D' Alembert formula for infinite string.
(Problem 18c at Sect. 13.4).

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0.$$

$u(x,0) = f(x)$

$u'_t(x,0) = g(x)$.

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(x)\,dx$$

$g \equiv 0 \iff$ travelling waves.

An initial wave moves with the speed $\underset{\sim}{a}$ at opposite directions.

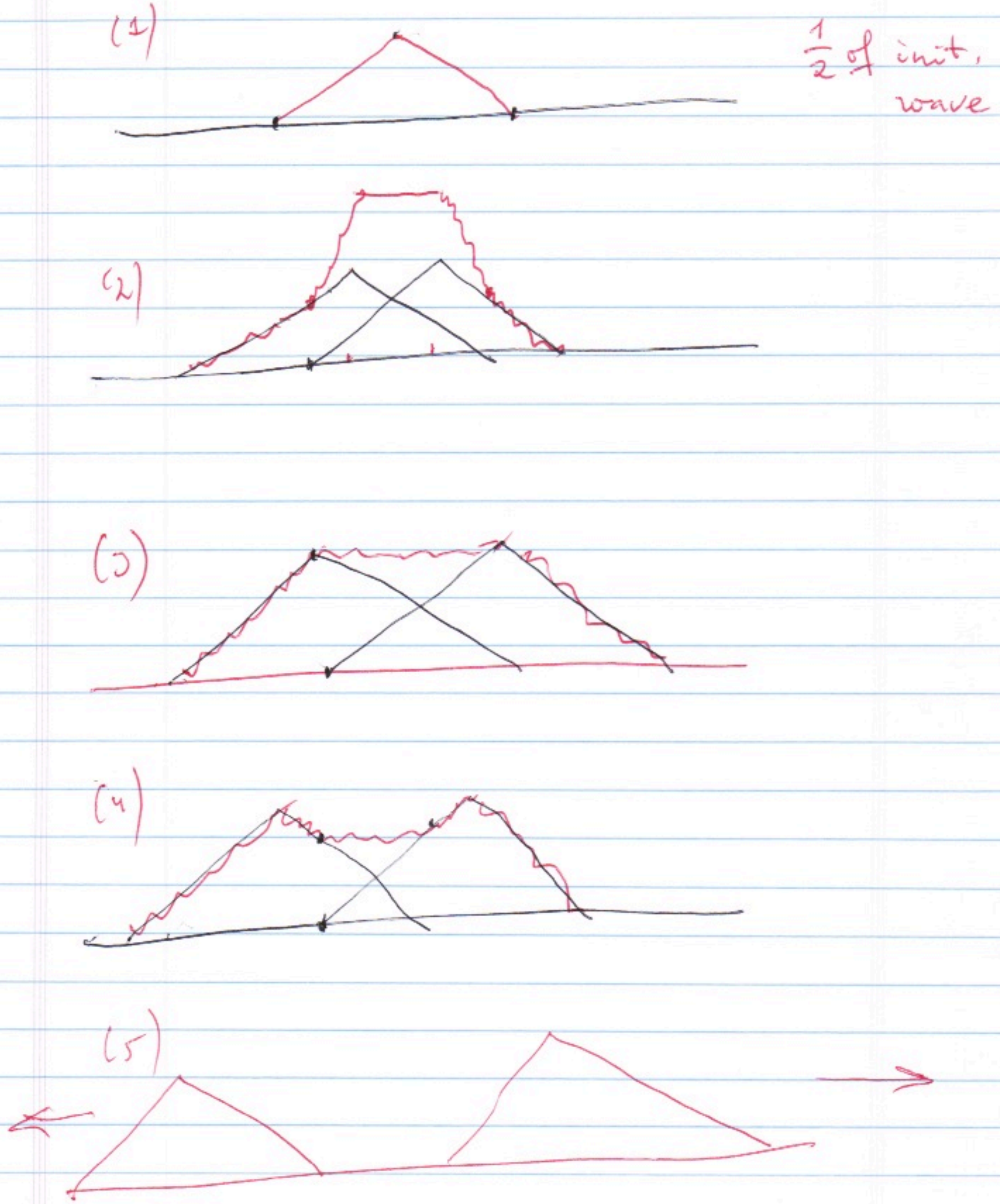
Ex. $f = \cancel{\sin}\, \sin x$, $g = 0$ $\quad u(x,t) = \frac{1}{2}[\sin(x+at) + \sin(x-at)]$
$= \sin x \cos at$

$f = 0$, $g = \sin x$ $\quad u = \frac{1}{2a}\int_{x-at}^{x+at} \sin x\, dx =$

$= \frac{1}{2a}\left(\cos(x-at) - \cos(x+at)\right) = \frac{1}{a} \sin x \sin at$

Lecture 23.

Plucked infinite string.



Laplace's Equation (Sect. 13.5)

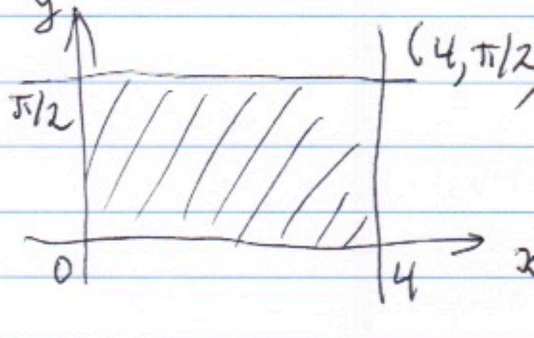
The steady-state temperature of a plate.

(1) $\Delta u = u_{xx} + u_{yy} = 0$;

(2) $u'_x(0,y) = u'_x(4,y) = 0$;

(3) $u(x,0) = 3$;

(4) $u(x, \frac{\pi}{2}) = x$.



Step 1. Separation of variables at (1).

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \qquad u = XY$$

$$(5)\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

This system is equivalent to (2) for separated variables.

$$u_\lambda(x,y) = X_\lambda(x)\, Y_\lambda(y).$$

Step 2. Boundary conditions (2).

$$X'' + \lambda X = 0$$
$$X'(0) = X'(4) = 0 \qquad [0,4]$$
$$X_n = \cos\left(\frac{n\pi}{4}x\right), \quad n = 0, 1, 2, \ldots$$
$$\lambda_n = \frac{n^2\pi^2}{16}$$

$Y_n(y) = a_n \cosh\left(\frac{n\pi}{4} y\right) + b_n \sinh\left(\frac{n\pi}{4} y\right), \; n = 1, 2, \ldots$

(eigen value $\left(-\frac{n^2\pi^2}{16}\right)$),

$Y_0(y) = a_0 + b_0 y.$ $\quad u_n = X_n Y_n, \; n = 0, 1, \ldots$

$$u(x,y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4}x\right)\left(a_n \cosh\left(\frac{n\pi}{4}y\right) + b_n \sinh\left(\frac{n\pi}{4}y\right)\right).$$

satisfies (1), (2). $\quad 0 \le x \le 4.$

Step 3. ~~Initial~~ Inhomogeneous conditions (3), (4).

$$(3) \; u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{4}x\right) \equiv 3;$$

So $a_0 = 3$, $a_n = 0$, $n > 0$,

$$(4) \; u\left(x, \frac{\pi}{2}\right) = 3 + b_0 \frac{\pi}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4}x\right) \cdot b_n \sinh\left(\frac{n\pi^2}{8}\right)$$

$$\equiv x, \; 0 < x < 4$$

Cosine Fourier Series:

$$x = 2 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos\left(\frac{n\pi}{4}x\right),$$

$$0 < x < 4$$

$3 + b_0 \frac{\pi}{2} = 2$; $b_0 \frac{\pi}{2} = -1$, $b_0 = -\frac{2}{\pi}$

$$b_n \sinh\left(\frac{n\pi^2}{8}\right) = -\frac{8}{\pi^2}\,\frac{1-(-1)^n}{n^2}$$

$$b_n = \frac{8}{\pi^2}\,\frac{\left((-1)^n - 1\right)}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)}.$$

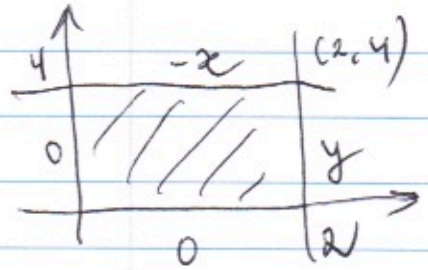
$$u(x,y) = 3 - \frac{2}{\pi} y + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)} \times$$

$$\times \cos\left(\frac{n\pi}{4} x\right) \sinh\left(\frac{n\pi}{4} y\right).$$

$0 < x < 4$, $0 < y < \frac{\pi}{2}$

Superposition principle.

23.5



$\Delta u = u_{xx} + u_{yy} = 0,\ 0<x<2,\ 0<y<4$

$u(0,y)=0,\ u(2,y)=y$

$u(x,0)=0,\ u(x,4)=-x,$

Divide the problem on 2 auxiliary problems

Problem 1.

$$\begin{cases} \Delta v = 0 \\ v(0,y)=v(2,y)=0 \\ v(x,0)=0,\ v(x,4)=-x. \end{cases} \qquad 0<x<2,\ 0<y<4.$$

Step 1.

$$v(x,y) = X(x)Y(y)$$

$X'' + \lambda X = 0$

$Y'' - \lambda Y = 0$

$\cancel{X'(0)=X'(2)=0}$

$X(0)=X(2)=0$

$X_n = \sin\left(\frac{n\pi}{2}x\right),\ \lambda_n = \frac{n^2\pi^2}{4},\ n=1,2,\ldots$

$$Y_n(x,y) = a_n \cosh\left(\frac{n\pi}{2}y\right) + b_n \sinh\left(\frac{n\pi}{2}y\right),$$

$$v_n(x,y) = X_n Y_n =$$

$$= \sin\left(\frac{n\pi}{2}x\right)\left(a_n \cosh\left(\frac{n\pi}{2}y\right) + b_n \sinh\left(\frac{n\pi}{2}y\right)\right)$$

$$v(x,y) = \sum_{n=1}^{\infty} v_n(x,y).$$

$$v(x,0) = 0 \Rightarrow a_n = 0 \text{ for all } n.$$

$$v(x,4) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right) \sinh(2n\pi) = -x, \quad 0<x<$$

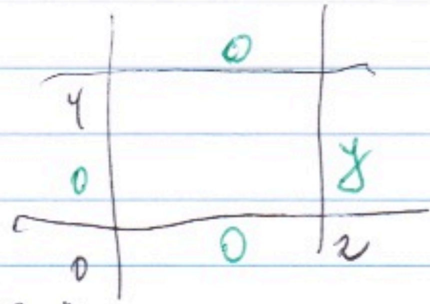
$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}x\right)$$

$$b_n \sinh(2n\pi) = \frac{4}{\pi}\frac{(-1)^n}{n}$$

$$b_n = \frac{4}{\pi}\frac{(-1)^n}{\sinh(2n\pi)}$$

$$v(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{\sinh(2n\pi)} \sin\left(\frac{n\pi}{2}x\right) \sinh\left(\frac{n\pi}{2}y\right)$$

Problem 2.



$\Delta w = 0$

$w(x,0) = w(x,4) = 0$

$w(0,y) = 0, \; w(2,y) = y$

Step 1. $w(x,y) = X(x)\,Y(y)$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

$$Y'' + \lambda Y = 0 \qquad Y(0) = Y(4) = 0$$

$$X'' - \lambda X = 0$$

$$Y_n(y) = \sin\left(\frac{n\pi}{4}y\right), \quad n = 1, 2, \dots$$

$$X_n = a_n \cosh\left(\frac{n\pi}{4}x\right) + b_n \sinh\left(\frac{n\pi}{4}x\right)$$

$$w_n(x,y) = X_n Y_n$$

$$w(x,y) = \sum_{n=1}^{\infty} \underline{\sin\left(\frac{n\pi}{4}y\right)}\left(a_n \cosh\left(\frac{n\pi}{4}x\right) + b_n \sinh\left(\frac{n\pi}{4}x\right)\right)$$

$w(0,y) \equiv 0 \Rightarrow a_n = 0$ for all n.

$$w(2,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{4}y\right), \quad 0 < y < 4$$

$$\equiv y.$$

$$y = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{4} y\right)$$

$$b_n \sinh\left(\frac{n\pi}{2}\right) = \frac{8}{\pi} \frac{(-1)^{n+1}}{n}$$

$$b_n = \frac{8}{\pi} \frac{(-1)^{n+1}}{n \sin\left(\frac{n\pi}{2}\right)}$$

$$w(x,y) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sin\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi}{4} x\right) \sin\left(\frac{n\pi}{4} y\right),$$

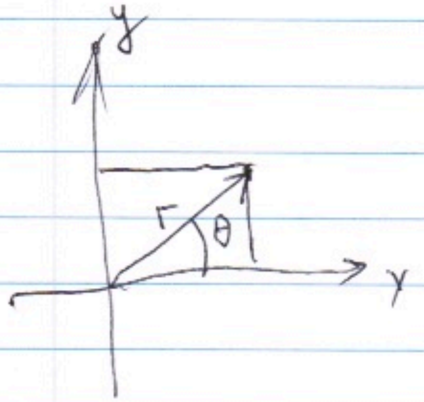
$$0 < x < 2, \quad 0 < y < 4.$$

Superposition:

$$u(x,y) = v(x,y) + w(x,y)$$

is the unique solution of the initial problem

Polar coordinates. (Sect. 14.1).

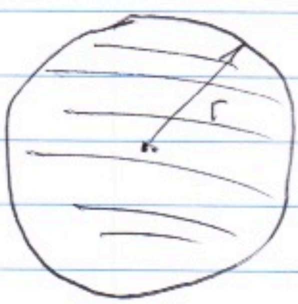


$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

Laplacian in Polar coordinates.

$$(1)\quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Aim: Problems in curved domain.



$\Delta u = 0.$

Separation of variables in polar coordinates.

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$(2)\quad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$$

(3) $\Theta'' + \lambda \Theta = 0, \quad 0 < \theta < 2\pi$

$\Theta(\theta) = \Theta(\theta + 2\pi), \; \Theta'(\theta) = \Theta'(\theta + 2\pi).$

Periodic ~~cod~~ condition - Singular Sturm - Liouville Problem.

$\lambda \geq 0 \quad \Theta_0 = c$

$\Theta_n(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta), \; \lambda = n^2$
$n = 0, 1, 2,$

(4) $R(r)$

$r^2 R'' + r R' - \lambda R = 0. \qquad \lambda \geq 0$

Cauchy - Euler equation

$\lambda = \alpha^2 \qquad R_\alpha(r) = c_1 r^\alpha + c_2 r^{-\alpha}, \; \alpha > 0,$

$R_0(r) = c_1 + c_2 \ln r. \quad , \alpha = 0$

λ is joint at (2): $\lambda = n^2, \; n = 0, 1, 2, - -$

$(\lambda = n^2 \text{ at } (3)),$

$u_n(r, \theta) = R_n(r) \Theta_n(\theta)$

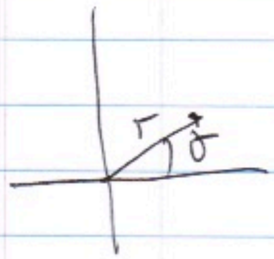
$= (c_n r^n + d_n r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta)$

$n = 1, 2$

$u_0(r, \theta) = c_0 + d_0 \ln r.$

Lecture 25

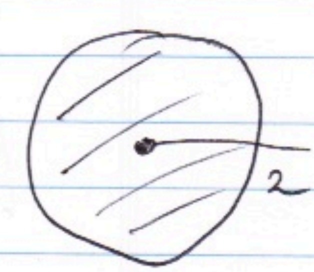
Polar coordinates (Sect. 14.1.)



$u(r,\theta)$

(1) $\Delta u = u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} = 0$

$r \leq 2$



(2) $u(2,\theta) = \theta, \quad -\pi < \theta < \pi$

Only one boundary condition

Step 1. Separation of variables

$u(r,\theta) = R(r)\,\Theta(\theta)$

(3) $\begin{cases} \Theta'' + \lambda \Theta = 0 \\ r^2 R'' + r R' - \lambda R = 0 \end{cases}$

$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, \dots$

$\lambda = n^2$

$R_n(r) = c_n r^n + d_n r^{-n}, n = 1, \dots; \; R_0(r) = c_0 + d_0 \ln r$

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta).$$

We didn't use yet the boundary condition!

Step 2.

The solution $u(r,\theta)$ is smooth at $r=0$

$$\Downarrow$$

$$d_0 = 0,\ d_n = 0,\ n \geq 1.$$

$$(4)\quad u(r,\theta) = c_0 + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin n\theta).$$

$$= \theta,\ -\pi < \theta < \pi$$

$$u(2,\theta) = \theta = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta),\ \cancel{\theta\epsilon}\ -\pi<\theta<\pi$$

$$c_0 = 0,\ a_n = 0,\ n \geq 1$$

&

$$b_n = \frac{(-1)^{n+1}}{n\,2^{n-1}}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^{n-1}} r^n \sin(n\theta).$$

is the solution of (1), (2). $\quad -\pi<\theta<\pi,\ r \leq 2$

The periodicity and the regularity at the origin replace some boundary conditions!