MATH 421. ADVANCED CALCULUS FOR ENGINEERING. FALL 2014. QUIZ 2

1. (90 points)

a) Find given Laplace transforms

$\mathcal{L}\{2t\cos(2t)\}$;

$\mathcal{L}\{\mathcal{U}(t-\pi/2)\sin(t-\pi/2)\}$;

$\mathcal{L}\{\mathcal{U}(t-1)t^2\}$.

b) Find given inverse Laplace transform

$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-3s+2}\right)$;

$\mathcal{L}^{-1}\left(\frac{3s^2-2s}{s^2-6s+13}\right)$.

a) 1) $\mathcal{L}\{2t\cos(2t)\} = -2\,\mathcal{L}(\cos 2t)' = -2\left(\frac{s}{s^2+4}\right)'$

$= -2\,\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{+2s^2-8}{(s^2+4)^2}$

2) $\mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2})\sin(t-\pi/2)\} = e^{-\frac{\pi s}{2}}\mathcal{L}(\sin t) =$

$= \frac{e^{-\pi s/2}}{s^2+1}$,

3) $\mathcal{L}\{\mathcal{U}(t-1)t^2\} = e^{-s}\mathcal{L}((t+1)^2) =$

$a=1,\ g(t)=t^2$

$= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

b) $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-3s+2}\right) = u(t-3)\left(e^{2(t-3)} - e^{t-3}\right) =$

$= 2u(t-3)\, e^{3/2(t-3)} \cancel{2} \sinh\left(\frac{t-3}{2}\right).$

$$\frac{1}{s^2-3s+2} = \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\mathcal{L}^{-1} \frac{\overset{\parallel}{1}}{(s-\frac{3}{2})^2 - \frac{1}{4}} = 2e^{3/2t} \sinh\left(\frac{t}{2}\right).$$

$$\mathcal{L}^{-1}\left(\frac{3s^2-2s}{s^2-6s+13}\right) = \mathcal{L}^{-1}\left(3 + \frac{\overset{16s-39}{16(s-3)+9}}{(s-3)^2+4}\right) =$$

$$= 3\delta(t) + e^{3t}\left(16\cos 2t + \frac{9}{2}\sin 2t\right).$$

Quest. $\mathcal{L}^{-1}\left(e^{-2s}\frac{3s^2-2s}{s^2-6s+13}\right) =$

$$= 3\delta(t-2) + u(t-2)\, e^{3(t-2)}\left(16\cos(2(t-2)) + \frac{9}{2}\sin 2(t-2)\right)$$

2. (60 points) Solve given initial-value problems using the Laplace transform
$y'' - 4y' + 3y = 2,\ y(0) = 2,\ y'(0) = -2;$

$$(s^2-4s+3)Y = \frac{2}{s} + sy(0) + y'(0) - 4y(0) =$$

$$= \frac{2}{s} + 2s - 10$$

$$(s-3)(s-1)Y = \frac{2}{s} + 2(s-3) - 4.$$

$$Y = \frac{2-4s}{s(s-3)(s-1)} + \frac{2}{s-1}$$

$$\frac{2-4s}{s(s-3)(s-1)} = \frac{a}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

$$2-4s = a(s-2)(s-1) + B\,s(s-1) + C\,s(s-3)$$

$$\left.\begin{array}{l} s=0 \\ s=3 \\ s=1 \end{array}\right| \begin{array}{ll} 2 = 3a, & a = \frac{2}{3} \\ -10 = 6B, & B = -5/3 \\ -2 = -2C, & C = 1 \end{array}$$

$y(0) = 2$

$$y(t) = a + Be^{3t} + (C+2)e^{t}$$

$$= 3e^{t} - \frac{5}{3}e^{3t} + \frac{2}{3}$$

$$= e^{2t}\left(\frac{4}{3}\cosh t - \frac{14}{3}\sinh t\right) + \frac{2}{3}$$

2. (60 points) Solve given initial-value problems using the Laplace transform
$y'' - 4y' + 3y = 2, y(0) = 2, y'(0) = -2;$

$$(s^2 - 4s + 3)Y = \frac{2}{s} + sy(0) + y'(0) - 4y(0) =$$

$$= \frac{2}{s} + 2s - 10$$

$$(s-3)(s-1)Y = \frac{2}{s} + 2(s-3) - 4.$$

$$Y = \frac{2 - 4s}{s(s-3)(s-1)} + \frac{2}{s-1}$$

$$\frac{2-4s}{s(s-3)(s-1)} = \frac{a}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

$$2 - 4s = a(s-2)(s-1) + B\,s(s-1) + C\,s(s-3)$$

$$\begin{array}{l|l} s=0 & 2 = 3a, \quad a = \frac{2}{3} \\ s=3 & -10 = 6B, \quad B = -5/3 \\ s=1 & -2 = -2C, \quad C = 1 \end{array}$$

$y(0) = 2$

$$y(t) = a + Be^{3t} + (C+2)e^{t}$$

$$= 3e^{t} - \frac{5}{3}e^{3t} + \frac{2}{3}$$

$$= e^{2t}\left(\frac{4}{3}\cosh t - \frac{14}{3}\sinh t\right) + \frac{2}{3}$$

3. (50 points) Solve given integral equation
$y'(t) = e^t + \int_0^t y(\tau)d\tau, y(0) = 1.$

$Y = \mathcal{L}y$

$$sY = \frac{1}{s-1} + \frac{Y}{s} + 1$$

$$\left(s - \frac{1}{s}\right)Y = \frac{1}{s-1} + 1 = \frac{s}{s-1}$$

$$\cancel{\left(\frac{s^2-1}{s}\right)}Y = \frac{s^2}{(s^2-1)(s-1)} = \frac{s^2}{(s-1)^2(s+1)}$$

$$\frac{a(s-1)+B}{(s-1)^2} + \frac{C}{s+1} = \frac{s^2}{(s-1)^2(s+1)}$$

$$a(s^2-1) + B(s+1) + C(s-1)^2 = s^2$$

$a + C = 1$
$B - 2C = 0$
$-a + B + C = 0$

$C = \frac{1}{4}, \quad B = \frac{1}{2}, \quad a = \frac{3}{4}$

$$y = ae^t + Be^t \cdot t + Ce^{-t}$$

$$= \frac{3}{4}e^t + \frac{1}{2}te^t + \frac{1}{4}e^{-t}$$

$y(0) = 1$

Lecture 9.

Part 2.
Orthogonal functions and Fourier Series.

Ch. 12.

Geometrical view on functions and series.

- Linear space of functions (infinite dimensional) ~~over~~ $[a,b]$.
- functions $\Leftrightarrow$ vectors
- Series $\Leftrightarrow$ expansion on a base

Orthogonality: $[a,b]$

The inner product

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx$$

$$f_1 \perp f_2 \Leftrightarrow (f_1, f_2) = 0$$

orthogonal

$$\|f\|^2 = (f,f) = \int_a^b f^2(x)\, dx, \quad \|f\| = \sqrt{\int_a^b f^2 dx}$$

norm

Orthogonal set $\{\varphi_1(x), \dots \varphi_n(x), \dots\}$

$\varphi_i \perp \varphi_j,\ i \neq j$. $(\varphi_i, \varphi_j) = 0$ $i \neq j$.

Examples.

1) $\{\sin x, \sin 2x, \dots\}$ is orthog. set

$\varphi_n(x) = \sin(nx)$ on $[-\pi, \pi]$

$$(\varphi_n, \varphi_m) = \int_{-\pi}^{\pi} \sin(nx) \sin(mx)\, dx =$$

$n \neq m$

$$\frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(n-m)x - \cos(n+m)x \right) dx$$

$n \neq m$

$$= \frac{1}{2} \left(\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right) \Bigg|_{-\pi}^{\pi}$$

$$= 0 \quad (\sin k\pi = 0)$$

if $n \neq m$

What is for $n = m$?

$$\|\varphi_n\|^2 = (\varphi_n, \varphi_n) = \int_{-\pi}^{\pi} \sin^2(nx)\, dx =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left(1 - \underbrace{\cos(2nx)}_{0}\right) dx$$

$$= \pi$$

$\|\varphi_n\| = \sqrt{\pi}$

2) $\{\cos nx,\ n=0,1,2,\ldots\} = \{\psi_n\}$

$$(\cos nx, \cos mx) = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$n \neq m$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(n-m)x + \cos(n+m)x)\,dx$$

$$= 0 \quad (n \neq m)$$

$\|\psi_0\|^2 = 2\pi$ $\qquad \|1\| = \sqrt{2\pi}$

$\|\psi_n\|^2 = \pi$ $\qquad \|\cos nx\| = \sqrt{\pi}$
$n \neq 0$ $\qquad n \neq 0$

3) $\varphi_n \perp \psi_m$

$$(\varphi_n^{*}, \psi_m) = \int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0$$

We can integrate as above or ...

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$$

We can remark that we integrate an odd function along a symmetric interval.

So $\{\sin mx, \cos nx\}$ is orthogonal on $[-\pi,\pi]$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$