Lecture 4

We can now to compute

$$\mathcal{L}^{-1}\left(\frac{as+b}{s^2+cs+d}\right)$$

using the translation 1.

Application.

$$\frac{P(s)}{Q(s)} \sim \text{rational fraction}$$

3,4

4,2

Example. Solve the d.e.

$$y''+2y'+2y=e^{2t},\quad y(0)=-1,\quad y'(0)=1.$$

$$Y=\mathcal{L}y$$

$$(s^2+\underbrace{2s+2}_{(s+1)^2+1})Y=\frac{1}{s-2}+sy(0)+y'(0)+2y(0)$$

$$=\frac{1}{s-2}-s-1=\frac{-s^2+s+3}{s-2}$$

$$Y=\frac{-s^2+s+3}{(s-2)((s+1)^2+1)}=\frac{a}{s-2}+\frac{B+C(s+1)}{(s+1)^2+1},$$

$$\tilde{s}=s+1$$

$$y=ae^{2t}+e^{-t}(B\sin t+C\cos t)\quad(*)$$

$$a(s^2+2s+2)+B(s-2)+C(s+1)(s-2)=-s^2+s+3$$

$$\begin{array}{l|l} s^2 & a+C=-1 \\ s & 2a+B-C=1 \\ 1 & 2a-2B-2C=3 \end{array}$$

$C=-a-1$

$3a+B=0$

$4a-2B=1 \quad 10a=1$

$$a=\frac{1}{10},\quad B=-\frac{3}{10},\quad C=-\frac{11}{10}$$

$(a,B,C)\Rightarrow(*)$

$$y(t)=\frac{1}{10}e^{2t}+e^{-t}\left(-\frac{3}{10}\sin t-\frac{11}{10}\cos t\right)$$

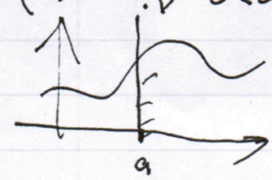
$$y(0)=-1$$

Translation 2.

The unit step function $U(t)$. (The Heaviside function)

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \le 0. \end{cases}$$

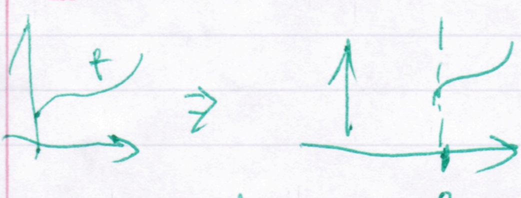
$$U(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t \le a \end{cases}$$ (cutting for f $t<a$)



2nd Transation Theorem.

$$\mathcal{L}(\underbrace{u(t-a) f(t-a)}) = \int_0^\infty u(t-a) f(t-a) e^{-st} dt =$$

$\boxed{a>0}$ What is it? $= \int_a^\infty f(t-a) e^{-st} dt$



$\boxed{u = t-a}$ $t = u+a$

Translation and cutting. $= \int_0^\infty f(u) e^{-s(u+a)} du = e^{-sa}(\mathcal{L}f)(s)$

$$\boxed{\mathcal{L}(u(t-a) f(t-a)) = e^{-sa} \mathcal{L}f, \ a \geq 0}$$

3rd line at Table 2.

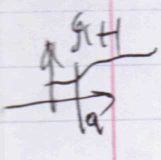
Ex. 1, $\mathcal{L}(u(t-1)(t-1)^2) = e^{-s} \cdot \frac{2}{s^3}$

2. $\mathcal{L}^{-1}\left(\frac{s e^{-2s}}{s^2+9}\right) = u(t-2) \cos(3(t-2))$.

~~3.6~~

4.4

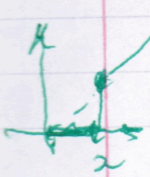
Another form of 2nd Translation Theorem.



$$\boxed{\mathcal{L}(g(t)\,u(t-a)) = e^{-as}\,\mathcal{L}g(t+a)\,,\quad a>0}$$

$f(t)=g(t+a)$ $\qquad$ $g(t)=f(t-a)$

Examples.



1. $\mathcal{L}(t^2\,u(t-2)) = e^{-2s}\,\mathcal{L}((t+2)^2) = e^{-2s}\,\mathcal{L}(t^2+4t+4)$

$a=2$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

2. $\mathcal{L}(\sin t\; u(t-\frac{\pi}{2})) = e^{-\pi/2\,s}\,\mathcal{L}(\sin(\frac{\pi}{2}+t)) =$

$g(t)=\sin t$

$a=\pi/2$

$$= e^{-\frac{\pi}{2}s}\,\mathcal{L}(\cos t) = e^{-\frac{\pi}{2}s}\,,\ \frac{s}{s^2+1}.$$

3. $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s+1)^3}\right) = \frac{1}{2}\,u(t-3)\,e^{-t+3}\,(t-3)^2.$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{1}{2}e^{-t}\,t^2$$

4. $\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+9}\right) = \frac{1}{3}\,u(t-5)\,\sin 3(t-5).$

Derivatives of the Laplace transform (4.4.1)

$$\left(\mathcal{L}f(s)\right)' = \left(\int_0^\infty e^{-st} f(t)\,dt\right)' = -\int_0^\infty e^{-st} f(t)\overset{t}{\vee}\,dt = \cancel{= -s\,\mathcal{L}f(s)}.$$

$$\mathcal{L}(tf(t)) = -\left(\mathcal{L}f(s)\right)'.$$

The generalization:

$$\mathcal{L}\left(t^k f(t)\right) = (-1)^k \left(\mathcal{L}f(s)\right)^{(k)}$$

Example.

$$\mathcal{L}\{t\sin t\} = -\left(\frac{1}{s^2+1}\right)' = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = -\left(\frac{s}{s^2+1}\right)' = -\frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$$= \frac{1}{s^2+1} - \frac{2}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) =$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = -\frac{1}{2}t\sin t + \cancel{}$$

$$= \frac{1}{2}\left(\sin t - t\cos t\right).$$

Table 2.

$$(\mathcal{L}f^{(k)})(s) = s^k \mathcal{L}f(s) - s^{k-1}f(0) - s^{k-2}f'(0) - \dots - f^{(k-1)}(0).$$

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}f\,(s-a), \quad s > \max(0,a)$$

$$\mathcal{L}(\mathcal{U}(t-a)f(t-a)) = e^{-sa}\mathcal{L}f, \quad a \geq 0$$

$$\mathcal{L}(g(t)\,\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t+a)), \quad a > 0$$

$$\mathcal{L}(t^k f(t)) = (-1)^k (\mathcal{L}f(s))^{(k)}.$$

Some problems from Ma 1.

4.1. §

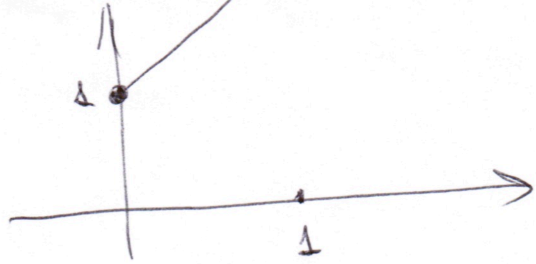
#37. $f(t) = \sin 2t \cos 2t = \frac{1}{2} \sin 4t$

$$\mathcal{L}f(s) = \frac{2}{s^2+16}$$

#38. $f(t) = \cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$$\mathcal{L}f(s) = \frac{s}{2} + \frac{1}{2} \frac{s}{s^2+4}$$

5 4. Примеры из HG1.



#4 $f(t) = \begin{cases} 2t+1 & 0 \le t < 1 \\ 0 \end{cases}$

$$G = \int_0^1 (2t+1)\, e^{-st}\, dt$$

$$\int_0^1 e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^1 = -\frac{1}{s}\left(e^{-s} - 1\right).$$

$$2\int_0^1 t\, e^{-st}\, dt = 2\left(-\frac{t}{s} e^{-st} \Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\, dt \right)$$

$$= 2\left(\frac{1}{s} - \frac{1}{s^2}\left(e^{-s} - 1\right)\right)$$

$$G = \frac{3}{s} - \frac{1}{s} e^{-s} - \frac{1}{s^2}\left(e^{-s} - 1\right)).$$

#5 $f = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 \end{cases}$

$$\int_0^{\pi} = \int_0^{\infty} - \int_{\pi}^{\infty} = \frac{1}{t^2+1} - \int_{\pi}^{\infty} \sin t\, e^{-st}\, dt =$$

$$= \frac{1}{t^2+1} - \int_0^{\infty} \sin(t+\pi)\, e^{-s(t+\pi)}\, dt$$

$\sin(t+\pi) = -\sin t$

$$= \frac{1}{t^2+1} + \int_0^{\infty} \sin(t)\, e^{-st}\, dt \cdot e^{-s\pi}$$

$$= \frac{1}{t^2+1}\left(1 + e^{-s\pi}\right)$$