

Example 2. The same problem, but

$$u(r, \theta) = |\theta|, \quad -\pi < \theta < \pi,$$

$$|\theta| = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\theta.$$

At (4)

$$c_0 = \frac{\pi}{2}, \quad b_n = 0, \quad n \geq 1$$

$$a_n = \frac{1}{\pi} \frac{(-1)^n - 1}{2^n n^2}, \quad n \geq 1$$

$$u(r, \theta) = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{2^n n^2} r^n \cos(n\theta)$$

Example 3.

$$(2) u(\cancel{0} 2, \theta) = \sin 3\theta, \quad -\pi < \theta < \pi$$

$$c_0 = 0; \quad a_n = 0, \quad n \geq 1, \quad b_n = 0, \quad n \neq 3,$$

$$u(r, \theta) = \frac{r^3}{12} \sin 3\theta.$$

Ex. 4.

$$(3) \Delta u = 0, \quad 0 \leq r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$$(1) u(3, \theta) = \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$(2) u'_\theta(r, \theta) = u'_\theta(r, \frac{\pi}{2}) = 0$$



$$u(r, \theta) = R(r) \Theta(\theta)$$

$$\Theta'' + \lambda \Theta = 0$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\Theta'(0) = \Theta'(\frac{\pi}{2}) = 0$$

$$\Theta_n(\theta) = \cos(2n\theta), \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad n=0, 1, 2, \dots$$

$$\lambda_n = 4n^2$$

$$R_n(r) = r^{2n}, \quad n=1, 2, \dots \quad \left. \begin{array}{l} \text{Regularity} \\ \text{at } r=0. \end{array} \right\}$$

$$R_0(r) = c_0$$

$$u(r, \theta) = c_0 + \sum_{n=1}^{\infty} c_n r^{2n} \cos(2n\theta), \quad r \leq 3, \quad 0 < \theta < \frac{\pi}{2}$$

$$u(3, \theta) = c_0 + \sum c_n 3^{2n} \cos(2n\theta) = \theta$$

$$\theta = \frac{\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(2n\theta)$$

Compare coefficients:

$$c_0 = \frac{\pi}{4}, \quad c_n = \frac{1}{\pi} \frac{1 - (-1)^n}{3^{4n} n^2}$$

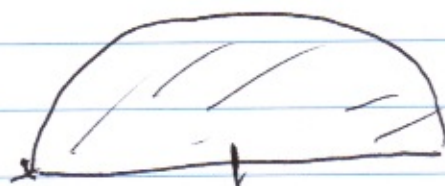
$$u(r, \theta) = \frac{\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \left(\frac{r}{3}\right)^{4n} \cos(4n\theta),$$

$r \leq 3, -\pi < \theta < \pi$

Ex. 5

$$\begin{cases} \Delta u = 0 \\ u(4, \theta) = 2 \\ u(r, 0) = u(r, \pi) = 0 \end{cases}$$

$$0 < r < 4, \quad 0 < \theta < \pi$$



$$u'' + \lambda u = 0$$

$$u(0) = u(\pi) = 0 \quad 0 < \theta < \pi$$

$$u_n(\theta) = \sin(n\theta), \quad \lambda_n = n^2, \quad n=1, 2, \dots$$

$$R_n(r) = r^n, \quad (\text{regularity at } r=0).$$

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n r^n \sin(n\theta)$$

$$u(2, \theta) = \sum_{n=1}^{\infty} c_n 2^n \sin(n\theta) = 2$$

$$2 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin n\theta$$

$$c_n = \frac{4}{\pi} \frac{1 - (-1)^n}{2^n n}$$

$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{1 - (-1)^n}{n} \left(\frac{r}{2}\right)^n \sin(n\theta)$$

$0 < r < 2,$