Lecture 5.

Each rational fraction

$$\frac{P(s)}{Q(s)}, \quad \deg P < \deg Q,$$

can be decomposed at a sum of

$$\frac{c_1(s+a)^{m-1} + \dots + c_{m-1}(s+a) + c_m}{(s+a)^m},$$

$$\frac{c_1 s^{2k-1} + c_2 s^{2k-2} + \dots + c_{2k}}{(s^2 + as + b)^k}$$

We need to use several formulas from Table 1.

Ex. Find

$$\mathcal{L}\left(3te^{-2t}\sin 3t\right) = \frac{6(s+2)}{((s+2)^2+9)^2} = \frac{6(s+2)}{(s^2+4s+13)}$$

Step 1.

$$\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$$

Step 2

$$\mathcal{L}(t\sin 3t) = -3\left(\frac{1}{s^2+9}\right)' = \frac{6s}{(s^2+9)^2}$$

Transforms of integrals.

The convolution

$$\boxed{f*g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.}$$

It's a "2nd product" of functions

$f*g = g*f$ (commutative)

$(f*g)*h = f*(g*h)$ (associative)

Convolution Theorem

$$\boxed{\mathcal{L}(f*g) = \mathcal{L}f \cdot \mathcal{L}g}$$

(without proof)

Applications:

1) Compute the convolution

$$f(t) = \int_0^t \tau \cos(t-\tau)\,d\tau = 1 - \cos t$$

$$\mathcal{L}f = \mathcal{L}(\tau)\,\mathcal{L}(\cos\tau) = \frac{1}{s^2}\cdot\frac{s}{s^2+1} = \frac{1}{s(s^2+1)} =$$

$$= \frac{a}{s} + \frac{Bs+C}{s^2+1}$$

Special case of $f*g$ if $g \equiv 1$,

$$f*1 = \int_0^t f(\tau)\,d\tau \leftarrow \text{antiderivative}$$

Corollary.

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{(\mathcal{L}f)(s)}{s}.$$

What is integral equation?

Example. Volterra integral equations,

$$f(t) = g(t) + \int_0^t h(t-\tau) f(\tau)\,d\tau = g(t) + (h*f)(t)$$

- f - unknown function

(g, h) are known

$F = \mathcal{L}f$, $G = \mathcal{L}g$, $H = \mathcal{L}h$

$F = G + H F$ $\quad F = G + HF$

$$F = \frac{G}{1-H}, \quad f = \mathcal{L}^{-1}\left(\frac{G}{1-H}\right)$$

Table 2.

$$(\mathcal{L}f^{(k)})(s) = s^k \mathcal{L}f(s) - s^{k-1}f(0) - s^{k-2}f'(0) - \dots - f^{(k-1)}(0).$$

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}f(s-a), \quad s > \max(0,a)$$

$$\mathcal{L}(\mathcal{U}(t-a)f(t-a)) = e^{-sa}\mathcal{L}f, \quad a \geq 0$$

$$\mathcal{L}(g(t)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t+a)), \quad a > 0$$

$$\mathcal{L}(t^k f(t)) = (-1)^k (\mathcal{L}f(s))^{(k)}.$$

$$\mathcal{L}(f * g) = \mathcal{L}f \, \mathcal{L}g$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{(\mathcal{L}f)(s)}{s}$$

$$\begin{array}{l|l} a+B=0 & a=1,\ B=-1,\ C=0 \\ C=0 & \\ a=1 & \end{array}$$

$$\mathcal{L}f=\frac{1}{s}-\frac{s}{s^2+1}.$$

We need to return back.

$$f=\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)=1-\cos t.$$

2. Compute

$$f(t)=\int_0^t e^{\tau}\cos(t-\tau)\,d\tau=\frac{1}{2}\left(e^t-\cos t+\sin t\right)$$

$$F=\mathcal{L}f=\frac{1}{s-1}\cdot\frac{s}{s^2+1}=\frac{s}{(s-1)(s^2+1)}=\frac{a}{s-1}+\frac{Bs+C}{s^2+1}$$

$$a(s^2+1)+(s-1)(Bs+C)=s$$

$$\begin{array}{c|l} s^2 & a+B=0 \\ s & -B+C=1 \\ 1 & a-C=0 \end{array}\qquad \begin{array}{l} 2a=1 \\ a=\frac{1}{2},\ B=-\frac{1}{2},\ C=\frac{1}{2} \end{array}$$

$$\mathcal{L}^{-1}F=ae^t+B\cos t+C\sin t$$

$$=\frac{1}{2}\left(e^t-\cos t+\sin t\right)$$

Ex.

$$f(t) = e^{3t} + \int_0^{\infty} e^{2(t-\tilde{\tau})} f(\tilde{\tau})\, d\tilde{\tau}$$

$g = e^{3t}$ $\quad h = e^{2t}$ $\quad G = \frac{1}{s-3}, H = \frac{1}{s-2},$

$F = \mathcal{L}f$

$$F = \frac{1}{s-3} + \frac{1}{s-2} F$$

$$F = \frac{\frac{1}{s-3}}{1 - \frac{1}{s-2}} = \frac{s-2}{(s-3)^2} = \frac{1}{s-3} + \frac{1}{(s-3)^2}$$

$$\underline{f(t) = e^{3t}(1+t).}$$

$\quad \overset{\shortparallel}{\mathcal{L}^{-1}F}$

Integrodifferential equations

$$y' + 4\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 1$$

$Y = \mathcal{L}y$

$$\left(s + \frac{4}{s}\right) Y = \frac{1}{s} + 1$$

$$\left(\frac{s^2+4}{s}\right) Y = \frac{s+1}{s}$$

$$Y = \frac{s+1}{s^2+4}$$

$$y(t) = \cos 2t + \frac{1}{2}\sin 2t$$

$\overset{\shortparallel}{\mathcal{L}^{-1}Y}$