constructions appear in the geometry of real pseudo Riemannian symmetric spaces in connection with analysis on such spaces.

2000 Mathematics Subject Classification. 22E30, 22E45, 32M15

Geometry of symmetric spaces continues to be a source of beautiful geometrical problems. For me, two things are typical for most existing problems in this area: complex geometrical constructions are responsible for a geometry of real symmetric spaces (it goes back to the great time of Poncelet-Plücker) and the appearance of nonhomogeneous complex invariant manifolds at some crucial moments of geometry of symmetric spaces and analysis on them. Both these circumstances are inherent in the problem which we start to discuss.

We need acknowledge that we can not start from a formal definition of crowns of Riemann symmetric spaces. There is only a confidence that for each such space X there is a canonical Stein neighborhood Crown(X), on which it can be extended to all natural geometrical constructions on X and all essential analytic objects on X admit holomorphic extensions on Crown(X) [G]. We can not formalize this intuition, but there is a number of results which support it. Let us start with the parametrical space of compact submanifolds in flag domains. J.Wolf with collaborators has investigated it for many years. He proved in particular [W] that it is a Stein manifold, and it is evident that it contains the corresponding Riemann symmetric space. The idea of the crown includes also a moment of universality: parametrical manifolds for a fixed group are the same for all flag domain (there are a few degenerate situations for groups of Hermitian type). There are several known other problems which bring Stein neighborhoods of Riemann symmetric spaces: holomorphic extensions of Szego kernels and solutions of Schmid equations, joint holomorphy domains of zonal spherical functions and other eigenfunctions of Laplace-Beltrami operators.

Let $X = G_{\mathbb{R}}/K_{\mathbb{R}}$ be a Riemann symmetric space (let $G_{\mathbb{R}}$ be a real semisimple Lie group and $K_{\mathbb{R}}$ be its maximal compact subgroup). If G, K are complexifications of these groups correspondingly, then $Y = G/K$ is a Stein symmetric (pseudo Riemann) space, but Y is too big to be the crown (if X is not compact). Akhiezer and myself [AG] tried to investigate directly invariant Stein neighborhoods Ω of X in Y. Firstly, we excluded full Y from our considerations through the addition of the condition that the action of $G_{\mathbb{R}}$ on Ω must be proper. We gave an example of

Supported in part by NSF grant DMS97-06836

a proper $G_{\mathbb{R}}$-invariant Stein neighborhood Ω (such domains are not unique). The principal construction of this paper is an invariant (but nonhomogeneous) domain D_0, which I believe is just Crown(X). The domain D_0 can be characterized by several equivalent conditions. In particular, it is an union of "good" $G_{\mathbb{R}}$-orbits: it is the maximal $G_{\mathbb{R}}$-invariant domain in Y which contains X and isotropy subgroups of all its points are compact. Such orbits can be parameterized in such a way.

Let $U \subset Y$ be the transversal to X compact symmetric space in Y, $T = \exp i\mathfrak{a}$ be a maximal torus in U, $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ for a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$. We parameterize orbits of $G_{\mathbb{R}}$ in Ω by intersections with T. Let D_0 be the union of orbits for $\exp(i\omega_0)$, where

$$\omega_0 = \{H \in \mathfrak{a}; |\alpha(H)| < \pi/2, \forall \alpha \in \Sigma\},$$

Σ is the restricted root system.

We could not prove

Conjecture 1. *D_0 is a Stein manifold.*

For a long time this conjecture was proven only in some special cases; now it is proven for all classical Riemann symmetric spaces (cf.below). It is not simple to check that an union of orbits of a noncompact group is Stein. In some cases D_0 admits a simple explicit description. The case of groups of Hermitian type (when the space X is Hermitian) is simplest. Then

$$D_0 = X \times \bar{X}$$

For the group $SO(1,n)$ the space X can be realized as the hyperboloid

$$\Box(y) = y_0^2 - y_1^2 - \cdots - y_n^2 = 1, \quad y_0 > 0, \quad y \in \mathbb{R}^{n+1}.$$

Then the domain D_0 is defined by the condition

$$\Box(z) = 1, \Box(y) > 0, y_0 > 0, \quad z = x + iy \in \mathbb{C}^{n+1}.$$

This domain is biholomorphicaly equivalent to the n-dimensional future tube, that is to the Hermitian symmetric space corresponding to the group $SO(2,n)$. In these examples D_0 turned out to be a symmetric space for a group bigger than $G_{\mathbb{R}}$, but this is not a general phenomena.

Many Riemann symmetric spaces X can be represented as real forms of Hermitian ones $\hat{X}$. Then it is natural to suggest that $\hat{X}$ will coincide with D_0. It is true for the examples considered above. In the general situation, probably, $\hat{X} \subset D_0$, but usually this inclusion is strict (in [GM] it is proved for $SO(p,q), p, q > 2$) We believe that for this class of symmetric spaces the Shilov boundaries of D_0 are causal symmetric spaces (of noncompact type).

Let us consider one more example of an explicit computation of D_0 [AG]. For $G = SL(n;\mathbb{R})$ the space X can be realized as the space of symmetric positive matrixes Q of the order n with the determinant 1, Y consists of complex unimodular symmetric matrices Q and D_0 is the connected component of the set of such Q that corresponding quadrics at $\mathbb{C}P^n$ have no real points. Apparently, X is contained in this set.

It would very interesting to understand the geometry of D_0 in cases when they are not symmetric domains for other groups.

Let us consider another approach to complex crowns. Let $D = D_\chi$ be the holomorphic function on Y such that

$$D(zb) = D(z)\chi(b), z \in Y, b \in B,$$

where B is a Borel subgroup and χ is its character. We will call by determinant functions such functions D and their translations $D(zg)$ by elements of the real group $g \in G_{\mathbb{R}}$. We can consider them as sections $D(z|u)$ of line bundles on $Y \times F, F = G/B$. They can also be interpreted in the language of spherical functions. Let us consider in Y the set where all determinant functions are different from 0. This set contains X (by the Iwasawa theorem); let D_1 be its connected component containing X. Then, of course, D_1 is a Stein manifold.

Conjectrure 2. $D_1 = D_0$.

Conjecture 1 is a corollary of Conjecture 2. We believe that the determinant functions are an important object of analysis on symmetric spaces and flag manifolds. In the case of Hermitian spaces they include norm-functions of Jordan algebras.

We explain now one way to prove these conjectures as a part of a more broad considerations of an extension of geometry of symmetric spaces on their complex crowns. There is a very deep Matsuki duality [M] between K and $G_{\mathbb{R}}$-orbits on flag manifolds $F = G/P$ where P is a parabolic subgroup ($P = B$ corresponds to complete flags). Then both groups K and $G_{\mathbb{R}}$ have on F finite number of orbits S_i and S'_j correspondingly. Matsuki duality says that there is such one-to-one correspondence $S_i \leftrightarrow S'_i$ that corresponding orbits have non-empty compact intersections. Special cases of this correspondence: all compact K-orbits are contained in some open $G_{\mathbb{R}}$-orbits and a unique open K-orbit contains a unique closed $G_{\mathbb{R}}$-orbit.

In our joint paper with Matsuki [GM] we consider the following problem. For which $z \in Y$ the Matsuki duality holds if one were to replace S_i with corresponding orbits $S_i(z)$ of the isotropy subgroups $K(z)$ (K corresponds to $x \in X \subset Y$). We believe that

Conjecture 3. *If $z \in D_0$ and P is a parabolic subgroup, then there is the similar correspondence $S_i(z) \leftrightarrow S'_i$ such that corresponding orbits have nonempty compact intersections. Moreover, D_0 is a minimal set which keeps this property for all orbits and all parabolic subgroups.*

We proved this conjecture for all classical groups and for all groups of Hermitian type. There is a stronger version of this conjecture. Let S, S' be corresponding K and $G_{\mathbb{R}}$-orbits on $F = G/P$. Let then $C(S, P) \supset X$ be the connected component of the set of such $z \in Y$ that the corresponding $K(z)$-orbit $S(z)$ has non-empty compact intersection with S'.

Conjecture 4. *$C(S, P) = D_0$ if one excludes degenerate cases when G is a group of Hermitian type and $C(S, P)$ coincides either with X or with $\bar{X}$.*

This conjecture in the case of closed orbits S (compact) and open orbits S' (flag domains) transforms in the problem to the parameterization of compact cycles in flag domains. It was proved in [WZ] for classical group of Hermitian type.

Another important special case of this conjecture is the case of open S and close S'. Let us recall that on flag manifolds there are such unique orbits. The supplement to $S(z), z \in Y$ will be defined by an equation $D(z|\cdot) = 0$ for some determinant function D and we have $D(z|u) \neq 0, z \in D_0, u \in S'$. It shows that if conjecture 4 is true for some P and open S then Conjecture 2 (and Conjecture 1) are true also. In [GM] conjecture 4 is proved for classical groups G and $P = B$. In this way Conjectures 2 and 1 are proved for classical groups.

Appendix. The extended duality on compact Hermitian symmetric spaces.

We will consider here the extension of the Matsuki duality in the simplest case when the flag manifold is a compact Hermitian symmetric space.

1.Let us start from the group $SU(p,q)$. Let $p \leq q, p+q = n$, $H(z,z)$ be a Hermitian form of the signature (p,q) in $\mathbb{C}^n$, $F = Gr(p,n)$, $G_{\mathbb{R}} = SU(p,q)$, corresponding to H. Then $G_{\mathbb{R}}$- orbits $F_{r,s}, r \leq p, s \leq p, r+s \leq p$, in F consist of such p-subspaces π, that restriction of H on them has the signature (r,s). Their closures $\tilde{F}_{r,s}$ consist of subspaces with signature $(k,l), k \leq r, l \leq s$. They are compact. Let us remark that $F_{r,s}, r+s = p$ (cases when H is nondegenerate on π) are domains, and $F_{p,0}$ is the Hermitian domain;$F_{0,0}$ is the joint edge of all these domains. Orbits $F_{r,s}$ are $G_{\mathbb{R}}$-orbits of S' type above.

Let us construct K-orbits S for this example. Let (u,v) be a pair of p-subspace and q-subspaces on which the form H is correspondingly positive (negative). We will consider the set of such pairs as a domain $D_0 = F_{p,0} \times F'_{0,q}$ in $Y = SL(n;\mathbb{C})/S(L(p;\mathbb{C}) \times L(q;\mathbb{C}))$ which is an open dense set in the direct product of dual Grassmanians $Gr(p,n) \times Gr(q,n)$. Let $K = K(u,v)$ be the isotropy subgroup of this pair (u,v). Its orbits on $F = Gr(p,n)$ are

$$C_{r,s} = \{\pi \in F; \dim(\pi \cap u) = r, \dim(\pi \cap v) = s\}.$$

The closure $\tilde{C}_{r,s}$ of $C_{r,s}$ is the union of $C_{k,l}, k \geq r, l \geq s$ and is compact. It is clear that

$$\tilde{C}_{r,s} \cap \tilde{F}_{r,s} = C_{r,s} \cap F_{r,s} \tag{1}$$

is compact and consists of such elements of $C_{r,s}$ on which the restriction of H has the signature (r,s).

If u lies on the boundary of $F_{p,0}$, then (1) is false since the signature of the restriction of H is $(k,0), k < r$. Then $C_{r,s} \cap F_{r,s}$ is not closed, since then $C_{r,s}$ contains elements $F_{k,s}, k < r$. The similar phenomena happens when v lies on the boundary of $F'_{0,q}$. If $r = p, s = 0$, then $C_{p,0}$ is one point u; for $r = 0$ the orbit $C_{0,p}$ coincides with the Grassmanian of p-subspaces in v. If one excludes these degenerate cases, then $F_{p,0} \times F'_{0,q}$ is the connected component of such pairs for which the intersection of orbits is compact. We have Conjecture 4 for this case.

2.In this proof we essentially used the realization of the symmetric space F as a space of planes. It is possible to use a similar realizations of other classical domains.

To illuminate the dependence from these realizations it is possible to use a language of realizations of Hermitian symmetric spaces as generalized matrices, for example, using Jordan algebras. Let us describe the structure of such considerations in the case of Hermitian symmetric spaces of tube type. The general case of Hermitian spaces can be reduced to this case using realizations as Siegel domains of 2nd type (Jordan triple systems), since orbits essentially depend on the tube type part.

Let $V_{\mathbb{R}}$ be a simple Euclidean Jordan algebra of rank l, V be its complexification, and $\|z\|$ be the norm-function on V (the generalized determinant). Then V is a dense coordinate chart of the corresponding compact Hermitian symmetric space F. Using the action of the complex structural group we can define the rank of elements V (ranks of generalized matrices) such that $\mathrm{rk}(z) = n$ on dense (Zariski open) set where $\|z\| \neq 0$ and also define analogues of the principal minors. It is possible to define the signature $(r, s), r, s \leq l, r = \mathrm{rk}_+(y), s = \mathrm{rk}_-(y)$ of $y \in V_{\mathbb{R}}$ which correspond to orbits $\Gamma_{r,s}$ of the real structural group. The real domains $\Gamma_{l,0}, \Gamma_{0,l}$ are self-dual cones; the tube sets $T_{r,s} = \{y = \Im z \in \Gamma_{r,s}\}$ are intersections of V with orbits $F_{r,s}$ of the conformal real group $G_{\mathbb{R}}$ on the symmetric space F (elements of $G_{\mathbb{R}}$ are holomorphic automorphisms of Hermitian tube domains $T_{l,0}, T_{0,l}$which lie in V).The domains $T_{l,0}, T_{0,l}$ coincide with the orbits $F_{l,0}, F_{0,l}$ correspondingly. It is important that (r, s) are the invariants of orbits $F_{r,s}$ (independent from a choice of the chart V). The sets $T_{r,s}$ are domains for $r + s = l$; $T_{0,0}$ is their joint edge. The closure $\tilde{T}_{r,s}$ of $T_{r,s}$ is the union if $T_{k,m}, k \leq r, m \leq s$.

Let $u \in T_{l,0}, v \in T_{0,l}$ and $K = K(u, v)$ be the isotropy subgroup of (u, v) in complex conformal group G (of holomorphic automorphisms of the compact Hermitian symmetric space F). Let us describe intersections of K-orbits $C_{r,s}$ with V. They are

$$D_{r,s} = \{z; \mathrm{rk}(z - u) = l - r, \mathrm{rk}(z - v) = l - s\}.$$

The pair (r, s) is an invariant of the orbit $C_{r,s}$.The closure $\tilde{D}_{r,s}$ contains $D_{k,m}, k \leq r, s \leq s$. The basic fact is that

$$\tilde{D}_{r,s} \cap \tilde{T}_{r,s} = D_{r,s} \cap T_{r,s} \tag{2}$$

which implies immediately (1) for this general case, which in turn gives compactness of the intersections of corresponding $G_{\mathbb{R}}$ and $K(u, v)$-orbits $F_{r,s}$ and $C_{r,s}$. The identity (2) follows from the implication for $u \in T_{l,0}$:

$$\mathrm{rk}(u - z) \leq k \Rightarrow \mathrm{rk}_+(\Im z) \geq l - k. \tag{3}$$

or from the equivalent implication

$$\mathrm{rk}_+(\Im z) < l - k \Rightarrow \mathrm{rk}(u - z) > k. \tag{4}$$

These implications follow from the generalized Siegel theorem:

$$z \in T_{l,0} \Rightarrow \|z\| \neq 0, \tag{5}$$

that is $\mathrm{rk}(z) = l$. Let us recall an idea from one of the proofs of the generalized Siegel theorem. Firstly, by an action of the real structural group (of the cone $\Gamma_{l,0}$)

each element of $T_{l,0}$ can be transformed to the form $\lambda+ie$, where e is a unit element of the real Jordan algebra $V_{\mathbb{R}}$ and λ is a diagonal element (the generalization of the possibility to transform two quadratic forms, one of which is positive, simultaneously to the sum of squares).The action of the structural group is linear and preserves the condition $\|z\| \neq 0$ and the norm of diagonal elements is equal to the product of its diagonal elements and we have $\|\lambda + ie\| \neq 0$.

Let us recall that the Siegel theorem admits an inversion: if $\|u + z\| \neq z$ for all $z \in T_{l,0}$, then either $u \in T_{l,0}$ or $u \in T_{0,l}$.

The Siegel theorem gives immediately (4) for $k = l-1$. The condition $\mathrm{rk}_+(y) < 1$ means that $y \in \Gamma_{0,l}$ and $(u - z) \in T_{l,0}$ and $\|u - z\| \neq 0$. So $\mathrm{rk}(u - z) = l$. For smaller k we need to consider by a similar way an appropriate principal minor. If u will be a boundary point of $T_{l,0}$, then similar considerations show that (2) is not true and the intersection in the second part is closed.

If either $s = 0$ or $r = 0$ the intersection of orbits $F_{r,s} \cap K_{r,s}(u, v)$ reduces to one point: either u or v. If one were to exclude these degenerate cases, the connected component of the set of pairs (u, v), for which this intersection of orbits is compact, coincides with $T_{l,0} \times T_{0,l}$. This is exactly Conjecture 4 for this case.

References

[AG] D. N. Akhiezer and S. G. Gindikin, *On Stein extensions of real symmetric spaces.*, Math. Ann. **286** (1990), 1–12.

[G] S.Gindikin, *Tube domains in Stein symmetric spaces.*, Positivity in Lie theory: Open problems, W. de Gruyter, 1998, pp. 81–98.

[GM] [redacted]Gindikin and T.Matsuki, *Stein extensions of Riemann symmetric spaces and dualities of orbits on flag manifolds*, preprint (2001).

[M] T.Matsuki, *Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits*, Hiroshima Math. J. **18** (1988), 59–67.

[W] J.A.Wolf, *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Ann.of Math. **136** (1992), 541–555.

[WZ] J. A. Wolf and R.Zierau, *Linear cycle spaces in flag domains*, Math. Ann. **316** (2000), 529–545.

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