Lecture 22.

1. Exercise (BVP for Heat equation).

equat. (1) $9\frac{\partial^2}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < \frac{\pi}{2}$, $t > 0$

bound. cond. (2) $u'_x(0,t) = u'_x(\frac{\pi}{2}, t) = 0$, $t > 0$

initial cond. (3) $u(x,0) = 2 - x$, $0 < x < \frac{\pi}{2}$.

a rod with isolated ends.

Step 1. Separation of variables.

$$u(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{T'}{9T} = -\lambda \quad (\text{constant}).$$

$$X'' + \lambda X = 0 \qquad 0 < x < \tfrac{\pi}{2}$$

(4) $$T' + 9\lambda T = 0 \qquad t > 0$$

$u_\lambda(x,t) = T_\lambda(t)\,X_\lambda(x)$ is a solution of (

$$T_\lambda = \exp(-9\lambda t)$$

Step 2. Boundary conditions (2).

$$(2) \Leftrightarrow X_\lambda'(0) = X_\lambda'\left(\frac{\pi}{2}\right) = 0$$

$$X_n(x) = c_n \cos(2nx), \ n = 0, 1, 2, \ldots \qquad \lambda_n = 4n^2.$$

$$u_n(x,t) = c_n \exp(-36 n^2 t) \cos(2nx),$$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \exp(-36 n^2 t) \cos(2nx).$$

Step 3. The initial condition (3),

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos(2nx) \equiv 2 - x, \ 0 < x < \frac{\pi}{2}.$$

$$2 - x = \left(2 - \frac{\pi}{4}\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(2nx), \ 0 < x < \frac{\pi}{2}$$

$$c_0 = 2 - \frac{\pi}{4}, \quad c_n = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2}, \ n > 0.$$

$$c_n = 0, \text{ even } n.$$

$$u(x,t) = 2 - \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{2} \exp(-36 n^2 t) \cos(2nx),$$

$$t > 0, \ 0 < x < \frac{\pi}{2}.$$

Wave equation Sect. 13.4

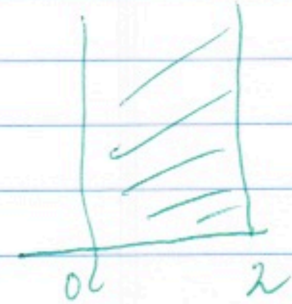
(String's eq.)

(1) $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $t>0$, $0<x<2$.

(2) $u(0,t) = u(2,t) = 0, t>0$

(3) $u(x,0) = 1$

(4) $\dfrac{\partial u}{\partial t}(x,t)\Big|_{t=0} = x$



Step 1. Separation o variables

$u(x,t) = X(x)\,T(t)$.

$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda$$

$$(5)\begin{cases} X'' + \lambda X = 0 & 0<x<2,\ t>0 \\ T'' + 4\lambda T = 0 \end{cases}$$

$$u_\lambda(x,t) = X_\lambda(x)\,T_\lambda(t)$$

Important: λ is the same at 2 equations

Step 2. Boundary conditions (2).

$$(2) \Leftrightarrow X_\lambda(0) = X_\lambda(2) = 0$$

(No conditions on $T_\lambda(t)$!).

Eigen functions.

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad n = 1, 2, \ldots$$

$$\lambda_n = \frac{n^2\pi^2}{4}.$$

Conclusion:

$$a_n \cos(n\pi t) + b_n \sin(n\ldots$$

$$u_n(x,t) = \sin\left(\frac{n\pi x}{2}\right)\left(\cancel{a_n \cos \frac{n\pi t}{\ldots} + b_n \sin \frac{n\pi t}{\ldots}}\right)$$

are solutions of (1), (2)

as well as

$$(6) \quad u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left(a_n \cos\left(\frac{n\pi t}{\cancel{2\ell}}\right) + b_n \sin\left(\frac{n\pi t}{\cancel{2\ell}}\right)\right)$$

We have the freedom in the choice of $\{a_n\}$, $\{b_n\}$.

Step 3. Initial conditions (3),(4).

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right), \quad 0<x<2.$$

It must be

$$u(x,0) \equiv 1, \quad 0<x<2$$

Let us extend 1 at a Sine Fourier series on $[0,2]$:

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right), \quad 0<x<2$$

So a_n at (6) are

$$a_n = \frac{2}{\pi}\,\frac{1-(-1)^n}{n}$$

$$u'_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \frac{n\pi}{2} \cancel{\cos} \sin\left(\frac{n\pi x}{2}\right).$$

It must be

$$u'_t(x,0) \equiv x, \quad 0<x<2.$$

Let us consider the Sine Fourier series for $f(x)=x$ on $[0,2]$:

$$x = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right), \quad 0<x<2.$$

So at (6)

$$b_n \frac{n\pi}{\cancel{2}} = \frac{4}{\pi}\frac{(-1)^{n+1}}{n};$$

$$b_n = \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

Finaly:

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left(\frac{1-(-1)^n}{n}\cos\left(\frac{n\pi t}{\cancel{2}}\right) + \right.$$

$$\left. + \frac{2(-1)^{n+1}}{\pi n^2}\sin\left(\frac{n\pi t}{\cancel{2}}\right)\right)$$

is the unique solution of (1)-(4).

D' Alembert formula for infinite string.
(Problem 18c at Sect. 13.4).

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0.$$

$$u(x,0) = f(x)$$

$$u'_t(x,0) = g(x).$$

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(x)\,dx$$

$g \equiv 0 \iff$ travelling waves.

An initial wave moves with the speed $\underset{\sim}{a}$ at opposite directions.

Ex. $f = \sin x$, $g = 0$ $\quad u(x,t) = \frac{1}{2}[\sin(x+at) + \sin(x-a$
$= \sin x \cos a$

$f = 0$, $g = \sin x$ $\quad u = \frac{1}{2a}\int_{x-at}^{x+at} \sin x\,dx =$

$= \frac{1}{2a}\left(\cos(x-at) - \cos(x+at)\right) = \frac{1}{a}\sin x \sin at$