Lecture 3.

Verification: $y(0)=2$

Problem 2.

$$y' + 2y = \cos 3t, \quad y(0) = -1$$

$y(t)$

Step 1. $Y = \mathcal{L}y$

$$(s+2)Y - y(0) = \frac{s}{s^2+9}$$

$$(s+2)Y = \frac{s}{s^2+9} - 1$$

Step 2. $$Y(s) = -\frac{1}{s+2} + \frac{s}{(s^2+9)(s+2)}$$

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Step 3. $y(t) = \mathcal{L}^{-1} Y$

$$y(t) = -\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{s}{(s^2+9)(s+2)}\right)$$

$$= -e^{-2t} + \dots$$

$$\frac{s}{(s^2+9)(s+2)} = \frac{a}{s+2} + \frac{Bs}{s^2+9} + \frac{C}{s^2+9}$$!

$$a(s^2+9) + Bs(s+2) + C(s+2) = s$$

$$\begin{array}{c|l} s^2 & a+B=0 \\ s & 2B+C=1 \\ 1 & 9a+2C=0 \end{array} \qquad B=-a \qquad \left. \begin{array}{l} a=-\frac{2}{13} \\ B=\frac{2}{13} \\ C=\frac{9}{13} \end{array} \right]$$

$$y(t) = (a-1)e^{-2t} + B\cos 3t + \frac{C}{3}\sin 3t.$$

$$= -\frac{15}{13}e^{-2t} + \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t.$$

Substitution: $t=0 \Rightarrow y(0) = -1.$

Lecture 3.

Translations 1.

Let we know $\mathcal{L}f = F$.

Find $\mathcal{L}(e^{at}f(t))$.

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt =$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = (\mathcal{L}f)(s-a).$$

Sign of a? $\begin{cases} a<0 & \text{- no problems for } s>0 \\ a>0 & \text{- only for } s>a \end{cases}$

$$\mathcal{L}(e^{at}f(t)) = (\mathcal{L}f)(s-a), \quad s > \max(0,a)$$

Ex. 1) $\mathcal{L}(e^{3t}\cos 5t) = \dfrac{s-3}{(s-3)^2+25} = \dfrac{s-3}{s^2-6s+34}$.

$\mathcal{L}(\cos 5t) = \dfrac{s}{s^2+25}$

Applications to $\mathcal{L}^{-1}$

2) $\mathcal{L}^{-1}\left(\frac{5}{(s-5)^3}\right) = \frac{5}{2} t^2 e^{5t}$, $s > 5$

$3(s+1)-4$

3) $\mathcal{L}^{-1}\left(\frac{3s-1}{(s+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{3}{s+1} - \frac{4}{(s+1)^2}\right) =$

$\tilde{s} = s+1$

$= 3e^{-t} - 4e^{-t} t = e^{-t}(3-4t)$.

4) $\mathcal{L}^{-1}\left(\frac{2s+3}{s^2+6s+13}\right) = \mathcal{L}^{-1}\left(\frac{2(s+3)-3}{(s+3)^2+4}\right)$

$\tilde{s} = s+3$

No real roots

Start of denominator

$= e^{-3t}\left(2\cos 2t - \frac{3}{2}\sin 2t\right)$.

corresponds to the transl.

start from denominator!

5) $\mathcal{L}^{-1}\left(\frac{3s-1}{s^2+4s+1}\right) = \mathcal{L}^{-1}\left(\frac{3(s+2)-7}{(s+2)^2-3}\right) =$

Roots! $2 \pm \sqrt{3}$

$= e^{-2t}\left(3\cosh(\sqrt{3}t) - \frac{7}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$.

Another solution.

$\mathcal{L}^{-1}\left(\frac{3s-1}{(s+2-\sqrt{3})(s+2+\sqrt{3})}\right) = \mathcal{L}^{-1}\left(\frac{A}{s+2-\sqrt{3}} + \frac{B}{s+2+\sqrt{3}}\right)$

$$\begin{cases} a+B=3 \\ a(2+\sqrt{3})+B(2-\sqrt{3}) = 2(a+B)+\sqrt{3}(a-B) = -1 \end{cases}$$

$$\sqrt{3}(a-B)-7=0 \Rightarrow a-B=\frac{7}{\sqrt{3}}=\frac{7\sqrt{3}}{3}$$

$$a=\frac{9+7\sqrt{3}}{6}, \quad B=\frac{9-7\sqrt{3}}{6}$$

$$\mathcal{L}^{-1}\left(\qquad\right) = a e^{(-2+\sqrt{3})t} + B e^{(-2-\sqrt{3})t}$$

We have 2 different answers!

Are they different?

No. Use formulas for cosh, sinh.

Conclusion: If $\Delta>0$ we can write the answer either through hyperbolic or through exponents!

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Problem 3,

$$y'' - 4y' = e^{-t}, \quad y(0)=1, \quad y'(0)=-1.$$

$$(s^2-4s)Y - sy(0) - y'(0) + 4y(0) = \frac{1}{s+1}$$

$$s(s-4)Y = \frac{1}{s+1} + s - 5$$

$$Y = \frac{1}{(s+1)s(s-4)} + \frac{1}{s-4} - \frac{5}{s(s-4)}$$

$$= \frac{1}{s-4} + \frac{-5s-4}{s(s-4)(s+1)} = \frac{1}{s-4} + \frac{a}{s} + \frac{B}{s-4} + \frac{C}{s+1}$$

$$y(t) = a + (B+1)e^{4t} + ce^{-t}$$

$$a(s-4)(s+1) + Bs(s+1) + Cs(s-4) = -5s-4$$

$$\begin{array}{l|l|l|l}
s^2 & a+B+C=0 & a=1 & a=1 \\
s & -3a+B-4C=-5 & B+C=-1 & B=-\frac{6}{5} \\
1 & -4a=-4 & B-4C=-2 & C=\frac{1}{5}
\end{array}$$

$$y(t) = 1 - \frac{1}{5}e^{4t} + \frac{1}{5}e^{-t}$$

$y(0)=1$

$y'(0)=-1$