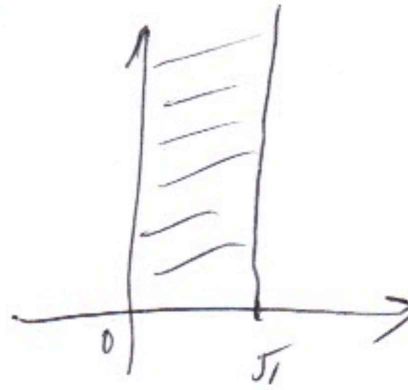
MATH 421. ADVANCED CALCULUS FOR ENGINEERING. FALL 2014. QUIZ 5



1. (100 points) Solve the wave equation

$$(1) \quad 9\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < \pi, t > 0$$

subject to the given conditions

$$(2) \quad u(0,t) = u(\pi,t) = 0,$$

$$(3),(4) \quad u(x,0) = 1, \frac{\partial u}{\partial t}(x,0) = -1. \qquad u = XT$$

Step 1. $\frac{X''}{X} = \frac{T''}{9T} = -\lambda$; $X'' + \cancel{9}\lambda X = 0$, $T'' + 9\lambda T = 0$

$X'' + \lambda X = 0$, $X(0) = X(\pi) = 0$ | $X_n = \sin nx$, $n = 1, 2, \ldots$, $\lambda_n = n^2$

$T_n = a_n \cos 3nt + b_n \sin 3nt$

$u_n(x,t) = \sin nx\,(a_n \cos 3nt + b_n \sin 3nt)$, $n = 1, 2$

are solutions of (1)
- with separated variables
- satisfying (2)

$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,t)=\sum_{n=1} \sin nx\,(a_n \cos 3nt + b_n \sin$$

satisfy (1), (2).

Find a_n, b_n from initial conditions (3), (4):

$$u(x,0)=\sum_{n=1}^{\infty} a_n \sin nx \equiv 1, \quad 0<x<\pi,$$

$$1=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx \Rightarrow$$

$$a_n=\frac{2}{\pi}\,\frac{1-(-1)^n}{n}$$

$$u'_t=(x,0)=\sum_{n=1}^{\infty} 3n\, b_n \sin nx \equiv -1,$$

$$b_n=-\frac{2}{\pi}\,\frac{1-(-1)^n}{3n^2}.$$

$$u(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx\left(\cos 3nt-\frac{\sin 3nt}{3n}\right.$$

$0<x<\pi,$

$t>0$

2. (100 points) Solve the heat equation

$$\pi^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < 2, t > 0$$

subject to the given conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2,t) = 0,$$

$$u(x,0) = -x.$$

$$X'' + \lambda X = 0,$$

$$T' + \pi^2 \lambda T, \quad T_\lambda(t) = \exp(-\pi^2 \lambda t),$$

$$X'(0) = X'(2) = 0,$$

$$X_n = \cos\left(\frac{n\pi x}{2}\right), n = 0, 1, 2, \ldots, \lambda_n = \frac{n^2\pi^2}{4},$$

$$T_n = \exp\left(-\frac{n^2\pi^4}{4} t\right); \quad u_n(x,t) = X_n T_n.$$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^4}{4} t\right) \cos\left(\frac{n\pi x}{2}\right).$$

$$u(x,0) \simeq -x \simeq -1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi}{2} x\right), 0 < x < 2$$

$$u(x,t) = -1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi}{2} x\right) \exp\left(-\frac{n^2\pi^4}{4} t\right),$$

$$0 < x < 2, t > 0.$$

Lecture 26.

Applications of polar coordinates.

Ex. 1.

$\Delta u = 0$

$$(1)\quad \Delta u = 0,\ 2 < r < 3,\ -\pi < \theta < \pi$$

$$(2)\quad u(2,\theta) = \theta$$

$$(3)\quad u(3,\theta) = |\theta|$$

$$u_n(r,\theta) = R_n(r)\,\Theta_n(\theta)$$

$$(4)\quad u(r,\theta) = c_0 + c_1 \ln r + \sum_{n=1}^{\infty} \left\{ r^n (a_n \cos n\theta + b_n \sin n\theta) + r^{-n} (d_n \cos n\theta + e_n \sin n\theta) \right\}$$

$$R_n(r) = c_n r^n + c_{-n} r^{-n},\ n > 0$$
$$= c_0 + c_1 \ln r,\ n = 0$$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta,\ n = 0, 1, \ldots$$

We minimize the number of undefinite coefficients at

$$u(r,\theta) = \sum u_n(r,\theta)$$

Computation of coefficients,

$$u(2,\theta) = c_0 + c_1 \ln 2 + \sum_{n=1}^{\infty}\left\{(2^n a_n + 2^{-n} d_n)\cos n\theta + (2^n b_n + 2^{-n} e_n)\sin n\theta\right\}$$

$$= \theta = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta, \quad -\pi < \theta < \pi$$

$$c_0 + c_1 \ln 2 = 0$$

$$2^n a_n + 2^{-n} d_n = 0$$

$$2^n b_n + 2^{-n} e_n = \frac{2(-1)^{n+1}}{n}$$

$$\cancel{u(3,\theta) = \frac{\pi}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n}}$$

$$u(3,\theta) = c_0 + c_1 \ln 3 + \sum_{n=1}^{\infty}\left\{(3^n a_n + 3^{-n} d_n)\cos n\theta + (3^n b_n + 3^{-n} e_n)\sin n\theta\right\}$$

$$= |\theta| = \frac{\pi}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos n\theta$$

$$c_0 + c_1 \ln 3 = \frac{\pi}{2}$$

$$3^n a_n + 3^{-n} d_n = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$3^n b_n + 3^{-n} e_n = 0$$

26.3

$$c_0 + c_1 \ln 2 = 0 \qquad c_0 = -c_1 \ln 2$$

$$c_0 + c_1 \ln 3 = \frac{\pi}{2} \qquad c_0 (\ln 3/2) = \frac{\pi}{2}$$

$$\boxed{c_0 = \frac{\pi}{2 \ln(3/2)}}$$

$$\boxed{c_1 = -\frac{\pi}{2 \ln(3/2) \ln 2}}$$

$$\begin{cases} 2^n a_n + 2^{-n} d_n = 0 \\ 3^n a_n + 3^{-n} d_n = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2} \end{cases}$$

$$d_n = -4^n a_n$$

$$a_n \left(3^n \cancel{4^n} - \frac{4^n}{3^n}\right) = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2}$$

$$a_n = \frac{1}{\pi} \frac{\left((-1)^n - 1\right) 3^n}{n^2 (9^n - 4^n)}, \quad d_n = -\frac{1}{\pi} \frac{\left((-1)^n - 1\right) 12^n}{n^2 (9^n - 4^n)}$$

$$3^n b_n + 3^{-n} e_n = 0$$

$$2^n b_n + 2^{-n} e_n = \frac{2(-1)^{n+1}}{n}$$

$$e_n = -9^n b_n$$

$$b_n \left(2^n - \frac{9^n}{2^n}\right) = \frac{2(-1)^{n+1}}{n}$$

$$b_n = 2 \frac{(-1)^{n+1} 2^n}{n (4^n - 9^n)}, \quad e_n = 2 \frac{(-1)^n 18^n}{n (4^n - 9^n)}$$

Substitute at (4).

$$u(r,\theta) = \frac{\pi}{2\ln(3/2)}\left(1 - \frac{\ln r}{\ln 2}\right) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi} \cos n\theta \left(\frac{((-1)^n - 1)\, 3^n}{n^2 (9^n - 4^n)} \left(r^n - 4^n r^{-n}\right)\right)$$

$$+ 2 \sin n\theta \left(\frac{(-1)^{n+1}\, 2^n}{n(4^n - 9^n)} \left(r^n - 9^n r^{-n}\right)\right).$$

$2 < r < 3$, $-\pi < \theta < \pi$