MATH 421. ADVANCED CALCULUS FOR ENGINEERING. FALL 2014. QUIZ 6

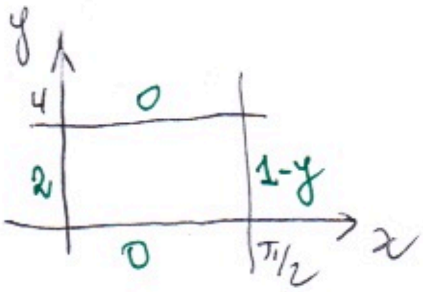
1. (75 points) Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi/2, 0 < y < 4,$$

subject to the given conditions

$$\frac{\partial u}{\partial y} u(x,0) = \frac{\partial u}{\partial y} u(x,4) = 0,$$

$$u((0,y) = 2, u(\tfrac{\pi}{2}, y) = 1 - y.$$



$$u = XY$$

$$Y'' + \lambda Y = 0, \quad Y'(0) = Y'(4) = 0$$

$$X'' - \lambda X = 0$$

$$Y_n = \cos\left(\frac{n\pi}{4} y\right), \quad n = 0, 1, 2, \ldots ; \quad \lambda_n = \frac{n^2\pi^2}{16}.$$

$$X_n = a_n \cosh\left(\frac{n\pi}{4} x\right) + b_n \sinh\left(\frac{n\pi}{4} x\right), \quad n > 0$$

$$X_0 = a_0 + b_0 x;$$

$$u(x,y) = a_0 + b_0 x + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4} y\right)\left(a_n \cosh\left(\frac{n\pi}{4} x\right) + b_n \sinh\left(\frac{n\pi}{4} x\right)\right)$$

$$u(0,y) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{4} y\right) = 2 \Rightarrow a_0 = 2, \; a_n = 0, \; n \geq 1.$$

$$u(x,y) = 2 + b_0 x + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{4} y\right) \sinh\left(\frac{n\pi}{4} x\right);$$

$$u\left(\frac{\pi}{2}, y\right) = 1 - y = \qquad 0 < y < 4$$

$$= 2 + b_0 \frac{\pi}{2} + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi^2}{8}\right) \cos\left(\frac{n\pi}{4} y\right)$$

$$= -1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi}{4} y\right)$$

$$2 + b_0 \frac{\pi}{2} = -1 \qquad\qquad b_0 = -\frac{6}{\pi}$$

$$b_n \sinh\left(\frac{n\pi^2}{8}\right) = \frac{8}{\pi^2} \frac{(-1)^n - 1}{n^2} \qquad\qquad b_n = \frac{8}{\pi^2} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)}$$

$$u(x,y) = 2 - \frac{6}{\pi} x + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)} \cos\left(\frac{n\pi}{4} y\right) \sinh\left(\frac{n\pi}{4} x\right)$$

$$0 < x < \frac{\pi}{2}, \quad 0 < y < 4$$

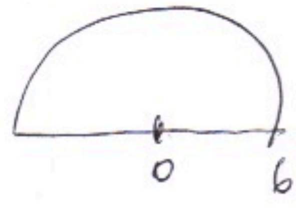
2. (75 points) Solve Laplace's equation in the semidisk (in radial coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 < \theta < \pi, 0 < r < 6$$

subject to the given conditions

$$u(6,\theta) = \theta - 1, 0 < \theta < \pi,$$

$$\frac{\partial u}{\partial \theta}(r,0) = 0, \frac{\partial u}{\partial \theta}(r,\pi) = 0.$$



$$\Theta'' + \lambda \Theta = 0$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\Theta'(0) = \Theta'(\pi) = 0$$

$$\Theta_n(\theta) = \cos(n\theta), \quad \lambda_n = n^2, \ n = 0, 1, \ldots$$

$$\Theta_0(\theta) = 1$$

$$R_n(r) = c_n r^n + d_n r^{-n} \qquad \Big| \text{ regularity: } d_n = 0$$

$$R_0(r) = c_0 + d_0 \ln r$$

$$u(r,\theta) = c_0 + \sum_{n=1}^{\infty} c_n r^n \cancel{\sin} \cos(n\theta)$$

$$u(6,\theta) = \theta - 1$$

$$= c_0 + \sum_{n=1}^{\infty} c_n 6^n \cos(n\theta)$$

$$= \frac{\pi}{2} - 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(n\theta)$$

$$C_0 = \frac{\pi}{2} - 1$$

$$C_n 6^n = -\frac{2}{\pi} \frac{(-1)^n - 1}{n^2}, \quad n > 0$$

$$C_n = -\frac{2}{\pi} \frac{(-1)^n - 1}{n^2 \, 6^n}$$

$$u(r,\theta) = \frac{\pi}{2} - 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \left(\frac{r}{6}\right)^n \cos(n\theta) \qquad \begin{array}{l} 0 < r < 6 \\ 0 < \theta < \pi \end{array}$$

3. (50 points) Using d'Alambert's formula solve the initial problem for the wave equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, -\infty < x < \infty, t > 0$$

subject to the given conditions

$$u(x,0) = \cos x,$$

$$\frac{\partial u}{\partial t}(x,0) = x^2.$$

$a = 2$

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(x)\,dx$$

$a=2$, $f(x) = \cos x$, $g(x) = x^2$.

$$u(x,t) = \frac{1}{2}\left[\cos(x+2t) + \cos(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} x^2\,dx$$

$$= \cos x \cos 2t + \frac{x^3}{12}\Big|_{x-2t}^{x+2t} = \cos x \cos 2t + \frac{1}{12}\left(12x^2t + 16t^3\right)$$

$$= \cos x \cos 2t + t\left(x^2 + \frac{4}{3}t^2\right).$$

$u(x,0) = \cos x$

$u'_t(x,0) = x^2.$

The final exam is

cumulative.

Pay the attention:

- Laplace transform and its inversion
- Applications to differential equations
- Convolution; integral and integrodifferential equations
- All types of Fourier ~~transforms~~ series
- Sturm - Liouville problems
- Separation of variables for boundary problems of partial diffential equatons.
Polar coordinates

Refer on all quizes, midterms and lecture notes on webpage.